

# Evacuating Robots via Unknown Exit in a Disk

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**Abstract.** Consider  $k$  mobile robots inside a circular disk of unit radius. The robots are required to evacuate the disk through an unknown exit point situated on its boundary. We assume all robots having the same (unit) maximal speed and starting at the centre of the disk. The robots may communicate in order to inform themselves about the presence (and its position) or the absence of an exit. The goal is for all the robots to evacuate through the exit in minimum time.

We consider two models of communication between the robots: in *non-wireless* (or *local*) *communication* model robots exchange information only when simultaneously located at the same point, and *wireless communication* in which robots can communicate one another at any time.

We study the following question for different values of  $k$ : what is the optimal evacuation time for  $k$  robots? We provide algorithms and show lower bounds in both communication models for  $k = 2$  and  $k = 3$  thus indicating a difference in evacuation time between the two models. We also obtain almost-tight bounds on the asymptotic relation between evacuation time and team size, for large  $k$ . We show that in the local communication model, a team of  $k$  robots can always evacuate in time  $3 + \frac{2\pi}{k}$ , whereas at least  $3 + \frac{2\pi}{k} - O(k^{-2})$  time is sometimes required. In the wireless communication model, time  $3 + \frac{\pi}{k} + O(k^{-4/3})$  always suffices to complete evacuation, and at least  $3 + \frac{\pi}{k}$  is sometimes required. This shows a clear separation between the local and the wireless communication models.

## 1 Introduction

Consider a team of mobile robots inside an environment represented by a circular disk of unit radius. The robots need to find an exit being a point at an unknown position on the boundary of the disk in order to evacuate through this point. The exit is recognized when visited by a robot. The robots may communicate in order to exchange the knowledge about the presence (or the absence) of the exit acquired through their previous movements. We consider two communication models. In the *non-wireless* (or *local*) model, communication is possible between

robots which arrive at the same point (in the environment) at the same moment, while the *wireless* model allows broadcasting a message by a robot, which is instantly acquired by other robots, independently of their current positions in the environment. The robots start at the centre of the disk and they can move with a speed not exceeding their maximum velocity (which is the same for all robots). The objective is to plan the movements of all robots, which result in the shortest worst-case time needed for all robots to evacuate.

### 1.1 Related work

Mobile agents are autonomous entities traveling within geometric or graph-modeled environments. Besides *mobility*, agents possess the ability to *perceive* the environment, *compute*, and *communicate* among themselves. They collaborate in order to perform tasks assigned to them. When agents operate in geometric environments (then they are usually called robots) their performance is measured by the geometric distance travelled, most often disregarding their computing, communicating and environment-perceiving activities.

When the geometric environment is not known in advance by the mobile robots, in many papers their task consisted in exploring the environment [1, 2, 13, 17]. The coordination of exploration between multiple robots has been mainly studied by the robotics community [10, 25, 26]. However even if the main objective assigned to the robots is different from exploration, often part of their activity is devoted to the recognition or mapping of the terrain and/or the position of the robots within it [20, 22, 24]. When the map of the environment is known to the robots, a lot of research was devoted to search games, when the searchers usually try to minimize the time to find an immobile or a moving hider [3, 4, 21]. The literature of the case of mobile fugitives, often known as cops and robbers or pursuit-evasion games is particularly rich [12, 15], with numerous variations related to the type of environment, speed of evasion and pursuit, robots visibility and many others [23]. The searching for a motionless point target in the simple environment presented in our paper has some similarities with the lost at sea problem, [16, 18], the cow-path problem [8, 9], and with the plane searching problem [5, 6].

The problem of evacuation has been studied for grid polygons from the perspective of constructing centralized evacuation plans, resulting in the fastest possible evacuation from the rectilinear environment [14]. Previously, [7] considered evacuation planning as earliest-arrival flows with multiple sources giving the first algorithm strongly polynomial in input/output size.

Evacuation in a distributed setting, when the mobile robots (know the simple environment but not the exit positions) has been recently asked in [11] for the case of a line. They proved that evacuation of multiple uniform agents is as hard as the cow-path problem. Evacuation of two robots without wireless communication was discussed with the research group of M. Yamashita during the visit of the second co-author at Kyushu University [19]. The discussion focused on laying the foundations for the lower bound presented in this paper and seeking ways to improve the respective upper bound. However, the main objective of

our problem is to find a compromise between, on one hand, spreading sufficiently the robots so that they can find the exit point fast in parallel, and, on the other hand, not to spread them too far so that, when one robot finds the exit, the escape route to it of the other robots is not too long.

## 1.2 Preliminaries

The environment is a disk of unit radius. The robots start their movement at the centre of the disk. We assume that the perception device of the robot permits to recognize a boundary point of the environment when the robot arrives there. Similarly, we assume that a robot recognizes the presence of other robots at the same position as well the fact that the robot is currently at the exit point. We also assume that the robots are labeled, i.e. they may execute different algorithms. Each such algorithm instructs the robot to make the moves with a speed not exceeding its maximal speed. In particular, the algorithm may ask the robot to move towards the centre of the disk or a chosen point on its boundary or to follow the boundary clockwise or counterclockwise. The movement may be changed when the perception mechanism allows the robot to acquire some knowledge about the environment (e.g. the exit point, boundary point, a meeting point with another robot). The robots are allowed to stay motionless at the same point. If  $A$  and  $B$  are points on the perimeter of the disk, by  $\widehat{AB}$  we will denote arc from  $A$  to  $B$  in the clockwise direction and by  $\overline{AB}$  we will denote the cord connecting  $A$  and  $B$ . The length of  $\widehat{AB}$  will be denoted by  $|\widehat{AB}|$  and the length of  $\overline{AB}$  will be denoted by  $|\overline{AB}|$ .

## 1.3 Outline and results of the paper

In Section 2 we consider the evacuation problem for two robots, while Section 3 analyzes the case of three robots. Section 4 proves tight asymptotic bounds for  $k$  robots. Each section is divided into two parts consisting of the analysis for the non-wireless and wireless models, respectively. The complexity details corresponding to the three sections are in Table 1.

**Table 1.** Upper and Lower bounds for  $k \geq 2$  robots

<i>Model</i>	<i>Bound</i>	$k = 2$	$k = 3$	$k \geq 4$
Non-wireless	Upper	$\sim 5.74$ (Th 1)	$\sim 5.09$ (Th 8)	$3 + \frac{2\pi}{k} < 4.58$ (Th 8)
	Lower	$\sim 5.199$ (Th 2)	$\sim 4.519$ (Th 5)	$3 + \frac{2\pi}{k} - O(k^{-2})$ (Th 9)
Wireless	Upper	$\sim 4.83$ (Th 3)	$\sim 4.22$ (Th 6)	$3 + \frac{\pi}{k} + O(k^{-4/3})$ (Th 10)
	Lower	$\sim 4.83$ (Th 4)	$\sim 4.159$ (Th 7)	$3 + \frac{\pi}{k} > 3.785$ (Th 11)

These results establish a clear separation between the non-wireless and the wireless communication models. Due to lack of space the detail of missing proofs of some theorems and lemmas are deferred to the full version of the paper.

## 2 Two Robots

Consider a disk centered at  $K$ . Two robots, say  $r_1, r_2$ , start at  $K$  moving with constant speed, say 1, searching for an exit located at an unknown point on the perimeter of the disk. In the sequel we prove upper and lower bounds for the two robot case in the non-wireless and wireless cases.

### 2.1 Non-wireless communication

Algorithm  $\mathcal{A}_1$  indicates the robot trajectory for evacuation without wireless communication.

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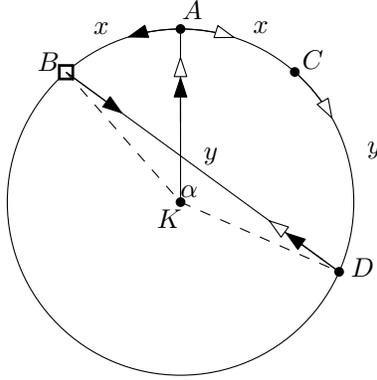
**Algorithm  $\mathcal{A}_1$**  [for two robots without wireless communication]

1. Both robots move to an arbitrary point  $A$  on the perimeter.
  2. At  $A$  the robots move along the perimeter of the disk in opposite directions; robot  $r_1$  moves counter-clockwise and robot  $r_2$  moves clockwise until one of the two robots, say  $r_1$ , finds the exit at  $B$ .
  3. Now robot  $r_1$  is at point  $B$  and  $r_2$  is at point  $C$  (symmetric to  $B$ ). Robot  $r_1$  chooses a point  $D$  such that the length of the chord  $BD$  is equal to the length of the arc  $\widehat{CD}$  and moves towards  $D$ .
  4. Since the length of the chord  $BD$  is equal to the length of the arc  $\widehat{CD}$ , both robots arrive at  $D$  at the same time. Robot  $r_1$  has knowledge about the location of the exit thus both robots can now follow the straight line  $DB$  and exit.
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In the following theorem we give a bound on the worst-case evacuation time of algorithm  $\mathcal{A}_1$ .

**Theorem 1.** *There is an algorithm for evacuating the robots from an unknown exit located on the perimeter of the disk which takes time  $1 + \alpha/2 + 3 \sin(\alpha/2)$ , where the angle  $\alpha$  satisfies the equation  $\cos(\alpha/2) = -1/3$ . It follows that the evacuation algorithm takes time  $\sim 5.74$ .*

*Proof.* (Theorem 1) We calculate the time required until both robots from algorithm  $\mathcal{A}_1$  reach the exit. Denote  $x = |\widehat{BA}| = |\widehat{AC}|$ ,  $y = |BD| = |\widehat{CD}|$  and  $\alpha = |\widehat{BD}|$ . According to the definition of the above algorithm  $\mathcal{A}_1$  the total time required is  $f(\alpha) = 1 + x + 2y$ . Observe that  $\alpha = 2x + y$ , and  $y = 2 \sin(\alpha/2)$ , because  $y$  is a chord of the angle  $\alpha$ . By substituting  $x$  and  $y$  in the definition of the function  $f$  we can express the evacuation time as a function of the angle  $\alpha$  as follows.  $f(\alpha) = 1 + \frac{\alpha - y}{2} + 2y = 1 + \frac{\alpha}{2} + \frac{3y}{2} = 1 + \frac{\alpha}{2} + 3 \sin(\alpha/2)$ . Now we



**Fig. 1.** Evacuation of two robots without wireless communication.

differentiate with respect to  $\alpha$  and we obtain:  $\frac{df(\alpha)}{d\alpha} = \frac{1}{2} + \frac{3}{2} \cos(\alpha/2)$ . It is easy to see that this derivative equals 0 for the maximum of function  $f(\alpha)$ , which yields as value for  $\alpha$  the solution of  $\cos(\alpha/2) = -1/3$ . This proves Theorem 1.

We remark however that algorithm  $\mathcal{A}_1$  is not optimal. Suppose that we modify algorithm  $\mathcal{A}_1$  by making the robot arriving at point  $D$  on Fig. 1 walk along a small segment  $z$  from  $D$  towards  $K$  and back, before continuing the circular arc following  $D$  (similarly for the other robot arriving at the point symmetric to  $D$ ). Our experiments show that, for sufficiently small length of such segment  $z$ , the maximum cost of 5.74 may be slightly reduced.

In the sequel we state and prove a lower bound.

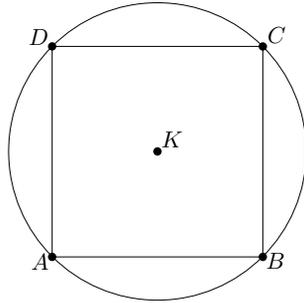
**Theorem 2.** *It takes at least  $3 + \frac{\pi}{4} + \sqrt{2}$  ( $\sim 5.199$ ) time units for two robots to evacuate from an unknown exit located in the perimeter of the disk.*

*Proof.* (Theorem 2) At the beginning, both robots are located at the center  $K$  of the disk. It takes at least 1 time unit for both of them to move to the perimeter of the disk.

In less than an additional  $\pi/4$  time units the two robots cover at most a length of  $\pi/2$  of the perimeter. The main idea is to observe, that until that time of the movement we can always construct a square  $ABCD$  with sides equal to  $\sqrt{2}$  whose all vertices are not yet visited by neither of the two robots. The vertices represent positions where an adversary can place an exit. Using an adversary argument it can be shown that an additional  $2 + \sqrt{2}$  time units are required for robot evacuation. We give details of this argument in the following two lemmas.

**Lemma 1.** *For any  $\epsilon > 0$ , at time  $1 + \frac{\pi}{4} - \epsilon$  there exists a square inscribed in the disk none of whose vertices has been explored by a robot.*

*Proof.* (Lemma 1) The proof is easily derived by rotating a square inscribed in the disk continuously for an angle of  $\pi/2$ . More precisely assume on the



**Fig. 2.** Forming a square  $ABCD$  of positions not yet explored by the robots.

contrary that such an inscribed square does not exist. Consider a partition of perimeter of the disk into four arcs of length  $\pi/2$ ,  $E_1, E_2, E_3, E_4$ . Any point  $e_1 \in E_1$  uniquely defines an inscribed square with vertices  $e_1 \in E_1, e_2 \in E_2, e_3 \in E_3, e_4 \in E_4$ . Moreover for a different  $e'_1 \in E_1, e'_1 \neq e_1$  vertices of the inscribed square  $\{e'_1, e'_2, e'_3, e'_4\}$  are different  $e'_i \neq e_i$  for all  $i \in 1, 2, 3, 4$ . By the assumption, for any  $e_1 \in E_1$  at least one of the vertices  $\{e_1, e_2, e_3, e_4\}$  of the inscribed square has to be explored (denote it by  $e^*$ ). Thus for any  $e_1$  we can identify an explored vertex  $e^*(e_1)$ . Since for different  $e_1$ , the inscribed square is different then the function  $e^*(e_1)$  is an injection. Thus the image of the function  $e^*(e_1)$  is a set of length  $\pi/2$  of explored points. But such set does not exist because at time  $1 + \pi/4 - \epsilon$  the total length of the set of explored points less than  $\pi/2$ . Therefore we obtain a contradiction at time  $1 + \frac{\pi}{4} - \epsilon$  that an inscribed square, none of whose vertices has been explored by a robot, does exist.

**Lemma 2.** *For any square inscribed in the disk none of whose vertices has been explored by a robot it takes more than  $2 + \sqrt{2}$  time to evacuate both robots from a vertex of the square.*

*Proof.* (Lemma 2) Take the square  $ABCD$  with unexplored vertices. Consider any evacuation algorithm  $\mathcal{A}$ . We allow the algorithm to place the robots on arbitrary positions of the disk (possibly also on vertices of the square). The adversary can run the algorithm with undefined position of the exit and place the exit depending on the behaviour of the robots. The adversary will run the algorithm from perspective of a fixed robot  $r$  and will place the exit at a some point  $P$ . The placement of the exit at point  $P$  in time  $t$  is possible if robot  $r$  has no information whether the exit is located in  $P$ . Formally we say that a point  $P$  is *unknown* to robot  $r$  at time  $t$  if for any time moment  $t' \in [0, t]$  robot  $r$  is at distance more than  $t'$  from  $P$ . This means that even if other robot started at  $P$  it could not meet  $r$  at any time in the interval  $[0, t]$ . Take a robot  $r$  and the first time moment  $t$  when the third vertex of the square is visited by a robot. Consider two cases

*Case 1.*  $\sqrt{2} \leq t < 2$ .

Denote the vertex visited by  $r$  in time  $t$  by  $A$ . The adversary places the exit in the antipodal point  $C$ . Observe that point  $C$  is unknown to  $r$  at time  $t$ . This is because if  $r$  was at distance at most  $t'$  from  $C$  at some time  $t' \in [0, t]$  then it would be at distance  $2 - t'$  from  $A$  and would reach  $A$  no sooner than at time  $2$ , which is a contradiction as  $t < 2$ . Thus placement of the exit in  $C$  cannot affect movement of  $r$  until time  $t$ . Therefore, the adversary can place the exit in  $C$  and the evacuation time in this case will be at least  $t + 2 \geq 2 + \sqrt{2}$ .

*Case 2.*  $2 \leq t$ .

Time moment  $t$  is the first time when three vertices of the square are explored (it is possible that in  $t$  both robots explore a new vertex). Therefore, at time  $t$ , some robot  $r$  has knowledge about at most three vertices. The adversary simply places the exit in the vertex unknown to  $r$  and the evacuation time of  $r$  will be at least  $t + \sqrt{2} \geq 2 + \sqrt{2}$ .

Observe that  $t$  cannot be smaller than  $\sqrt{2}$  because within time  $t$  at least one robot has to traverse at least one side of the square. This proves Lemma 2.

Clearly, the proof of Theorem 2 is an immediate consequence of Lemmas 1 and 2.

## 2.2 Wireless communication

Algorithm  $\mathcal{A}_2$  indicates the robot trajectory for evacuation with wireless communication.

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**Algorithm  $\mathcal{A}_2$**  [for two robots with wireless communication]

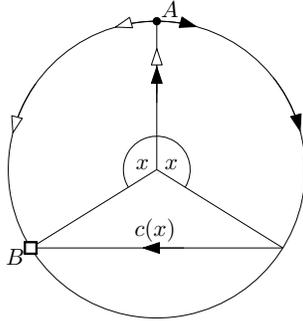
1. Both robots move to an arbitrary point  $A$  on the perimeter.
  2. At  $A$  the robots start moving along the perimeter of the disk in opposite directions: robot  $r_1$  moves counter-clockwise and robot  $r_2$  moves clockwise until one of the robots, say  $r_1$ , finds the exit at  $B$ .
  3. Robot  $r_1$  notifies  $r_2$  using wireless communication about the location of the exit and robot  $r_2$  takes the shortest chord to  $B$ .
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**Theorem 3.** *There is an algorithm for evacuating two robots from an unknown exit located on the perimeter of the disk which takes time at most  $1 + \frac{2\pi}{3} + \sqrt{3}$ .*

*Proof.* (Theorem 3) Consider the maximum evacuation time of algorithm  $\mathcal{A}_2$ . If the angular distance between  $A$  and  $B$  equals  $x$ , then the length of the chord taken by the robot  $r_2$  equals to  $c(x) = 2 \sin(\pi - x)$  (see Figure 3). Thus the evacuation time  $T$  satisfies  $T \leq \max_{0 \leq x \leq \pi} \{1 + x + 2 \sin(\pi - x)\} = \max_{0 \leq x \leq \pi} \{1 + x + 2 \sin x\}$ . The function  $f(x) = 1 + x + 2 \sin x$  in the interval  $[0, \pi]$  is maximized at the point  $x^* = 2\pi/3$  and  $f(x^*) = 1 + 2\pi/3 + \sqrt{3}$ . This proves Theorem 3.

We now state the main lower bound.

**Theorem 4.** *For any algorithm it takes at least  $1 + \frac{2\pi}{3} + \sqrt{3}$  time in the worst case for two robots to evacuate from an unknown exit located in the perimeter of the disk.*



**Fig. 3.** Evacuation of two robots with wireless communication.

### 3 Three Robots

In this section we analyze evacuation time for three robots in both non-wireless and wireless models.

#### 3.1 Non-wireless communication

The first lemma provides a lower bound which is applicable for any  $k$  robots in the non-wireless model.

**Lemma 3.** *For any  $k \geq 3$  and  $1 < \alpha < 2$ , it takes in the worst case at least  $\min \{3 + \frac{\alpha\pi}{k}, 3 + 2 \sin(\pi - \frac{\alpha\pi}{2})\}$  time to evacuate from an unknown exit located on the perimeter of the disk in the model without wireless communication.*

*Proof.* (Lemma 3) Take any evacuation algorithm  $\mathcal{A}$ . Denote by  $\mathcal{A}_r^p(t)$  the position of robot  $r$  in time  $t$  if the exit is located at point  $p$ . Since we are considering the worst case, we need to show that there exists a point  $p^*$  on the perimeter such that if the exit is located at  $p^*$  then the evacuation time of the algorithm  $\mathcal{A}$  is at least  $3 + \frac{2\pi}{k} - O(k^{-2})$ . Consider the following three time intervals:  $I_1 = [0, 1)$ ,  $I_2 = [1, 1 + \frac{\alpha\pi}{k})$ ,  $I_3 = [1 + \frac{\alpha\pi}{k}, 3)$ . Since algorithm  $\mathcal{A}$  is deterministic, the robots will follow a fixed trajectory, independent of the location of the exit until finding the exit or being notified about it by some other robot. Denote these trajectories by  $p_1(t), p_2(t), \dots, p_k(t)$ . Consider two cases:

*Case 1.* *There exists a robot  $r$  and time  $t^* \in I_3$  such that point  $p = p_r(t^*)$  of the trajectory of the robot  $r$  is on the perimeter of the disk.*

We will argue that the adversary can place the exit at point  $p^*$  being antipodal of  $p$ . We need to prove that if the exit is at point  $p^*$  then until time  $t^*$  robot  $r$  will be unaware of the location of the exit and will follow the trajectory  $p_r(t)$ . Consider the trajectory followed by robot  $r$  in algorithm  $\mathcal{A}$  if the exit is at point  $p^*$ . Robot  $r$  is following the trajectory  $p_r(t)$  until finding the exit or being notified about it. We want to show that robot  $r$  cannot be notified about the exit until time

$t^*$ . Assume on the contrary that  $1 \leq t' < t^*$  is the first moment in time when  $r$  either discovered the exit or met a robot carrying information about the location of the exit. Thus we have that  $\mathcal{A}_r^{p^*}(t) = p_r(t)$ , for all  $t \in [0, t']$ . First note that since  $p = p_r(t^*)$  we have that  $\text{dist}(\mathcal{A}_r^{p^*}(t'), p^*) = \text{dist}(p_r(t'), p^*) > t' - 1$ . The last inequality is true because if the distance between  $p_r(t')$  and  $p^*$  would be at most  $t' - 1$  then the distance to  $p$  would be at least  $3 - t'$  (because  $p$  and  $p^*$  are antipodal) and robot  $r$  following trajectory  $p_r(t)$  would not be able to reach  $p$  until time  $t^*$  (recall  $t^* < 3$ ), which is a contradiction since  $p_r(t^*) = p$ . Now observe that in algorithm  $\mathcal{A}$  if the exit is located at  $p^*$  then for any time moment  $t' \leq 3$ , any robot carrying information about the location of the exit is at distance at most  $t' - 1$  from  $p^*$  (it is because robots can exchange informations only when they meet and the maximum speed of a robot is 1). Thus it is not possible that robot  $r$  in time  $t'$  obtain the information about the exit by meeting another robot. It is also not possible that  $p_r(t') = p^*$ , because robot  $r$  following trajectory  $p_r(t)$  would not be able to reach  $p$  until time  $t^*$ . Thus such  $t'$  does not exist and we have:  $\mathcal{A}_r^{p^*}(t) = p_r(t)$ , for all  $t \in [0, t^*]$ . In time moment  $t^*$  robot  $r$  following algorithm  $\mathcal{A}$  is at distance 2 from the exit located at  $p^*$ . Thus the total evacuation time is at least  $t^* + 2 \geq 3 + \alpha\pi/k$ , since  $t^* \geq 1 + \alpha\pi/k$  (because  $t^* \in I_3$ ).

*Case 2. None of the trajectories  $p_1(t), p_2(t), \dots, p_k(t)$  in the interval  $I_3$  is equal to a point on the perimeter.*

In this case we consider robots following the trajectories  $p_1(t), p_2(t), \dots, p_k(t)$  in the time interval  $[0, 3)$ . We need the following lemma.

**Lemma 4.** *If a perimeter of a disk whose subset of total length  $u + \epsilon > 0$  has not been explored for some  $\epsilon > 0$  and  $\pi \geq u > 0$ , there exist two unexplored boundary points between which the distance along the perimeter is at least  $u$ .*

The set of points  $U$  on the perimeter of the disk that were not visited by any robot following such trajectories satisfies  $|U| \geq 2\pi - \alpha\pi$  because in this case robots can explore the perimeter only in time interval  $I_2$  of length  $\alpha\pi/k$ . Thus by Lemma 4 there exists a pair of unexplored points at distance at least  $2\pi - \alpha\pi - \epsilon$  for any  $\epsilon > 0$ . The chord connecting these two points has length at least  $2 \sin(\pi - \alpha\pi/2 - \epsilon/2)$ . Take this chord and denote its endpoints by  $u_1$  and  $u_2$ . The adversary can run the algorithm  $\mathcal{A}$  until moment  $t'$  when one of the points  $u_1, u_2$  is visited and the adversary can place the exit in the other one. Note that until moment  $t'$  robots are following trajectories  $p_r(t)$  because none of the robots has any information about the exit, thus  $t' \geq 3$ . Now the first robot that visited one of the points  $u_1, u_2$  still needs to travel at least  $2 \sin(\pi - \alpha\pi/2 - \epsilon/2)$  because the exit is on the other end of the chord. Thus exploration time is in this case at least  $3 + 2 \sin(\pi - \alpha\pi/2 - \epsilon/2)$ . We showed that the worst case time of evacuation  $T$  for any correct algorithm satisfies  $T \geq \min \left\{ 3 + \frac{\alpha\pi}{k}, 3 + 2 \sin \left( \pi - \frac{\alpha\pi}{2} - \frac{\epsilon}{2} \right) \right\}$ , for any  $\epsilon > 0$ . The claim of the lemma follows by passing to the limit as  $\epsilon \rightarrow 0$ .

**Theorem 5.** *It takes at least 4.519 time in the worst case to evacuate three robots from an unknown exit located in the perimeter of the disk in the model without wireless communication.*

*Proof.* (Theorem 5) We have by Lemma 3 that the evacuation time  $T$  of any evacuation algorithm  $\mathcal{A}$  satisfies  $T \geq \min\{3 + \frac{\alpha\pi}{k}, 3 + 2\sin(\pi - \alpha\pi/2)\}$  for any  $k \geq 3$ . To prove the statement we numerically find such  $\alpha$  that  $\frac{\alpha\pi}{3} = 2\sin(\pi - \frac{\alpha\pi}{2})$ . If we set  $\alpha = 1.408$ , we obtain  $T \geq \min\{3 + \frac{\alpha\pi}{3}, 3 + 2\sin(\pi - \frac{\alpha\pi}{2})\} > 4.519$ . This proves Theorem 5.

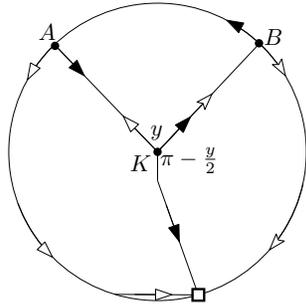
### 3.2 Wireless communication

We have three robots  $r_1, r_2, r_3$  and consider the following algorithm.

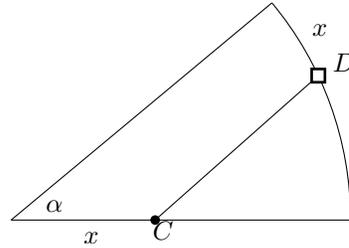
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**Algorithm  $\mathcal{A}_3$**  [for three robots with wireless communication]

1. Robot  $r_1$  moves to an arbitrary point  $A$  of the perimeter, robots  $r_2$  and  $r_3$  move together to the point  $B$  at angle  $y = 4\pi/9 + 2\sqrt{3}/3 - 401/300$  in the clockwise direction to the radius taken by robot  $r_1$ .
  2. Robot  $r_1$  moves in the counter-clockwise direction. Robot  $r_2$  moves in the clockwise direction. Robot  $r_3$  moves in the counter-clockwise direction for time  $y$ . Then  $r_3$  moves towards the center. Then  $r_3$  moves towards the perimeter at angle  $\pi - y/2$  in the clockwise direction to radius  $RB$ .
  3. A robot that discovers the exit sends notification to other robots.
  4. Upon receiving notification a robot walks to the exit using the shortest path.
- 



**Fig. 4.** Evacuation of three robots with wireless communication.



**Fig. 5.**  $|CD| = \sqrt{1 - 2x \cos(\alpha - x) + x^2}$

The upper bound is proved in the following theorem.

**Theorem 6.** *It is possible to evacuate three robots from an unknown exit located on the perimeter of the disk in time at most  $\frac{4\pi}{9} + \frac{2\sqrt{3}+5}{3} + \frac{1}{600} < 4.22$  in the model with wireless communication.*

The lower bound is proved in the following theorem.

**Theorem 7.** *Any algorithm takes at least  $1 + \frac{2}{3} \arccos(-\frac{1}{3}) + \frac{4\sqrt{2}}{3} \sim 4.159$  time in the worst case for three robots to evacuate from an unknown exit located in the perimeter of the disk.*

## 4 $k$ Robots

We prove asymptotically tight bounds for  $k$  robots in both the non-wireless and wireless models.

### 4.1 Non-wireless communication

The trajectory of the robots is given in algorithm  $\mathcal{A}_4$ .

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**Algorithm  $\mathcal{A}_4$**  [for  $k$  robots with wireless communication]

1. The  $k$  robots “spread” at equal angles  $2\pi/k$  and they all reach the perimeter of the disk in time 1.
  2. Upon reaching the perimeter, they all move clockwise along the perimeter for  $2\pi/k$  time units.
  3. In one time unit, all robots move to the center of the disk. Since at least one robot has found the exit it can inform the remaining robots.
  4. In one additional time unit all robots move to the exit.
- 

**Theorem 8.** *It is possible to evacuate  $k$  robots from an unknown exit located on the perimeter of the disk in time  $3 + \frac{2\pi}{k}$  in the model with local communication.*

*Proof.* (Theorem 8) Clearly the algorithm  $\mathcal{A}_4$  is correct and attains the desired upper bound.

The following technical lemma provides bounds on the sin and cos functions based on their corresponding Taylor series expansions.

**Lemma 5.** *For any  $x \geq 0$  the following bound on values of  $\sin x$  and  $\cos x$  hold:*

- (1)  $\sin x \geq x - x^3/3!$
- (2)  $\cos x \leq 1 - x^2/2! + x^4/4!$

**Theorem 9.** *It takes time at least  $3 + \frac{2\pi}{k} + O(k^{-2})$  in the worst case to evacuate  $k$  robots from an unknown exit located on the perimeter of the disk in the model without wireless communication.*

*Proof.* (Theorem 9) We have by Lemma 3 that the evacuation time  $T$  of any evacuation algorithm  $\mathcal{A}$  satisfies  $T \geq \min\{3 + \frac{\alpha\pi}{k}, 3 + 2 \sin(\pi - \alpha\pi/2)\}$ . If we set  $\alpha = 2k/(k+1)$  then taking into account Lemma 5 we obtain:

$$\begin{aligned} T &\geq \min \left\{ 3 + \frac{\pi}{k+1}, 3 + 2 \sin \left( \frac{\pi}{k+1} \right) \right\} \geq 3 + \frac{\pi}{k+1} - \frac{\pi^3}{3!(k+1)^3} \\ &= 3 + \frac{\pi}{k} - \frac{\pi}{k(k+1)} - \frac{\pi^3}{3!(k+1)^3} = 3 + \frac{\pi}{k} - O(k^{-2}), \end{aligned}$$

This proves the theorem.

For  $k \geq 3$  robots we conjecture that the time  $T$  required to find a exit on the perimeter of a disk is exactly  $3 + \frac{2\pi}{k}$ .

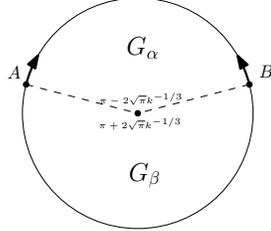
## 4.2 Wireless communication

The trajectory of the robots is given in algorithm  $\mathcal{A}_5$ .

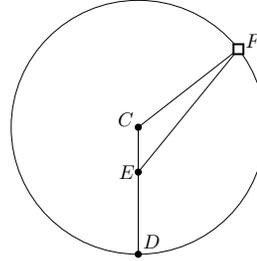
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**Algorithm  $\mathcal{A}_5$**  [for  $k$  robots with wireless communication]

1. Divide the team of robots into two groups: Group  $G_\alpha$  of size  $k_\alpha = \lceil k^{2/3} \rceil$ , and Group  $G_\beta$  of size  $k_\beta = k - k_\alpha$ .
  2. Assign a continuous arc  $\widehat{AB}$  of length  $\pi - 2\sqrt{\pi}k^{-1/3}$  to group  $G_\alpha$  and remaining part of the perimeter denoted by  $\widehat{BA}$  (of length  $\pi + 2\sqrt{\pi}k^{-1/3}$ ) to group  $G_\beta$ .
  3. Divide arcs  $\widehat{AB}$  and  $\widehat{BA}$  equally between members of groups. Each robot belonging to  $G_\alpha$  is assigned an arc of length  $a_\alpha = |\widehat{AB}|/k_\alpha$ . Each robot from group  $G_\beta$  receives an arc of length  $a_\beta = |\widehat{BA}|/k_\beta$ .
  4. Each robot goes from the center to the perimeter and explores an assigned arc. Extremal robots from group  $G_\alpha$  when exploring the assigned arcs go towards each other (see Figure 6). All other robots explore assigned arcs in any direction. A robot that discovers the exit sends notification to all other robots using wireless communication.
  5. Upon receiving a notification about the position of the discovered exit, a robot takes the shortest chord to the exit.
  6. Robots from group  $G_\beta$  after finishing exploration of their arcs start moving towards the center.
- 



**Fig. 6.** Extremal (leftmost and rightmost) robots from group  $G_\alpha$  are moving towards the interior of the arc  $\widehat{AB}$ .



**Fig. 7.**  $|DE| + |EF| < |DC| + |CF|$

**Theorem 10.** *If  $k \geq 100$  then it is possible to evacuate  $k$  robots from an unknown exit located in the perimeter of the disk in time  $3 + \frac{\pi}{k} + O(k^{-4/3})$ , in the model with wireless communication.*

*Proof.* (Theorem 10) Consider the evacuation time of the algorithm  $\mathcal{A}_5$ . Note that since  $k \geq 100$  then  $k - \lceil k^{2/3} \rceil \geq \lceil k^{2/3} \rceil$  implying that  $a_\alpha > a_\beta$ . Thus robots from  $G_\beta$  finish exploration first and start going towards the center while robots from  $G_\alpha$  are still exploring (point 6. in the pseudocode). We will show an upper bound on evacuation time  $T$  of the algorithm. Consider two cases:

Case 1. The exit is located within the arc  $\widehat{AB}$ .

Consider the evacuation time  $T_\beta$  of robots from group  $G_\beta$ . Observe that since  $\epsilon > 1$ , then  $a_\alpha < 1$  thus the exit is discovered while robots from  $G_\beta$  are walking towards the center (before they reach the center). Robots from  $G_\beta$  start moving towards the center at time  $1 + a_\beta$ . At some time  $t'$  satisfying  $2 + a_\beta > t' > 1 + a_\beta$  the exit is discovered by a robot from group  $G_\alpha$ . Consider a trajectory taken by a robot  $r$  from group  $G_\beta$  starting from time  $1 + a_\beta$ . If  $r$  would simply walk to the center and then from the center to the exit (location of the exit would be known by the time when  $r$  reaches the center). The time would be not more than  $3 + a_\beta$ . By the triangle inequality the path taken by robot  $r$  acting according to the algorithm is shorter (see Figure 7). Thus the evacuation time  $T_\beta$  for robots belonging to team  $G_\beta$  is at most

$$\begin{aligned} T_\beta &\leq 3 + a_\beta \leq 3 + \frac{\pi + 2\sqrt{\pi}k^{-1/3}}{k - k_\alpha} \\ &= 3 + \frac{\pi + 2\sqrt{\pi}k^{-1/3}}{k} + \frac{(\pi + 2\sqrt{\pi}k^{-1/3})\lceil k^{2/3} \rceil}{k(k - \lceil k^{2/3} \rceil)} = 3 + \frac{\pi}{k} + O(k^{-4/3}). \end{aligned}$$

Consider now the evacuation time of robots from group  $G_\alpha$ . Assume that the exit is discovered at time  $1 + x$  for some  $0 \leq x \leq a_\alpha$ . Since the extremal robots from group  $G_\alpha$  are walking towards each other at the time moment  $1 + x$  two arcs of length  $x$  has been explored starting from each endpoint of arc  $\widehat{AB}$ . Thus the distance on the perimeter between extremal unexplored points of arc  $\widehat{AB}$  is  $\pi - 2\sqrt{\pi}k^{-1/3} - 2x$ . Hence the maximum length of a chord connecting two unexplored points of arc  $\widehat{AB}$  in this moment is  $2 \sin((\pi - 2\sqrt{\pi}k^{-1/3} - 2x)/2)$ . Therefore the time  $T_\alpha$  until evacuation of all robots from group  $G_\alpha$  is at most

$$\begin{aligned} T_\alpha &\leq \max_{0 \leq x \leq a_\alpha} \left\{ 1 + x + 2 \sin \left( \frac{\pi - 2\sqrt{\pi}k^{-1/3} - 2x}{2} \right) \right\} \\ &= \max_{0 \leq x \leq a_\alpha} \left\{ 1 + x + 2 \cos(\sqrt{\pi}k^{-1/3} + x) \right\}. \end{aligned}$$

The function  $f(x) = 1 + x + 2 \cos(\sqrt{\pi}k^{-1/3} + x)$  has derivative  $f'(x) = 1 - 2 \cos(\sqrt{\pi}k^{-1/3} + x)$ . For  $k \geq 100$  we have that  $2\sqrt{\pi}k^{-1/3} + a_\alpha \leq \pi/6$ . Thus  $\cos(\sqrt{\pi}k^{-1/3} + x) \leq 1/2$  for all  $x \in [0, a_\alpha]$ , which implies that the function  $f(x)$  is non-decreasing in the considered set. In order to find the maximum it is sufficient to consider its value at the extremal point  $a_\alpha$ .

$$\begin{aligned} T_\alpha &\leq 1 + a_\alpha + 2 \sin(\pi/2 - (\sqrt{\pi}k^{-1/3} + a_\alpha)) \\ &= 1 + \frac{\pi - 2\sqrt{\pi}k^{-1/3}}{\lceil k^{2/3} \rceil} + 2 \cos \left( \sqrt{\pi}k^{-1/3} + \frac{\pi - 2\sqrt{\pi}k^{-1/3}}{\lceil k^{2/3} \rceil} \right) \end{aligned}$$

$$\begin{aligned}
&\leq 1 + \frac{\pi - 2\sqrt{\pi}k^{-1/3}}{\lceil k^{2/3} \rceil} + 2 - \left( \sqrt{\pi}k^{-1/3} + \frac{\pi - 2\sqrt{\pi}k^{-1/3}}{\lceil k^{2/3} \rceil} \right)^2 \\
&\quad + \left( \sqrt{\pi}k^{-1/3} + \frac{\pi - 2\sqrt{\pi}k^{-1/3}}{\lceil k^{2/3} \rceil} \right)^4 / 12 \\
&\leq 3 + O(k^{-4/3})
\end{aligned}$$

Thus in this case the evacuation time  $T \leq \max\{T_\alpha, T_\beta\} \leq 3 + \frac{\pi}{k} + O(k^{-4/3})$ .

*Case 2. The exit is located within arc  $\widehat{BA}$ .*

Each robot from group  $G_\beta$  explores an arc of length  $(\pi + 2\sqrt{\pi}k^{-1/3})/(k - k_\alpha)$ . Thus time until the exit is discovered is at most  $1 + (\pi + 2\sqrt{\pi}k^{-1/3})/(k - \lceil k^{2/3} \rceil)$ . Since we are in the wireless communication model, each robot is notified immediately and needs additional time at most 2 to go to the exit. Thus the total evacuation time in this case is at most

$$\begin{aligned}
T &\leq 3 + \frac{\pi + 2\sqrt{\pi}k^{-1/3}}{k - k^{2/3} - 1} \\
&= 3 + \frac{\pi + 2\sqrt{\pi}k^{-1/3}}{k} + \frac{(\pi + 2\sqrt{\pi}k^{-1/3})(k^{2/3} + 1)}{k(k - k^{2/3} - 1)} \\
&= 3 + \frac{\pi}{k} + O(k^{-4/3})
\end{aligned}$$

This completes the proof of Theorem 10.

**Lemma 6.** *For any  $k \geq 2$  and  $x$  satisfying  $\pi/k \leq x < 2\pi/k$  and any evacuation algorithm it takes time at least  $1 + x + 2\sin(xk/2)$  to evacuate from an unknown exit located in the perimeter of the disk.*

**Theorem 11.** *It takes at least  $3 + \frac{\pi}{k}$  time in the worst case to evacuate  $k \geq 2$  robots from an unknown exit located on the perimeter of the disk in the model with wireless communication.*

*Proof.* (Theorem 11) This is a direct consequence of Lemma 6 where  $x = \pi/k$ .

## 5 Conclusion

We studied the evacuation problem for  $k$  robots in a disk of unit radius and provided several algorithms in both non-wireless and wireless communication models for  $k = 2$  and  $k = 3$  robots. For the case of  $k$  robots we were able to give asymptotically tight bounds thus indicating a clear separation between the non-wireless and the wireless communication models. There are many interesting open questions. An interesting challenge would be to tighten our bounds or even determine optimal algorithms for  $k = 2, 3$  robots. Another interesting class of problems is concerned with evacuation from more than one exit, or with robots having distinct maximal speeds. Finally, the geometric domain being considered, the starting positions of the robots, as well as the communication model provide challenging variants of the questions considered in this paper.

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