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Efficient Local Search in Coordination Games on Graphs

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Abstract

We study strategic games on weighted directed graphs, where the payoff of a player is defined as the sum of the weights on the edges from players who chose the same strategy augmented by a fixed non-negative bonus for picking a given strategy. These games capture the idea of coordination in the absence of globally common strategies. Prior work shows that the problem of determining the existence of a pure Nash equilibrium for these games is NP-complete already for graphs with all weights equal to one and no bonuses. However, for several classes of graphs (e.g. DAGs and cliques) pure Nash equilibria or even strong equilibria always exist and can be found by simply following a particular improvement or coalition-improvement path, respectively. In this paper we identify several natural classes of graphs for which a finite improvement or coalition-improvement path of polynomial length always exists, and, as a consequence, a Nash equilibrium or strong equilibrium in them can be found in polynomial time. We also argue that these results are optimal in the sense that in natural generalisations of these classes of graphs, a pure Nash equilibrium may not even exist.

1 Introduction

Nash equilibrium is an important solution concept in game theory which has been widely used to reason about strategic interaction between rational agents. Although Nash's theorem guarantees existence of a mixed strategy Nash equilibrium for all finite games, pure strategy Nash equilibria need not always exist. In many scenarios of strategic interaction, apart from the question of existence of pure Nash equilibria, an important concern is whether it is possible to compute an equilibrium outcome and whether a game always converges to one. The concept of *improvement paths* is therefore fundamental in the study of strategic games. Improvement paths are essentially maximal paths constructed by starting at an arbitrary joint strategy and allowing players to improve their choice in a unilateral manner. At each stage, a single player who did not select a best response is allowed to update his choice to a better strategy. By definition, every finite improvement path terminates in a Nash equilibrium. In a seminal paper, Monderer and Shapley [18] studied the class of games in which every improvement path is guaranteed to be finite, which was coined as the *finite improvement property* (FIP). They showed that games with the FIP are precisely those games to which we can associate a generalised ordinal potential function. Thus FIP not only guarantees the existence of pure Nash equilibria but also ensures that it is possible to converge to an equilibrium outcome by performing *local search*. This makes FIP a desirable property to have in any game. An important class of games that have the FIP is *congestion games* [21]. However, the requirement that *every* improvement path is finite, turns out to be a rather strong condition and there are very restricted classes of games that have this property.

Young [23] proposed weakening the finite improvement property to ensure the *existence* of a finite improvement path starting from any initial joint strategy. Games for which this property hold are called *weakly acyclic games*. Thus weakly acyclic games capture the possibility of reaching pure Nash equilibria through unilateral deviations of players irrespective of the starting state. Milchtaich [16] showed that although congestion games with player specific payoff functions do not have the FIP, they are weakly acyclic. Weak acyclicity of a game also ensures that certain modifications of the traditional no-regret algorithm yields almost sure convergence to a pure Nash equilibrium [15].

Although finite improvement path guarantees the existence of a Nash equilibrium, it does not necessarily provide an efficient algorithm to compute an equilibrium outcome. In many situations, improvement paths could be exponentially long. In fact, Fabrikant *et al.* [10] showed that computing a pure Nash equilibrium in congestion games is PLS-complete. Even for symmetric network congestion games, where it is known that a pure Nash equilibrium can be efficiently computed [10], there are classes of instances where any best response improvement path is exponentially long [1]. Thus identifying classes of games that have finite improvement paths in which it is possible to converge to a Nash equilibrium in a polynomial number of steps is of obvious interest.

In game theory, coordination games are often used to model situations in which players attain maximum payoff when agreeing on a common strategy. In this paper, we study a simple class of coordination games in which players try to coordinate within a certain neighbourhood. The neighbourhood structure is specified by a finite directed graph whose nodes correspond to the players. Each player chooses a colour from a set of available colours. The payoff of a player is defined as the sum of the weights on the edges from players who choose the same colour and a fixed bonus for picking that particular colour. These games constitute a natural class of strategic games, which capture the following three key characteristics. *Join the crowd property:* the payoff of each player weakly increases when more players choose her strategy. *Asymmetric strategy sets:* players may have different strategy sets. *Local dependency:* the payoff of each player depends only on the choices made by the players in its neighbourhood.

A similar model of coordination games on graphs was introduced in [2] where the authors considered undirected graphs. However, the transition from undirected to directed graphs drastically changes the status of the games. For instance, in the case of undirected graphs, coordination games have the FIP. While in the directed case, Nash equilibria may not exist. Moreover, the problem of determining the existence of Nash equilibria is NP-complete for coordination games on directed graphs [4]. However, if the underlying graph is a directed acyclic graph (DAG), a complete graph or a simple cycle, then pure Nash equilibria always exist. These proofs can easily be adapted to show that weighted DAGs and weighted simple cycles have finite improvement paths.

Related work. Although the class of potential games are well studied and has been a topic of extensive research, weakly acyclic games have received less attention. Engelberg and Schapira [7] showed that certain Internet routing games are weakly acyclic. In a recent paper Kawald and Lenzner [14] show that certain classes of network creation games are weakly acyclic and moreover that a specific scheduling of players can ensure that the resulting improvement path converges to a Nash equilibrium in $\mathcal{O}(n \log n)$ steps. Brokkelkamp and Vries [6] improved Milchtaich's result [16] on congestion games with player specific payoff functions by showing that a specific scheduling of players is sufficient to construct an improvement path that converges to a Nash equilibrium.

Unlike in the case of exact potential games, there is no neat structural characterisation of weakly acyclic games. Some attempts in this direction has been made in the past. Fabrikant *et al.* [8] proved that the existence of a unique (pure) Nash equilibrium in every sub-game implies that the game is weakly acyclic. A comprehensive classification of weakly acyclic games in terms of schedulers is done in [3]. Finally, Milchtaich [17] showed that every finite extensive-form game with perfect information is weakly acyclic.

The model of coordination games are related to various well-studied classes of games. Coordination games on graphs are *polymatrix games* [13]. In these games, the payoff for each player is the sum of the payoffs from the individual two player games he plays with every other player separately. Hoefer [12] studied clustering games that are also polymatrix games based on undirected graphs. However, in this setup each player has the same set of strategies and it can be shown

Graph Class	improvement path	c-improvement path
weighted DAGs	$\mathcal{O}(n)$ [4]	$\mathcal{O}(n)$ [4]
weighted simple cycles with 2 bonuses	$\mathcal{O}(n)$ [Thm. 5]	$\mathcal{O}(n)$ [Thm. 7]
open chains of cycles	$\mathcal{O}(nm^2)$ [Thm. 9]	$\mathcal{O}(nm^3)$ [Cor. 14]
closed chains of cycles	$\mathcal{O}(nm^2)$ [Thm. 11]	$\mathcal{O}(nm^3)$ [Thm. 13]
weighted open chains of cycles	$\mathcal{O}(nm^3)$ [Thm. 10]	??
weighted closed chains of cycles	Nash equilibrium may not exist [Example 12]	
partition-cycles	$\mathcal{O}(n(n-k))$ [Thm. 18]	??
partition-cycles+bonuses	$\mathcal{O}(kn(n-k))$ [Thm. 20]	??
weighted partition-cycles	Nash equilibrium may not exist [Example 19]	

Table 1: An upper bound on the length of the shortest improvement and c-improvement path for a given class of graphs. All edges are unweighted and there are no bonuses unless the name of the class says otherwise. For simple cycles and chains of cycles we assume that each cycle has n nodes and the number of cycles in the chain is m. For partition-cycles, n is the total number of nodes and $1 \leq k < n$ is the number of nodes in the top part of the cycle (set V_T).

that these games have the FIP. A model that does not assume all strategies to be the same, but is still based on undirected graphs, was shown to have the FIP in [20]. When the graph is undirected and complete, coordination games on graphs are special cases of the monotone increasing congestion games that were studied in [22].

Our contributions. In this paper, we identify some natural classes of polymatrix games based on the coordination game model, which even though do not have the FIP (cf. Example 4 in [4]), are weakly acyclic. We also show that for these games a finite improvement path of polynomial length can be constructed in a uniform manner. Thus not only do these games have pure Nash equilibria, but they can also be efficiently computed by local search.

We start by analysing coordination games on simple cycles. Even in this simple setting, improvement paths of infinite length may exist. However, we show that there always exists a finite improvement path of polynomial length. We then extend the setting of simple cycles in two directions. First we consider chains of simple cycles where we show that polynomial length improvement paths exist. We then consider simple cycles with cross-edges and show the existence of polynomial length improvement paths. We also demonstrate that these results are optimal in the sense that most natural generalisations of these structures may result in games in which a Nash equilibrium may not even exists. Most of our constructions involve a common proof technique: we identify a specific scheduling of players using which, starting at an arbitrary initial joint strategy, we can reach a joint strategy in which at most two players are not playing their best response. We argue that such a joint strategy can then be updated to converge to a Nash equilibrium. We also identify a structural condition on coalitional deviation once a Nash equilibrium is attained. This property is then used to show the existence of a finite "coalitional" improvement path which terminates in a strong equilibrium. Our results also imply an almost sure convergence, although not necessarily in a polynomial number of steps, to a Nash equilibrium when the order of deviations is random, but "fair". Fairness requires that for any deviation, there is a fixed nonzero lower bound on the probability of it taking place from any state of the game where it can be taken. Note that this implies that the same holds for any finite sequence of deviations. A Nash equilibrium is reached almost surely with such a random order of deviations, because when starting at any state we either follow a finite improvement path to a Nash equilibrium with a nonzero probability or that path stops in some new state from where we can follow another finite improvement path with a nonzero probability. As this repeats over and over again, almost surely one such finite improvement path will succeed.

Table 1 summarises most of our results.

Potential applications. Coordination games constitute an abstract game model which is well studied in game theory and has been shown to model many practical scenarios. The game model that we consider in this paper is an extension of coordination games to the network setting (in which the neighbourhood relation is specified using a directed graph) where common strategies are not guaranteed to exist and payoffs are not necessarily symmetric.

The graph classes that we consider are typical for network topologies, e.g. token ring local area networks are organised in directed simple cycles, open chains topology is supported by recommendation G.8032v2 on Ethernet ring protection switching, and closed chains are used in multi-ring protocols. The basic technique that we use to show convergence to Nash equilibria is based on finite improvement paths of polynomial length. The concept of an improvement path is fundamental in the study of games and it has been used to explain and analyse various real world applications. One such example is the border gateway protocol (BGP) which establishes routes between competing networks on the Internet. Over the years, there has been extensive research, especially in network communications literature, on how stable routing states are achieved and maintained in BGP in spite of strategic concerns. Fabrikant and Papadimitriou [9] and independently, Levin and others [11] observed that BGP can be viewed as best-response dynamics in a class of routing games and finite improvement paths that terminates in a pure Nash equilibria essentially translates to stable routing states. Following this observation, Engelberg and Schapira [7] presents a game theoretic analysis of routing on the Internet where they show weak acyclicity of various routing games.

The coalition formation property inherent to coordination games on graphs also make the game model relevant to cluster analysis. In cluster analysis, the task is to organise a set of objects into groups according to some similarity measure. Here, the strategies can be viewed as possible cluster names and a pure NE naturally corresponds to a 'satisfactory' clustering of the underlying graph. Clustering from a game theoretic perspective was for instance applied to car and pedestrian detection in images, and face recognition in [19]. This approach was shown to perform very well against the state of the art.

Structure of the paper. In Section 3 we introduce the game model and make an important observation on the structure of coalition deviation from a Nash equilibrium in coordination games on directed graphs. In Section 4 we analyse games whose underlying graphs are simple cycles. In Section 5 we study chains of cycles and in Section 6 we consider simple cycles with cross edges.

2 Preliminaries

A ***strategic game*** $\mathcal{G} = (S_1, \ldots, S_n, p_1, \ldots, p_n)$ with $n > 1$ players, consists of a non-empty set S_i of ***strategies*** and a ***payoff function*** $p_i : S_1 \times \cdots \times S_n \to \mathbb{R}$, for each player i. We denote $S_1 \times \cdots \times S_n$ by S, call each element $s \in S$ a ***joint strategy*** and abbreviate the sequence $(s_j)_{j \neq i}$ to s_{-i}. Occasionally we write (s_i, s_{-i}) instead of s. We call a strategy s_i of player i a ***best response*** to a joint strategy s_{-i} of his opponents if for all $s'_i \in S_i$, $p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i})$.

We call a non-empty subset $K := \{k_1, \ldots, k_m\}$ of the set of players $N := \{1, \ldots, n\}$ a ***coalition***. Given a joint strategy s we abbreviate the sequence $(s_{k_1}, \ldots, s_{k_m})$ of strategies to s_K and $S_{k_1} \times \cdots \times S_{k_m}$ to S_K. We occasionally write (s_K, s_{-K}) instead of s. If there is a strategy x such that $s_i = x$ for all players $i \in K$, we also write (x_K, s_{-K}) instead of s.

Given two joint strategies s' and s and a coalition K, we say that s' is a ***deviation of the players in*** K from s if $K = \{i \in N \mid s_i \neq s'_i\}$. We denote this by $s \xrightarrow{K} s'$. If in addition $p_i(s') > p_i(s)$ holds for all $i \in K$, we say that the deviation s' from s is ***profitable***. Further, we say that a coalition K ***can profitably deviate from*** s if there exists a profitable deviation of the players in K from s. Next, we call a joint strategy s a ***k-equilibrium***, where $k \in \{1, \ldots, n\}$, if no coalition of at most k players can profitably deviate from s. Using this definition, a ***Nash equilibrium*** is a 1-equilibrium and a ***strong equilibrium***, see [5], is an n-equilibrium.

A ***coalitional improvement path***, in short a ***c-improvement path***, is a maximal sequence $\rho = (s^1, s^2, \ldots)$ of joint strategies such that for every $k > 1$ there is a coalition K such that s^k is a profitable deviation of the players in K from s^{k-1}. If ρ is finite then by $last(\rho)$ we denote the last element of the sequence. Clearly, if a c-improvement path is finite, its last element is a strong equilibrium. We say that $\mathcal{G}$ is ***c-weakly acyclic*** if for every joint strategy there exists a finite c-improvement path that starts at it. Note that games that are c-weakly acyclic have a strong equilibrium. We call a c-improvement path an ***improvement path*** if each deviating coalition consists of one player. The notion of a game being ***weakly acyclic*** [23; 16], is then defined by referring to improvement paths instead of c-improvement paths.

3 Coordination games on directed graphs

We now define the class of games we are interested in. Fix a finite set of colours M. A weighted directed graph (G, w) is a structure where $G = (V, E)$ is a graph without self loops over the vertices $V = \{1, \ldots, n\}$ and w is a function that associates with each edge $e \in E$, a non-negative weight w_e. We say that a node j is a ***neighbour*** of the node i if there is an edge $j \to i$ in G. Let N_i denote the set of all neighbours of node i in the graph G. A ***colour assignment*** is a function $C : V \to 2^M$ which assigns to each node of G a finite non-empty set of colours. We also introduce the concept of a ***bonus***, which is a function β that to each node i and a colour $c \in M$ assigns a natural number $\beta(i, c)$. Note that bonuses can be modelled by incoming edges from fixed colour source nodes, i.e. nodes with no incoming edges and only one colour available to them. When stating our results, bonuses are assumed to be not present, unless we explicitly state that they are allowed. Bonuses are extensively used in our proofs because a coordination game restricted to a given subgraph can be viewed as a coordination game with bonuses induced by the remaining nodes of the graph.

Given a weighted graph (G, w), a colour assignment C and a bonus function β a strategic game $\mathcal{G}(G, w, C, \beta)$ is defined as follows: the players are the nodes,

- the set of strategies of player (node) i is the set of colours $C(i)$; we occasionally refer to the strategies as ***colours***.
- the payoff function $p_i(s) = \sum_{j \in N_i,\, s_i = s_j} w_{j \to i} + \beta(i, s_i)$.

So each node simultaneously chooses a colour and the payoff to the node is the sum of the weights of the edges from its neighbours that chose its colour augmented by the bonus to the node from choosing the colour. We call these games ***coordination games on directed graphs***, from now on just ***coordination games***. When the weights of all the edges are 1, we obtain a coordination game whose underlying graph is unweighted. In this case, we simply drop the function w from the description of the game. Similarly if all the bonuses are 0 then we obtain a coordination game without bonuses. Likewise, to denote this game we omit the function β. In a coordination game without bonuses where the underlying graph is unweighted, each payoff function is defined by $p_i(s) := |\{j \in N_i \mid s_i = s_j\}|$.

Finally, given a directed graph G and a set of nodes K, we denote by $G[K]$ the subgraph of G induced by K.

We now show a structural property of a coalition deviation from a Nash equilibrium in our coordination games. This will be used later to prove c-weak acyclicity for a class of games based on their weak acyclicity. Note that this cannot be done for all classes of graphs, because there exist a coordination game on undirected graph which is weakly acyclic, but has no strong equilibrium [2].

Lemma 2. *Any profitable coalition deviation from a Nash equilibrium includes a unicoloured directed simple cycle.*

Proof. Let s be any Nash equilibrium in the game and let coalition K have a profitable deviation, s', from s. It suffices to show that each node in K has a predecessor in K deviating to the same colour. Assume that for some player $i \in K$ it is not the case. We then have the following: $p_i(s) < p_i(s'_K, s_{-K}) = \sum_{j \in K : s'_j = s'_i} w_{j \to i} + \sum_{j \notin K : s_j = s'_i} w_{j \to i} + \beta(i, s'_i) \leq 0 + \sum_{j : s_j = s'_i} w_{j \to i} + \beta(i, s'_i) = p_i(s'_i, s_{-i})$, so player i would also be able to improve his payoff by unilaterally switching to s'_i in s, which contradicts the fact that s is a Nash equilibrium. □

Example 1. *([4]) Consider the directed graph and the colour assignment depicted in Figure 1. Take the joint strategy s that consists of the underlined strategies. Then the payoffs are as follows:*

- *0 for the nodes 1, 7, 8 and 9,*
- *1 for the nodes 2, 4, 5, 6,*
- *2 for the node 3.*

Note that the above joint strategy is not a Nash equilibrium. For example, node 1 can profitably deviate to colour a. □

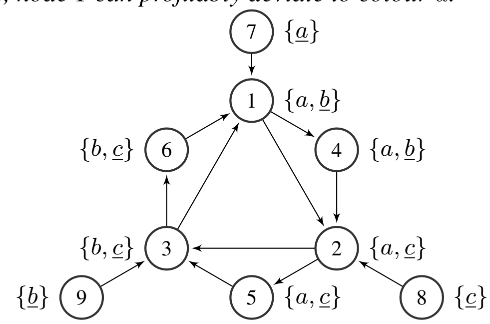

Figure 1: A directed graph with a colour assignment.

4 Simple cycles

In this section we focus on the case when the game graph is a directed simple cycle. Despite the simplicity of this model the problems we consider are already nontrivial for such a basic graph structure. We first restate a result from [4] where unweighted graphs are considered. To fix the notation, suppose that the considered graph is $1 \to 2 \to \ldots \to n \to 1$. Below for $i \in \{2, \ldots, n\}$, $i \ominus 1 = i - 1$, and $1 \ominus 1 = n$.

Theorem 3. *([4]) Every coordination game with bonuses on an unweighted simple cycle has a c-improvement path of length $\mathcal{O}(n)$.*

We would like to extend this result to weighted graphs with bonuses. However as the following example demonstrates, if in a simple cycle, we allow non-trivial weights on at least three edges and associate bonuses with at least three nodes then there are coordination games that need not even have a Nash equilibrium.

Example 4. *Consider the simple cycle on three nodes 1, 2 and 3 in which all the edges have weight 2. Let $C(1) = \{a, b\}$, $C(2) = \{a, c\}$ and $C(3) = \{b, c\}$. Let the bonus be defined as $\beta(1, a) = \beta(2, c) = \beta(3, b) = 1$ and equal to 0 otherwise. The structure essentially corresponds to the one shown in Figure 1. The resulting coordination game does not have a Nash equilibrium. Below we list all the joint strategies and we underline a strategy that is not a best response to the choice of other players: $(\underline{a}, a, b)$, $(a, a, \underline{c})$, $(a, c, \underline{b})$, $(a, \underline{c}, c)$, $(b, \underline{a}, b)$, $(\underline{b}, a, c)$, $(b, c, \underline{b})$ and $(\underline{b}, c, c)$.* □

We show here that this counterexample is essentially minimal, i.e. if only two nodes have bonuses or only two edges have weights then the coordination game is weakly acyclic.

Theorem 5. *Every coordination game on a weighted simple cycle in which at most two nodes have bonuses has an improvement path of length $\mathcal{O}(n)$.*

Proof. Assume without loss of generality that one of the nodes which has a bonus is node 1 (otherwise we can re-label the nodes on the cycle). Let the other vertex with a bonus be some $k \in N$. Let s be an arbitrary joint strategy. We perform the following sequence of best response updates.

We proceed around the cycle in the order $1, \ldots, n$ and let players switch to any of their best responses. We argue that in at most three rounds, the resulting improvement path terminates in a Nash equilibrium. At the end of the first round, players $2, \ldots, n$ are playing their best response. If the resulting joint strategy s^1 is a Nash equilibrium, then we stop. Otherwise player 1 strategy s_1^1 is not a best response to s_{-1}^1. Let player 1 update his strategy, denote the resulting joint strategy s^2. There are two cases:

- Suppose $s_1^2 = s_n^1$. We proceed around the cycle in the cyclic order up until the node $k-1$ and update the strategy of each player. Note that if at some point in between we reach a Nash equilibrium then we stop, otherwise the only colour that is propagated along the cycle until node $k-1$ is s_n^1. Let the resulting joint strategy be s^3. Now suppose s_k^3 is not a best response to s_{-k}^3. Let player k update his strategy and call the resulting joint strategy s^4. If $s_k^4 = s_{k-1}^3 (= s_n^1)$ then we continue around the cycle making players update to their best response. This improvement path is guaranteed to terminate since the only colour which is propagated is s_n^1. If $s_k^4 \neq s_{k-1}^3$, then $s_k^4 = c^k$ for some $c^k \in C(k)$. Continue in the cyclic order from $k+1, \ldots, n$ making players update to their best response. Let the resulting joint strategy be s^5. Note that in this sequence if a player switches then it is to the colour $s_k^4 = c^k$.

  Suppose s_1^5 is not a best response to s_{-1}^5, then let player 1 update and call the resulting joint strategy s^6. If $s_1^6 = s_n^6 (= c^k)$ then continue in the cyclic order from $2, \ldots, k-1$. The only colour which is propagated is c^k and this improvement path is finite since the colour chosen by k is c^k. If $s_1^6 \neq s_n^6$ then $s_1^6 = c^1$ for some $c^1 \in C(1)$.

  Now let players update to their best response in the cyclic order $2, \ldots, n$. Either the improvement path terminates before player k updates since his best response remains c^k or player k updates to c^1 and then the improvement path also terminates since the only colour which is propagated in the cycle is c^1.

- Suppose $s_1^2 \neq s_n^1$, then $s_1^2 = c^1$. Proceed around the cycle in the cyclic order and let players update to their best responses. If player k switches to c^1 then the only colour which is propagated is c^1 and the improvement path terminates in one round. Otherwise, player k eventually updates to c^k. As in the earlier case, let players $k+1, \ldots, n, 1, \ldots, k-1$ update to their best response in that order. The resulting improvement path is finite.

□

This proof can easily be adapted to show the same result for graphs with at most two weighted edges.

Theorem 6. *Every coordination game on a simple cycle with bonuses where at most two edges have non-trivial weights (i.e. weights greater than 1) has an improvement path of length $\mathcal{O}(n)$.*

The above results are optimal due to Example 4. We can also show that if a game played on a simple cycle is weakly acyclic, then it is c-weakly acyclic.

Theorem 7. *In a coordination game played on a weighted simple cycle with bonuses, any finite improvement path can be extended to a finite c-improvement path just by adding one profitable coalition deviation step at the end of it.*

Proof. Let us denote by s a Nash equilibrium that this game reaches via some finite improvement path. If s is a strong equilibrium then we are done. Otherwise there exists a coalition K with a profitable deviation, s', from s. Due to Lemma 2, the coalition K has to include all players, because there is only one cycle in the game graph, and all of them have to switch to the same colour in s'. We argue that $(s', s_{-K}) = s'$ is a Nash equilibrium. Suppose there is a player i, that can switch to colour x and improve his payoff. Then, $s'_{i \ominus 1} \neq x$, because all players play the same colour in s'. We have $p_i(s) < p_i(s') < p_i((x, s'_{-i})) = \beta(i, x) \leq p_i((x, s_{-i}))$; a contradiction with the assumption that s is a Nash equilibrium.

Finally, let $\rho = s, s^1, s^2, \ldots$ be any c-improvement path. Due to the above observations every s^i is a Nash equilibrium where all players play the same colour. Note that it cannot be $s^i = s^j$ for any $i \neq j$, because every s^{i+1} is a profitable deviation from s^i. Therefore any c-improvement path starting at a Nash equilibrium is finite and its length is at most equal to the number of colours in the game. However, we can cut this path short by choosing as the first coalition deviation step the last colouring in ρ. This would still be a profitable deviation for all the players, because for all i we have $p_i(s) < p_i(s^1) < p_i(s^2) < \ldots$. □

Corollary 8. *Every coordination game on a weighted simple cycle in which at most two nodes have bonuses (or with bonuses but in which at most two edges have non-trivial weights) has a c-improvement path of length $\mathcal{O}(n)$.*

5 Sequence of simple cycles

Next we look at the graph structure which consists of a chain of $m \geq 2$ simple cycles. Formally, for $j \in \{1, 2, \ldots, m\}$, let $\mathcal{C}_j$ be the cycle $1^j \to 2^j \ldots \to n^j \to 1^j$. For simplicity, we assume that all the cycles have the same number of nodes. The results that we show hold for arbitrary cycles as long as each cycle has at least 3 nodes. An open chain of cycles, $\mathcal{N}$ is the structure in which for all $j \in \{1, \ldots, m-1\}$ we have $1^j = k^{j+1}$ for some $k \in \{2, \ldots, n\}$. In other words, it is a chain of m cycles. First, we have the following result.

Theorem 9. *Every coordination game on an unweighted open chain of cycles has an improvement path of length $\mathcal{O}(nm^2)$.*

Proof sketch. We provide a proof sketch, the details can be found in the appendix.

The idea behind the proof is to view the open chain of cycles as a sequence of simple cycles with bonuses. Here at most two nodes in each cycle have non-trivial bonuses. We then apply Theorem 3 to construct a finite improvement path for each cycle and argue that these paths can be composed in a certain manner to obtain a finite improvement path in the open chain of cycles that terminates in a Nash equilibrium.

Let $\{\mathcal{C}_j \mid j \in \{1, 2, \ldots, m\}\}$ be the set of simple cycles which constitute the open chain of cycles $\mathcal{N}$. The maximum in-degree of any node in $\mathcal{N}$ is two and in each $\mathcal{C}_j$, there are at most two nodes u and v with in-degree two with one of the incoming edges $x \rightarrow u$ from a node x in $\mathcal{C}_{j+1}$ if $j < m$ and the other $y \rightarrow v$ from a node y in $\mathcal{C}_{j-1}$ if $j > 1$. Given a joint strategy s, we can view these external incoming edges into $\mathcal{C}_j$ as bonuses to the nodes u and v. That is, $\beta_j^s(u, c) = 1$ if $s_x = c$ and 0 otherwise, $\beta_j^s(v, c) = 1$ if $s_y = c$ and 0 otherwise. For all $i \in \{1^j, \ldots, n^j\} \setminus \{u, v\}$, for all c, $\beta_j^s(i, c) = 0$.

For each $j \in \{1, 2, \ldots, m\}$ and a joint strategy s, consider the cycle $\mathcal{C}_j$ along with the bonus β_j^s. This induces a coordination game on a (unweighted) simple cycle with bonuses. By Theorem 3 such a coordination game is weakly acyclic.

Given a joint strategy s, for $j \in \{1, \ldots, m\}$, call the node 1^j a *break point* in s if the following two conditions are satisfied:

(C1) $\forall k \leq j$ and $i \in \{1^k, \ldots n^k\}$, s_i is a best response to s_{-i},

(C2) $s_{1^j} = s_{n^j}$.

For a joint strategy s, let $guard(s)$ be the largest $j \in \{1, \ldots, m-1\}$ such that 1^j is a break point in s, if no such j exists then $guard(s) = 0$. Let s^0 be an arbitrary joint strategy in the game whose underlying graph is $\mathcal{N}$. We construct a finite improvement path inductively as follows. Initially, the improvement path consists of the joint strategy s^0. Suppose we have constructed an improvement path ρ' such that $last(\rho') = s'$. Choose the least $j \in \{1, \ldots, m\}$ such that there is a node in $\mathcal{C}_j$ which is not playing its best response in s'. Apply Theorem 3 to the game induced by $\mathcal{C}_j$ and $\beta_j^{s'}$ to extend the improvement path.

We can then argue that each time a cycle $\mathcal{C}_{j+1}$ is updated, either the number of cycles playing the best response strictly goes up or if that quantity decreases, then the value of $guard$ strictly increases. Note that the value of $guard$ is always weakly increasing. Thus if we consider the pair, the value of $guard$ and the number of cycles playing the best response, then this pair under lexicographic ordering forms a progress measure for the specific scheduling of nodes defined above. The value of $guard$ is bounded by $m - 1$ and the number of cycles is bounded by m.

The improvement path constructed in Theorem 3 is of length $\mathcal{O}(n)$. Each time the number of cycles playing the best response increase, a single colour can be propagated down the entire chain of cycles. In the worst case, the value of the guard can increase by 1 at the end of each phase. Thus in the worst case, the length of the improvement path that is constructed is $\mathcal{O}(nm^2)$. □

Weighted open chain of cycles. We say that an open chain of cycles is weighted if at least one of the component cycle has an edge with non-trivial weights (i.e. an edge with weight at least 2). We now show that Theorem 9 can be extended to the setting of weighted open chain of cycles.

As in the proof of Theorem 9, the idea behind the proof is to view the weighted open chain of cycles as a sequence of *weighted* simple cycles with bonuses. The crucial observation is that at most two nodes in each cycle have bonuses. We can then apply Theorem 5 to construct a finite improvement path for each cycle and argue that these paths can be composed in a specific manner.

Let $\{\mathcal{C}_j \mid j \in \{1, 2, \ldots, m\}\}$ be the set of simple cycles which constitute the open chain of cycles $\mathcal{N}$. By the definition of $\mathcal{N}$, for all $j \in \{1, \ldots, m-1\}$, the node 1^j has in-degree two with an edge $n^j \rightarrow 1^j$ of weight w_1^j and an edge $(k-1)^{j+1} \rightarrow 1^j$ with weight w_2^j. To simplify the presentation of the proof, we assume that for all j, $w_1^j \neq w_2^j$. We first show the following two restricted results.

Lemma 1. *In the open chain of cycles $\mathcal{N}$ consisting of cycles $\mathcal{C}_1, \ldots, \mathcal{C}_m$, if for all $j \in \{1, 2, \ldots, m-1\}$ we have $w_1^j > w_2^j$ then $\mathcal{N}$ is weakly acyclic.*

Proof sketch. We provide a proof sketch, the details can be found in the appendix. As in the proof of Theorem 9, given a joint strategy s, we can view the external incoming edges into $\mathcal{C}_j$ as bonuses to the corresponding nodes. The only difference in this case is that the value of the bonus instead of being 1, is the weight of the corresponding edge. Let β_j^s denote this bonus function. The cycle $\mathcal{C}_j$ along with β_j^s defines a coordination game on a weighted simple cycle with at most two nodes having non-trivial bonuses. By Theorem 5, such a coordination game is weakly acyclic.

Let s^0 be an arbitrary joint strategy in the game whose underlying graph is $\mathcal{N}$. We construct a finite improvement path inductively as follows. Initially, the improvement path consists of the joint strategy s^0. Suppose we have constructed an improvement path ρ' such that $last(\rho') = s'$. Choose the least $j \in \{1, \ldots, m\}$ such that there is a node in $\mathcal{C}_j$ which is not playing its best response in s'. Apply Theorem 5 to the game induced by $\mathcal{C}_j$ and $\beta_j^{s'}$ to extend the improvement path. Since $w_1^j > w_2^j$ for all j, we can show that the partial improvement path ρ that is constructed in this manner satisfies the following invariant:

(I) Let $last(\rho) = s$ and let j be the largest index $j \in \{1, \ldots m-1\}$ such that for all $k \leq j$, $i \in \{1^k, \ldots n^k\}$, s_i is a best response to s_{-i}. If $s_{1^k} \neq s_{n^k}$ then $s_{n^k} \notin C(1^k)$.

The invariant asserts that if in the strategy s, the choice of the nodes 1^k and its unique predecessor n^k in $\mathcal{C}_k$ are not the same then the colour chosen by n^k is not in the available colours for 1^k. Using this, We can argue that the above procedure terminates in a Nash equilibrium. □

Lemma 2. *In the open chain of cycles $\mathcal{N}$ consisting of cycles $\mathcal{C}_1, \ldots, \mathcal{C}_m$, if for all $j \in \{1, 2, \ldots, m-1\}$ we have $w_1^j < w_2^j$ then $\mathcal{N}$ is weakly acyclic.*

Proof. Let s^0 be an arbitrary joint strategy in the game whose underlying graph is $\mathcal{N}$. We construct a finite improvement path inductively as follows. Initially, the improvement path consists of the joint strategy s^0. Suppose we have constructed an improvement path ρ' such that $last(\rho') = s'$. Choose the greatest $j \in \{1, \dots, m\}$ such that there is a node in $\mathcal{C}_j$ which is not playing its best response in s'. Apply Theorem 5 to the game induced by $\mathcal{C}_j$ and $\beta_j^{s'}$ to extend the improvement path. Since $w_1^j < w_2^j$ for all j, we can show that the partial improvement path ρ that is constructed in this manner satisfies the following invariant:

- Let $last(\rho) = s$ and let j be the smallest index $j \in \{1, \dots m-1\}$ such that for all $l \geq j$, $i \in \{1^l, \dots n^l\}$, s_i is a best response to s_{-i}. If $s_{1^l} \neq s_{(k-1)^{l+1}}$ then $s_{(k-1)^{l+1}} \notin C(1^k)$.

Due to the invariant above and the fact that $w_1^j < w_2^j$ for all j, we can use an argument very similar to that of the proof of Lemma 1, to show that a finite improvement path can be constructed. □

Theorem 10. *Every coordination game on a weighted open chain of cycles has an improvement path of length $\mathcal{O}(nm^3)$.*

Proof sketch. Let $\mathcal{N}$ be the open chain of cycles consisting of the sequence of weighted cycles $\mathcal{C}_1, \dots, \mathcal{C}_m$. The idea is to split this sequence of cycles into blocks. A block B_j is simply a sequence of simple cycles in $\mathcal{N}$, say $\mathcal{C}_p, \dots, \mathcal{C}_l$ such that one of the following conditions hold,

- for all $k \in \{p, \dots, l-1\}$ either $w_1^k > w_2^k$,
- for all $k \in \{p, \dots, l-1\}$, $w_2^k > w_1^k$.

We can then repeatedly apply Lemma 1 and 2 and compose the resulting improvement paths in a specific manner to construct a finite improvement path for $\mathcal{N}$.

The improvement path constructed by applying Lemma 1 and Lemma 2 can be of length $\mathcal{O}(nm^2)$. While composing this path we might have to propagate colours down the chain. We can argue that we always make progress by at least one block. Thus in worst case, the length of the improvement path can be $\mathcal{O}(nm^3)$. The details can be found in the appendix. □

If we allow both weights and bonuses in the underlying graph which constitutes an open chain of cycles, then it follows from Example 4 that there are coordination games that do not have a Nash equilibrium.

Closed chain of cycles. As earlier, let $\mathcal{C}_j$ be the cycle $1^j \to 2^j \dots \to n^j \to 1^j$ for $j \in \{1, \dots, m\}$. Consider the structure in which for all $j \in \{1, \dots, m-1\}$, we have $1^j = k^{j+1}$ for some $k \in \{2, \dots n\}$ and $1^m = k^1$. In other words, instead of having a chain of simple cycles, we now have a "cycle" of simple cycles. We can argue that if these simple cycles are unweighted then the coordination game whose underlying graph is such a structure remains weakly acyclic. However, if we allow the simple cycles to have non-trivial weights then the resulting game need not have a Nash

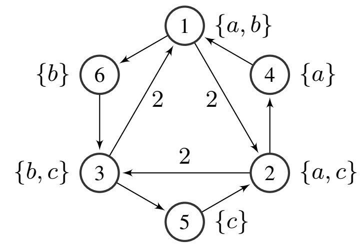

Figure 2: A weighted closed chain of cycles with no Nash equilibrium.

equilibrium as demonstrated in Example 12. Note that to construct a counter example (Figure 2), we only need three cycles each containing three nodes and a single edge in each cycle with weight 2.

Theorem 11. *Every coordination game on an unweighted closed chain of cycles has an improvement path of length $\mathcal{O}(nm^2)$.*

Proof sketch. Let $\{\mathcal{C}_j \mid j \in \{1, 2, \dots, m\}\}$ be the set of simple cycles which constitute the graph $\mathcal{C}$. In each $\mathcal{C}_j$, there is exactly two nodes with in-degree two. By the definition of $\mathcal{C}$, the simple cycles $\mathcal{C}_1$ and $\mathcal{C}_m$ share one node $1^m = k^1$ for some $k \in \{1, \dots n\}$. Let s^0 be an arbitrary joint strategy in the game whose underlying graph is $\mathcal{C}$. The idea of the proof is to view the sequence of cycles $\mathcal{C}_1, \dots, \mathcal{C}_{m-1}$ as an open chain of cycles. By Theorem 9, there is a finite improvement path starting at s^0 and terminating in s^1 such that for all nodes in cycles $\mathcal{C}_1, \dots, \mathcal{C}_{m-1}$ are playing their best response in s^1. We can argue that this path can be extended to a finite improvement path in $\mathcal{C}$. The improvement path constructed by applying Theorem 9 has length $\mathcal{O}(nm^2)$. And this can be extended to a finite improvement path in $\mathcal{C}$ with a constant number of updates of nodes in the cycles. The details can be found in the appendix. □

Example 12. *Consider the coordination game with the underlying graph given in Figure 2. Here, the nodes 4, 5, and 6 do not have a choice of colours and so in any joint strategy they need to choose the unique colour in their respective colour set. The set of joint strategies that we need to consider is then the same as given in Example 4. It follows that the game does not have a Nash equilibrium.* □

As in the case of simple cycles, we can show that unweighted closed chains of cycles and open chains of cycles are c-weakly acyclic. This implies the existence of strong equilibria in coordination games played on such graph structures.

Theorem 13. *Every coordination game on an unweighted closed chain of cycles has a c-improvement path of length $\mathcal{O}(nm^3)$.*

Proof sketch. We provide a proof sketch, the details can be found in the appendix. Let s be a Nash equilibrium that this game reaches via an improvement path of length $\mathcal{O}(nm^2)$ as constructed in Theorem 11. If s is not a strong equilibrium, then there exists a coalition K with a profitable deviation,

s', from s. Due to Lemma 2, the coalition K has to include at least one simple cycle, $\mathcal{C}$, switching to the same colour in s'. This can be one of the cycles $\mathcal{C}_i$ or one of the two cycles going around the whole game graph containing the set of nodes $A = \{1^j \mid j \in \{1, \ldots, m\}\}$. Note that s' may not be a Nash equilibrium but because the game is weakly acyclic there is a finite improvement path which leads to a Nash equilibrium s'' from s'. We can argue that $s'_i = s''_i$ for all $i \in \mathcal{C}$.

Now again, if s'' is not a strong equilibrium, then there exists a new coalition K' with a profitable deviation, s''', from s''. We can show that either $\mathcal{C} = A$ or no node from $\mathcal{C}$ can be part of K'. This implies that we can construct a c-improvement path by appropriately composing the improvement paths from Theorem 11 along with deviations by simple cycles. At least one simple cycle changes colour in each such deviation and none of its nodes change colour afterwards. This shows that the number of non-unilateral coalition deviation is at most equal to the number of different simple cycles in the game graph, which is equal to m. Thus there is a c-improvement path of length $\mathcal{O}(nm^3)$. □

Corollary 14. *Every coordination game on an unweighted open chain of cycles has a c-improvement path of length $\mathcal{O}(nm^3)$.*

Proof. Any open chain of cycles can be converted to a closed chain of cycles by adding one additional colour, c^*, and one simple cycle with four nodes: one node from $\mathcal{C}_1$ different from 1^1 and n^1, one node from $\mathcal{C}_m$ different from 1^m and n^m, two extra nodes between these two with c^* as the only colour available to them. It is easy to see that a coordination game played on this closed chain of cycles has essentially the same behaviour as a game played on the original open chain of cycles. In particular there is a one-to-one mapping between their c-improvement paths. □

Finally, so far we assumed that we know the decomposition of the game graph into a chain of cycle in advance. In general the input may be an arbitrary graph and we would need to find this decomposition first. Fortunately this can be done in linear time.

Proposition 15. *Checking whether a given graph G is a open chain of cycles or closed chain of cycles, and if so partitioning G into simple cycles $\mathcal{C}_1, \ldots, \mathcal{C}_m$ can be done in $\mathcal{O}(|G|)$.*

Proof. Assuming that G is a closed chain of cycles, the set $A = \{1^j | 1 \leq j \leq m\}$ is just the set of all nodes in G with outdegree 2. denoted by B. We perform a depth first search starting from any node of G and list the nodes with outdegree 2 as we encounter them. We identify the j-th node on this list with 1^j. We can then easily identify the remaining nodes in each cycle C_j for $1 \leq j \leq m$. Argument is similar for open chains of cycles. □

6 Simple cycles with cross-edges

In this section we consider coordination games whose underlying graph forms simple cycles along with some additional "non-cyclic" edges between nodes. We say that the graph $G = (V, E)$ is a simple cycle with cross-edges if $V = \{1, 2, \ldots, n\}$ and the edge set E can be partitioned into two sets E_c and E_p such that $E_c = \{i \to i \oplus 1 \mid i \in \{1, \ldots n\}\}$ and $E_p = E \setminus E_c$. In other words, E_c contains all the cyclic edges and E_p all the additional cross-edges in G.

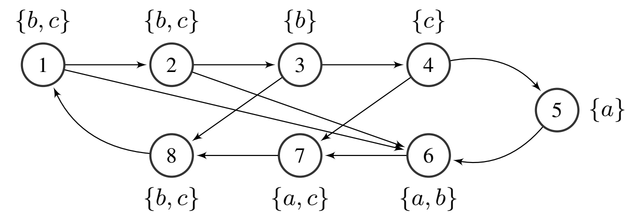

Figure 3: A partition-cycle.

The results in the previous section show that simple cycles are quite robust in terms of maintaining the property of being weakly acyclic. Even with weighted edges and chains of simple cycles, the resulting coordination games remain weakly acyclic. In this section, we study the same question: whether simple cycles with cross-edges are weakly acyclic. We first show that if we allow arbitrary (unweighted) cross-edges, then there are games that may not have a Nash equilibrium (Example 16). We then identify a restricted class of cycles with cross-edges for which the game is weakly acyclic.

Example 16. *Consider the graph G' which we obtain by adding the following edges to the graph in Figure 1: $6 \to 7$, $4 \to 8$ and $5 \to 9$. Thus G' defines a simple cycle: $1 \to 4 \to 8 \to 2 \to 5 \to 9 \to 3 \to 6 \to 7 \to 1$ along with the cross-edges represented in Figure 1 (the nodes in G' can be easily renamed if required to form the cyclic ordering $1 \to 2 \ldots 8 \to 9$). Note that in the resulting graph G', for any joint strategy, the payoff for node 7 is always 0 since $C(7)$ and $C(6)$ are disjoint. Same holds for node 8 and node 9. Also, note that the best response for nodes 4, 5 and 6 is to always select the same strategy as nodes 1, 2 and 3 respectively. Therefore, to show that the game does not have a Nash equilibrium, it suffices to consider the strategies of nodes 1, 2 and 3. We can denote this by the triple (s_1, s_2, s_3). The joint strategies are then the same as those listed in Example 4. It follows that the game does not have a Nash equilibrium.* □

Partition-cycle. Let $G = (V, E)$ be a simple cycle with cross-edges where $E = E_c \cup E_p$. We call G a *partition-cycle* if (V, E_c) forms a simple cycle and the vertex set V can be partitioned into two sets V_T and V_B such that $V_T, V_B \neq \emptyset$ and the following conditions are satisfied: $E_p \subseteq V_T \times V_B$,

- $E_c \cap (V_T \times V_T)$ forms a path in (V, E_c),
- $E_c \cap (V_B \times V_B)$ forms a path in (V, E_c).

Example 17. *The directed graph in Figure 3 is an example of a partition-cycle. One possible partition of the vertex set would be $V_T = \{1, 2, 3, 4, 5\}$ and $V_B = \{6, 7, 8\}$. The edge set E_c consists of the edges $1 \to 2, 2 \to 3, \ldots, 8 \to 1$ whereas $E_p = \{1 \to 6, 2 \to 6, 3 \to 8, 4 \to 7\}$.* □

We first show that every coordination game whose underlying graph is an unweighted partition cycle is weakly acyclic. For the sake of simplicity, we fix the following notation: the partition-cycle is given by $G = (V, E)$ where $V = \{1, \dots n\}$, $V_T = \{1, 2, \dots, k\}$ and $V_B = \{k+1, k+2, \dots, n\}$. If $E_p = \emptyset$ then we get a simple cycle without cross-edges on n nodes. For $i \in V_B$, $c \in C(i)$ and a joint strategy s, let $\mathcal{S}(i, c, s) = \{j \in V_T \mid j \to i \text{ and } s_j = c\}$. We also define the set $MC(i, s) = \{c \in C(i) \mid |\mathcal{S}(i, c, s)| \geq |\mathcal{S}(i, c', s)| \text{ for all } c' \in C(i)\}$. Given a player i and a joint strategy of the other players s_{-i} let $BR(i, s_{-i})$ denote the set of best responses of player i to s_{-i}.

Theorem 18. *Every coordination game without bonuses on an unweighted partition-cycle has an improvement path of length $\mathcal{O}(n(n-k))$.*

Proof. Consider an initial joint strategy s^0. We construct a finite improvement path starting in s^0 as follows. We proceed around the cycle and consider the players $1, 2, \dots, n$ in that order. For each player i, in turn, for the corresponding joint strategy s, if s_i is not a best response to s_{-i}, we update it to a best response respecting the following property:

(P1) If $s_{i \ominus 1} \in BR(i, s_{-i})$ and there exists a $c \in MC(i, s)$ such that $p_i(c, s_{-i}) = p_i(s_{i \ominus 1}, s_{-i})$ then player i switches to c (in this case $c \in BR(i, s_{-i})$ as well).

Let s^1 be the resulting joint strategy at the end of the first round. It follows that the players $2, \dots, n$ are playing their best response in s^1. If s^1 is a Nash equilibrium then the improvement path is constructed. If not then the only player who is not playing its best response is player 1. This implies that $s^1_n \neq s^0_n$. Let l_1 be the least index in $V_B = \{k+1, \dots n\}$ such that for all $j \in \{l_1, \dots, n\}$, $s^1_{l_1} \neq s^0_{l_1}$ and $s^1_{l_1} = s^1_n$. Let $X = \{l_1, l_1 + 1, \dots, n\}$. Note that $X \neq \emptyset$ since $n \in X$. We repeatedly let players update to their best response strategies in the cyclic order in multiple rounds. We can argue that in each round $|X|$ strictly increases. By definition, $|X| \leq |V_B|$ and therefore the improvement path constructed in this manner eventually terminates in a Nash equilibrium.

In the second round starting at the joint strategy s^1, we let players update to their best response following the cyclic order $1, 2, \dots k$. Let s^2 be the resulting joint strategy. Note that in this sequence, if a player is not playing its best response then the best response strategy is simply to switch to the current strategy of its unique predecessor on the cycle (recall that all nodes in V_T have exactly one incoming edge). Thus the only colour which is propagated is s^1_n. Now starting at s^2, let players update to their best response following the cyclic order $k+1, \dots n$ and let s^3 be the resulting joint strategy. If s^3 is a Nash equilibrium then we have a finite improvement path. If not, then player 1 is the unique player not playing its best response and $s^3_n \neq s^1_n$. We know that for all $j \in X$, $s^1_j = s^1_n$. By the above argument we also have $|\mathcal{S}(j, s^1_n, s^3)| \geq |\mathcal{S}(j, s^1_n, s^1)|$. Thus if $s^3_n \neq s^1_n$ then for all $j \in X$, $s^3_j = s^3_n$. Now consider the node l_1 and let t and t' be the joint strategies in the improvement path constructed where $t = (s^1_{l_1}, t_{-l})$ and $t' = (s^3_{l_1}, t_{-l})$. For all $m \in V_T$, we have $t_m = s^2_m (= s^3_m)$ and $|\mathcal{S}(l_1, s^1_{l_1}, s^3)| \geq |\mathcal{S}(l_1, s^1_{l_1}, s^1)|$. Thus if t_{l_1} is not a best response of player l_1 then $s^3_{l_1} = s^3_{l_1 - 1}$ and $s^3_{l_1 - 1} \neq s^1_{l_1 - 1}$. Now let l_2 be the least index in $V_B = \{k+1, \dots n\}$ such that for all $j \in \{l_2, \dots, n\}$, $s^3_{l_1} \neq s^1_{l_1}$ and $s^3_{l_1} = s^3_n$. Let $X' = \{l_2, l_2 + 1, \dots, n\}$. Clearly, $l_2 < l_1$ and therefore, $|X'| > |X|$ and $X \subseteq X'$. Let $X := X'$ and we repeat this process. In each successive round, $|X|$ strictly increases and by definition, $|X| \leq |V_B|$. Therefore, in at most $|V_B|$ rounds, either we reach a Nash equilibrium or we reach a joint strategy s' where for all $j, m \in V_B$, $s'_j = s'_m$. In this case we go around in the cyclic order $1, 2, \dots, k$ and update players to their best response. As earlier we can argue that the only colour which is propagated is s'_n and therefore this improvement path terminates in a Nash equilibrium.

The above proof shows that starting from the second round, the size of the set X strictly increases and we know that $|X| \leq |V_B| = n - k$. Each time, in the worst case, we might have to update all the nodes in the cyclic order. Thus in the worst case the length of this improvement path is at most $\mathcal{O}((n-k) \cdot n)$ $\square$

From Theorem 5, we know that simple cycles even with weighted edges are weakly acyclic. However, partition-cycles with weighted edges need not always have a Nash equilibrium (Example 19). On the other hand, we show in Theorem 20 that unweighted partition-cycles with bonuses remain weakly acyclic. Thus Theorem 3 can be extended to partition-cycles.

Example 19. *Consider the partition-cycle G given in Figure 3 and suppose we add weight 2 to edges $6 \to 7$ and $7 \to 8$. The resulting game does not have a Nash equilibrium. Note that in any joint strategy, nodes 3, 4 and 5 have to choose the colour b, c and a respectively. Therefore, it suffices to consider strategies of nodes 6, 7, 8, 1 and 2. Also note that in any joint strategy s, the best response for players 1 and 2 is s_8 (the strategy of player 8 in s). Thus we can also restrict attention to joint strategies s in which $s_1 = s_2 = s_8$. So let us denote a joint strategy s by the triple (s_6, s_7, s_8). Below we list all such joint strategies and we underline a strategy that is not a best response: $(\underline{a}, a, b)$, $(a, a, \underline{c})$, $(a, c, \underline{b})$, $(a, \underline{c}, c)$, $(b, \underline{a}, b)$, $(\underline{b}, a, c)$, $(b, c, \underline{b})$ and $(\underline{b}, c, c)$.* $\square$

Theorem 20. *Every coordination game with bonuses on an unweighted partition-cycle has an improvement path of length $\mathcal{O}(kn(n-k))$.*

Proof sketch. The main idea is to enforce the players to update their strategy based on a specific priority over colours induced by the bonuses. For each node in V_T to satisfy the priority of updates over colours induced by the bonuses, we might have to cycle through each node and construct the improvement path as given in the proof of Theorem 18. Thus in the worst case, the length of the improvement path which is constructed is $\mathcal{O}(k \cdot (n-k)n)$. Details are provided in the appendix. $\square$

Note that Example 19 shows that with just two weighted edges between nodes in V_B, it is possible to construct games which may not have a Nash equilibrium. We now show that if the weights are only present on edges between nodes in V_T or on the cross-edges E_p then the resulting game remains weakly acyclic. If we allow bonuses on nodes then we can add weights to the cross-edges E_p and the resulting game remains weakly acyclic. On the other hand, from Example 4 we already know that if we allow both weights and bonuses, even without cross-edges, there are graphs in which the resulting game need not have a Nash equilibrium.

Given a partition cycle $G = (V_T \cup V_B, E_c \cup E_p)$, let $E_T = (V_T \times V_T) \cap E_c$. That is, the set E_T consists of all the cyclic edges between nodes in V_T.

Theorem 21. *Every coordination game without bonuses on a partition-cycle with weights on edges in $E_T \cup E_p$ is weakly acyclic.*

Proof. Let $G = (V_T \cup V_B, E_c \cup E_p)$ be a partition-cycle and $E_T = (V_T \times V_T) \cap E_c$. We first show that for each weighted edge in E_p we can add a set of unweighted edges and obtain a new partition-cycle G' such that every improvement path in G' can be converted into an improvement path in G. Let $u \to v$ be an edge in E_p with weight w. Note that by definition of G, $u \in V_T$ and $u \in V_B$. Let $x \to u$ and $u \to y$ be the cyclic edges in E_c associated with the node u. We replace the node u with w new nodes $u_1, \dots, u_w$ and for all $j \in \{1, \dots, w\}$ we set $C(u_j) = C(u)$. We also add the following unweighted edges to the edge set E. For all $j \in \{1, \dots, w-1\}$, $u_j \to u_{j+1} \in E_c$, $u_j \to v \in E_p$, $u_w \to v \in E_p$ and $\{x \to u_1, u_w \to y\} \subseteq E_c$. In any joint strategy s, the best response of nodes $u_2, \dots u_w$ would be to choose the same colour as u_1. Which implies the following: the node v had an incoming edge of weight w supporting the colour s_u in G iff in the modified graph in any joint strategy in which the nodes $u_2, \dots, u_w$ are playing their best response, the node v has w edges supporting the colour s_u.

The proof of Theorem 18 shows that it is possible to construct a finite improvement path by updating players in the cyclic order. A crucial property which was used is that in each successive rounds, while updating players in V_T, the only colour which is propagated is s_n, the colour chosen by node n in the end of the previous round. Even if the edges in E_T are weighted, the property continues to hold since the best response for each node $i \in V_T$ is still to choose the same colour as it unique predecessor $i \ominus 1$ on the cycle, provided the colour is in $C(i)$. Note that the edges in $E_c \setminus E_T$ are unweighted. Thus by using a similar argument as in the proof of Theorem 18, we can conclude that the game is weakly acyclic. □

Theorem 22. *Every coordination game with bonuses on a partition-cycle with weights on edges in E_p is weakly acyclic.*

Proof. Each weighted edge in E_p can be converted into a set of unweighted edges such that the resulting graph G' is still a partition-cycle. From every finite improvement path in the coordination game whose underlying graph is G', we can construct a finite improvement path in G. Thus by Theorem 20, the result follows. □

Finally, we assumed that the decomposition and ordering of the nodes in the input partition-cycle graph is given in advance. The decomposition can be computed in linear time as well.

Proposition 23. *Checking whether a given graph G is a partition-cycle and if so finding its V_T, V_B and suitable ordering on these subsets of nodes can be done in $\mathcal{O}(|G|)$.*

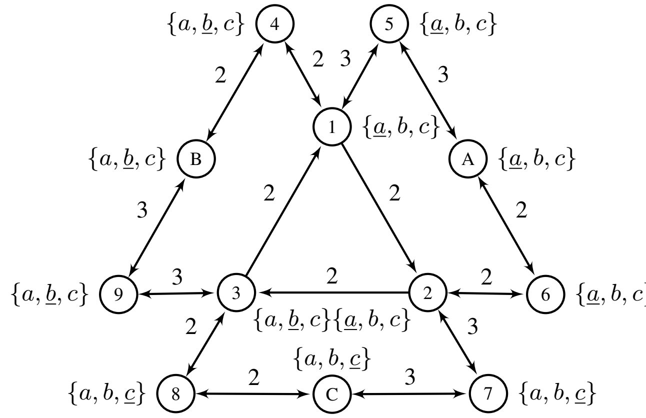

Figure 4: A coordination game with trivial strong equilibria unreachable from the given initial joint strategy.

Proof. Note that the ordering of $G = (V, E)$ we are looking for defines a Hamiltonian path in G with particular properties. We start by selecting only the nodes in G with outdegree 1; these are all the nodes that can potentially be in V_B. Next, we look at the graph $G' = (V_B, E \cap V_B \times V_B)$. First, we remove any edges from G' that form a cycle using, e.g. depth-first search. We then topologically sort the resulting DAG. We obtain several disjoint paths $B_1, \dots, B_k$ as candidates for V_B. We assume that there is at least one cross-edge in G, because otherwise the problem is trivial. We check to which of these disjoint paths this cross-edge leads to and we set that path as V_B and the rest of the nodes are set as V_T. The order on V_B is given by the topological order. We then look at $G'' = (V_T, E \cap V_T \times V_T)$. If G'' has a cycle then G is a not a partition-cycle. Otherwise, topologically sorting G'' gives us the order of nodes in V_T. Finally, it is straightforward to test whether V_T and V_B satisfy the remaining requirements for the graph G to be a partition-cycle. □

7 Conclusions

We presented natural classes of graphs for which coordination games have improvement or c-improvement paths of polynomial size. We also showed that for most natural extensions of these classes, the resulting coordination game may not even have a Nash equilibrium. Note that although we defined bonuses as natural numbers, our results also hold for any integer bonuses, because after increasing all bonuses by a fixed amount, all players' incentives stay the same.

In general, local search may not be an efficient technique to find a Nash equilibrium or a strong equilibrium in coordination games even when the game graph is strongly connected. In fact, a coordination game can have trivial strong equilibria which cannot be reached from some of its initial joint strategies. For example, the game in Figure 4 has three trivial strong equilibria in which all players pick the same colour. However, every improvement or c-improvement path from the initial joint strategy (given by the underlined strategies) is infinite. Moreover, although the game graph is weighted, the weighted edges can easily be replaced by unweighted ones just by adding auxiliary nodes (see Example 24 in the appendix). Therefore, the non-existence of a finite improvement or c-improvement path in coordination games even for strongly connected unweighted graphs does not imply the non-existence of Nash equilibria or strong equilibria.

In proving our results, we used various generalised potential techniques, and exploited structural properties of the classes of graphs studied. It would be interesting to see whether there is a common progress measure that works for all the classes of graphs that we consider as well as for more general ones. In particular, we conjecture that coordination games on unweighted graphs with indegree at most two are c-weakly acyclic. Extensive computer simulations seem to support this conjecture. This class of graphs strictly generalises the unweighted open chains of cycles and closed chains of cycles that we showed to be c-weakly acyclic. We also leave open the existence of finite c-improvement paths in weighted open chains of cycles and partition-cycles. Although they seem likely to exist, unicoloured simple cycles introduced by coalition deviations from Nash equilibria can disappear when trying to reach a new Nash equilibrium after them, so a detailed analysis of the interplay between these two steps is required to prove their c-weak acyclicity.

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Appendix A – Sequence of simple cycles

Theorem 9. *Every coordination game on an unweighted open chain of cycles has an improvement path of length* $\mathcal{O}(nm^2)$.

Proof. Let $\{\mathcal{C}_j \mid j \in \{1, 2, \ldots, m\}\}$ be the set of simple cycles which constitute the open chain of cycles $\mathcal{N}$. The maximum in-degree of any node in $\mathcal{N}$ is two and in each $\mathcal{C}_j$, there are at most two nodes u and v with in-degree two with one of the incoming edges $x \to u$ from a node x in $\mathcal{C}_{j+1}$ if $j < m$ and the other $y \to v$ from a node y in $\mathcal{C}_{j-1}$ if $j > 1$. Given a joint strategy s, we can view these external incoming edges into $\mathcal{C}_j$ as bonuses to the nodes u and v. That is, $\beta_j^s(u, c) = 1$ if $s_x = c$ and 0 otherwise, $\beta_j^s(v, c) = 1$ if $s_y = c$ and 0 otherwise. For all $i \in \{1^j, \ldots, n^j\} \setminus \{u, v\}$, for all c, $\beta_j^s(i, c) = 0$.

For each $j \in \{1, 2, \ldots, m\}$ and a joint strategy s, consider the cycle $\mathcal{C}_j$ along with the bonus β_j^s. This induces a coordination game on a (unweighted) simple cycle with bonuses. By Theorem 3 such a coordination game is weakly acyclic.

Given a joint strategy s, for $j \in \{1, \ldots, m\}$, call the node 1^j a *break point* in s if the following two conditions are satisfied:

(C1) for all $k \leq j$, for all $i \in \{1^k, \ldots n^k\}$, s_i is a best response to s_{-i},

(C2) $s_{1^j} = s_{n^j}$.

For a joint strategy s, let $guard(s)$ be the largest $j \in \{1, \ldots, m-1\}$ such that 1^j is a break point in s, if no such j exists then $guard(s) = 0$.

Let s^0 be an arbitrary joint strategy in the game whose underlying graph is $\mathcal{N}$. We construct a finite improvement path inductively as follows. Initially, the improvement path consists of the joint strategy s^0. Suppose we have constructed an improvement path ρ' such that $last(\rho') = s'$. Choose the least $j \in \{1, \ldots, m\}$ such that there is a node in $\mathcal{C}_j$ which is not playing its best response in s'. Apply Theorem 3 to the game induced by $\mathcal{C}_j$ and $\beta_j^{s'}$ to extend the improvement path.

In other words, the procedure works as follows: Suppose there is a node in $\mathcal{C}_1$ which is not playing its best response in s^0. Start with the coordination game induced by $\mathcal{C}_1$ and β_1^s. By Theorem 3, there is a finite improvement path that terminates in a joint strategy s^1 such that for all $i \in \{1^1, \ldots n^1\}$, s_i^1 is a best response to s_{-i}^1.

Now suppose we have constructed a partial improvement path ρ where $last(\rho) = s$ and let j be the largest index $j \in \{1, \ldots m-1\}$ such that for all $k \leq j$, $i \in \{1^k, \ldots n^k\}$, s_i is a best response to s_{-i}. Consider the coordination game induced by the $\mathcal{C}_{j+1}$ and $\beta_{j+1}^{s^j}$. By Theorem 3, there is a finite improvement path that terminates in a joint strategy s^{j+1} such that for all $i \in \{1^{j+1}, \ldots n^{j+1}\}$, s_i^{j+1} is a best response to s_{-i}^{j+1}. Now since the two cycles $\mathcal{C}_j$ and $\mathcal{C}_{j+1}$ share a node, i.e. $1^j = k^{j+1}$. It is possible that in the joint strategy s^{j+1}, the node 2^j is not playing its best response any longer. To avoid multiple subscripts, let us denote the node 2^j by 2, 1^j by 1. So we have that s_2^{j+1} is not a best response to s_{-2}^{j+1}. Note that by assumption the node 2^j was playing its best response in the joint strategy s^j. And the only node in $\mathcal{C}_j$ that could possibly change its strategy in s^{j+1} is 1^j. Assume that node 2 has a unique predecessor (or the in-degree of node 2 is 1). Then we also have, $s_1^{j+1} \neq s_2^{j+1}$, $s_1^{j+1} \in C(s_2^{j+1})$ and $p_2(s_1^{j+1}, s_{-2}^{j+1}) \geq p_2(c, s_{-2}^{j+1})$ for all $c \in C(2^j)$. Let the node 2^j switch to the colour s_1^{j+1}. We then update the nodes in the cyclic order in $\mathcal{C}_j$ successively if they are not playing their best response. We then do the same procedure for cycles in the order $\mathcal{C}_{j-1}, \ldots, \mathcal{C}_{guard(s^{j+1})}$.

Now suppose in this sequence of updates the only colour which is propagated is s_1^{j+1}. Then we have reached a joint strategy in which all nodes on cycles $\mathcal{C}_{j+1}, \mathcal{C}_j, \ldots, \mathcal{C}_{guard(s^{j+1})}$ are playing their best response. So the number of cycles playing their best response has strictly increased. If while propagating down the sequence $\mathcal{C}_{j-1}, \ldots, \mathcal{C}_1$ a new colour is introduced then note that this new colour can only be introduced by a node of indegree 2. Suppose the first instance in the improvement path a new colour is introduced is by node 1^l for $l < j$. Let s^1 be the joint strategy before node $(k-1)^{l+1}$ updates to its best response and s^2 be the joint strategy before node 1^l changes to the colour $c \neq s_1^{j+1}$ (which is its best response). Recall that the node 1^l is same as the node k^{l+1} for some $k \in \{1, \ldots, n\}$. Since we proceed in the cyclic order in $\mathcal{C}_l$, we know that in s^1 node 1^l was playing a best response where as by assumption in s^2 node 1^l is not. The only difference between s^1 and s^2 is in the strategy of node $(k-1)^{l+1}$ and by assumption, the strategy of node $(k-1)^{l+1}$ in s^2 is same as s_1^{j+1}. Since $c \neq s_1^{j+1}$, $c \neq s_{1^l}^2$ and c is a best response for node 1^l to the joint strategy $s_{-1^l}^2$, it implies that $s_{n^l}^2 = c$. Let s^3 be the joint strategy obtained from s^2 by having $s_{1^l}^3 = c$. This implies that node 1^l satisfies condition (C2) in the joint strategy s^3. If it also satisfies condition (C1) then we have identified a break point. It also follows that $guard(s^3) > guard(s^{j+1})$ and we have strictly reduced the number of cycles we need to consider.

If condition (C1) is not satisfied in s^3 then the only node not on its best response in $\mathcal{C}_l$ is 2^l. Apply the same propagation and let 1^k be the last node in this sequence which introduces a new colour c' and let s^4 be the joint strategy obtained after node 1^k switches. By the same argument it holds that condition (C2) is satisfied by node 1^k in the joint strategy s^4. Now update the nodes in the cyclic order and in the sequence $\mathcal{C}_k, \mathcal{C}_{k-1}, \ldots, \mathcal{C}_{guard(s^{j+1})}$. It can be verified that the only colour propagated is c'. Let s^5 be the resulting strategy in which all the nodes in the cycles $\mathcal{C}_k, \mathcal{C}_{k-1}, \ldots, \mathcal{C}_{guard(s^{j+1})}$ are playing their best response. This implies that the node 1^k is a break point in s^5. Thus we have $guard(s^5) > guard(s^{j+1})$ and we can repeat the same procedure inductively for the cycles $\{\mathcal{C}_{guard(s^5)+1}, \ldots, \mathcal{C}_m\}$.

The improvement path constructed in Theorem 3 is of length $\mathcal{O}(n)$. Each time the number of cycles playing the best response increase, a single colour can be propagated down the entire chain of cycles. In the worst case, the value of the guard can increase by 1 at the end of each phase. Thus in the worst case, the length of the improvement path that is constructed is $\mathcal{O}(nm^2)$. □

Lemma 1. *In the open chain of cycles $\mathcal{N}$ consisting of cycles $\mathcal{C}_1, \ldots, \mathcal{C}_m$, if for all $j \in \{1, 2, \ldots, m-1\}$ we have $w_1^j > w_2^j$ then $\mathcal{N}$ is weakly acyclic.*

Proof. As in the proof of Theorem 9, given a joint strategy s, we can view the external incoming edges into $\mathcal{C}_j$ as bonuses to the corresponding nodes. The only difference in this case is that the value of the bonus instead of being 1, is the weight of the corresponding edge. Let β_j^s denote this bonus function. The cycle $\mathcal{C}_j$ along with β_j^s defines a coordination game on a weighted simple cycle with at most two nodes having non-trivial bonuses. By Theorem 5, such a coordination game is weakly acyclic.

Let s^0 be an arbitrary joint strategy in the game whose underlying graph is $\mathcal{N}$. We construct a finite improvement path inductively as follows. Initially, the improvement path consists of the joint strategy s^0. Suppose we have constructed an improvement path ρ' such that $last(\rho') = s'$. Choose the least $j \in \{1, \ldots, m\}$ such that there is a node in $\mathcal{C}_j$ which is not playing its best response in s'. Apply Theorem 5 to the game induced by $\mathcal{C}_j$ and $\beta_j^{s'}$ to extend the improvement path. Since $w_1^j > w_2^j$ for all j, we can show that the partial improvement path ρ that is constructed in this manner satisfies the following invariant:

(I) Let $last(\rho) = s$ and let j be the largest index $j \in \{1, \ldots m-1\}$ such that for all $k \leq j$, $i \in \{1^k, \ldots n^k\}$, s_i is a best response to s_{-i}. If $s_{1^k} \neq s_{n^k}$ then $s_{n^k} \notin C(1^k)$.

The invariant asserts that if in the strategy s, the choice of the nodes 1^k and its unique predecessor n^k in $\mathcal{C}_k$ are not the same then the colour chosen by n^k is not in the available colours for 1^k.

To see how the above process works, suppose there is a node in $\mathcal{C}_1$ which is not playing its best response in s. Start with the coordination game induced by $\mathcal{C}_1$ and β_1^s. By Theorem 5, there is a finite improvement path that terminates in a joint strategy s^1 such that for all $i \in \{1^1, \ldots n^1\}$, s_i^1 is a best response to s_{-i}^1. Since $w_1^1 > w_2^1$ the invariant (I) holds for the node 1^1.

Now suppose we have constructed a improvement path ρ where $last(\rho) = s$ and let j be the largest index $j \in \{1, \ldots m-1\}$ such that for all $k \leq j, i \in \{1^k, \ldots n^k\}$, s_i is a best response to s_{-i} and the invariant (I) holds. Consider the coordination game induced by the $\mathcal{C}_{j+1}$ and $\beta_{j+1}^{s^j}$. By Theorem 5, there is a finite improvement path that terminates in a joint strategy s^{j+1} such that for all $i \in \{1^{j+1}, \ldots n^{j+1}\}$, s_i^{j+1} is a best response to s_{-i}^{j+1}. Now since the two cycles $\mathcal{C}_j$ and $\mathcal{C}_{j+1}$ share a node, i.e. $1^j = k^{j+1}$. It is possible that in the joint strategy s^{j+1}, the node 2^j is not playing its best response any longer. To avoid multiple subscripts, let us denote the node 2^j by 2, 1^j by 1, n^j by n and $(k-1)^{j+1}$ by $k-1$. So we have that s_2^{j+1} is not a best response to s_{-2}^{j+1}. Note that by assumption the node 2^j was playing its best response in the joint strategy s^j. And the only node in $\mathcal{C}_j$ that could possibly change its strategy in s^{j+1} is 1^j. If 1^j changes its strategy then this means that $s_1^{j+1} \neq s_n^{j+1}$. By invariant (I), this means $s_n^{j+1} \notin C(1)$ and so $s_1^{j+1} = s_{k-1}^{j+1}$. We also have, $s_1^{j+1} \neq s_2^{j+1}$, $s_1^{j+1} \in C(s_2^{j+1})$ and $p_2(s_1^{j+1}, s_{-2}^{j+1}) \geq p_2(c, s_{-2}^{j+1})$ for all $c \in C(2^j)$. Let the node 2^j switch to the colour s_1^{j+1}. We then update the nodes in the cyclic order in $\mathcal{C}_j$ successively if they are not playing their best response. It can be verified that for every node which is not playing its best response, the colour s_1^{j+1} is a best response. Therefore the only colour which is propagated is s_1^{j+1}. So this sequence of updates terminate in a joint strategy in which all the nodes in $\mathcal{C}_j$ and $\mathcal{C}_{j+1}$ is playing their best response.

In this resulting joint strategy it could be that the node 2^{j-1} is not playing the best response (since the node $1^{j-1} = k^j$ switched). Again by the same reasoning, and by invariant (I), we can argue that in this case we can update the strategies of the players such that only the colour s_1^{j+1} is propagated. Continuing in this manner we arrive at a joint strategy in which all nodes on cycles $\mathcal{C}_1, \ldots, \mathcal{C}_{j+1}$ are playing their best response. Since $w_1^{j+1} > w_2^{j+1}$ the invariant (I) continues to hold. In case the weights on the incoming edges are not distinct, then depending on the initial joint strategy s^0 it is possible that a new colour is introduced when we propagate down the chain $\mathcal{C}_j, \ldots, \mathcal{C}_1$. In this case we can identify *break points* and use a similar technique as done in the proof of Theorem 9 to identify a progress measure. □

Theorem 10. *Every coordination game on a weighted open chain of cycles has an improvement path of length $\mathcal{O}(nm^3)$.*

Proof. Let $\mathcal{N}$ be the open chain of cycles consisting of the sequence of weighted cycles $\mathcal{C}_1, \ldots, \mathcal{C}_m$.to combine A *block* B_j is a sequence of simple cycles in $\mathcal{N}$. We can represent $\mathcal{N}$ as a sequence of blocks which we define inductively as follows: The block B_1 consists of the sequence of cycles $\mathcal{C}_1, \ldots, \mathcal{C}_l$ such that for all $k \in \{1, \ldots, l-1\}$, $w_1^k > w_2^k$ or for all $k \in \{1, \ldots, l\}$, $w_1^k < w_2^k$.

Suppose we have inductively constructed the block B_j and let $\mathcal{C}_p$ be the last cycle in B_j. Then B_{j+1} consists of the sequence of cycles $\mathcal{C}_{p+1}, \ldots, \mathcal{C}_q$ such that one of the following conditions hold,

- for all $k \in \{p+1, \ldots, q-1\}$, $w_1^k > w_2^k$ and if $q \neq m$ then $w_1^q < w_2^q$,
- for all $k \in \{p+1, \ldots, q-1\}$, $w_1^k < w_2^k$ and if $q \neq m$ then $w_1^q > w_2^q$,

Thus the open chain of cycles consisting of the sequence of weighted cycles can now be represented as a sequence of blocks $B_1, \dots, B_l$. The pair of blocks B_i and B_{i+1} share a node in common. Let s^0 be an arbitrary joint strategy in the game whose underlying graph is $\mathcal{N}$. We construct a finite improvement path inductively as follows. Initially, the improvement path consists of the joint strategy s^0. Suppose we have constructed an improvement path ρ' such that $last(\rho') = s'$. Choose the least $j \in \{1, \dots, l\}$ such that there is a node in the block B_j which is not playing its best response in s'. Let B_j consists of the cycles $\mathcal{C}_p, \dots, \mathcal{C}_q$. If for all $k \in \{p, \dots, q-1\}$, $w_1^k > w_2^k$ then apply Lemma 1 to the sequence $\mathcal{C}_p, \dots, \mathcal{C}_q$ with the possibility of bonus to a node in $\mathcal{C}_p$ and $\mathcal{C}_q$ to extend the improvement path. If for all $k \in \{p, \dots, q-1\}$, $w_1^k < w_2^k$ then apply Lemma 2 to extend the improvement path.

The proof that this procedure constructs a finite improvement path is similar to the proof of Theorem 9. Suppose we have constructed a partial improvement path ρ where $last(\rho) = s$ and j is the largest index such that all nodes in blocks $B_1, \dots, B_j$ are playing the best response in s. Consider the block B_{j+1}, by applying either Lemma 1 or Lemma 2 (depending on the case), we can extend the improvement path to ρ^1 such that in $s^1 = last(\rho^1)$. Let us assume that the block B_{j+1} consists of the cycles $\mathcal{C}_{p+1}, \dots, \mathcal{C}_q$. Then B_j and B_{j+1} share a common node, 1^p. If the strategy of the node 1^p in s and s^1 is the same, then in s^1 we have strictly increased the number of blocks playing the best response. Suppose $s^1_{1^p} \neq s_{1^p}$, then there are two cases to analyse. Suppose in the block B_j, for all cycles $\mathcal{C}_k$, $w_1^k > w_2^k$. Then we can argue that the only colour which is propagated is $s^1_{1^p}$ and therefore, after applying Lemma 1, the number of blocks playing the best response increases. Suppose in the block B_j, for all cycles $\mathcal{C}_k$, $w_1^k < w_2^k$. By the procedure explained in Lemma 2 we can reach a joint strategy s^2 such that all nodes in B_j is playing their best response. Now if $s^2_{1^p} = s^1_{1^p}$ then all the nodes in B_{j+1} is also playing their best response and therefore, the number of blocks playing the best response increases. Suppose $s^2_{1^p} \neq s^1_{1^p}$ then it has to be the case that $s^2_{1^p} = s^2_{n^p}$. Note that by definition of blocks, $w_1^p > w_2^p$. Like in the proof of Theorem 9 we can define 1^p to be a break-point in s^2 since $w_1^p > w_2^p$, $s^2_{1^p} = s^2_{n^p}$ and all nodes in B_j are playing their best response. Similar to the proof of Theorem 9 we can argue that after each such phase, either the number of block playing the best response strictly increases or the value of the maximal break point strictly increases.

The improvement path constructed by applying Lemma 1 and Lemma 2 can be of length $\mathcal{O}(nm^2)$. While composing this path we might have to propagate colours down the chain. However, we can argue that we always make progress by at least one block. Thus in worst case, the length of the improvement path can be $\mathcal{O}(nm^3)$. □

Theorem 11. *Every coordination game on an unweighted closed chain of cycles has an improvement path of length* $\mathcal{O}(nm^2)$.

Proof. Let $\{\mathcal{C}_j \mid j \in \{1, 2, \dots, m\}\}$ be the set of simple cycles which constitute the graph $\mathcal{C}$. In each $\mathcal{C}_j$, there is exactly two nodes with indegree two. By the definition of $\mathcal{C}$, the simple cycles $\mathcal{C}_1$ and $\mathcal{C}_m$ share one node $1^m = k^1$ for some $k \in \{1, \dots n\}$. Let s^0 be an arbitrary joint strategy in the game whose underlying graph is $\mathcal{C}$. The idea of the proof is the view the sequence of cycles $\mathcal{C}_1, \dots, \mathcal{C}_{m-1}$ as an open chain of cycles. By Theorem 9, there is a finite improvement path starting at s^0 and terminating in s^1 such that for all nodes in cycles $\mathcal{C}_1, \dots, \mathcal{C}_{m-1}$ are playing their best response in s^1. Now we update the strategies of nodes in $\mathcal{C}_m$ in the cyclic order, let the resulting joint strategy be s^2 if the nodes 1^m and 1^{m-1} choose the same strategy in both s^1 and s^2 then we have constructed the finite improvement path.

Suppose $s^2_{1^{m-1}} \neq s^1_{1^{m-1}}$ and $s^2_{1^{m-1}} = s^2_{(k-1)^m}$ (where the nodes 1^{m-1} and k^m are the same). In s^2 the node 2^{m-1} may no longer be playing the best response. We proceed in the reverse order and update the nodes in the cycles $\mathcal{C}_{m-1}, \mathcal{C}_{m-2}, \dots, \mathcal{C}_1, \mathcal{C}_m$. If no new colour is introduced and the only colour which is propagated is $s^2_{1^{m-1}}$ then the improvement path terminates after updating nodes in $\mathcal{C}_m$. If a new colour $c' \neq s^2_{(k-1)^m}$ is introduced then let 1^q be the first time this happens while updating players in the order of cycles $\mathcal{C}_{m-1}, \mathcal{C}_{m-2}, \dots, \mathcal{C}_1$ (note that a new colour can be introduced only by a node with indegree 2). Let s^3 be the resulting joint strategy, then due to the order of scheduling nodes, it follows that $s^3_{(k-1)^{q+1}} \notin C(1^q)$ and $s^3_{1^q} = s^3_{n^q}$. Each time a new colour is introduced, for the node involved, the above condition is satisfies. In other words, the node forms a *break point* for that particular joint strategy as defined in the proof of Theorem 9. The important observation is that, since the new colour of node 1^q is supported by the node n^q, the payoff for 1^q is at least 1 and therefore while we update nodes in the reverse order of cycles, if no more new colours are introduced, then the only colour which is propagated further down the chain is $s^3_{1^q}$ and then the path terminates at $(k-1)^{(q-1)}$. Other new colours could be introduced in this propagation. However, the node which introduces the new colour is then a break point. Let 1^r be the last node where a new colour is introduced and the resulting joint strategy be s^4. This implies that $s^4_{(k-1)^{r+1}} \notin C(1^r)$ and $s^3_{1^r} = s^3_{n^r}$. Now we schedule the cycles $\mathcal{C}_r, \mathcal{C}_{r-1}, \dots \mathcal{C}_{q+1}$. The only colour which is propagated is the colour of 1^r and the node 1^q does not update its strategy since the colour chosen by 1^q and n^q is the same. So after this, all nodes in the cycles $\mathcal{C}_r, \mathcal{C}_{r-1}, \dots \mathcal{C}_{q+1}$ are on their best response. Let the resulting joint strategy be s^5. It could still be that 2^r is not on its best response (since 1^r updated the colour to a new colour). We now update the nodes in the order of cycles $\mathcal{C}_r, \mathcal{C}_{r+1}, \dots, \mathcal{C}_q$, this propagates the colour $s^4_{1^r} (= s^5_{1^r})$. If there is a node 1^l such that the colour chosen by 1^l is same as that of n^l and $s^4_{1^r} \notin C(1^l)$ then the propagation stops. If not, then the same colour $s^4_{1^r}$ is propagated and the improvement path terminates at the cycle $\mathcal{C}_{r+1}$.

If $s^2_{1^{m-1}} = s^1_{1^{m-1}}$ it could still be that $s^2_{1^1} \neq s^1_{1^1}$. In this case we update the players in the increasing order of cycles $\mathcal{C}_1, \dots \mathcal{C}_n$ and using a similar argument as above, we can show that a finite improvement path can be constructed. □

Theorem 13. *Every coordination game on an unweighted closed chain of cycles has a c-improvement path of length*

$\mathcal{O}(nm^3)$.

Proof. Let s be a Nash equilibrium that this game reaches via an improvement path of length $\mathcal{O}(nm^2)$ as constructed in Theorem 11. If s is a strong equilibrium then we are done. Otherwise there exists a coalition K with a profitable deviation, s', from s. Due to Lemma 2, the coalition K has to include at least one simple cycle, $\mathcal{C}$, switching to the same colour in s'. This can be one of the cycles $\mathcal{C}_i$ or one of the two cycles going around the whole game graph containing the set of nodes $A = \{1^j | 1 \leq j \leq m\}$.

Note that s' may not be a Nash equilibrium but because the game is weakly acyclic there is a finite improvement path which leads to a Nash equilibrium s'' from s'. We now show that $s''|_\mathcal{C} = s'|_\mathcal{C}$. Let s^* be a strategy profile along the path from s' to s'' when for the first time a node, i, from $\mathcal{C}$ switches its colour. We have $p_i(s^*) \leq 1$ because i has at most two incoming edges and one of them is from a node in $\mathcal{C}$. At the same time, $p_i(s^*) \geq p_i(s') + 1 \geq p_i(s) + 2 \geq 2$, because the deviation of i to s^*_i is assumed to be profitable and so is the deviation, as part of coalition K, to s'; a contradiction.

Now again, either s'' is a strong equilibrium, and we are done, or there exists a new coalition K' with a profitable deviation, s''', from s''. We claim that either $\mathcal{C} = A$ or no node from $\mathcal{C}$ can be part of K'. Any node $i \in \mathcal{C} \setminus A$ has only one incoming edge and so node i cannot be part of K' and improve any further from $p_i(s'') = 1$. Moreover, any successor of i in $\mathcal{C}$ cannot be part of K' either, because it would need to switch to a different colour than i and so cannot improve his payoff of 1 in s''. It follows that either $\mathcal{C} \cap K' = \emptyset$ or $\mathcal{C} \setminus A$ is empty, which implies $\mathcal{C} = A$. In the latter case, every simple cycle, which has to be part of coalition K', has a nonempty intersection with A. Such a node, i, would need to improve its payoff to 2, because $p_i(s'') \geq 1$, so both of its predecessors have to belong to K'. In particular, its predecessor in A. It follows that $A \subseteq K'$. Furthermore, all predecessors of nodes in A should belong to K', but this includes all nodes of the game. Therefore, all nodes in the game have to switch to the same colour which would form a strong equilibrium. It follows that there can be at most one profitable coalition deviation after coalition $\mathcal{C} = A$ deviates. So we can safely ignore this special case in the analysis below and assume that always $\mathcal{C} \cap K' = \emptyset$.

Finally, we construct a finite c-improvement path $\rho = s^{0,0}$, $s^{1,0}, s^{1,1}, \dots, s^{1,k_1}, s^{2,0}, s^{2,1}, \dots, s^{2,k_1}, s^{3,0}, \dots$ as follows. It starts with $s^{0,0} = s$ and we stipulate $k_0 = 0$. For any $j \geq 1$, strategy profile $s^{j,0}$ is a result of a profitable deviation by any coalition from $s^{j-1,k_{j-1}}$. If there is no such deviation the path is finished and $s^{j-1,k_{j-1}}$ is a strong equilibrium. Otherwise, although $s^{j,0}$ may not be a Nash equilibrium, the game is weakly acyclic and thanks to Theorem 11 there exist an improvement path $s^{j,1}, s^{j,2}, \dots, s^{j,k_j}$ of length $\mathcal{O}(nm^2)$ which reaches a Nash equilibrium s^{j,k_j}. We know that in each $s^{j,0}$ at least one simple cycle changes colour and none of its nodes change colour afterwards. This shows that the number of non-unilateral coalition deviation is at most equal to the number of different simple cycles in the game graph, which is equal to m. Therefore, ρ is a c-improvement path of length $\mathcal{O}(nm^3)$. □

Appendix B – Partition cycle

Theorem 20. *Every coordination game with bonuses on an unweighted partition-cycle has an improvement path of length $\mathcal{O}(kn(n-k))$.*

Proof. Consider the initial joint strategy s^0. We construct a finite improvement path starting in s^0 by proceeding in the cyclic order and updating players' strategies. The argument that this results in a finite improvement path, is very similar to the proof of Theorem 18. The main idea is to enforce the players to update their strategy based on a specific priority over colours induced by the bonuses. Let us define $MB(i) = \{c \in C(i) \mid \text{ for all } c' \in C(i), \beta(i,c) \geq \beta(i,c')\}$ and $Max(i,s) = \{c \in C(i) \mid \text{ for all } c' \in C(i), \beta(i,c) + \mathcal{S}(i,c,s) \geq \beta(i,c') + \mathcal{S}(i,c',s)\}$.

Given a partial improvement path ρ with $last(\rho) = s$, if s_i is not a best response to s_{-i} and $i \in V_T$, then we update the strategy of i such that it satisfies the following property (P2):

(P2) if $s_{i\ominus 1} \in BR(i, s_{-i})$ and there exists a $c \in MB(i)$ such that $p_i(c, s_{-i}) = p_i(s_{i\ominus 1}, s_{-i})$ then player i switches to c (clearly, in this case $c \in BR(i, s_{-i})$ as well).

For $i \in V_B$, we update the strategy of i such that it satisfies the property (P3):

(P3) If $s_{i\ominus 1} \in BR(i, s_{-i})$ and there exists a $c \in Max(i,s)$ such that $p_i(c, s_{-i}) = p_i(s_{i\ominus 1}, s_{-i})$ then player i switches to c (clearly, in this case $c \in BR(i, s_{-i})$ as well).

For all $i \in V_T$, if in an improvement path, player i updates its strategy then by (P2), it switches to a colour in $MB(i)$. Due to (P2) and the fact that i has a unique incoming edge, we can verify that in any subsequent joint strategy s^1, if i updates its strategy to a colour c' then it has to be that $s^1_{i\ominus 1} = c'$ and $c' \in MB(i)$. Thus we can assume that after some finite prefix of the improvement path constructed by updating players in the cyclic ordering, for all nodes $i \in V_T$, i is choosing a strategy in $MB(i)$.

Let s^2 be the resulting joint strategy. Consider the set X constructed at this stage as defined in the proof of Theorem 18. Let $X = \{l_1, \dots n\}$. By the construction of X, we have that for all $j,k \in X$, $s^2_j = s^2_k$. By definition of the set X, all the nodes $j \in X$ have updated its strategy, and therefore, they conform to property (P3). For a node $j \in X$, let $s^2_j = c^1$. From (P3), it follows that for all $j \in \{l_1 + 1, \dots n\}$, for all $c^2 \in C(j), \beta(j,c^1) + \mathcal{S}(j,c^1,s^2) + 1 > \beta(j,c^2) + \mathcal{S}(i,c^2,s^2)$. For node l_1, if $s^2_{l_1 - 1} \neq s^2_{l_1}$ then $s^2_{l_1} \in Max(l_1, s^2)$ otherwise, $s^2_{l_1}$ satisfies the same property as above. This implies that in the next cyclic round of updates, for each node $j \in X$, either j updates to the same strategy as its unique predecessor on the cycle or j is already on a best response strategy which implies that the resulting joint strategy is a Nash equilibrium. Thus following the argument given in the proof of Theorem 18 a finite improvement path can be constructed. □

Example 24. *Consider the coordination game graph in Figure 4. This game graph is strongly connected and in fact there are only three edges missing to turn it into an undirected graph. Also, although the game graph is weighted, the* *weighted edges can easily be replaced by unweighted ones just by adding auxiliary nodes without affecting the strong connectedness of the graph. At the same time, the behaviour of the game on this new unweighted graph will essentially be the same as on the original one. Note that coordination games on undirected unweighted graphs are known to have FIP [2]. If we do not require strong connectedness of the game graph, this example can be slightly simplified by removing nodes A, B, C and turning bidirectional edges from nodes 4–9 into outgoing edges.*

First, let us notice that nodes A, B, and C in this game would never like to switch their colour; all of them already have the maximum possible payoff of 5. This implies that nodes 5 – 9 will never change their colour either, e.g. node 5 has at least payoff of 2 for picking b *and no matter the colour node 1 chooses, node 5 will never be better off switching to a different colour. Therefore, the only nodes that can ever switch colours are nodes 1–3.*

Now, let us analyse the initial colouring in Figure 4. The payoff of node 2 is 4 and his maximum possible payoff is 5. However, he can only get payoff 5 if he switches to c *and node 1 switches to* c*. The latter is not possible because node 1 gets payoff of at most 2 for picking* c *while picking* a *gives him at least 3. In conclusion, node 2 cannot be part of a deviating coalition in this colouring. Node 3 will not change his colour either because he gets payoff 3 while the other colours give him payoff 2. Therefore, the only node which can switch in any coalition is node 1 and his only profitable deviation is switching to colour* b*.*

Once this switch happens, he gets payoff 4 in the new colouring, while his maximum payoff is 5. It can be argued as before that node 1 cannot be part of a deviating coalition in this new colouring. However, there are two possible deviating coalitions: either node 2 unilaterally switches to colour c*, or nodes 2 and 3 switch to colour* c *together. In the former, the game will be in essentially the same situation as with the initial colouring in Figure 4; one just need to rotate the colours, numbers and the game to the "left", i.e. colour* b *is* a *and node* 2 *is node* 1 *etc. While in the latter, the games will be in the situation essentially the same as in the colouring encountered after the first switch from the initial colouring.*

It is easy to see now that eventually this game arrives at the initial colouring and the whole process will repeat forever. Therefore, this game is not weakly acyclic nor c-weakly acyclic. On the other hand, it has three trivial strong equilibria in which all players pick the same colour. □