Young diagrams, the Homfly skein of the annulus and unitary invariants.

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Introduction. The aim of this article is to describe how the algebra of Young diagrams underlies both the Homfly and the unitary invariants of links, and how it features in the construction of manifold invariants from either route. This, and the rather fuller accounts of parts of the material in [1] and [3], can be taken as a sequel to the discussion in [10].

The algebra of Young diagrams. The Young diagram algebra Y is well-known classically in the context of representations of the unitary groups $SU(N)$. A Young diagram λ is determined by positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ and is usually viewed as a left-justified array of cells, with λ_i cells in row i. Write $|\lambda| = \sum \lambda_i$ for the total number of cells, $c(\lambda) = \lambda_1$ for the number of columns and $\rho(\lambda) = k$ for the number of rows in λ. The algebra Y consists of linear combinations of Young diagrams. The product of two diagrams λ and μ is defined as an integer linear combination of diagrams by the Littlewood-Richardson rules for multiplication, as described for example in [7], which is an excellent reference for much of the classical material on Young diagrams and representation theory. The algebra Y itself is isomorphic to the polynomial algebra $\mathbf{Z}[c_1, c_2, \ldots, c_j, \ldots]$, where c_j is the Young diagram consisting of a single column with j cells. There is an explicit determinantal formula for any Young diagram λ as a polynomial in $\{c_j\}$. This formula, known variously as the Giambelli formula or the Jacobi-Trudi identity, presents λ in Y as the determinant of a $c(\lambda) \times c(\lambda)$ matrix M. The entries m_{ij} in M are all single-column diagrams given, under the convention that $c_0 = 1$ and $c_r = 0$ for $r < 0$, by $m_{ij} = c_{\lambda_i^\vee + j - i}$ where the ith column of λ has $\lambda_i^\vee$ cells. Thus the columns of λ appear on the diagonal of M, and along each row of M the entries are columns of steadily increasing length.

A very similar description of Y as a polynomial algebra in variables $\{d_j\}$ can be given, where d_j is the Young diagram consisting of a single row with j cells. The transition from variables $\{c_i\}$ to $\{d_j\}$ is readily made by the following power series relation. Write $c(X) = \sum_{i=0}^{\infty} (-1)^i c_i X^i$ and $d(X) = \sum_{j=0}^{\infty} d_j X^j$. Then $c(X)d(X) = 1$ as formal power series with coefficients in Y.

The representation ring $\mathcal{R}_N$ of $SU(N)$ is an algebra in which multiplication comes from the tensor product of representations. It can be described in terms

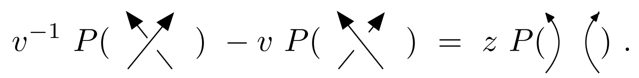
* Second author supported by EPSRC grant GR/J72332.

of the algebra Y by means of a surjective homomorphism $Y \to \mathcal{R}_N$ in which $c_N \mapsto 1$ and $c_j \mapsto 0$ for $j > N$, giving $\mathcal{R}_N \cong \mathbf{Z}[c_1, c_2, \ldots, c_{N-1}]$ as a polynomial algebra. In this setting the fundamental N-dimensional representation corresponds to the Young diagram c_1 with a single cell, and its jth exterior power to the Young diagram c_j. Each irreducible representation is the image of a single Young diagram with at most $N-1$ rows. The Littlewood-Richardson rules essentially stem from the description of the tensor product of irreducibles in $\mathcal{R}_N$.

The framed Homfly skein of the annulus.

We give here a brief account of the skein theory determined by the 2-variable Homfly polynomial, where the use of one further variable allows framed links to be handled readily. This is the natural context in which to discuss both satellite knots and quantum group invariants, as each requires for their definition that a choice of framing has been made for the link components.

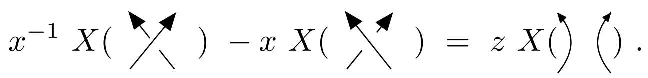
The Homfly polynomial $P_L(v,z)$ of an oriented link L is an ambient isotopy invariant of L. It is determined, up to some fixed scalar multiple, by the skein relation



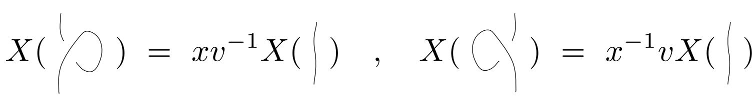
Here we shall choose the scalar so that $P = 1$ for the empty knot.

There is a framed version X of the Homfly polynomial, depending on a choice of framing of L, which is given by $X_L(x,v,z) = (xv^{-1})^{w(D)} P_L(v,z)$. Here $w(D)$ is the writhe of any diagram D of the framed link L which realises the chosen framing by means of the 'blackboard parallel'.

The skein relation for X, based on correctly framed diagrams, is then

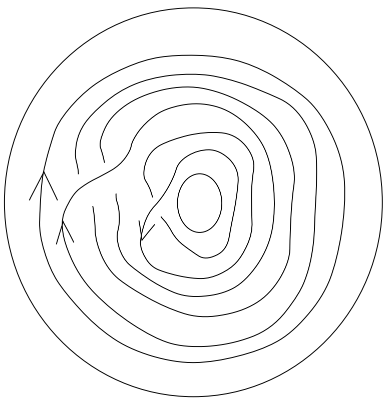


Altering the framing by insertion or deletion of curls changes X by multiples of xv^{-1}, as shown.



The framed Homfly skein of the annulus $\mathcal{C}$ consists of linear combinations of framed diagrams in the annulus, with coefficient ring $\Lambda = \mathbf{C}[x^{\pm 1}, v^{\pm 1}, z^{\pm 1}]$, up to the skein relation for X and the framing relation for curls inside the annulus. When a framed knot K is decorated by a pattern Q in the annulus the polynomial X_{K*Q} of the resulting satellite gives an invariant of K which depends on Q only as an element of $\mathcal{C}$. Write $X(L;Q_1,\ldots,Q_k)$ for the invariant of a k-component link L with patterns $Q_1,\ldots,Q_k$ applied to its components. The invariants for L constructed from this skein-theoretic approach by varying the choices of pattern are equivalent to the family of quantum group invariants based on the irreducible representations of the quantum groups $SU(N)_q$, as N varies. An early account of this can be found in [17], while there is a description along the lines of the present paper in [1]. There are also related developments in a series of papers by Akutsu, Deguchi and Wadati, particularly [4,5], which include many explicit details. The algebra Y allows some helpful transitions between the viewpoints to be made by relating it on one hand to a subalgebra of $\mathcal{C}$ and on the other to the representation ring for each $SU(N)_q$.

For the moment we shall continue purely in the skein theory context. The skein $\mathcal{C}$ forms a commutative algebra under the product induced by placing patterns side by side in parallel annuli. Write $\mathcal{C}_{\pm j}$ for the subspaces spanned by the closures of braids on j strings, oriented either in the same or the opposite sense to the core of the annulus. The algebra $\mathcal{C}$ has been studied by Turaev [16] who showed it to be the free polynomial algebra on generators $\{A_j\}, j\in \mathbf{Z}$ for an explicit choice of elements $A_j\in\mathcal{C}_j$; an alternative account by Hoste and Kidwell can be found in [8]. For $j>0$ the element A_j can be represented by the closure of the j-string braid $\sigma_{j-1}\cdots\sigma_2\sigma_1$, with strings oriented in the direction of the core of the annulus, while the same closed braid with orientation reversed represents A_{-j}. The diagram shown here represents the element $A_1A_3A_{-2}$.



The identity element of the algebra, taken as A_0, can be represented by the empty diagram. In what follows we shall restrict attention to the subalgebra $\mathcal{C}^+$ generated by $\{A_j\}$, $j\geq 0$, which contains $\mathcal{C}_j$ for each $j\geq 0$. As a linear space $\mathcal{C}^+$ is graded by

$$\mathcal{C}^+\cong\bigoplus_{j\geq 0}\mathcal{C}_j.$$

This grading is respected by the product.

The linear subspace $\mathcal{C}_j$ can itself be described in terms of another algebra. Linear combinations of braids on j strings, modulo the skein relation for X, form an algebra under braid composition which is isomorphic to the Hecke algebra H_j, [12]. Braid closure induces a surjective linear map from this algebra to the subspace $\mathcal{C}_j \subset \mathcal{C}$. At this stage it is helpful to introduce the variable s, with $z = s - s^{-1}$. Later specialisations will involve substituting $s = e^{h/2}$, to correspond with the use of quantum invariants having $q = e^h$. In the present generic setting we shall take the coefficient ring as $\Lambda_1 = \mathbf{Z}[x^{\pm 1}, v^{\pm 1}, s^{\pm 1}, (s - s^{-1})^{-1}]$, the ring of Laurent polynomials in x, v and s, extended by denominators $s - s^{-1}$. At some point we need to invert certain 'quantum integers', $[k] = \dfrac{s^k - s^{-k}}{s - s^{-1}}$, when we shall use the extended ring $\Lambda_j = \mathbf{Z}[x^{\pm 1}, v^{\pm 1}, s^{\pm 1}, (s^k - s^{-k})^{-1}; 0 < k \le j]$, for a suitable choice of j.

In H_j the elementary braids σ_i satisfy the quadratic equation $x^{-1}\sigma - x\sigma^{-1} = z$ which has roots $-xs^{-1}, xs$. Two weighted sums, a_j and b_j, of the positive permutation braids, one for each choice of root, are defined by replacing each σ_i in the sum $E_j(\sigma)$ of all $j!$ positive permutation braids with $(xs^{-1})^{-1}\sigma_i$ or $-(xs)^{-1}\sigma_i$ respectively.

Thus a_j and b_j have the form

$$\sum_{\pi \in S_j} c^{l(\pi)} \beta_\pi,$$

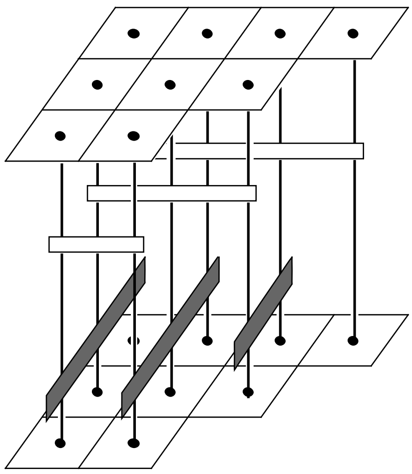
where β_π is the positive permutation braid with permutation π, $l(\pi) = wr(\beta_\pi)$ is its writhe, and $c = \begin{cases} x^{-1}s & \text{for } a_j \\ -(xs)^{-1} & \text{for } b_j \end{cases}$. These elements of H_j are quasi-idempotent, satisfying $b_j^2 = s^{-(j^2-j)/2}[j]!\ b_j$ and $a_j^2 = s^{(j^2-j)/2}[j]!\ a_j$, where $[j]! = \prod_{1 \le k \le j} [k]$. They become the anti-symmetriser and the symmetriser respectively in the symmetric group algebra on setting $x = s = 1$, [10]. Write

$$C_j = \frac{s^{(j^2-j)/2}}{[j]!}\,\hat{b}_j, \ D_j = \frac{1}{s^{(j^2-j)/2}[j]!}\,\hat{a}_j$$

for the closures of the corresponding idempotents, as elements of the skein $\mathcal{C}_j$. It is possible to use Turaev's result to show that $\mathcal{C}^+$ is also freely generated by $\{C_k\}, k \ge 0$, or equally by $\{D_k\}, k \ge 0$ as in [1], provided that the coefficient ring Λ is extended to $\Lambda_\infty = \mathbf{Z}[x^{\pm 1}, v^{\pm 1}, s^{\pm 1}, (s^k - s^{-k})^{-1}; k > 0]$. We can write Turaev's element A_j in terms of $\{C_k\}$ using only $k \le j$ and so we only need the ring Λ_j to deal with $\mathcal{C}_j$.

We may then formally define an isomorphism from Y to $\mathcal{C}^+$ by $c_i \mapsto C_i$, when the coefficient ring is Λ_∞. It is satisfying to find that this isomorphism carries each single-row Young diagram d_j to D_j. This can be proved using skein theory to establish the relation $C(X)D(X) = 1$, where $C(X) = \sum(-1)^i C_i X^i$ and $D(X) = \sum D_j X^j$ are formal power series with coefficients in $\mathcal{C}$, combined with the result $c(X)d(X) = 1$ above. Indeed the isomorphism maps a general Young diagram λ to an element $Q_\lambda \in \mathcal{C}_{|\lambda|}$ which can be constructed very appealingly from a template in the shape of λ using the two sorts of quasi-idempotent, as described below. More details of these elements, based on a skein-theory version of Gyoja's idempotents for the Hecke algebras, can be found in [3], while closely related versions are described by Yokota in [18].

The most satisfactory view of the elements Q_λ is in a more 3-dimensional context of combinations of diagrams in a solid torus, rather than an annulus. We show below a diagrammatic view of a linear combination of 3-dimensional braids, whose endpoints lie on the cells of a Young diagram λ, rather than in the conventional straight line.

$e_\lambda \quad =$ 

This should be regarded as an element of the Homfly skein of $D^2 \times I$, with endpoints at the top and bottom on the template λ, and some implicit choice of parallel for each strand to determine a framing. The white boxes, following the rows of λ, contain the braid combination a_j when the box has length j, while the grey boxes, following the columns, similarly contain combinations b_j. The whole combination will be denoted by e_λ. (The notation e_λ is used in [1] for a closely related element of the Hecke algebra $H_{|\lambda|}$ given by making a specific arrangement of the $|\lambda|$ endpoints in a straight line.) In either context the element e_λ can naturally be composed with itself, and satisfies the relation $e_\lambda^2 = \alpha_\lambda e_\lambda$ for some scalar $\alpha_\lambda \in \Lambda_1$. The element Q_λ is then defined by

$$Q_\lambda = \frac{1}{\alpha_\lambda} \hat{e}_\lambda,$$

where the closure of e_λ is an element of the skein of the solid torus. It is the same as the element arising from the closure of the straightened braids, provided only that the same choice of homeomorphism of D^2 is used for both the top and the bottom in lining up the cells of λ.

In defining Q_λ in this way we have to extend the coefficient ring Λ_1 to include denominators α_λ. This has already proved necessary if we want to use the elements C_j to generate $\mathcal{C}^+$. Under the isomorphism from the Young diagram algebra Y to $\mathcal{C}^+$ the image of λ would then be expected, from the Giambelli formula, to need only denominators of the form $s^k - s^{-k}$, with a bound on k in terms of the combinatorics of λ. Calculations of α_λ, using a lemma of Yokota, confirm this very nicely, and give an explicit formula for α_λ as a product over the cells of λ.

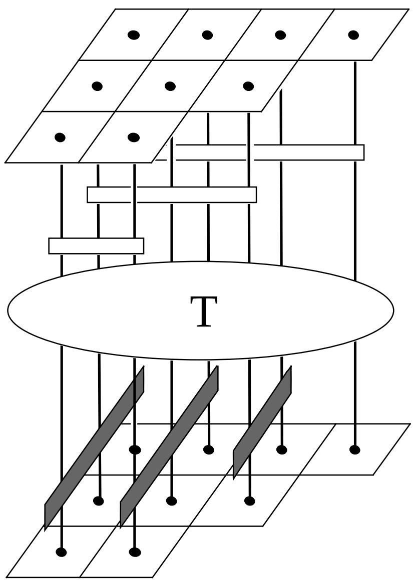
To exhibit this formula, and some later ones as well, we shall define the *hook length* and *content* of cells in a Young diagram. The cell in position (i,j) of λ has *hook length* $hl = \lambda_i + \lambda_j^\vee - i - j + 1$ and *content* $cn = j - i$. Note that the content of a cell is an absolute function of its position, while the hook length depends heavily on λ.

Then

$$\alpha_\lambda = \prod_{\text{cells}} s^{cn} \frac{s^{hl} - s^{-hl}}{s - s^{-1}}.$$

Thus the denominators in Q_λ are indeed of the form $s^k - s^{-k}$, where the largest value of k is the largest hook length of any cell. This occurs for the cell in position $(1,1)$, at the top left of λ, where its value is $\rho(\lambda) + c(\lambda) - 1$.

One striking feature of the elements e_λ, shown in [3], is their 'internal stability', namely that if any tangle T is inserted in $D^2 \times I$ between the white and the grey boxes, as shown schematically here, the resulting element of the skein is just some scalar multiple of e_λ.



The scalar T_λ, for any tangle T, is an element of Λ_1, and thus a rational function of x, v and s with only powers of $s-s^{-1}$ as denominators. When a Λ_j-linear combination of tangles is used in place of T then the resulting scalar lies in Λ_j. The fact that $e_\lambda^2 = \alpha_\lambda e_\lambda$ for some $\alpha_\lambda \in \Lambda_1$ is an immediate consequence of this, although we need to know also that $\alpha_\lambda \neq 0$ in order to construct Q_λ.

An important case is when T is the complete right-hand curl on $|\lambda|$ strings. The resulting scalar $f_\lambda \in \Lambda$ is known as the *framing factor* for λ. When the invariant $X(L; \ldots, Q_\lambda, \ldots)$ is calculated with one component of the link L decorated by Q_λ, and the framing on that component is then increased by 1, keeping the decorations of all other components unchanged, then the value of the invariant is multiplied by f_λ. This can be readily seen because the two invariants to be compared can be calculated from diagrams which differ only in having e_λ with or without the full curl inside it as one part of the complete diagram.

Calculations for f_λ have been described in a slightly different context in [11]. A direct skein theory calculation, as in [3], gives a cell-based formula

$$f_\lambda = x^{|\lambda|^2} v^{-|\lambda|} s^{n_\lambda}, \text{ where } n_\lambda = 2\sum_{\text{cells}} cn,$$

twice the sum of the content of the cells. The sum can be evaluated to give n_λ as the sum of the squares of the row lengths minus the squares of the column lengths, i.e.

$$n_\lambda = \sum \lambda_i^2 - \sum (\lambda_j^\vee)^2.$$

Alternative formulae for f_λ in terms of the weight lattice and the Casimir operator are known in the context of the specialisation of variables corresponding to the quantum group $SU(N)_q$.

Besides the full curl, another interesting central element of the Hecke algebra H_j which is less well-known from the geometric viewpoint is the sum of the Murphy operators [6], [9]. This element M, constructed as a weighted sum of the $\binom{j}{2}$ positive permutation braids coming from transpositions, determines a scalar m_λ for each λ with $|\lambda| = j$ when placed inside e_λ. These scalars m_λ distinguish all the Young diagrams, unlike the corresponding framing factors f_λ. A skein theory description and calculation is given in [3] for the cell-based formula

$$m_\lambda = \sum_{\text{cells}} s^{cn}[cn].$$

A further important consequence of the internal stability of e_λ arises in the nature of the scalar $X(L; \ldots, Q_\lambda, \ldots)$, where one component L_j of L is decorated by Q_λ. Suppose that each other component of L is decorated by some diagram in the annulus. Then there exists a scalar $\kappa \in \Lambda_1$, depending on

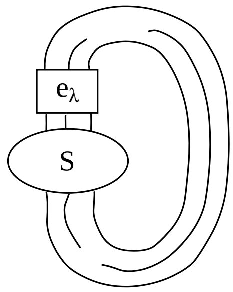
L and the choice of decorations, such that

$$X(L;\ldots,Q_\lambda,\ldots)=\kappa X(U_0;Q_\lambda),$$

where U_0 is the unknot with trivial framing. This can be seen by writing

$$X(L;\ldots,Q_\lambda,\ldots)=\frac{1}{\alpha_\lambda}X(L;\ldots,\hat{e}_\lambda,\ldots)$$

and drawing a diagram of the decorated link with the copy of e_λ inserted in the neighbourhood of the component L_j as indicated.



Write S for the tangle consisting of all the rest of the decorated diagram. Because S contains $|\lambda|$ parallel strings around the component L_j it commutes with any tangle on $|\lambda|$ strings and so we can move the lower part of e_λ consisting of the grey boxes through S to get an equivalent element of the skein which has the tangle S inside e_λ, and is still equivalent to the composite tangle $e_\lambda S$. From the internal stability of e_λ it follows that $e_\lambda S=\kappa e_\lambda$ in the skein. Then

$$X(L;\ldots,\hat{e}_\lambda,\ldots)=\kappa X(U_0;\hat{e}_\lambda)=\alpha_\lambda\kappa X(U_0;Q_\lambda),$$

giving the result. We use this later when we make substitutions in the invariants, to note that if $X(U_0;Q_\lambda)$ becomes zero after the substitution then so does $X(L;\ldots,Q_\lambda,\ldots)$, for all L.

Quantum invariants. Reshetikhin and Turaev showed [14] how to use a finite-dimensional module V over a suitable quantum group to construct an invariant $J(K;V)$ of a framed knot K which is a power series in the quantum group parameter h. It can usually be expressed easily in terms of $q=e^h$ or $s=e^{h/2}$.

The construction extends to determine an invariant of framed oriented links when ‘coloured’ by a choice of module for each component. The invariants are multilinear under direct sums of modules, while a framed knot K coloured with a tensor product $V\otimes W$ of two modules has the same invariant as the link $K^{(2)}$ made up of two parallel copies of K when coloured by V and W respectively on the two components. It is thus usual to regard the invariants for a framed link L with k components as elements $J(L;w_1,\ldots,w_k)$ of the power series ring $\mathbf{C}[[h]]$ parametrised by a choice of $w_1,\ldots,w_k$ in the representation ring of the quantum group. Further results allow the quantum invariants of a satellite $K*Q$ when coloured by a module V to be calculated in terms of the quantum invariants of K itself, coloured by a suitable linear combination of summands of tensor product $V^{\otimes j}$ where the pattern Q is the closure of an oriented j-braid, or (j,j) tangle T. This combination can be interpreted as an element $\varphi_V(T)$ of the representation ring of the quantum group, giving $J(K*Q;V)=J(K;\varphi_V(T))$. The element $\varphi_V(T)$ depends only on Q, rather than the choice of T, although this is not immediately clear unless T is a braid.

Reshetikhin and Turaev [13,15,14] established a direct connection between the invariants determined by the quantum unitary groups $SU(N)_q$ and the Homfly polynomial invariants. For a link coloured by the fundamental N-dimensional $SU(N)_q$-module $V_\square$, they showed that the invariant $J(L;V_\square,\ldots,V_\square)$ equals $e_N(X_L)$, where $e_N:\Lambda\to\mathbf{C}[[h]]$ is the substitution in which $v\mapsto s^{-N}=e^{-Nh/2}$, $x\mapsto e^{-h/2N}$ and $z\mapsto s-s^{-1}=e^{h/2}-e^{-h/2}$. This result makes use of the fact that the R-matrix for $V_\square$ satisfies the relation $x^{-1}R-xR^{-1}=(s-s^{-1})Id$, while change of framing for $V_\square$ involves the scalar factor xv^{-1}, with the substitution above.

When $V=V_\square$ the element $J(K*Q;V)$ depends only on Q as an element of $\mathcal{C}$, once the substitution e_N is made, as indeed does $\varphi_V(T)$ as an element of the representation ring of $SU(N)_q$, which is isomorphic to $\mathcal{R}_N$ for generic q. We thus have a map $\varphi_N:\mathcal{C}^+\to\mathcal{R}_N$ with the property that

$$e_N(X(L;Q_1,\ldots,Q_k))=J(L;\varphi_N(Q_1),\ldots,\varphi_N(Q_k)).$$

This map $\varphi_N:\mathcal{C}^+\to\mathcal{R}_N$, with coefficients altered by e_N, is an algebra homomorphism and carries the element Q_λ to the irreducible $SU(N)_q$-module V_λ.

Substitutions. The element $Q_\lambda\in\mathcal{C}$, when written as a polynomial in Turaev's generators $\{A_j\}$, has coefficients in Λ_k with $k=\rho(\lambda)+c(\lambda)-1$, where λ has $\rho(\lambda)$ rows and $c(\lambda)$ columns. Skein calculations show immediately that $X(K;Q_\lambda)\in\Lambda_k$ for every choice of K, since $\alpha_\lambda X(K;Q_\lambda)=X(K;\hat{e}_\lambda)\in\Lambda_1$.

Write $\delta(\lambda)$ for the rational function $\delta(\lambda)=X(U_0;Q_\lambda)$, where U_0 is the unknot with zero framing. The induced map $\delta:\mathcal{C}\to\Lambda_\infty$ is a ring homomorphism, determined by the values $\delta(C_i)$. Direct skein calculation in [1] shows that

$$\delta(C_i)=(-1)^i\frac{v-v^{-1}}{s-s^{-1}}\ \frac{vs-v^{-1}s^{-1}}{s^2-s^{-2}}\cdots\frac{vs^{i-1}-v^{-1}s^{-i+1}}{s^i-s^{-i}}.$$

This gives $\delta(C(X))=\sum(-1)^i\delta(C_i)X^i=\displaystyle\prod_{k=0}^{\infty}\frac{1-vs^{2k+1}X}{1-v^{-1}s^{2k+1}X}$ and hence

$$\delta(D_j)=(-1)^j\frac{v-v^{-1}}{s-s^{-1}}\ \frac{vs^{-1}-v^{-1}s}{s^2-s^{-2}}\cdots\frac{vs^{-j+1}-v^{-1}s^{j-1}}{s^j-s^{-j}},$$

from the relation $C(X)D(X)=1$.

The value of $\delta(\lambda)$ for single rows and columns is thus established. For general λ it can then be found using the Giambelli formula. There is however a nice alternative formula as a product over the cells of λ, which can be proved using skein theory and induction on the number of cells to calculate $X(U_0;\hat{e}_\lambda)$, [2]. When combined with the formula for α_λ we get

$$\delta(\lambda)=\prod_{\text{cells}}\frac{v^{-1}s^{cn}-vs^{-cn}}{s^{hl}-s^{-hl}},$$

where the cell in position (i,j) of λ has hook length $hl=\lambda_i+\lambda^\vee_j-i-j+1$ and content $cn=j-i$ as above

The substitution e_N gives $e_N(\delta(\lambda))=J(U_0;V_\lambda)$ which is the 'quantum dimension' of the $SU(N)_q$-module V_λ. The alternative formula above for $\delta(\lambda)$ then gives a quantised version of the hook length formula for the module dimension in terms of quantum integers $[k]=(s^k-s^{-k})/(s-s^{-1})$, as in [13]. Thus

$$e_N(\delta(\lambda))=\prod_{\text{cells}}\frac{[N+cn]}{[hl]}.$$

Because $e_N(\delta(C_i))$ is a Laurent polynomial in s for each i then so is $e_N(\delta(\lambda))$, for each λ, using, for example the Giambelli formula. Thus all the denominators $[hl]$ in the formula above must cancel out.

We have already noted that $X(L;\ldots,Q_\lambda,\ldots)=\kappa\delta(\lambda)$, where $\kappa\in\Lambda_1$ if the other decorations of L are Λ_1-linear combinations of diagrams in the annulus, and generally $\kappa\in\Lambda_j$ when the other decorations use Λ_j-linear combinations of diagrams. Suppose that $e:\Lambda_\infty\to\Lambda'$ is a ring homomorphism, determined by the values $e(x),e(v)$ and $e(s)$ for x,v and s, such that none of the denominators evaluates to 0. It follows that if $e(\delta(\lambda))=0$ then $e(X(L;\ldots,Q_\lambda,\ldots))=0$. In such cases the evaluation $e(X(L;\ldots,Q,\ldots))$ depends only on Q modulo the ideal in $\mathcal{C}^+$ generated by Q_λ.

If $e(v)=e(s^{-N})$ then $e(\delta(C_j))=0$ for $j>N$, and the ideal generated by $C_j,j>N$ has no effect on the value of $e(X(L;\ldots))$. Further skein calculation shows that if $e(x^{-N})=e(s)$ then the ideal generated by C_N-1 can also be factored out of the decorating algebra $\mathcal{C}^+$. Write $\mathcal{I}_N\subset\mathcal{C}^+$ for the ideal generated by $\{C_j\},j>N$ and C_N-1. An important example occurs when $\Lambda'=\mathbf{Z}[s^{\pm1/N},(s^k-s^{-k})^{-1};k\in\mathbf{Z}]$ and $e_N:\Lambda_\infty\to\Lambda'$ is defined by $e_N(x)=s^{-1/N}$, $e_N(s)=s$ and $e_N(v)=s^{-N}$, giving essentially the power-series substitution above. Let Q and Q' be elements of $\mathcal{C}^+$ with $Q-Q'\in\mathcal{I}_N$. Then $e_N(X(L;\ldots,Q,\ldots))=e_N(X(L;\ldots,Q',\ldots))$, where the decorations of the other components of L agree in the two cases. Since $\mathcal{C}^+/\mathcal{I}_N\cong\mathbf{Z}[C_1,\ldots,C_{N-1}]$ we may replace all decorations in $X(L;\ldots)$ by suitable polynomials in $C_1,\ldots,C_{N-1}$ without altering the evaluation under e_N. This evaluation will therefore lie in the subring $e_N(\Lambda_{N-1})\subset\Lambda'$ whose only denominators are $s^k-s^{-k}, 0<k<N$. This fact will be used in the next section where we want to substitute a root of unity for s in certain link invariants.

Roots of unity.

Potential problems arise if we want to substitute a root of unity for s in the ring Λ_∞. However we shall show that we can avoid difficulties with denominators when making substitutions in the Homfly invariants of links by restricting coefficients to a suitable Λ_k.

The most striking example where this can be done effectively gives a direct lead to the construction of 3-manifold invariants based on the quantum group $SU(N)_q$, via the Homfly skein. At the level of link invariants we first choose a value of N, and then pick a root of unity $s=e^{m\pi i/r}$ with $r>N$ and $(m,r)=1$. We shall then make the substitution e given by $e(x)=e^{-m\pi i/rN}$, $e(s)=e^{m\pi i/r}$ and $e(v)=e^{-Nm\pi i/r}$. This can be seen as a composition of $e_N:\Lambda_\infty\to\Lambda'$ followed by the substitution of $s=e^{m\pi i/r}$ to reach $\mathbf{C}$, although it is not properly defined for *all* elements of Λ_∞. However if we start with any link invariant $X(L;\ldots)$ where the decorations are by Λ_{r-1}-linear combinations of Young diagrams (or more properly elements $Q_\lambda\in\mathcal{C}^+$) then the result of the last section shows that $e_N(X(L;\ldots))\in\Lambda'$ lies in the image of Λ_{r-1}. We hence get a well-defined complex number $e(X(L;\ldots))$ on substituting $s=e^{m\pi i/r}$.

The decorations of L can again be altered in certain ways without affecting the final evaluation. We know already that elements of the ideal $\mathcal{I}_N$ will not affect the evaluation e_N. In addition we can see from the formula for $\delta(D_j)$ that $e(\delta(D_j))=0$ for the $N-1$ values $l+1\le j\le r-1$, where $l=r-N$ is known as the *level* for the substitution.

Write $\mathcal{I}_{N,l}$ for the ideal in $\mathcal{C}^+$ generated by $\{D_j\}, l+1\le j\le r-1$ and $\mathcal{I}_N$. Assuming again that we use only Λ_{r-1}-linear combinations of Q_λ in $\mathcal{C}^+$ then $e(X(L;\ldots))$ is well-defined and depends only on the choice of decoration in $\mathcal{C}^+$ modulo $\mathcal{I}_{N,l}$.

[Some such restriction on coefficients is needed to exclude elements such as $(s^r-s^{-r})^{-1}D_{r-1}$ from $\mathcal{I}_{N,l}$].

For suppose that Q and Q' are appropriate decorations in $\mathcal{C}^+$ with $Q-Q'\in\mathcal{I}_{N,l}$. Write $Q-Q'=\sum R_jD_j+RI$ where R and R_j are Λ_{r-1}-linear combinations of Q_λ, and $I\in\mathcal{I}_N$. Then

$$X(L;\ldots,Q,\ldots)-X(L;\ldots,Q',\ldots)=\sum_j X(L^{(2)};\ldots,R_j,D_j,\ldots)$$
$$+X(L^{(2)};\ldots,R,I,\ldots),$$

where $L^{(2)}$ is given from L by doubling the component decorated by Q, and decorations on other components are the same in all cases. Now

$$e_N(X(L^{(2)};\ldots,R,I,\ldots))=0$$

and we can also change all the decorations except D_j to polynomials in $C_1,\ldots,C_{N-1}$ in the other cases without altering the evaluation. We finally note that

$$X(L^{(2)};\ldots,D_j,\ldots)=\kappa\delta(D_j),$$

where $\kappa\in\Lambda_{r-1}$ if the decorations are by polynomials in $C_1,\ldots,C_{N-1}$ with coefficients in Λ_{r-1}. Hence $e(X(L;\ldots,Q,\ldots))=e(X(L;\ldots,Q',\ldots))$.

The natural parameter space for $SU(N)_q$ invariants when they are to be evaluated at $s=e^{m\pi i/r}$, with $r=N+l$ is thus the *Verlinde algebra* $V_{N,l}=\mathcal{C}_+/\mathcal{I}_{N,l}$. This is a finite-dimensional algebra of dimension $\binom{r-1}{N-1}$, seen by first factoring out $\mathcal{I}_N$ to get $\mathbf{C}[C_1,\ldots,C_{N-1}]$ and then the $N-1$ polynomials $D_{l+1},\ldots,D_{r-1}$, as in [1].

The Verlinde algebra $V_{N,l}$ has a basis which can be represented by the Young diagrams λ with $\rho(\lambda)\le N-1$ and $c(\lambda)\le l$. In terms of this basis the structure constants $b^{\nu}_{\lambda\mu}$, where $\lambda\mu=\sum b^{\nu}_{\lambda\mu}\ \nu$, are integers with important symmetry properties which can be established quickly within the context of the Young diagram algebra, Y. To explain this we shall start by making a fixed choice of N. Then each Young diagram λ has a dual λ^*, corresponding to the dual of the irreducible $SU(N)_q$-module V_λ. It can be defined purely combinatorially as the complement of λ in an $N\times\lambda_1$ rectangular array of cells, after rotation through π. In the representation ring $\mathcal{R}_N$ we may write $\lambda\mu=\sum a_{\lambda\mu\nu}\nu^*$, summed over all ν with at most $N-1$ rows. The coefficients $a_{\lambda\mu\nu}$ are non-negative integers which are symmetric under all permutations of λ, μ and ν, as $a_{\lambda\mu\nu}$ can be interpreted as the multiplicity of the trivial representation in the product $\lambda\mu\nu$.

When we pass to the quotient $V_{N,l}=\mathcal{R}_N/\mathcal{I}_{N,l}$ we can write $\lambda\mu=\sum b_{\lambda\mu\nu}\nu^*$ where λ,μ and ν are now diagrams lying inside an $(N-1)\times l$ rectangle. A combinatorial calculation on Young diagrams modulo $\mathcal{I}_{N,l}$ shows that the integers $b_{\lambda\mu\nu}$ are again symmetric under *all* permutations of λ,μ and ν. (It is enough to show firstly that the diagrams in the rectangle, which span $V_{N,l}$, are linearly independent, so that $b_{\lambda\mu\nu}$ is well-defined, and secondly that the coefficient of the empty diagram in $V_{N,l}$ for the product $\nu\rho$ of diagrams in the rectangle is 1 when $\rho=\nu^*$ and 0 otherwise.)

Manifold invariants.

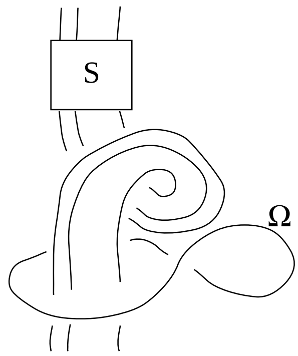
Construction of an invariant of a 3-manifold M presented by surgery on a framed link L uses the symmetry property above, [1]. Starting from the choice of N, and $s = e^{m\pi i/r}$ as above we can define the element $\Omega = \sum \delta(\mu^*)Q_\mu$, where μ runs over the Young diagram basis for $V_{N,l}$. Then decorate all components of L with Ω. The element $X(L;\Omega,\ldots,\Omega)$, when evaluated with the substitution e above gives a complex number which, after a simple normalisation, can be shown by the symmetry property to depend only on M.

A central part of the proof, using the Kirby moves, is to establish that $\lambda\Omega = \delta(\lambda)\Omega$ modulo $\mathcal{I}_{N,l}$, after the substitution e. Working modulo $\mathcal{I}_{N,l}$ in the Young diagram algebra Y we can start with $\Omega = \sum \delta(\mu^*)\mu$, where sums are over Young diagrams which lie in an $(N-1)\times l$ rectangle. In this range all the coefficients $\delta(\mu^*)$ belong to Λ_{r-1}. Then

$$\begin{aligned}
\lambda\Omega &= \sum \delta(\mu^*)\lambda\mu \\
&= \sum_{\mu,\nu} \delta(\mu^*) b_{\lambda\mu\nu}\nu^* \text{ modulo } \mathcal{I}_{N,l} \\
&= \sum (b_{\lambda\nu\mu}\delta(\mu^*))\nu^* \text{ by symmetry} \\
&= \sum_{\nu} \delta(\lambda\nu)\nu^*,
\end{aligned}$$

after evaluation of coefficients, since $\lambda\nu = \sum b_{\lambda\nu\mu}\mu^*$ modulo $\mathcal{I}_{N,l}$ and the ring homomorphism δ is zero on $\mathcal{I}_{N,l}$. Then $\lambda\Omega = \sum \delta(\lambda\nu)\nu^* = \sum \delta(\lambda)\delta(\nu)\nu^* = \delta(\lambda)\Omega$ modulo $\mathcal{I}_{N,l}$.

The other major step involves showing that if L and L' are related by a positive Kirby move then, after evaluation, $X(L;\Omega,\ldots,\Omega) = c_+X(L';\Omega,\ldots\Omega)$, where the constant c_+ is independent of L. A diagram for L when decorated by Ω can be presented as the unknot U_0 with zero framing decorated by a suitable element $\hat{S}\in\mathcal{C}$, while L' similarly decorated can be presented as the closure of the tangle shown below.



This diagram can also be viewed as the unknot U_1 with framing 1 decorated by the product $\hat{S}\Omega \in \mathcal{C}$. Since we are to apply the evaluation e we can work modulo $\mathcal{I}_{N,l}$ and use the result above to replace $\hat{S}\Omega$ by $\delta(\hat{S})\Omega$ in the calculation. Thus

$$e(X(L;\Omega,\ldots,\Omega)) = e(X(U_0;\hat{S})) = e(\delta(\hat{S}))$$

and

$$\begin{aligned} e(X(L';\Omega,\ldots,\Omega)) &= e(X(U_1;\hat{S}\Omega)) \\ &= e(X(U_1;\delta(\hat{S})\Omega)) \\ &= e(X(U_1;\Omega)\delta(\hat{S})). \end{aligned}$$

We can then take $c_+ = e(X(U_1;\Omega)) = e(\sum f_\mu \delta(\mu *)\delta(\mu))$, with a similar constant c_- for the negative Kirby move.

In the description here it is primarily the properties of the Young diagram algebra which govern the behaviour of the manifold invariant; it is clear that it can be calculated using either a choice of Homfly invariants or of generic $SU(N)_q$ invariants and then making a suitable choice of substitution. A little care, as described above, is needed in making some of the replacements and substitutions in the right order to ensure that zero denominators can legitimately be avoided.

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