§0. In this paper, we exploit the close interplay between hyperbolic 3-manifolds and pseudo-Anosov maps which arises via the mapping torus construction. This enables us to prove results in the theory of surface automorphisms which appear to be difficult to obtain directly. In this way we extend and unify some examples of Bonahon [1] and Morton [7], which are concerned with examining how the monodromy of a fibred knot changes when composed with the Dehn twist in some simple closed curve in the fibre. The interest in this construction is motivated by the study of so-called 'Stallings twists' which generate new fibred knots from old. We show:

Theorem A

Let \( \theta : F \to F \) be a pseudo-Anosov map and \( T : F \to F \) be a Dehn twist about an essential simple closed curve, \( c \), in \( F \).

Then

(i) for all but finitely many \( r \), \( T^r \theta \) is isotopic to a pseudo-Anosov map.

(ii) There is a \( K \), such that for all \( r \geq K \), the maps \( T^r \theta \) are all nonconjugate. (And similarly for all \( r \leq -K \)).

This is the content of §1. Our proof relies on certain theorems concerning hyperbolic 3-manifolds and is independent of considerations such as measured laminations and traintracks.

* Supported by SERC research grant.
§2 is devoted to providing information about the invariant laminations and dilatations of the maps $T^r\theta$ and their behaviour as $r \to \infty$. This behaviour is very sensitive to how often the number $i(c, \theta^j c) = 0$, where $i(\ldots)$ denotes the geometric intersection number of simple closed curves.

Our result on invariant laminations then becomes:

**Theorem B**  
With the notation of Theorem A, let $L_r$ denote the unstable lamination of $T^r\theta$, where this is defined.

Suppose that $i(c, \theta^j c) = 0 \quad 1 \leq j \leq t - 1$ and $i(c, \theta^t c) \neq 0$.

Then given any neighbourhood $V$ in projective measured lamination space of the convex hull of $\{c, \theta c, \ldots, \theta^{t-1} c\}$, there is a $K$, such that $L_r \in V$, for all $|r| \geq K$.

In the 'generic' case, namely that $i(c, \theta c) \neq 0$, this says that $L_r \to c$ in projective measured lamination space, which is the expected behaviour. We have been unable to prove in general that the sequence $\{L_r\}$ converges.

The result on the dilatation is of a similar flavour:

**Theorem C**  
With the notation of Theorem B, let $\lambda_r$ the dilatation of $T^r\theta$, where this is defined. Then there are constants $K$ and $L$ such that:

$$|r|.i(\theta^t c, c) - K \leq |\lambda_r^t| \leq L|r| + K$$

for all $|r|$ sufficiently large. In the generic case, when $i(\theta c, c) \neq 0$, and so $t = 1$, we can take $L = i(\theta c, c)$, and we can choose $K$ arbitrarily close to $\frac{i(\theta^2 c, c)}{i(\theta c, c)}$. 

Because of the generality involved we are unable to appeal to the methods of traintracks, which almost invariably require a careful alignment of C with an invariant traintrack for θ. Our proof uses formulae of Fathi which express how i(., .) changes with Dehn twisting and requires the use of Theorem B.

We use without proof many standard properties of pseudo-Anosov maps and measured lamination spaces, both projectivised and unprojectivised. We refer the reader to [2] for expositions and terminology.

The authors would like to thank G. P. Scott for many useful conversations.

Remark
Subsequent work of Fathi, [15], provides a satisfying generalisation of these results.
§1.

Throughout, all manifolds will be assumed to be orientable and all maps orientation preserving. A hyperbolic manifold will always be taken to be complete and of finite volume.

Given an automorphism of a closed, orientable surface $\theta : \Sigma \to \Sigma$, we form the mapping torus $M(\theta)$ by taking $\Sigma \times I$ and glueing $\Sigma \times \{0\}$ to $\Sigma \times \{1\}$ via $\theta$. This gives a surface bundle over $S^1$ whose homeomorphism type determines its holonomy map, $\theta$, up to conjugacy and isotopy.

The following construction, first introduced by Stallings in [10], relates a composition of the holonomy map and a Dehn twist, to a Dehn surgery on the mapping torus. Formally, if $T : \Sigma \to \Sigma$ denotes a Dehn twist in an essential simple closed curve $c$ in $\Sigma$, then it is easily seen (for example, see [5], [10]) that the manifold $M(T\theta)$ is obtained from $M(\theta)$ by doing a $(1, r)$ surgery on $M(\theta) \setminus N(c)$, relative to a suitable choice of longitude in $\partial N(c)$.

This construction has been used extensively in the theory of fibred knots, although in that case it is more usual to make a mapping torus using a bounded surface. In this context, it is frequently assumed that $\theta$ fixes $\partial \Sigma$ and one makes the further identifications of $\{x\} \times I$ for each $x \in \partial \Sigma$, so that $M(\theta)$ is a closed manifold, usually $S^3$.

The construction is then constrained to require that $\Sigma$ be unknotted and have zero linking with a push-off. These conditions are designed to ensure that if $M(\theta) = S^3$, then $M(T\theta) = S^3$. This will not concern us, consequently we make no such restriction.
We specialise now to the case that $\theta$ is pseudo-Anosov.

(Observable this carries an implicit assumption that genus $(F) \geq 2$.)

Lemma 1.1 Let $c$ be an essential simple closed curve on the surface $F$ and $\theta : F \to F$ a pseudo-Anosov map. Define

$W = M(\theta) \setminus c \times \{\frac{1}{2}\}$. Then $W$ contains no incompressible, nonperipheral embedded tori.

Proof Since $c$ is essential, $W$ is irreducible. Let $F' = F \setminus \hat{c}$, which we identify with the $\frac{1}{2}$-level in $W$. Let $T$ be an embedded incompressible torus in $W$.

Consider the components of $T \setminus F'$. Since $F$ is not a torus, $F \times I$ contains no incompressible torus, and so $T \cap F' \neq \emptyset$. Using the irreducibility of $W$ and the incompressibility of $F'$, we may isotope $T$ to arrange that all these components are annuli.

Suppose there is an annulus component $A$ which misses some fibre. Then by cutting $W$ open along $F'$ and the fibre which is avoided, we see that $A$ lies in a 'product' $F \times I \setminus c \times \{0\}$ with $\partial A$ lying in one boundary component of this product. Thus the components of $\partial A$ are freely homotopic in $F \times I \setminus c \times \{0\}$. There are then two possibilities.

Either the components of $\partial A$ are freely homotopic in $F'$, in which case they bound an annulus in $F'$, or they are parallel to $c$ and on opposite sides of some small neighbourhood of $c \subset F$. All the former intersections may be removed by an isotopy of $T$.

Thus we may assume that all components of $T \setminus F'$ meet $F'$ in this latter way, or run around the $S^1$ factor in $M(\theta)$. 
Denoting the free homotopy class by $[\cdot]$, a pseudo-Anosov map has the property that for all essential simple closed curves $P$, $[\theta^k P] = [\theta^l P]$ if and only if $k = l$.

Further, observe that the only incompressible $\partial$-compressible annuli in $F \times I$ are isotopic to products $P \times I$. Thus any annulus running around the $S^1$ factor has boundary components $P$ and $\partial P$.

Putting together the remarks of the previous two paragraphs, it is now easy to see that there can be no annuli running around the $S^1$-factor, since these can never close up.

Thus the only annuli possible miss a fibre and have boundary components parallel to and on opposite sides of $c$. It is now clear that there are only two such components and that they close up to a peripheral torus.

\[ \square \]

**Theorem 1.2**

Let $\theta : F \to F$ be a pseudo-Anosov map, and $T : F \to F$ the Dehn twist about some essential simple closed curve $c \subset F$.

Then for all but finitely many $r$, $T^r \theta$ is isotopic to a pseudo-Anosov map.

**Proof**

Form the mapping torus $M(\theta)$. Then we claim that $W = M(\theta) \setminus c \times \{\frac{1}{2}\}$ can be made into a complete, finite volume hyperbolic 3-manifold. For, $W$ is Haken, so by a theorem of Thurston [13], $W$ admits a decomposition into geometric pieces and Lemma 1.1 shows that in fact only one geometric piece is involved. $W$ cannot be a torus bundle. If $W$ were Seifert fibred, then since the surface $F$ is incompressible, it can be made horizontal or vertical; since it is not a torus, it must be the former. But this then implies $W$ is a closed manifold, a contradiction. Thus we conclude that $W$ is hyperbolic. Let $W' = M(\theta) \setminus N(c)$. Then a theorem of [12] implies
that all but finitely many surgeries on $W'$ give rise to closed, hyperbolic 3-manifolds, in particular, all but finitely many of the mapping tori $M(T^6)$ are hyperbolic. We complete the proof by quoting [14], which states that a mapping torus is hyperbolic if and only if its holonomy is pseudo-Anosov. (Although we are only using the easy implication of this result.)

\[ \square \]

**Remarks**

(i) The natural 'companion' result to this, namely that all but finitely many of the maps $T_0^r$ are pseudo-Anosov, is a special case of a theorem of Papadopoulos [8]. Modulo some understanding of projective lamination space, this result is much easier to prove, the reason being that pseudo-Anosovs have better contraction properties on this space than Dehn twists. It appears to be difficult to prove 1.2 directly.

(ii) It would be of some interest to show there were a bound on the number of exceptional values of $r$, depending only on $F$, or perhaps universal. For example, a theorem of Gordon and Litherland [4] asserts that there are at most six surgeries giving rise to non-irreducible manifolds.

(iii) The restriction to one simple closed curve is, of course, unnecessary, though some care is needed in formulating the result for 1-submanifolds. The problem which can arise is that for some $\Theta$ and $c$, $\{c, \Theta c\}$ can be a 1-submanifold. In this case, the manifold $M(\Theta) \setminus \{c, \Theta c\}$ contains nonperipheral embedded incompressible tori, so cannot be hyperbolic. Nonetheless, we do have:
Theorem 1.3

Let $\theta : F \to F$ be a pseudo-Anosov map and $T_1 : F \to F$ be the Dehn twist on an essential simple closed curve $c_1$, $1 \leq i \leq n$, where $c_i$ is a 1-submanifold. Then infinitely many of the maps $T_1^{k_1} T_2^{k_2} \cdots T_n^{k_n}$ are pseudo-Anosov.

Proof

By 1.2, $T_1^{k_1} \theta$ is pseudo-Anosov for infinitely many $k$. Fix one such $k$; $k_1$ say. Then for infinitely many $k$, $T_2^{k}(T_1^{k_1} \theta)$ is pseudo-Anosov. And so on.

In order that 1.2 be useful, we need to be sure that infinitely many of the maps $T^r \theta$ are nonconjugate. One possible approach to this would be to examine the behaviour of the dilatation (which is a conjugacy invariant) as $r \to \infty$. However, there is a direct proof of nonconjugacy, closer to the spirit of this section:

Theorem 1.4

Let $\theta : F \to F$ be pseudo-Anosov and $T : F \to F$ denote the Dehn twist about an essential simple closed curve $c \subset F$.

Then there is a $K$, such that for all $r \geq K$, the maps $T^r \theta$ are all nonconjugate. (Also, for all $r \leq -K$, the maps $T^r \theta$ are all nonconjugate.)

Proof

Consider the hyperbolic manifold $W = M(\theta) \setminus \mathbb{P} \times \{\hat{1}\}$. This has some finite volume $v(W)$. Then the methods of [12] show that $v(M(T^r \theta)) \to v(W)$ as $r \to \infty$. Now $M(T^r \theta)$ is given by $(1,r)$ Dehn surgery on $w' = M(\theta) \setminus \mathbb{P}(c)$. Explicit estimates in [9] then show that $v(W) = v(M(T^r \theta)) + \frac{1}{Q(p,q)} + O(1/r^d)$, where $Q(p,q)$ is a positive definite quadratic form depending only on $W$. It follows that $v(M(T^r \theta))$ increases strictly with $r$ for large enough $r$, $r \geq K$, say.
Volume is a topological invariant, by Mostow Rigidity, and hence for $r, s > K$, $r \neq s$, the manifolds $M(T^r \Theta)$ and $M(T^s \Theta)$ are not homeomorphic. The maps $T^r \Theta$ and $T^s \Theta$ are then nonconjugate.

This result implies that there are at worst pairwise conjugacy repetitions among the pseudo-Anosovs $T^r \Theta$, $|r| \geq K$. Taking 1.2 and 1.4 together gives an alternative (although marginally less explicit) description of some examples of Bonahon [1] which is independent of calculations involving invariant train tracks. (Strictly speaking, we should prove 1.2 and 1.4 in the case of a bounded surface to obtain these examples; however, this presents no difficulty.)
§2. In this section we focus on the behaviour of the invariant laminations and dilatation of \( T^r \) as \( r \to \infty \). Our methods now draw more on the theory of surface automorphisms. To state our results we must introduce some terminology.

Let us denote measured lamination space for the closed orientable surface \( F \) of genus \( g \) by \( \text{ML} \), and its projectivisation by \( \mathbb{P}\text{ML} \), a compact space homeomorphic to \( S^{6g-7} \). Then, given a collection of disjoint, simple essential closed curves \( c_1, \ldots, c_t \) on \( F \), we define the convex hull of \( c_1, \ldots, c_t \), written \( \mathbb{H}[c_1, \ldots, c_t] \) to be the closure in \( \mathbb{P}\text{ML} \) of the set of measured laminations

\[ \{ s_1 [c_1] \# \ldots \# s_t [c_t] \mid 0 \leq s_j \leq 1, \sum s_j = 1 \} \].

Here \([c]\) is taken to be the point of \( \mathbb{P}\text{ML} \) corresponding to \( c \), and more generally, \([\cdot]\) is taken to denote the projective class of a measured lamination. Notice that \( \mathbb{H}[c_1] = [c_1] \).

We use the symbol \( i(\cdot, \cdot) \) to denote the geometric intersection number of two essential simple closed curves and observe that this admits a natural continuous extension to \( \text{ML} \). (See [14].)

We shall have need of the following lemma. A proof is contained in the Appendix to Exposé 4 of [3].

**Lemma 2.1** Let \( \alpha_1, \ldots, \alpha_k \) be disjoint essential simple closed curves on \( F \).

Let \( \beta, \gamma \) be any essential simple closed curves on \( F \) and let \( \Gamma(n_1, \ldots, n_k) \) denote \( \gamma \) Dehn twisted \( n_j \) times in \( \alpha_j \), where the \( n_j \)'s are all positive (or all negative).

Then

\[ |i(\Gamma(n_1, \ldots, n_k), \beta) - \sum_j n_j [i(\gamma, \alpha_j) i(\alpha_j, \beta)]| \leq i(\gamma, \beta). \]
Theorem 2.2  
Let \( \theta : F \to F \) be pseudo-Anosov and \( T : F \to F \) the Dehn twist about an essential simple closed curve \( c \subset F \). Let \( L_{\theta} \) denote the unstable lamination of \( T^k \theta \).

Suppose that \( i(c, \theta^k c) = 0 \) \( 1 \leq k \leq t - 1 \) and that \( i(c, \theta^t c) \neq 0 \).

Then given any neighbourhood \( V \subset \mathbb{P}ML \) of \( H(c, \theta c, \ldots, \theta^{t-1} c) \), there is a \( K \), such that for all \( |r| \geq K \), \( [L_r] \in V \).

Proof  
For brevity, we write \( c^i_1 = \theta^i c \). Observe that since \( \theta T c^{i-1}_1 = T \theta^i c \), we have that \( (T^r \theta)^t = T^{r_1}_{0 \theta^1} \cdots t^{r_t}_{t-1^t} \), where \( T_j \) denotes the Dehn twist about \( c_j \). Observe that the unstable lamination of \( (T^r \theta)^t \) is the same as that of \( T^r \theta \), so to prove the result, it is sufficient to work with the former map.

Consider now \( \theta^t H(c_0^t, \ldots, c_{t-1}^t) = H(\theta^t c_0^t, \ldots, \theta^t c_{t-1}^t) \). Each point in this set has the property that it assigns nonzero measure to the lamination \( c^t = c_0^t \cup c_1^t \cup \ldots \cup c_{t-1}^t \) i.e. \( i(c^t, L) \neq 0 \) for each \( L \) with \( [L] \in \theta^t H(c_0^t, \ldots, c_{t-1}^t) \). Hence there is a closed neighbourhood \( U \) in \( \mathbb{P}ML \) of \( H(\theta^t c_0^t, \ldots, \theta^t c_{t-1}^t) \) with this property. (Equivalently, every \( [L, \mu] \) in \( U \) has the property that \( L \) has a leaf in the support of \( \mu \) with transverse intersections with at least one of the set \( \{c_0^t, \ldots, c_{t-1}^t\} \).

Now we appeal to the formula of Lemma 2.1. This states in our terminology, that for all simple closed curves \( \gamma, \beta \),

\[
|i(T^{r_1}_{0 \theta^1} \cdots T^r_{t-1} \gamma, \beta) - r \sum_{j=0}^{t-1} i(\gamma, c^t_j) i(c^t_j, \beta)| \leq i(\gamma, \beta).
\]

Notice that this formula continues to hold for \( \beta, \gamma \in ML \). Now if \( M_r, M \in ML \) and \( i(M_r, \beta) \to i(M, \beta) \) as \( r \to \infty \) for each \( \beta \in ML \) then \( M_r \to M \) in \( ML \).
Given any neighbourhood \( W \subset \mathcal{PML} \) of \( H(c_0, \ldots, c_{t-1}) \), and any \([L] \in U\) we can find a \( k \) (depending on \([L]\)) such that
\[
T_r^0 \cdots T_r^{t-1}[L] \in W \quad \text{for all } |r| \geq k.
\]
For we can take \( \gamma = L \) in the formula above, and \( M_r = \frac{1}{r} T_r^0 \cdots T_r^{t-1} L \), to see that \( M_r \to M = \bigsqcup k_j c_j \) where \( k_j = i(L, c_j) \). Now \([M] \in H(c_0, \ldots, c_{t-1})\), since \( k_j \) are not all zero, so eventually all \([M_r]\) lie in \( W \).

An elementary argument based on the compactness of \( U \) now implies that there is a \( K \), such that for all \(|r| \geq K\), \( T_r^0 \cdots T_r^{t-1}(U) \subset W \).

Since \( \theta^t \) induces a continuous map of \( \mathcal{PML} \), there is a neighbourhood \( V \) of \( H(c_0, \ldots, c_{t-1}) \) with \( \theta^t(V) \subset \text{int } U \). Choose \( K \), so that
\[
T_r^0 \cdots T_r^{t-1}(U) \subset V \quad \text{for all } |r| \geq K.
\]
Because \((T_\theta^r)^t\) is pseudo-Anosov, if \([L, \mu]\) is any lamination in \( \mathcal{PML} \), save the stable lamination of \( T_\theta^r L \), we have \((T_\theta^r)^{tk}[L] \to [L_r] \) as \( k \to \infty \). In particular, if \(|r| \geq K\), we see that \([L] \in V\) implies \((T_\theta^r)^{tk}[L] \in V\) for all \( k > 0 \), so that \([L_r] \in \overline{V}\). This implies the result.

\[ \Box \]

Remarks

(i) By considering \((T_\theta^r)^{-1}\), a similar proof shows that the stable laminations lie close to \( H(c_{-1}, \ldots, c_{-t}) \) as \( r \to \infty \).

(ii) In the 'typical' case, namely that \( i(c, \phi c) \neq 0 \), this result can be interpreted as saying \([L_r] \to [c]\) in \( \mathcal{PML} \) as \( r \to \infty \). We do not know if this happens in all cases. Notice that since \( \mathcal{ML} \) is covered by open sets \( W_\tau \) corresponding to the weight cones of maximal traintracks \( \tau \), (that is, traintracks all of whose complementary regions are trigons - see [2] or [8] for an exposition) in this simple case, there is a traintrack \( \tau \) carrying \( c \) and all laminations \( L_r \) for
sufficiently large. If this traintrack is nicely aligned with respect to \( c \), results concerning the dilatation can be obtained from linear algebra.

(iii) We can now give a 'construction-free' proof of the fact that the set \( \{(L_{\theta}^p, \mu_{\theta}^p) | (L_{\theta}^p, \mu_{\theta}^p) \text{ is the unstable lamination of a pseudo-Anosov } \theta \text{ on } F \} \) is dense in \( \mathbb{PML} \).

For if \( c \) is any simple closed curve and \( \theta \) is any pseudo-Anosov, there is a \( p \), such that \( i(\theta^p c, c) \neq 0 \). Then setting \( L_{\theta}^p \) to be the unstable lamination of \( T_{\theta}^p \), 2.2 shows that \( [L_{\theta}^p] \to [c] \) in \( \mathbb{PML} \). Hence the closure of the above set contains all simple closed curves, and so is all of \( \mathbb{PML} \).

In fact an analogous argument can be used to show that the normal closure in the mapping class group of any pseudo-Anosov contains sufficient 'pseudo-Anosov laminations' to give a dense set in \( \mathbb{PML} \).

We now turn our attention to the behaviour of the dilatation of the pseudo-Anosov maps \( T_{\theta}^r \) as a function of \( r \), as \( r \to \infty \). Our result is that the growth is dependent on the first \( t \) such that \( i(c, \theta^t c) \neq 0 \).

Let us fix our notation as usual: \( \theta : F \to F \) will be a pseudo-Anosov and \( T : F \to F \) will be the Dehn twist about an essential simple closed curve \( c \subset F \). Suppose that \( i(c, \theta^j c) = 0 \) for \( 1 \leq j \leq t - 1 \) and \( i(c, \theta^t c) \neq 0 \).

Let \( \lambda_{\theta}^t \) denote the dilatation of \( T_{\theta}^t \), where this is defined.

**Theorem 2.3** There are constants, \( K \) and \( L \), such that for all sufficiently large \( |r| \) we have:

\[
|r|i(\theta^t c, c) - K \leq |\lambda_{\theta}^t| \leq L|r| + K.
\]
Proof

We again invoke the formula of Lemma 2.1. In the notation of the proof of 2.2, if we write \( T_0 T_1 \ldots T_{t-1} = \gamma \), then the dilatation of \( (T^r \theta)^t = \gamma^r \theta^t \) is \( \lambda^r \). Here, 2.1 becomes

\[
\left| i(\gamma^r, \beta) - \sum_{j=0}^{t-1} ri(\gamma, \theta^j c) i(\theta^j c, \beta) \right| \leq i(\gamma, \beta)
\]

for all laminations \( \gamma, \beta \in \mathcal{ML} \). For brevity, write

\[
\gamma = c \cup \ldots \cup \theta^{t-1} c.
\]

Take \( \gamma = \theta^{t_0} c \). Then we obtain

\[
\left| i(\gamma^r, \beta) - \sum_{j=0}^{t-1} i(\theta^{t_0} c, \theta^j c) i(\theta^j c, \beta) \right| \leq i(\theta^{t_0} c, \beta).
\]

For any lamination \( \beta \) with \( i(\gamma^r, \beta) \neq 0 \), we may divide through;

\[
\left| \frac{i(\gamma^r, \beta)}{i(\gamma^r, \beta)} - \sum_{j=0}^{t-1} \frac{i(\theta^{t_0} c, \theta^j c) i(\theta^j c, \beta)}{i(c, \beta)} \right| \leq \frac{i(\theta^{t_0} c, \beta)}{i(c, \beta)}.
\]

In particular, take \( \beta \) to be \( L_r^{st} \), the stable lamination of \( T^r \theta \), or rather, any representative lamination, since \( L_r^{st} \) is only defined in \( \mathcal{ML} \) up to a real multiple. The inequality then becomes

\[
\left| \frac{\lambda^r}{\lambda^r - \sum_{j=0}^{t-1} \frac{i(\theta^{t_0} c, \theta^j c) i(\theta^j c, L_r^{st})}{i(c, L_r^{st})}} \right| \leq \frac{i(\theta^{t_0} c, L_r^{st})}{i(c, L_r^{st})}
\]

independent of the choice of representative of \( L_r^{st} \) because of its homogeneity.

By the remark (i) following 2.2, for \(|r|\) sufficiently large, the projective class of \( L_r^{st} \) is very close to the convex hull of
However, the above formula is homogeneous in $\mathbb{R}^n_c$, thus for the purposes of estimation, it is sufficient to choose any representative in the projective class. Since every point in $H(\theta^{-1}, \ldots, \theta^{-t}c)$ ascribes nonzero measure to $\gamma_c$, we see that $i(\gamma_c, L^\text{st}_r)$ is bounded away from zero as $r \to \infty$ and thus there is a constant $K$, such that for all $|r|$ sufficiently large,

$$\frac{i(\theta^t, L^\text{st}_r)}{i(\gamma_c, L^\text{st}_r)} \leq K.$$ 

Similarly,

$$\sum_{j=0}^{t-1} \frac{i(\theta^{t_j}, \theta^j \gamma_c) i(\theta^j \gamma_c, L^\text{st}_r)}{i(\gamma_c, L^\text{st}_r)} \leq L$$

for all $|r|$ sufficiently large. Hence the triangle inequality implies

$$|\lambda_t| \leq L|r| + K.$$ 

For the other half of the inequality, notice that

$$i(\theta^t, \theta^j \gamma_c) \geq i(\theta^t \gamma_c, c) \quad \text{for } 0 \leq j \leq t-1.$$ 

Thus

$$|\lambda_t| + K \geq |r|i(\theta^t, \gamma_c) \sum_{j=0}^{t-1} \frac{i(\theta^j \gamma_c, L^\text{st}_r)}{i(\gamma_c, L^\text{st}_r)} = |r|i(\theta^t \gamma_c, c).$$

This completes the proof. \qed
Remark \[ \lambda_r - ri(\theta c, c) \leq \frac{i(\theta c, L^*_{t})}{i(c, L^*_{r})}. \]

Since \([L^*_{r}] = [\theta^{-1} c]\) in this case, the right hand side converges to
\[ \frac{i(\theta c, \theta^{-1} c)}{i(\theta c, \theta^{-1} c)} = \frac{i(\theta^2 c, c)}{i(\theta^2 c, c)}. \]

We now digress briefly with an example to illustrate the predicted behaviour in the case \(i(\theta c, c) = 0\).

Example 2.4 We shall construct \(\theta\) and \(c\) so that \(\theta\) has an invariant traintrack \(\tau\) which is also invariant under \(T = T_c\). The dilatation of \(\theta T^k\) can then be readily calculated from its action on the weight cone \(W_\tau\). By 1.2 it is enough to show that \(\theta\) is pseudo-Anosov to ensure that almost all \(\theta T^k\) are also pseudo-Anosov.

Consider the pair of filling submanifolds \(\theta = C_1 \cup C_2 \cup C_3\) and \(\mathcal{D} = D_1 \cup D_2\) of Figure 1. With the convention that a positive Dehn twist is a twist to the right, the traintrack \(\tau\) of Figure 2 is invariant under all elements of the semigroup generated by \(\{T_{C_1}, T_{C_2}, T_{C_3}, T_{D_1}, T_{D_2}\}\).

Moreover it is also invariant under the \(\gamma\)-rotation, \(\gamma\), about the Axis shown in Figure 1. If we now weight the traintrack we see that the weight cone \(W_\tau\) is 5-dimensional, with generators shown in Figure 2.

Set \(\gamma T_{D_2}^{-1} T_{D_1}^{-1} T_{C_1} T_{C_2} T_{C_3} = \theta = T_{D_2}^{-1} T_{c}\). Then
\[\theta^2 = \gamma T_{D_2}^{-1} T_{D_1}^{-1} T_{C_1} T_{C_2} T_{C_3} = T_{D_2}^{-1} T_{c} T_{D_1}^{-1} T_{c}\] since \(\gamma\) commutes with \(T_{c}\) and with \(T_{D_2}\). Now \(\theta^2\) is pseudo-Anosov, by Long's construction in [6], so \(\theta\) is pseudo-Anosov.

Now take \(T_{C_3} = T\) and consider the dilatation \(\lambda(\theta T^k)\) for large \(k\).

By the usual methods, the matrix of the action on \(W_\tau\) is given by
\[
\begin{bmatrix}
0 & 0 & 1 & 0 & k \\
0 & 1 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & k+2 \\
1 & 1 & 0 & 3 & 1 \\
\end{bmatrix}
\]

with respect to the ordered basis \{a, c, e, b, d\}. The largest eigenvalue of this matrix is the dilatation of the map \( \Theta^k \). A direct calculation shows that the characteristic polynomial is

\[
\chi(t) = (t - 1)(t^4 - 2t^3 - t^2(4k + 9) - 2t + 1).
\]

It is then straightforward to verify that

(a) \( \chi'(x) > 0 \) for \( x > 2\sqrt{k} \),

(b) \( \chi(2\sqrt{k}) < 0 \),

(c) \( \chi(2\sqrt{k} + 2) > 0 \); at least for \( k \geq 10 \), say. Hence the largest root of \( \chi(t) \) lies somewhere in the interval \((2\sqrt{k}, 2\sqrt{k} + 2)\), giving the expected growth in \( k \), since here we have

\[
\imath(\Theta c_3, c_3) = 0, \quad \imath(\Theta^2 c_3, c_3) \neq 0.
\]

One of the consequences of 2.3, first observed by Sullivan in [11], is that the volume of a hyperbolic mapping torus is not in general proportional to the entropy (= log \( \lambda \)) of its holonomy. For \( v(M(T^r \theta)) \) converges to \( v(M(\theta) \setminus c \times \{ 1 \}) < \infty \) and the entropy is unbounded. However, if \( \theta, \psi \) fix the same unstable lamination (hence the same stable lamination) the situation is particularly simple – \( M(\theta) \) and \( M(\psi) \) are commensurable. Thus in this case

\[
v(M(\theta)) = A(L) \log \lambda
\]

where \( A(L) \) is some constant depending on the stable lamination \( L \) of \( \theta \). It is of some interest to compute the behaviour of the set \( \{ A(L) | L \in \text{ML} \} \). For example, determine its smallest element.
References

1. F. Bonahon  Ribbon fibered knots, cobordism of surface
diffeomorphisms and pseudo-Anosov diffeomorphisms,

2. A. J. Casson  Automorphisms of surfaces after Nielsen and
Thurston, Lecture Notes, University of Texas (1982).

3. A. Fathi, F. Laudenbach et V. Poenaru  Travaux de Thurston sur

4. C. McA. Gordon and R. A. Litherland  Incompressible planar
surfaces in 3-manifolds, Preprint.

5. J. Harer  How to construct all fibered knots and links,

6. D. D. Long  Constructing pseudo-Anosov homeomorphisms,

7. H. R. Morton  Fibred knots with a given Alexander polynomial,
L'Enseignement Mathématique, Université de Genève no. 31 (1982).

8. A. Papadopoulos  ReSEAUX ferROVIAIRES, DIFFEOMORPHISMES pseudo-
Anosov et automorphismes symplectiques de l'homologie d'une

9. W. D. Neumann and D. Zagier  Volumes of hyperbolic 3-manifolds,
Preprint, University of Maryland.

11. D. Sullivan  
   Travaux de Thurston sur les groupes quasi-Fuchsiens et les variétés hyperboliques de dimension 3  
   fibrées sur $S^1$, Séminaire Bourbaki, 32e année (1979/80)  
   No. 554.

12. W. P. Thurston  
   The geometry and topology of 3-manifolds,  
   Preprint.

13. W. P. Thurston  
   Hyperbolic structures on 3-manifolds I:  
   Deformation of acylindrical manifolds, Preprint.

14. W. P. Thurston  
   Hyperbolic structures on 3-manifolds II:  
   Surface groups and 3-manifolds which fiber over the circle,  
   Preprint.

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