Murphy operators in knot theory

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The Murphy operators in the Hecke algebra H_n are commuting elements which arose originally in an algebraic setting in connection with representation theory. They can be represented diagrammatically in a Homfly skein theory version of H_n. Symmetric functions of the Murphy operators are known to lie in the centre of H_n. Diagrammatic views of these are given which demonstrate their algebraic properties readily, and how analogous central elements can be constructed diagrammatically in some related algebras.

Introduction

This article is based closely on a talk given at the meeting on Differential Geometric Methods in Theoretical Physics on the occasion of the opening of the new building for the Nankai Institute of Mathematics in August 2005. It appears as a chapter in Differential Geometry and Physics, Nankai Tracts in Mathematics volume 10, (2006). More detailed accounts of the results described during the talk can be found in the references noted.

I first heard about the Murphy operators on my previous visit to Nankai ten years ago for a statistical mechanics satellite meeting. At that meeting Chakrabarti gave a talk about the properties of what he termed the 'fundamental element' which generated the centre of the Hecke algebra H_n[3].

At that time Aiston and I had been studying geometrically based models for H_n in terms of the group B_n of n-string braids, and I initially expected that his fundamental element must be represented by the well-known generator for the centre of the braid group, namely the full twist braid Δ^2. However it soon became clear that Chakrabarti was referring to a different, and more useful, element of H_n, with the algebraic feature that it had distinct eigenvalues on the different irreducible submodules of H_n.

Chakrabarti then told me that this element was the sum of the Murphy elements (Murphy operators) in H_n. These are elements which have their origin in work of Jucys[2] and subsequently Murphy[8].

Having been introduced to these elements, Aiston and I looked at them in our geometrical model in order to understand them in that context, and to see if their algebraic properties could be readily established there.

While we were able to understand their basic appearance, and establish the eigenvalue property quite quickly[5] it was not until a few years later that I came across a more satisfactory geometric way to represent them, and a particularly striking way to produce their sum as an obviously central element in H_n[4].

This in turn led me to a natural description for other central elements, and similar descriptions of central elements in some natural extensions of the Hecke algebras.

A further consequence of the eigenvalue property led me also to a very helpful way of identifying the elements in a natural combinatorial model constructing 2-variable knot invariants which correspond neatly to the invariants produced by irreducible quantum $SL(N)$ modules.

I shall give here a brief account of the Jucys-Murphy elements in an algebraic context, before describing the geometric models for H_n and for the further construction.

1 Murphy operators in Hecke algebras

The Hecke algebra H_n is a deformed version of the group algebra $\mathbf{C}[S_n]$ of permutations. Jucys[2] and Murphy[8] studied certain sums of transpositions $m(j) \in \mathbf{C}[S_n]$.

$$
\begin{array}{rcl}
m(2) & = & (1\,2) \\
m(3) & = & (1\,3) + (2\,3) \\
m(4) & = & (1\,4) + (2\,4) + (3\,4) \\
\vdots & = & \\
m(j) & = & \displaystyle\sum_{i=1}^{j-i} (i\,j) \\
\vdots & = &
\end{array}
$$

These elements have the following two properties:

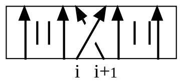
1. The elements $m(j)$ commute.
2. Every symmetric polynomial in them, for example their sum, or the sum of their squares, lies in the *centre* of the algebra.

Dipper and James[1] found corresponding elements $M(j)$ in H_n which they named the *Murphy operators*, having similar properties:

1. The elements $M(j)$ commute.
2. Every symmetric polynomial in them lies in the centre of H_n.

The Hecke algebra H_n can be readily presented as linear combinations of n-string braids subject to a simple linear relation depending on a single parameter z.

The elementary braids $\sigma_i^{\pm 1}$ when composed by placing one below another will generate all n-braids. Here

$\sigma_i \;=$ 

is the braid on n strings in which string i crosses string $i+1$ once in the positive sense.

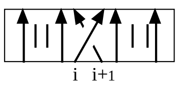
They satisfy Artin's braid relations

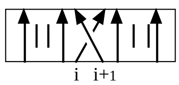
$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i, \quad |i-j| > 1, \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, \end{aligned}$$

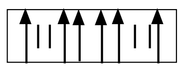
Elements of H_n can be regarded as linear combinations of braids on which we impose the further quadratic relations

$$\sigma_i^2 = z\sigma_i + 1.$$

These relations can be visualised in the form $\sigma_i - \sigma_i^{-1} = z$ as

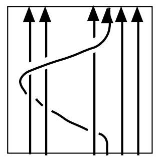


$-$


$= z$

.

Setting the parameter $z = 0$ gives $\sigma_i = \sigma_i^{-1}$ and reduces each braid to the permutation defined by following its strings, when σ_i becomes the transposition $(i\; i+1)$. The elements $M(j)$ were based on a choice of braids which each reduce to individual transpositions when $z = 0$.

Ram[9] pointed out that these could be combined into a single braid

$T(j) \ =$ 

to represent each $M(j)$, up to linear combination with the identity.

Explicitly $T(j) = 1 + zM(j)$. So long as $z \neq 0$ the elements $T(j)$ will do equally well in place of $M(j)$.

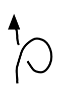
The geometric braids $T(j)$ clearly commute. Their product is the full twist braid Δ^2 which commutes with all braids, and so lies in the centre of H_n. It is not immediately clear however that their sum, or any other symmetric function of them is central.

2 A skein theory version

I shall now construct a model of H_n based on more general diagrams which will provide a simple representative for the sum. In this wider context, known as skein theory, we work with pieces of oriented knot diagrams, lying with some prescribed boundary conditions in a fixed surface F. Diagrams consist of arcs respecting the boundary conditions along with further closed curves, and may be altered by sequences of the standard Reidemeister moves R_{II} and R_{III}. The moves can be interpreted as the natural physical moves allowed on pieces of ribbon representing the curves.

The *skein* $\mathcal{S}(F)$ consists of formal linear combinations of diagrams in F (sometimes known as *tangles*) modulo two linear relations

(1)  $-$  $= \ (s - s^{-1})$ 

and (2)  $= \ v^{-1}$ 

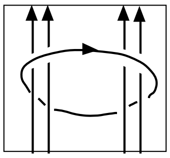
between diagrams which differ only as shown. The coefficient ring can be taken as $\Lambda = \mathbf{Z}[v^{\pm 1}, s^{\pm 1}]$ with powers of $s^k - s^{-k}$ in the denominators.

Theorem 2.1. *(Morton-Traczyk[7]) The skein of the rectangle with n input points at the bottom and n output points at the top is the Hecke algebra H_n, with scalars extended to Λ and $z = s - s^{-1}$.*

Any diagram in the rectangle can be reduced to a Λ-combination of braids by use of relations (1) and (2). For braids, the relation (1) becomes the algebraic relation $\sigma_i = \sigma_i^{-1} = z$.

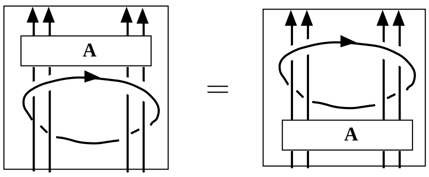
The algebra composition in the skein version of H_n is given by placing diagrams one below the other, as for braids. We can then exhibit lots of diagrams which belong to the centre of H_n in this model.

For a start the diagram

$T^{(n)} =$ 

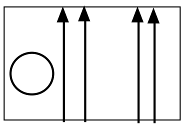
is central.

This can be readily seen, since any diagram A can be passed through using only Reidemeister moves II and III.

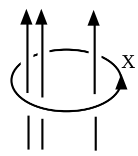


Theorem 2.2. *(Morton[4]) $T^{(n)}$ is the sum of the variant Murphy operators $T(j)$, up to linear combination with the identity.*

This result depends essentially on a repeated application of the skein relation (1), leading to the equation

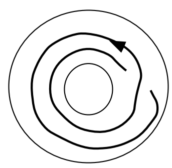
$T^{(n)} -$  $= v^{-1} z \sum_{j=1}^{n} T(j).$

Replacing the encircling curve in $T^{(n)}$ by a more complicated combination of diagrams



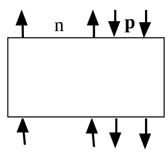
gives a huge range of further central elements.

The choices for X are best thought of as elements in the skein $\mathcal{C}$ of the annulus without prescribed boundary points, for example

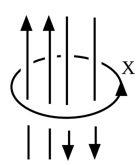
$X =$  .

There is a nice choice X_m for each m which gives the sum of the mth powers of the Murphy operators $T(j)$ in H_n no matter what n may be[4]. It is then possible to produce any symmetric polynomial in $T(j)$ from a suitable choice of X.

In the same spirit, algebras $H_{n,p}$ can be constructed as the skeins based on the rectangle with inputs and outputs arranged as shown,



where elements are again composed by placing one below the other. There is again a large choice of similarly constructed elements



in the centre of the algebra. These can all be expressed as supersymmetric polynomials in two families of commuting elements in the algebra which can be considered as an analogue of Murphy elements in this setting.

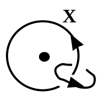
Even where the basic skein relation is altered, for example to Kauffman's 4-term relation on non-oriented diagrams, similar diagrams to these will give central elements. In this setting too these central elements may be interpreted as polynomials in some form of Murphy elements.

3 The annulus

Representation theory of Hecke algebras also leads naturally to the skein of the annulus. We are interested in finding trace functions on H_n, namely linear functions tr to a commutative algebra such that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. The character of a matrix representation has this property, but we need not restrict the image of the function to be the scalars.

Any diagram T in H_n can be *closed* to give a diagram $\hat{T}$ in the annulus, as shown, with the property that $\widehat{AB} = \widehat{BA}$. This procedure respects the skein relations, and so determines a Λ-linear map $\hat{}: H_n \to \mathcal{C}$ to the skein of the annulus. Now $\mathcal{C}$ is a commutative algebra, so the closure map is a trace function on H_n, and its composite with any linear function on $\mathcal{C}$ will determine further trace functions. Indeed it is possible to construct all irreducible characters of H_n by suitable linear functions on $\mathcal{C}$.

Write $\mathcal{C}_n \subset \mathcal{C}$ for the image of H_n, and define the *meridian map* $\varphi : \mathcal{C}_n \to \mathcal{C}_n$ diagrammatically by

$$\varphi(X) \; = $$  .

Thus if $X = \hat{A}$ then $\varphi(X) = \widehat{AT^{(n)}}$. If $AT^{(n)} = cA$ then $\hat{A}$ is an eigenvector of φ with eigenvalue c.

Theorem 3.1. *The meridian map φ has no repeated eigenvalues.*

Aiston and I[5] gave a direct proof of this by exhibiting suitable choices of A. The result can be interpreted as a different angle on Chakrabarti's observation about the action of the sum of the Murphy operators on H_n.

Indeed when the meridian map is extended over all diagrams in the annulus to give $\varphi : \mathcal{C} \to \mathcal{C}$ it still has no repeated eigenvalues[6]. In $\mathcal{C}_n$ the eigenvectors correspond to partitions of n, and the subspace of $\mathcal{C}$ spanned by the union of $\mathcal{C}_n$ for all n can be interpreted as the representation ring of

$SL(N)$ for large N. In this context the eigenvectors match up well with the irreducible representations, and give well-adapted skein theoretic elements Q_λ for each $\lambda \vdash n$. These can be used to provide a 2-variable invariant of a knot for each partition λ that yields the irreducible 1-variable quantum $SL(N)$ invariants for each N by a simple substitution. Eigenvectors for the meridian map in the whole skein of the annulus correspond to pairs λ, μ of partitions, and again give natural 2-variable invariants which are well-adapted to quantum group interpretations[6].

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