Amended Feb 1986 following referee's comments.
Version sent to Topology.

HYPERBOLIC 3-MANIFOLDS AND SURFACE AUTOMORPHISMS

D. D. Long* and H. R. Morton

§0. In this paper, we exploit the close interplay between hyperbolic 3-manifolds and pseudo-Anosov maps which arises via the mapping torus construction. This enables us to prove results in the theory of surface automorphisms which appear to be difficult to obtain directly. In this way we extend and unify some examples of Bonahon [1] and Morton [7], which are concerned with examining how the monodromy of a fibred knot changes when composed with the Dehn twist in some simple closed curve in the fibre. The interest in this construction is motivated by the study of so-called 'Stallings twists' which generate new fibred knots from old. We show:

<u>Theorem A</u> Let $\theta : F \to F$ be a pseudo-Anosov map and $T : F \to F$ be a Dehn twist about an essential simple closed curve, c, in F.

Then

(i) for all but finitely many r, $T^r\theta$ is isotopic to a pseudo-Anosov map.

(ii) There is a K, such that for all $r \geq K$, the maps $T^r\theta$ are all nonconjugate. (And similarly for all $r \leq -K$.)

This is the content of §1. Our proof relies on certain theorems concerning hyperbolic 3-manifolds and is independent of considerations such as measured laminations and traintracks.

---

* Supported by SERC research grant.

§2 is devoted to providing information about the invariant laminations and dilatations of the maps $T^r\theta$ and their behaviour as $r \to \infty$. This behaviour is very sensitive to how often the number $i(c, \theta^j c) = 0$, where $i(.,.)$ denotes the geometric intersection number of simple closed curves. Our result on invariant laminations then becomes:

<u>Theorem B</u> With the notation of Theorem A, let L_r denote the unstable lamination of $T^r\theta$ where this is defined.

Suppose that $i(c, \theta^j c) = 0 \quad 1 \le j \le t-1$ and $i(c, \theta^t c) \ne 0$. Then given any neighbourhood V in projective measured lamination space of the convex hull of $\{c, \theta c, \ldots, \theta^{t-1}c\}$, there is a K, such that $L_r \in V$, for all $|r| \ge K$.

In the 'generic' case, namely that $i(c, \theta c) \ne 0$, this says that $L_r \to c$ in projective measured lamination space, which is the expected behaviour. We have been unable to prove in general that the sequence $\{L_r\}$ converges.

The result on the dilatation is of a similar flavour:

<u>Theorem C</u> With the notation of Theorem B, let λ_r the dilatation of $T^r\theta$, where this is defined. Then there are constants K and L such that

$$|r|.i(\theta^t c, c) - K \le |\lambda_r^t| \le L|r| + K$$

for all $|r|$ sufficiently large. In the generic case, when $i(\theta c, c) \ne 0$, and so $t = 1$, we can take $L = i(\theta c, c)$, and we can choose K arbitrarily close to $\frac{i(\theta^2 c, c)}{i(\theta c, c)}$.

Because of the generality involved we are unable to appeal to the methods of traintracks, which almost invariably require a careful alignment of c with an invariant traintrack for θ. Our proof uses formulae of Fathi which express how $i(.,.)$ changes with Dehn twisting and requires the use of Theorem B.

We use without proof many standard properties of pseudo-Anosov maps and measured lamination spaces, both projectivised and unprojectivised. We refer the reader to [2] for expositions and terminology.

The authors would like to thank G. P. Scott for many useful conversations.

Remark

Subsequent work of Fathi, [15], provides a satisfying generalisation of these results.

§1.

Throughout, all manifolds will be assumed to be orientable and all maps orientation preserving. A hyperbolic manifold will always be taken to be complete and of finite volume.

Given an automorphism of a closed, orientable surface $\theta : F \to F$, we form the <u>mapping torus</u> $M(\theta)$ by taking $F \times I$ and glueing $F \times \{0\}$ to $F \times \{1\}$ via θ. This gives a surface bundle over S^1 whose homeomorphism type determines its holonomy map, θ, up to conjugacy and isotopy.

The following construction, first introduced by Stallings in [10], relates a composition of the holonomy map and a Dehn twist, to a Dehn surgery on the mapping torus. Formally, if $T : F \to F$ denotes a Dehn twist in an essential simple closed curve c in F, then it is easily seen (for example, see [5], [10]) that the manifold $M(T^r\theta)$ is obtained from $M(\theta)$ by doing a $(1\ r)$ surgery on $\overline{M(\theta) \setminus N(c)}$, relative to a suitable choice of longitude in $\partial N(c)$.

This construction has been used extensively in the theory of fibred knots, although in that case it is more usual to make a mapping torus using a bounded surface. In this context, it is frequently assumed that θ fixes ∂F and one makes the further identifications of $\{x\} \times I$ for each $x \in \partial F$, so that $M(\theta)$ is a closed manifold, usually S^3.

The construction is then constrained to require that C be unknotted and have zero linking with a push-off. These conditions are designed to ensure that if $M(\theta) = S^3$, then $M(T^r\theta) = S^3$. This will not concern us, consequently we make no such restriction.

We specialise now to the case that θ is pseudo-Anosov. (Observe this carries an implicit assumption that genus $(F) \geq 2$.)

__Lemma 1.1__ Let c be an essential simple closed curve on the surface F and $\theta : F \to F$ a pseudo-Anosov map. Define $W = M(\theta) \setminus c \times \{\frac{1}{2}\}$. Then W contains no incompressible, nonperipheral embedded tori.

__Proof__ Since c is essential, W is irreducible. Let $F' = F \setminus \mathring{c}$, which we identify with the $\frac{1}{2}$-level in W. Let T be an embedded incompressible torus in W.

Consider the components of $T \setminus F'$. Since F is not a torus, $F \times I$ contains no incompressible torus, and so $T \pitchfork F' \neq \emptyset$. Using the irreducibility of W and the incompressibility of F', we may isotope T to arrange that all these components are annuli.

Suppose there is an annulus component A which misses some fibre. Then by cutting W open along F' and the fibre which is avoided, we see that A lies in a 'product' $F \times I \setminus c \times \{0\}$ with ∂A lying in one boundary component of this product. Thus the components of ∂A are freely homotopic in $F \times I \setminus c \times \{0\}$. There are then two possibilities.

Either the components of ∂A are freely homotopic in F', in which case they bound an annulus in F', or they are parallel to c and on opposite sides of some small neighbourhood of $c \subset F$. All the former intersections may be removed by an isotopy of T.

Thus we may assume that all components of $T \setminus F'$ meet F' in this latter way, or run around the S^1 factor in $M(\theta)$.

Denoting the free homotopy class by $[.]$, a pseudo-Anosov map has the property that for all essential simple closed curves P, $[\theta^k P] = [\theta^\ell P]$ if and only if $k = \ell$.

Further, observe that the only incompressible ∂-compressible annuli in $F \times I$ are isotopic to products $P \times I$. Thus any annulus running around the S^1 factor has boundary components P and θP.

Putting together the remarks of the previous two paragraphs, it is now easy to see that there can be no annuli running around the S^1-factor, since these can never close up.

Thus the only annuli possible miss a fibre and have boundary components parallel to and on opposite sides of c. It is now clear that there are only two such components and that they close up to a peripheral torus. $\square$

Theorem 1.2 Let $\theta : F \to F$ be a pseudo-Anosov map, and $T : F \to F$ the Dehn twist about some essential simple closed curve $c \subset F$.

Then for all but finitely many r, $T^r\theta$ is isotopic to a pseudo-Anosov map.

Proof Form the mapping torus $M(\theta)$. Then we claim that $W = M(\theta) \setminus c \times \{\frac{1}{2}\}$ can be made into a complete, finite volume hyperbolic 3-manifold. For, W is Haken, so by a theorem of Thurston [13], W admits a decomposition into geometric pieces and Lemma 1.1 shows that in fact only one geometric piece is involved. W cannot be a torus bundle. If W were Seifert fibred, then since the surface F is incompressible, it can be made horizontal or vertical; since it is not a torus, it must be the former. But this then implies W is a closed manifold, a contradiction. Thus we conclude that W is hyperbolic. Let $W' = M(\theta) \setminus N(c)$. Then a theorem of [12] implies that all but finitely many surgeries on W' give rise to closed, hyperbolic 3-manifolds, in particular, all but finitely many of the mapping tori $M(T^r\theta)$ are hyperbolic. We complete the proof by quoting [14], which states that a mapping torus is hyperbolic if and only if its holonomy is pseudo-Anosov. (Although we are only using the easy implication of this result.) □

Remarks (i) The natural 'companion' result to this, namely that all but finitely many of the maps $T\theta^r$ are pseudo-Anosov, is a special case of a theorem of Papadopoulos [8]. Modulo some understanding of projective lamination space, this result is much easier to prove, the reason being that pseudo-Anosovs have better contraction properties on this space than Dehn twists. It appears to be difficult to prove 1.2 directly.

(ii) It would be of some interest to show there were a bound on the number of exceptional values of r, depending only on F, or perhaps universal. For example, a theorem of Gordon and Litherland [4] asserts that there are at most six surgeries giving rise to non-irreducible manifolds.

(iii) The restriction to one simple closed curve is, of course, unnecessary, though some care is needed in formulating the result for 1-submanifolds. The problem which can arise is that for some θ and c, $\{c, \theta c\}$ can be a 1-submanifold. In this case, the manifold $M(\theta) \setminus \{c, \theta c\}$ contains nonperipheral embedded incompressible tori, so cannot be hyperbolic. Nonetheless, we do have:

<u>Theorem 1.3</u> Let $\theta : F \to F$ be a pseudo-Anosov map and $T_i : F \to F$ be the Dehn twist on an essential simple closed curve c_i, $1 \le i \le n$, where c_i is a 1-submanifold. Then infinitely many of the maps $T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} \theta$ are pseudo-Anosov.

<u>Proof</u> By 1.2, $T_1^k \theta$ is pseudo-Anosov for infinitely many k. Fix one such k; k_1 say. Then for infinitely many k, $T_2^k (T_1^{k_1} \theta)$ is pseudo-Anosov. And so on. □

In order that 1.2 be useful, we need to be sure that infinitely many of the maps $T^r \theta$ are nonconjugate. One possible approach to this would be to examine the behaviour of the dilatation (which is a conjugacy invariant) as $r \to \infty$. However, there is a direct proof of nonconjugacy, closer to the spirit of this section:

<u>Theorem 1.4</u> Let $\theta : F \to F$ be pseudo-Anosov and $T : F \to F$ denote the Dehn twist about an essential simple closed curve $c \subset F$.

Then there is a K, such that for all $r \ge K$, the maps $T^r \theta$ are all nonconjugate. (Also, for all $r \le -K$, the maps $T^r \theta$ are all nonconjugate.)

<u>Proof</u> Consider the hyperbolic manifold $W = M(\theta) \setminus c \times \{\frac{1}{2}\}$. This has some finite volume $v(W)$. Then the methods of [12] show that $v(M(T^r \theta)) \to v(W)$ as $r \to \infty$. Now $M(T^r \theta)$ is given by $(1,r)$ Dehn surgery on $W' = M(\theta) \setminus N(c)$. Explicit estimates in [9] then show that $v(W) = v(M(T^r \theta)) + \dfrac{1}{Q(p,q)} + O(1/r^4)$, where $Q(p,q)$ is a positive definite quadratic form depending only on W. It follows that $v(M(T^r \theta))$ increases strictly with r for large enough r, $r \geqslant K$, say.

' — ' = overscore

Volume is a topological invariant, by Mostow Rigidity, and hence for $r,s \geq K$, $r \neq s$, the manifolds $M(T^r\theta)$ and $M(T^s\theta)$ are not homeomorphic. The maps $T^r\theta$ and $T^s\theta$ are then nonconjugate. □

This result implies that there are at worst pairwise conjugacy repetitions among the pseudo-Anosovs $T^r\theta$, $|r| \geq K$. Taking 1.2 and 1.4 together gives an alternative (although marginally less explicit) description of some examples of Bonahon [1] which is independent of calculations involving invariant train tracks. (Strictly speaking, we should prove 1.2 and 1.4 in the case of a bounded surface to obtain these examples; however, this presents no difficulty.)

§2. In this section we focus on the behaviour of the invariant laminations and dilatation of $T^r\theta$ as $r \to \infty$. Our methods now draw more on the theory of surface automorphisms. To state our results we must introduce some terminology.

Let us denote measured lamination space for the closed orientable surface F of genus g by ML, and its projectivisation by $\mathbb{P}ML$, a compact space homeomorphic to S^{6g-7} (see [21]). Then, given a collection of disjoint, simple essential closed curves $c_1, \ldots, c_t$ on F, we define the <u>convex hull</u> of $c_1, \ldots, c_t$, written $H\{c_1, \ldots, c_t\}$ to be the closure in $\mathbb{P}ML$ of the set of measured laminations

$$\{s_1[c_1] \amalg \ldots \amalg s_t[c_t] \mid 0 \le s_j \le 1 \ , \ \sum s_j = 1\}.$$

Here $[c]$ is taken to be the point of $\mathbb{P}ML$ corresponding to c, and more generally, $[.]$ is taken to denote the projective class of a measured lamination. Notice that $H\{c_1\} = [c_1]$.

We use the symbol $i(.,.)$ to denote the geometric intersection number of two essential simple closed curves and observe that this admits a natural continuous extension to ML. (See [14].)

We shall have need of the following lemma. A proof is contained in the Appendix to Exposé 4 of [3].

<u>Lemma 2.1</u> Let $\alpha_1, \ldots, \alpha_k$ be disjoint essential simple closed curves on F.

Let β, γ be any essential simple closed curves on F and let $\Gamma(n_1, \ldots, n_k)$ denote γ Dehn twisted n_j times in α_j, where the n_j's are all positive (or all negative).

Then

$$\left| i(\Gamma(n_1, \ldots, n_k), \beta) - \sum_j |n_j| i(\gamma, \alpha_j) i(\alpha_j, \beta) \right| \le i(\gamma, \beta).$$

<u>Theorem 2.2</u> Let $\theta : F \to F$ be pseudo-Anosov and $T : F \to F$ the Dehn twist about an essential simple closed curve $\bar{c} \subset F$. Let L_r denote the unstable lamination of $T^r\theta$.

Suppose that $i(c, \theta^k c) = 0 \quad 1 \leq k \leq t-1$ and that $i(c, \theta^t c) \neq 0$.

Then given any neighbourhood $V \subset \mathbb{P}ML$ of $H\{c, \theta c, \ldots, \theta^{t-1}c\}$, there is a K, such that for all $|r| \geq K$, $[L_r] \in V$.

<u>Proof</u> For brevity, we write $c_i = \theta^i c$. Observe that since $\theta T_c \theta^{-1} = T_{\theta c}$ we have that $(T^r\theta)^t = T_0^r T_1^r \ldots T_{t-1}^r \theta^t$, where T_j denotes the Dehn twist about c_j. Observe that the unstable lamination of $(T^r\theta)^t$ is the same as that of $T^r\theta$, so to prove the result, it is sufficient to work with the former map.

Consider now $\theta^t H\{c_0, \ldots, c_{t-1}\} = H\{\theta^t c_0, \ldots, \theta^t c_{t-1}\}$. Each point in this set has the property that it assigns nonzero measure to the lamination $\tilde{c} = c_0 \amalg c_1 \amalg \ldots \amalg c_{t-1}$ i.e. $i(\tilde{c}, L) \neq 0$ for each L with $[L] \in \theta^t H\{c_0, \ldots, c_{t-1}\}$. Hence there is a closed neighbourhood U in $\mathbb{P}ML$ of $H\{\theta^t c_0, \ldots, \theta^t c_{t-1}\}$ with this property. (Equivalently, every $[L, \mu]$ in U has the property that L has a leaf in the support of μ with transverse intersections with at least one of the set $\{c_0, \ldots, c_{t-1}\}$.)

Now we appeal to the formula of Lemma 2.1. This states in our terminology, that for all simple closed curves γ, β,

$$\left| i(T_0^r T_1^r \ldots T_{t-1}^r \gamma, \beta) - r \sum_{j=0}^{t-1} i(\gamma, c_j) i(c_j, \beta) \right| \leq i(\gamma, \beta).$$

Notice that this formula continues to hold for $\beta, \gamma \in ML$. Now if M_r, $M \in ML$ and $i(M_r, \beta) \to i(M, \beta)$ as $r \to \infty$ for each $\beta \in ML$ then $M_r \to M$ in ML.

Given any neighbourhood $W \subset \mathbb{P}ML$ of $H\{c_0, \ldots, c_{t-1}\}$, and any $[L] \in U$ we can find a k (depending on $[L]$) such that $T_0^r \ldots T_{t-1}^r[L] \in W$ for all $|r| \geq k$. For we can take $\gamma = L$ in the formula above, and $M_r = \frac{1}{r} T_0^r \ldots T_{t-1}^r L$, to see that $M_r \to M = \coprod k_j c_j$ where $k_j = i(L, c_j)$. Now $[M] \in H\{c_0, \ldots, c_{t-1}\}$, since k_j are not all zero, so eventually all $[M_r]$ lie in W.

An elementary argument based on the compactness of U now implies that there is a K, such that for all $|r| \geq K$, $T_0^r \ldots T_{t-1}^r(U) \subset W$.

Since θ^t induces a continuous map of $\mathbb{P}ML$, there is a neighbourhood V of $H\{c_0, \ldots, c_{t-1}\}$ with $\theta^t(V) \subset \text{int } U$. Choose K, so that

$$T_0^r \ldots T_{t-1}^r(U) \subset V \quad \text{for all } |r| \geq K.$$

Because $(T^r\theta)^t$ is pseudo-Anosov, if $[L, \mu]$ is any lamination in $\mathbb{P}ML$, save the stable lamination of $T^r\theta$, we have $(T^r\theta)^{tk}[L] \to [L_r]$ as $k \to \infty$. In particular, if $|r| \geq K$, we see that $[L] \in V$ implies $(T^r\theta)^{tk}[L] \in V$ for all $k > 0$, so that $[L_r] \in \overline{V}$. This implies the result.

□

<u>Remarks</u> (i) By considering $(T^r\theta)^{-1}$, a similar proof shows that the stable laminations lie close to $H\{c_{-1}, \ldots, c_{-t}\}$ as $r \to \infty$.

(ii) In the 'typical' case, namely that $i(c, \theta c) \neq 0$, this result can be interpreted as saying $[L_r] \to [c]$ in $\mathbb{P}ML$ as $r \to \infty$. We do not know if this happens in all cases. Notice that since ML is covered by open sets W_τ corresponding to the weight cones of maximal traintracks τ, (that is, traintracks all of whose complementary regions are trigons - see [2] or [8] for an exposition) in this simple case, there is a traintrack τ carrying c and all laminations L_r for

$|r|$ sufficiently large. If this traintrack is nicely aligned with respect to c, results concerning the dilatation can be obtained from linear algebra.

(iii) We can now give a 'construction-free' proof of the fact that the set $\{(L_\theta, \mu_\theta) | (L_\theta, \mu_\theta)$ is the unstable lamination of a pseudo-Anosov θ on $F\}$ is dense in $\mathbb{P}ML$.

For if c is any simple closed curve and θ is any pseudo-Anosov, there is a p, such that $i(\theta^p c, c) \neq 0$. Then setting L_r to be the unstable lamination of $T^r \theta^p$, 2.2 shows that $[L_r] \to [c]$ in $\mathbb{P}ML$. Hence the closure of the above set contains all simple closed curves, and so is all of $\mathbb{P}ML$.

In fact an analogous argument can be used to show that the normal closure in the mapping class group of any pseudo-Anosov

contains sufficient 'pseudo-Anosov laminations' to give a dense set in $\mathbb{P}ML$.

We now turn our attention to the behaviour of the dilatation of the pseudo-Anosov maps $T^r\theta$ as a function of r, as $r \to \infty$. Our result is that the growth is dependent on the first t such that $i(c, \theta^t c) \neq 0$. Let us fix our notation as usual: $\theta : F \to F$ will be a pseudo-Anosov and $T : F \to F$ will be the Dehn twist about an essential simple closed curve $c \subset F$. Suppose that $i(c, \theta^j c) = 0 \quad 1 \leq j \leq t - 1$ and $i(c, \theta^t c) \neq 0$.

Let λ_r denote the dilatation of $T^r\theta$, where this is defined.

<u>Theorem 2.3</u> There are constants, K and L, such that for all sufficiently large $|r|$ we have:

$$|r| i(\theta^t c, c) - K \leq |\lambda_r^t| \leq L|r| + K.$$

Proof We again invoke the formula of Lemma 2.1. In the notation of the proof of 2.2, if we write $T_0 T_1 \cdots T_{t-1} = \mathfrak{T}$, then the dilatation of $(T^r\theta)^t = \mathfrak{T}^r\theta^t$ is λ_r^t. Here, 2.1 becomes

$$\left| i(\mathfrak{T}^r\gamma,\ \beta) - \sum_{j=0}^{t-1} r i(\gamma,\ \theta^j c) i(\theta^j c,\ \beta)\right| \leq i(\gamma,\ \beta)$$

for all laminations $\gamma, \beta \in ML$. For brevity, write

'~' = tilde

$\tilde{c} = c \amalg \theta c \amalg \ldots \amalg \theta^{t-1} c$. Take $\gamma = \theta^t \tilde{c}$. Then we obtain

$$\left| i(\mathfrak{T}^r\theta^t\tilde{c},\ \beta) - r\sum_{j=0}^{t-1} i(\theta^t\tilde{c},\ \theta^j c)\, i(\theta^j c,\ \beta)\right| \leq i(\theta^t\tilde{c},\ \beta).$$

For any lamination β with $i(\tilde{c},\ \beta) \neq 0$, we may divide through;

$$\left| \frac{i(\mathfrak{T}^r\theta^t\tilde{c},\ \beta)}{i(\tilde{c},\ \beta)} - r\sum_{j=0}^{t-1} \frac{i(\theta^t\tilde{c},\ \theta^j c) i(\theta^j c,\ \beta)}{i(\tilde{c},\ \beta)}\right| \leq \frac{i(\theta^t\tilde{c},\ \beta)}{i(\tilde{c},\ \beta)}.$$

In particular, take β to be L_r^{st}, the stable lamination of $T^r\theta$, or rather, any representative lamination, since L_r^{st} is only defined in ML up to a real multiple. The inequality then becomes

$$\left| \lambda_r^t - r\sum_{j=0}^{t-1} \frac{i(\theta^t\tilde{c},\ \theta^j c) i(\theta^j c,\ L_r^{st})}{i(\tilde{c},\ L_r^{st})}\right| \leq \frac{i(\theta^t\tilde{c},\ L_r^{st})}{i(\tilde{c},\ L_r^{st})}$$

independent of the choice of representative of L_r^{st} because of its homogeneity.

By the remark (i) following 2.2, for $|r|$ sufficiently large, the projective class of L_r^{st} is very close to the convex hull of

$\{\theta^{-1}c, \ldots, \theta^{-t}c\}$. However, the above formula is homogeneous in L_r^{st}, thus for the purposes of estimation, it is sufficient to choose any representative in the projective class. Since every point in $H\{\theta^{-1}c, \ldots, \theta^{-t}c\}$ ascribes nonzero measure to $\tilde{c}$, we see that $i(\tilde{c}, L_r^{st})$ is bounded away from zero as $r \to \infty$ and thus there is a constant K, such that for all $|r|$ sufficiently large,

$$\frac{i(\theta^t\tilde{c}, L_r^{st})}{i(\tilde{c}, L_r^{st})} \leq K.$$

Similarly,

$$\sum_{j=0}^{t-1} \frac{i(\theta^t\tilde{c}, \theta^j c)\, i(\theta^j c, L_r^{st})}{i(\tilde{c}, L_r^{st})} \leq L$$

for all $|r|$ sufficiently large. Hence the triangle inequality implies

$$|\lambda_r^t| \leq L|r| + K.$$

For the other half of the inequality, notice that

$$i(\theta^t\tilde{c}, \theta^j c) \geq i(\theta^t c, c) \qquad \text{for} \quad 0 \leq j \leq t-1.$$

Thus

$$|\lambda_r^t| + K \geq |r|\, i(\theta^t c, c) \sum_{j=0}^{t-1} \frac{i(\theta^j c, L_r^{st})}{i(\tilde{c}, L_r^{st})} = |r|\, i(\theta^t c, c).$$

This completes the proof. □

Remark It follows from the proof, that in the 'generic' case when $i(c, \theta c) \neq 0$, we actually do a little better and obtain

$$|\lambda_r - ri(\theta c, c)| \leq \frac{i(\theta c, L_r^{st})}{i(c, L_r^{st})} .$$

Since $[L_r^{st}] \to [\theta^{-1}c]$ in this case, the right hand side converges to

$$\frac{i(\theta c, \theta^{-1}c)}{i(c, \theta^{-1}c)} = \frac{i(\theta^2 c, c)}{i(\theta c, c)} .$$

We now digress briefly with an example to illustrate the predicted behaviour in the case $i(\theta c, c) = 0$.

Example 2.4 We shall construct θ and c so that θ has an invariant traintrack τ which is also invariant under $T = T_c$. The dilatation of θT^k can then be readily calculated from its action on the weight cone W_τ. By 1.2 it is enough to show that θ is pseudo-Anosov to ensure that almost all θT^k are also pseudo-Anosov.

Consider the pair of filling submanifolds $\mathscr{C} = C_1 \cup C_2 \cup C_3$ and $\mathscr{D} = D_1 \cup D_2$ of Figure 1. With the convention that a positive Dehn twist is a twist to the right, the traintrack τ of Figure 2 is invariant under all elements of the semigroup generated by $\{T_{C_1}, T_{C_2}, T_{C_3}, T_{D_1}^{-1}, T_{D_2}^{-1}\}$. Moreover it is also invariant under the π-rotation, γ, about the Axis shown in Figure 1. If we now weight the traintrack we see that the weight cone W_τ is 5-dimensional, with generators shown in Figure 2.

Set $\gamma T_{D_2}^{-1} T_{D_1}^{-1} T_{C_1} T_{C_2} T_{C_3} = \theta = \gamma T_{\mathscr{D}}^{-1} T_{\mathscr{C}}$. Then $\theta^2 = \gamma T_{\mathscr{D}}^{-1} T_{\mathscr{C}} \gamma T_{\mathscr{D}}^{-1} T_{\mathscr{C}}, = T_{\mathscr{D}}^{-1} T_{\mathscr{C}} T_{\mathscr{D}}^{-1} T_{\mathscr{C}}$, since γ commutes with $T_{\mathscr{C}}$ and with $T_{\mathscr{D}}$. Now θ^2 is pseudo-Anosov, by Long's construction in [6], so θ is pseudo-Anosov.

Now take $T_{C_3} = T$ and consider the dilatation $\lambda(\theta T^k)$ for large k. By the usual methods, the matrix of the action on W_τ is given by

$$\left[\begin{array}{ccc|cc} 0 & 0 & 1 & 0 & k \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 1 & 1 & k+2 \\ 1 & 1 & 0 & 3 & 1 \end{array}\right]$$

with respect to the ordered basis $\{a, c, e, b, d\}$. The largest eigenvalue of this matrix is the dilatation of the map θT^k. A direct calculation shows that the characteristic polynomial is

$$\chi(t) = (t - 1)(t^4 - 2t^3 - t^2(4k + 9) - 2t + 1).$$

It is then straightforward to verify that (a) $\chi'(x) > 0$ for $x > 2\sqrt{k}$, (b) $\chi(2\sqrt{k}) < 0$, (c) $\chi(2\sqrt{k} + 2) > 0$; at least for $k \geq 10$, say. Hence the largest root of $\chi(t)$ lies somewhere in the interval $(2\sqrt{k}, 2\sqrt{k} + 2)$, giving the expected growth in k, since here we have

$$i(\theta c_3, c_3) = 0, \qquad i(\theta^2 c_3, c_3) \neq 0.$$

One of the consequences of 2.3, first observed by Sullivan in [11], is that the volume of a hyperbolic mapping torus is not in general proportional to the entropy $(= \log \lambda)$ of its holonomy. For $v(M(T^r\theta))$ converges to $v(M(\theta) \setminus c \times \{\frac{1}{2}\}) < \infty$ and the entropy is unbounded. However, if θ, ψ fix the same unstable lamination (hence the same stable lamination) the situation is particularly simple - $M(\theta)$ and $M(\psi)$ are commensurable. Thus in this case

$$v(M(\theta)) = A(L)\log \lambda$$

where $A(L)$ is some constant depending on the stable lamination L of θ. It is of some interest to compute the behaviour of the set $\{A(L) \mid L \in ML\}$. For example, determine its smallest element.

Figure 1

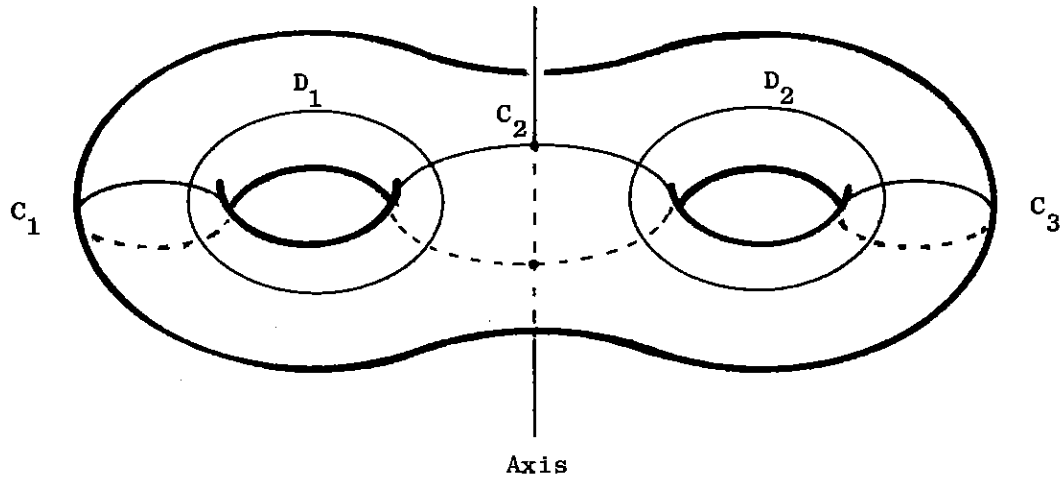
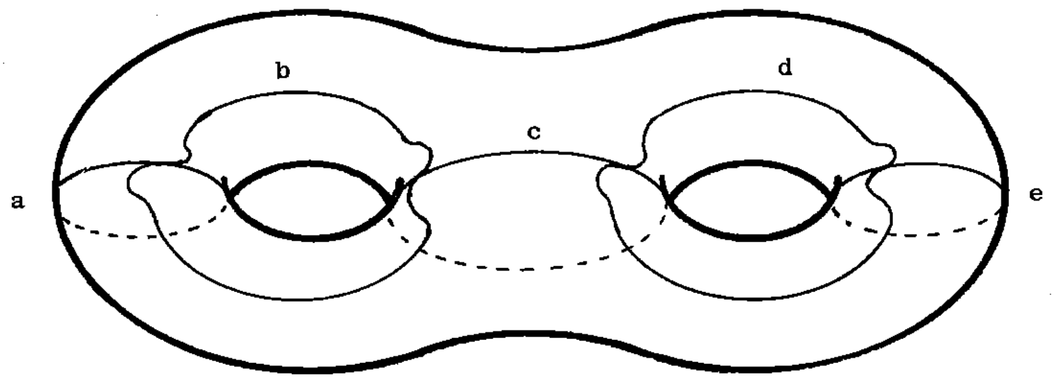


Figure 2



References

1. F. Bonahon Ribbon fibered knots, cobordism of surface diffeomorphisms and pseudo-Anosov diffeormorphisms, Math. Proc. Camb. Phil. Soc. 94 (1983), 235-251.

2. A. J. Casson Automorphisms of surfaces after Nielsen and Thurston, Lecture Notes, University of Texas (1982).

3. A. Fathi, F. Laudenbach et V. Poenaru. Travaux de Thurston sur les surfaces, Astérisque 66-67 (1979).

4. C. McA. Gordon and R. A. Litherland Incompressible planar surfaces in 3-manifolds, Preprint.

5. J. Harer How to construct all fibered knots and links, Topology (1982) vol. 21 no. 3 pp.263-280.

See over 6. D. D. Long Constructing pseudo-Anosov homeomorphisms, Proceedings of Vancouver Conference (1983), to appear.

7. H. R. Morton Fibred knots with a given Alexander polynomial, L'Enseignement Mathématique, Université de Genève no. 31 (1982).

8. A. Papadopoulos Reseaux ferroviaires, diffeomorphismes pseudo-Anosov et automorphismes symplectiques de l'homologie d'une surface, Thèse de 3e cycle Orsay (1983).

See over. 9. W. D. Neumann and D. Zagier Volumes of hyperbolic 3-manifolds, Preprint, University of Maryland.

10. J. Stallings Constructions of fibred knots and links, Proc. Sym. in Pure Math. AMS Vol. 32 Part II (1978) pp.55-60.

11. D. Sullivan Travaux de Thurston sur les groupes quasi-Fuchsiens et les variétés hyperboliques de dimension 3 fibrées sur S^1, Séminaire Bourbaki, 32e année (1979/80) No. 554.

12. W. P. Thurston The geometry and topology of 3-manifolds, Preprint.

See over 13. W. P. Thurston Hyperbolic structures on 3-manifolds I: Deformation of acylindrical manifolds, Preprint.

14. W. P. Thurston Hyperbolic structures on 3-manifolds II: Surface groups and 3-manifolds which fiber over the circle, Preprint.

15.

Department of Pure Mathematics,
The University of Liverpool,
P.O. Box 147, Liverpool. L69 3BX
England

Amended References

6. D. D. Long Constructing pseudo-Anosov homeomorphisms, Springer Lecture notes, 1144, Knot theory and manifolds, Ed. D. Rolfsen, 108-114.

9. W. D. Neumann and D. Zagier Volumes of hyperbolic three-manifolds, Topology 24 (1985), 307-332.

13. J. W. Morgan On Thurston's uniformization theorem for three-dimensional manifolds, contribution to 'The Smith conjecture', 37-125, Academic Press (1984), ed. J. W. Morgan and H. Bass.

15. A. Fathi **Dehn twists and pseudo-Anosov diffeomorphisms, Preprint, Orsay (1985).**