Arbitrary Arrow Update Logic☆

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Abstract

In this paper we introduce arbitrary arrow update logic (AAUL). The logic AAUL takes arrow update logic, a dynamic epistemic logic where the accessibility relations of agents are updated rather than the set of possible worlds, and adds a quantifier over such arrow updates.

We investigate the relative expressivity of AAUL compared to other logics, most notably arbitrary public announcement logic (APAL). Additionally, we show that the model checking problem for AAUL is PSPACE-complete. Finally, we introduce a proof system for AAUL, and prove it to be sound and complete.

Keywords: Modal Logic, Knowledge Representation, Arrow Update Logic, Dynamic Epistemic Logic, Arbitrary Arrow Update Logic, Arbitrary Announcement Logic

1. Introduction

In dynamic epistemic logic [16] various information changing events can be modeled, from modest public announcements, to powerful action models that can change an epistemic model beyond recognition. Here, we study arrow updates, a type of information changing event that is more powerful than public

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announcements but less powerful than action models. Roughly speaking, in public announcement logic (PAL [28]) one specifies which *states* in the model will remain as a result of the announcement, in arrow update logic (AUL [22]) one puts constraints on pairs in *relations* that endure the update (while in action model logic, also new states and new pairs can emerge as a result of the action). Let us emphasise at this point that although such relations can denote indistinguishability for an agent between states, they can also denote any kind of transition between states, or a temporal relation, a preference, etc. In other words, arrow update logic is relevant for many logics that are used in Artificial Intelligence, whether these logics model epistemic, doxastic or other attitudes of agents, dynamics, strategic interaction, or systems of norms (see also Section 2).

One line of dynamic epistemic logics adds quantifiers over information changing events, ranging from quantifiers over public announcements [10, 5], group announcements [2], to quantifiers over action models [5]. An overview of the literature on this topic is provided by [14]. These different "quantified operator" logics find their application in analyzing the concept of knowability [10], but also in, e.g., security where one can express properties like no information changing event can disclose certain information to some agent. Such logics with quantifiers over information change find an application in epistemic protocol logics [32, 15] that allow for protocol change or protocol declaration. For a different approach to quantification over information change see [8], where a first-order modal logic is used.

In this paper we introduce *arbitrary arrow update logic* (AAUL), which allows quantification over arrow updates. Like the other quantified logics, we can use AAUL to reason about knowability and security. Additionally, AAUL can be used to reason about protocol and rule design, as we will show in Section 2.

We establish three kinds of results concerning AAUL. The first concerns expressivity of the logic. We show that, under the usual assumptions that the set of agents is finite and the set of propositional variables is infinite, arbitrary public announcement logic and arbitrary arrow update logic are incomparable in expressivity over the class of all Kripke models. We also identify a case where

AAUL is more expressive than APAL. Finally, we compare arbitrary arrow update logic to a number of other logics, and conclude that it is incomparable to epistemic logic with common knowledge and that it is more expressive than basic epistemic logic (and therefore also more expressive than arbitrary action model logic and refinement modal logic [12]). Secondly, we show that the model checking problem for AAUL is PSPACE-complete. Finally, we introduce a proof system for AAUL, and prove it to be sound and complete with respect to our set of intended models.

To argue for the relevance of AAUL for Artificial Intelligence in general and knowledge representation in particular, it is helpful to also show why AUL is relevant, and to keep in mind that AAUL is to AUL what APAL is to PAL. Where in PAL, semantically (that is, on Kripke models), the object of study is the elimination of states that do not satisfy a given specification (the announcement), in APAL then the question is what kind of sets can be eliminated, and which properties are invariant under arbitrary elimination. As pointed out above, PAL and APAL are primarily studied in contexts where the states represent epistemic information of agents, so PAL and APAL are pre-dominantly used as formalisms to study dynamic epistemic phenomena, answering questions like what kind of information can be learned ('for which φ is $[\varphi]\Box_a\varphi$ true?'), and what kind of information is knowable ('for which φ is there an announcement ψ such that $[\psi]\Box_a\varphi$?'). But elimination of states is also relevant in other contexts then epistemic ones, like for instance in deontic reasoning, where some states may be (morally, or deontically) better than other states. In this context, the PAL construct $[\psi]\varphi$ would be interpreted as 'if a law guaranteeing ψ would be enforced, as a result, φ would be true'.

Where PAL and APAL focus on the elimination of specified or arbitrary sets of states, respectively, the focus of attention of AUL and AAUL is on the elimination of specified or arbitrary sets of *transitions*. For instance, where the deontic interpretation of (A)PAL addresses ought-to-be norms (*'Sein Sollen'*), a deontic interpretation of (A)AUL is about ought-to-do norms (*'Tun Sollen'*), see e.g., the chapter 'Deontic logic as I see it', by the founder of deontic logic, von

Wright, in [26] or [13] for a computer science perspective. So if the relations in the Kripke model represent transitions, AUL can be used to reason about social laws: is it the case that, by disallowing certain transitions, we can guarantee a particular property? Norms can relate to rationality for instance, and indeed, in AUL one can mimic backward induction in an extensive form game by requiring that all moves for agent i should be kept which do not affect his chances of winning the game. But then, under this perspective, AAUL is useful for the *Syntheses* problem in social laws, and the *mechanism design* problem in game theory, because it allows one to study questions like 'is there a social law (in the sense that only certain transitions are allowed) that guarantees a certain outcome?' Or, 'is there a game (in the sense that only certain moves in the extensive form of it are allowed) that only leaves a specified set of outcomes?'. The application of AUL and AAUL to social laws and mechanism design in further studied in [25, 23]. We return to the normative interpretation of arrow updates in Section 2.2.

Arrow updates also have epistemic interpretations, which reinforces their relevance for knowledge representation. As we will argue in Section 2.1, arrow updates are more general than public announcements, since one can model *semi-private* announcements. These are announcements where only a sub-group of all the agents learn certain information, while all agents are aware what the protocol is (like when all students in a class know that their teacher has sent their marks to the administration office). This implies that AAUL provides a formalism to reason about arbitrary semi-private announcements, making it possible to express properties that are relevant for epistemic planning, like 'there is a private announcement, such that everybody in Ag_1 knows what the password to the system is, while everybody in Ag_2 remains ignorant about this password'. The application of (A)AUL to doxastic logic would have a similar taste as that to epistemic logic. To give a simple example, removing a reflexive arrow in doxastic logic would correspond to a situation where an agent's belief are not necessary correct any more.

More generally, in every AI-context where Kripke models are used to repre-

sent information in a certain context, AUL and AAUL can be applied to reason about a dynamic representation of that context, where certain transitions between certain states can be removed. If the binary relations represent agents who can take moves, AUL enables us to reason about forbidding certain agents to act in certain situations, wheres AAUL can represent information about what can be achieved in principle, by restricting the moves that are available to the agents. If the accessibility relation represents the flow of time, AAUL can formulate questions of what is guaranteed to hold if certain transitions will not occur. The relation in a Kripke model could represent what the goals are of agents: AAUL in this case would provide a formalism to reason about agents dropping goals, which is considered to be an important aspect in agent programming languages (see for instance the programming language GOAL [21, 1]). Likewise AUL and AAUL provide tools to reason about intention revision ([31]) and hence, in principle for the dynamics of many agents' attitudes, including Beliefs, Desires and Intentions ([29]).

The arbitrary arrow update operator in AAUL adds *implicit* quantification over arrow updates. Recently, [9] used the capacity in second order modal logic to *explicitly* quantify over propositions. This makes it possible to define arbitrary announcements within the object language: $\forall p[p]\varphi$. Additionally, this also makes it possible to express properties like *preservation* ($\varphi \to \forall p[p]\varphi$) and *knowability* ($\varphi \to \exists p\langle p\rangle\Box_a\varphi$ within the object language. It would of course be interesting to do something similar for arrow updates. If we represent an arrow update by U, the analogue of preservation would express a *stability* condition of the environment, and, likewise, knowability could be studied with respect to arrow updates. There would also interesting generalisations outside the scope of epistemic logic that would become expressible, like for instance in agent normative languages: $Ought(\varphi) \to \exists U\langle U\rangle[\pi]\varphi$ (if φ ought to be the case, there is a social law such that, when implemented, the agent's program will achieve (or maintain) it). However, as will become clear from the next sections, updates U are not represented by a formula only, and hence quantifying over them in an object language would require much more than embedding it in second-order

modal logic. We leave studying such extensions of the object language for future work.

The rest of this paper is organised as follows. We start with an informal discussion of logics for arrow updates in Section 2. After that, in Section 3, we define arbitrary arrow update logic as well as the logics we want to compare AAUL to. Then, in Section 4 we prove a number of expressivity results related to AAUL. In Section 5 we show that the model checking problem for AAUL is PSPACE-complete. Finally, in Section 6 we provide a proof system for AAUL, and we show that it is sound and complete.

2. The Different Meanings of Arrow Updates

Public announcements and arrow updates were originally introduced as types of dynamic epistemic logic. As such, they were intended to be interpreted as so-called "epistemic events," which change the information state of one or more agents. But there are other interpretations of these operators that seem equally useful. We briefly discussed several of these interpretations in the Introduction. Here, we discuss two interpretations in more detail: the *epistemic* interpretation and the *normative* interpretation.[1] interpretation.

2.1. The Epistemic Interpretation

Consider the following situation: Alice and Bob are playing a simple card game. There are only two cards in play, the king of spades and the ace of spades. Both players are dealt one card, face down, so neither player knows which card they (or the other player) have been dealt. There are two possibilities: Alice has the king, which we denote by p, or Alice has the ace, which we denote by $\neg p$. Suppose that Alice has in fact been dealt the king, although neither Alice nor Bob knows this. We are interested in the information state of the two agents, which is usually represented as a Kripke model such as the model M_{Ep} shown

[1] Note that we call it the "normative interpretation" because we use the logics to reason about norms. We do not wish to take a position in the normative vs. descriptive debate.

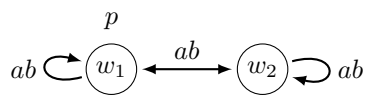


Figure 1: A simple epistemic model M_{Ep}.

in Figure 1. In this interpretation, arrows between worlds are used to represent the uncertainty of agents. There is an arrow labeled a from w to w' if and only if in world w agent a is uncertain about the state of the world and thinks it might be w' instead of w. In such a case we say that w' is *epistemically accessible* from w for a. We say that a *knows* in w that φ if and only if φ is true in every world w' that is epistemically accessible for a.

Note that M_{Ep} is an accurate representation of the simple card game. The case where Alice has the king is represented as world w_1 and the case where Alice has the ace is represented as world w_2. Regardless of who has which card, neither agent knows what card they have, so they consider both cases possible. But the agents *do* posses knowledge: in w_1 for instance, Alice knows that Bob does not know that Alice holds the king, written $M_{Ep}, w_1 \models \Box_a \neg \Box_b p$. In our example Alice holds the king, so w_1 is the actual state of the world while w_2 is an alternative that the agents consider possible.

In this setting, a public announcement represents any event that is publicly observed and that provides agents with more information. In particular, because it is publicly observed, all agents receive the same information. An example of a public announcement is a literal announcement that is made in public by a trusted source. Claire could walk in, look at both cards and state that "Alice has the king." But there are also other events that, while not literally being announcements, have the same effect. For example, Alice could turn her card, and place it face up on the table. Note that when Alice does this, all agents receive the same information: they learn that p is true.

In the model, the effect of a public announcement is quite simple: if φ is announced then all worlds that do not satisfy φ are removed, as well as all arrows that point to such worlds. In the example, if Alice turns her card face

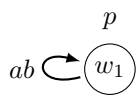


Figure 2: The updated model $M_{Ep} * p$.

up the model M_{Ep} is changed to the model $M_{Ep} * p$, which is shown in Figure 2. In this updated model only one world remains. In this world p holds, so both agents know that p. This makes sense: Alice just turned her card face up so obviously both agents now know that she has the king. We denote this by $M_{Ep}, w_1 \models [p](\Box_a p \wedge \Box_b p)$, which can be read as "after p is announced, a and b know that p."

A formula φ holds after an *arbitrary* public announcement, written $[!]\varphi$, if for every ψ, we have $[\psi]\varphi$. The dual, $\langle!\rangle\varphi$, denotes that for some ψ, it holds that $[\psi]\varphi$. In our example, we have for instance that after all announcements Bob considers it possible that Alice has the king of spades, and there is no announcement after which Alice learns her card while Bob does not:

$$M_{Ep}, w_1 \models [!]\neg\Box_b\neg p \wedge \neg\langle!\rangle(\Box_a p \wedge \neg\Box_b p) \tag{1}$$

Public announcements are very useful for modeling events that are observed by all agents. But they cannot model more complicated events. For these more complicated events we could instead turn to the extremely powerful action models [7]. Unfortunately, the great power of action models comes with a significant cost: the model checking problem for public announcement logic can be solved in polynomial time [11][Lemma 29], whereas the model checking problem for a logic with event models is PSPACE-complete [4].[2] Additionally, the increased power of action models requires a more complicated syntax, making them harder for humans to read and understand. Arrow updates fit in between: they are capable of modeling more events than public announcements—if not as many

[2]The satisfiability problem for action models is also more expensive than that of public announcements, taking NEXPTIME as opposed to PSPACE [4].

as action models—but their model checking problem can still be solved in polynomial time, as shown in [25]. Switching from public announcements to arrow updates does, however, still come with some cost: arrow updates are harder for humans to read and understand than public announcements, even though they are still easier to read than action models.

The kind of events that can be modeled by arrow updates are sometimes referred to as *semi-private* announcements. Like public announcements, semi-private announcements are events where agents learn new information. But unlike public announcements, this new information need not become available to all agents. However, while the new information can be private, the procedure or protocol through which the agents gain information is publicly known. It is this combination of private information and a public protocol that gives semi-private announcements their name.

Returning to our card scenario, suppose that instead of turning her card face up, Alice openly looks at her card without showing it to Bob. By doing this, Alice learns that she holds the king. Her action cannot be modeled as a public announcement, because the new information is not given to all agents. It can, however, be modeled as a semi-private announcement: Alice looking at her card can be seen as a run of the protocol "if Alice holds the king then she learns she has the king, if Alice holds the ace she learns that she has the ace." Bob does not learn that Alice holds the king, but he does know that the protocol has been executed. We can represent this semi-private announcement as the arrow update $U_{Ep} := (p, a, p), (\neg p, a, \neg p), (\top, b, \top)$. The triples in the arrow update are called *clauses*. A clause has the form (source, agent, target), and can be interpreted as "if *source* holds then *agent* learns that *target* holds."[3] Or perhaps it might be better to say that the agent learns that the target *used* to hold, as in some cases the very fact that the agent learns the truth of the target

[3]For technical reasons we allow multiple clauses with the same source and agent but different target. Such clauses can be read disjunctively. For example, $(p, a, \Box_b p), (p, a, \Box_a p)$ can be read as $(p, a, \Box_b p \vee \Box_a p)$.

can make it false. For example, if a learns that the Moore sentence $p \wedge \neg\Box_a p$ holds, then $p \wedge \neg\Box_a p$ becomes false.

Remark 1. There is another, more technical, difference between public announcements and arrow updates that is worth mentioning. Public announcements are usually assumed to be *truthful*. There is no such assumption for arrow updates. While the information contained in an arrow update can, of course, be truthful, there is no technical restriction on arrow updates that guarantees truthfulness. So arrow updates can be used to model events that, accidentally or by design, convey incorrect information to an agent.

Knowledge is, traditionally, assumed to be truthful. It might therefore be slightly more accurate to say that we are modeling the effect that events have on the agent's *beliefs*, rather than their *knowledge*.

Recall that in order to apply a public announcement $[\varphi]$ in a Kripke model we removed all worlds that did not satisfy φ. In order to apply an arrow update we do something similar: an arrow $(w_1, w_2) \in R(a)$ *satisfies a clause* $(\varphi_1, b, \varphi_2)$ if and only if w_1 satisfies φ_1, $a = b$ and w_2 satisfies φ_2. When applying an arrow update we remove all arrows that do not satisfy any of the clauses in the update.[4] Let us return to our example. If we apply the update $U_{Ep} = (p, a, p), (\neg p, a, \neg p), (\top, b, \top)$ to the model M_{Ep} we obtain the model shown in Figure 3. All arrows for b are retained, because they satisfy $(\top, b, \top)$. The arrow for a from w_1 to itself satisfies (p, a, p) and the arrow for a from w_2 to itself satisfies $(\neg p, a, \neg p)$, so both are retained. The arrows for a between w_1 and w_2 satisfy none of the clauses, so they are removed. Note that in $M_{Ep} * U_{Ep}$ Alice knows which card she holds and Bob knows that Alice knows this, but Bob does not know which card Alice holds. This is exactly as it should be, since Alice looked at her card openly but without showing it to Bob. Similar to public announcements, we use $M_{Ep}, w_1 \models [U_{Ep}]\psi$ to denote "after

[4] In order to emphasize the similarities between public announcements and arrow updates we specify the arrows that are to be retained. We could, alternatively, specify the arrows that are to be removed. The two types of specification are inter-definable.

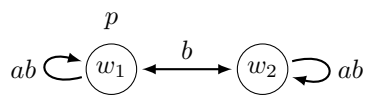


Figure 3: The updated model $M_{Ep} * U_{Ep}$.

U_{Ep} has happened, ψ holds." In our particular example, we have for instance $M_{Ep}, w_1 \models [U_{Ep}](\Box_a p \wedge \neg\Box_b p)$.

It should now be clear how the quantified construct $[\updownarrow]\varphi$ reads, namely that φ holds for every arrow update U.[5] We use $\langle\updownarrow\rangle$ as a dual of $[\updownarrow]$, so $\langle\updownarrow\rangle\varphi$ holds if and only if there is some U such that $[U]\varphi$. In the epistemic interpretation, $\langle\updownarrow\rangle\varphi$ means "there is some semi-private announcement U that will make φ true." A typical example is $\langle\updownarrow\rangle(\Box_a p \wedge \neg\Box_b p)$, which can be read as "it is possible to semi-privately inform a that p is true, without informing b." Indeed, for our example we have:

$$M_{Ep}, w_1 \models \langle\updownarrow\rangle\Box_b\neg p \wedge \langle\updownarrow\rangle(\Box_a p \wedge \neg\Box_b p) \tag{2}$$

The reader should compare (1) with (2). With public announcements and arbitrary public announcements, we can only remove access to a world for all agents at the same time, by removing the world entirely. Using arrow updates and arbitrary arrow updates, we can more subtly just remove access between two worlds for some agents, while leaving it intact for others. In words: using announcements in M_{Ep}, the two agents that start out knowing the same will always know the same, while using arrow updates, it is possible to reach a situation in which the two agents have different knowledge.

2.2. The Normative Interpretation

In the normative interpretation, we interpret arrows not as uncertainty but as the agents' ability to act; There is an a-arrow from w_1 to w_2 if and only if in w_1 there is some action agent a can take that would change the state of the

[5] There is a technical complication here, related to circularity. See Section 3 for details.

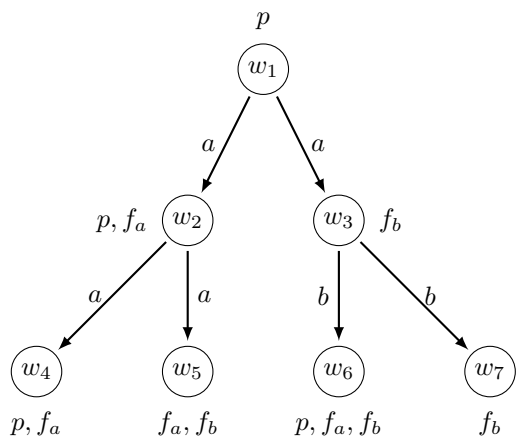


Figure 4: The model M_{No}.

world from w_1 to w_2. So we have $\Box_a \varphi$ if and only if φ will be true after every (single) action by a. Let us consider an example. Alice an Bob are together in the office, working late. Both need to use the printer, but printer time is a limited resource: there is only enough time to print two files. At each point in time, either Alice has control of the printer (p) or Bob has control ($\neg p$). The person in control of the printer can choose to print their own file, or they can print the other agent's file. If they print the other's file, then by doing so they also transfer control. If Alice has finished printing her file we represent this by f_a, and if Bob has finished printing his file we represent that by f_b. This situation can be represented by the model M_{No} shown in Figure 4.

In this example, like in most other situations, some of the available actions are more desirable than others. Ideally, both agents get the opportunity to print their files, so in w_2 Alice should not keep control of the printer to herself, and in w_3 Bob should not keep control to himself. The division of actions into "good" actions and "bad" actions can be referred to as a *norm*.[6]

[6]In the example, we considered a norm of fairness or perhaps morality, but we could also have considered a norm of legality, rationality or etiquette. Even an arbitrary set of actions can become a norm if the agents involved agree to abide by it.

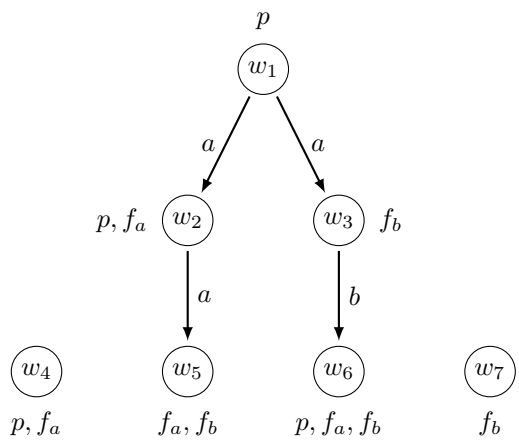


Figure 5: The updated model $M_{No} * U_{No}$.

In the normative interpretation, arrows in a Kripke model represent actions. Arrow updates allow us to specify a subset of the arrows, which we can interpret as those actions that are allowed. Returning to our example, the norm that agents should not keep the printer to themselves if they have finished printing their file can be represented by the arrow update $U_{No} := (\neg f_a, a, \top), (f_a, a, f_b), (\neg f_b, b, \top), (f_b, b, f_a)$.[7] If we apply U_{No} to M_{No} we get the model shown in Figure 5. In the updated model we only consider those actions that are not only possible but also allowed. Writing $\Box\varphi$ for $\Box_a\varphi \wedge \Box_b\varphi$, we have $M_{No} * U_{No}, w_1 \models \Box\Box(f_a \wedge f_b)$, which is equivalent to $M_{No}, w_1 \models [U_{No}]\Box\Box(f_a \wedge f_b)$. The latter can be read as "if everyone obeys the norm U_{No} then in two times steps $f_a \wedge f_b$ is guaranteed to hold." Readers familiar with the literature on Normative Systems may also note that the normative interpretation of arrow updates is in some ways very similar to Normative Temporal Logic, see for example [3].

In the normative interpretation, $[U]\varphi$ means "if everyone obeys the norm U, then φ will hold." As such, $\langle\updownarrow\rangle\varphi$ means "there is some norm/rule/protocol/law

[7]Note that U_{No} can be read as "if your file isn't printed yet, do whatever you want. If your file is printed, make sure the other file gets printed."

that, if obeyed, will guarantee the truth of φ." Returning to the example given above, $\langle\updownarrow\rangle\Box\Box(f_a \wedge f_b)$ means "there is some rule that, if followed, would guarantee that both files get printed. We saw that $[U_{No}]\Box\Box(f_a \wedge f_b)$ holds in M_{No}, w_1, so $\langle\updownarrow\rangle\Box\Box(f_a \wedge f_b)$ holds there as well.

Remark 2. Recall that, in the epistemic interpretation, arrow updates are not necessarily truthful. Semantically, this means that there is no guarantee that an arrow from a state to itself will be retained. In the normative interpretation, this property means that a norm U does not necessarily allow agents to remain in the same state—i.e. to do nothing. So arrow updates can be used to formalize norms that require agents to take action.

In sum, using logics for arrow updates, one can reason about the result of removing certain transitions from a model. This can be used to reason about ethics, rationality or planning. One interesting line of research (that we will not pursue further in this paper, but see [24] for some preliminary results), would be to enhance the capability of AAUL to reason about planning by adding more temporal operators. For example, we could use the CTL operators AG ("on every path, at all times in the future") and AF ("on every path, at some time in the future") to represent properties like *liveness*, *fairness* and *safety* in concurrent processes (see for instance [30]). Let *good* represent some kind of desirable state, *bad* an undesirable state, en_i the fact that agent i is enabled and ex_i the fact that agent i is allowed to execute. Then we can define $\mathit{live} := AG\,AF\,\mathit{good}$, $\mathit{fair} := \bigwedge_i(AG\,AF\,en_i \rightarrow AG\,AF(en_i \wedge ex_i))$ and $\mathit{safe} := AG\,\neg\mathit{bad}$. The formula $[U](\mathit{live} \wedge \mathit{fair} \wedge \mathit{safe})$ then means "the protocol U guarantees liveness, fairness and safety." As such, $\langle\updownarrow\rangle(\mathit{live} \wedge \mathit{fair} \wedge \mathit{safe})$ expresses that there is a way to constrain the overall system such that the desirable properties hold.

3. Language and semantics

In this paper we compare AAUL to a number of other logics. For the sake of brevity we only give full definitions for some of these logics. In addition to AAUL we give definitions for arbitrary public announcement logic (APAL), the

fragments arrow update logic (AUL) and public announcement logic (PAL) of AAUL and APAL respectively and a basic epistemic logic (EL). For definitions of the other logics we refer to publications that do give a complete definition.

This still leaves us with five logics to define. The most convenient way to do this is to consider them as fragments of one larger logic. This logic is a combination of APAL and AAUL, so we refer to it as APAUL. Let A be a nonempty finite set of agents and P a countably infinite set of propositional variables.

Definition 1 (Languages)**.** The language $L(A, P)$ of APAUL consists of all formulas and updates given by the following BNF:

$$\begin{array}{rcl} \varphi & ::= & p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_a\varphi \mid [\varphi]\varphi \mid [U]\varphi \mid [!]\varphi \mid [\updownarrow]\varphi \\ U & ::= & (\varphi, a, \varphi) \mid (\varphi, a, \varphi), U \end{array}$$

where $p \in P$ and $a \in A$. We write L for $L(A, P)$ where this should not cause confusion.

The language L_{EL} of epistemic logic is the fragment of L that does not contain the operators $[\varphi]$, $[U]$, $[!]$ and $[\updownarrow]$. The language L_{PAL} of public announcement logic is the fragment of L that does not contain the operators $[U]$, $[!]$ and $[\updownarrow]$. The language L_{APAL} of arbitrary public announcement logic is the fragment of L that does not contain the operators $[U]$ and $[\updownarrow]$. The language L_{AUL} of arrow update logic is the fragment of L that does not contain the operators $[\varphi]$, $[!]$ and $[\updownarrow]$. The language L_{AAUL} of arbitrary arrow update logic is the fragment of L that does not contain the operators $[\varphi]$ and $[!]$.

We use $\vee, \rightarrow, \leftrightarrow, \top, \bot, \bigwedge$ and $\bigvee$ in the usual way as abbreviations, and we abuse notation slightly by identifying the list $U = (\varphi_1, a_1, \psi_1), \cdots, (\varphi_n, a_n, \psi_n)$ with the set $U = \{(\varphi_1, a_1, \psi_1), \cdots, (\varphi_n, a_n, \psi_n)\}$. Furthermore, we use $\Diamond_a$, $\langle\varphi\rangle$, $\langle U\rangle$, $\langle!\rangle$ and $\langle\updownarrow\rangle$ as abbreviations for $\neg\Box_a\neg$, $\neg[\varphi]\neg$, $\neg[U]\neg$, $\neg[!]\neg$ and $\neg[\updownarrow]\neg$. Finally, if $B = \{a_1, \cdots, a_n\}$ we write (φ, B, ψ) for $(\varphi, a_1, \psi), \cdots, (\varphi, a_n, \psi)$.

These languages are all interpreted on Kripke models.

Definition 2. A *Kripke model* M is a triple (W, R, V) where

- $W \neq \emptyset$ is a set of worlds,
- $R : A \to \wp(W \times W)$ assigns an accessibility relation to each $a \in A$ and
- $V : P \to \wp(W)$ is a valuation.

A Kripke model $M = (W, R, V)$ is an *S5 model* if $R(a)$ is an equivalence relation for all $a \in A$.

We can now define the semantics of APAUL. The other logics simply inherit their semantics from APAUL.

Definition 3. Let $M = (W, R, V)$ be a Kripke model and let $w \in W$. The satisfaction relation $\models$ is given inductively as follows.

$$
\begin{array}{lll}
M, w \models p & \text{iff} & w \in V(p) \\
M, w \models \neg\varphi & \text{iff} & M, w \not\models \varphi \\
M, w \models (\varphi \wedge \psi) & \text{iff} & M, w \models \varphi \text{ and } M, w \models \psi \\
M, w \models \Box_a \varphi & \text{iff} & M, v \models \varphi \text{ for each } v \text{ such that } (w, v) \in R(a) \\
M, w, \models [\psi]\varphi & \text{iff} & M, w \not\models \psi \text{ or } (M * \psi), w \models \varphi \\
M, w \models [U]\varphi & \text{iff} & (M * U), w \models \varphi \\
M, w \models [!]\varphi & \text{iff} & M, w \models [\psi]\varphi \text{ for each } \psi \in L_{PAL} \\
M, w \models [\updownarrow]\varphi & \text{iff} & M, w \models [U]\varphi \text{ for each } U \in L_{AUL}
\end{array}
$$

where $(M * \psi)$ and $(M * U)$ are given by:

$$
\begin{array}{lcl}
M * \psi & = & (W^\psi, R^\psi, V^\psi) \\
W^\psi & = & \{w \in W \mid M, w \models \psi\} \\
R^\psi(a) & = & R(a) \cap (W^\psi \times W^\psi) \\
V^\psi(p) & = & V(p) \cap W^\psi \\
M * U & = & (W, R^U, V) \\
R^U(a) & = & \{(v, v') \in R(a) \mid \exists(\varphi, a, \varphi') \in U : \\
 & & \qquad (M, v \models \varphi \text{ and } M, v' \models \varphi')\}
\end{array}
$$

A formula φ is *true on M*, denoted $M \models \varphi$, if $M, w \models \varphi$ for all $w \in W$. A formula φ is *valid*, denoted $\models \varphi$, if $M \models \varphi$ for every Kripke model M. A formula φ is *valid on S5*, denoted $\models_{S5} \varphi$, if $M \models \varphi$ for every S5 model M.

Remark 3. Note that the arbitrary public announcement operator [!] quantifies only over $\psi \in L_{PAL}$. So the formulas that [!] quantifies over do not themselves contain [!] (or [↕], for that matter). Likewise, [↕] only quantifies over arrow updates U that do not contain [↕] and [!]. Restricting [!] and [↕] in this way is necessary in order to avoid circularity.

The price we pay for solving the circularity problem this way is that [!] and [↕] do not quite have the informal meaning we would like to associate with them. We would like $[!]\varphi$ to mean "for every announcement $[\psi]$, we have $[\psi]\varphi$." And when we say *every* announcement, that includes those announcements $[\psi]$ where ψ contains [!]. But, as shown in [24], there are φ and ([!]-containing) ψ such that $\not\models [!]\varphi \to [\psi]\varphi$. So [!] does not quite have the desired informal meaning. A similar proof for [↕] does not yet exist, but we believe that the U and φ containing [↕] can be constructed such that $\not\models [\updownarrow]\varphi \to [U]\varphi$.

Remark 4. Unlike public announcements, arrow updates do not preserve S5: updating an S5 model M with an arrow update U may result in a non-S5 model $M * U$.[8] We can, of course, evaluate AAUL on S5 models. We just have to keep in mind that it is possible that, during the evaluation of a formula, we may move from an S5 model to a non-S5 model. As a result, necessitation fails for $[U]$ on S5. For example, we have $\models_{S5} \Box_a\varphi \to \varphi$ but $\not\models_{S5} [U](\Box_a\varphi \to \varphi)$.

Necessitation for $[U]$ does still hold on the class of all Kripke models: if $\models \psi$ then $\models [U]\psi$. It therefore seems fair to say that the class of all Kripke models is the "natural habitat" of arrow updates.

The main reason why arrow updates do not preserve S5 is that, as discussed above, they are not guaranteed to be truthful.

Now that we have defined the semantics of AAUL, let us consider a few equivalences that will be useful later.

Lemma 1. *For every pointed model M, w, every $p \in P$, every $a \in A$ and every*

[8] The same holds for other commonly used subclasses of the class of all Kripke models, such as S4 and KD45.

$\varphi, \psi, U, U' \in L_{AAUL}$ *we have*

$$\begin{array}{lcl} M, w \models [U]p & \Leftrightarrow & M, w \models p \\ M, w \models [U]\neg\varphi & \Leftrightarrow & M, w \models \neg[U]\varphi \\ M, w \models [U](\varphi \wedge \psi) & \Leftrightarrow & M, w \models [U]\varphi \wedge [U]\psi \\ M, w \models [U]\Box_a\varphi & \Leftrightarrow & M, w \models \bigwedge_{(\psi,a,\chi)\in U}(\psi \to \Box_a(\chi \to [U]\varphi)) \\ M, w \models [U][U']\varphi & \Leftrightarrow & M, w \models [U \times U']\varphi \end{array}$$

where $U \times U' = \{(\psi_1 \wedge [U]\psi_2, a, \chi_1 \wedge [U]\chi_2) \mid (\psi_1, a, \chi_1) \in U, (\psi_2, a, \chi_2) \in U'\}$.

Proof. The first three statements are easy to prove. Arrow updates do not change the valuation of a model, so $M, w \models [U]p$ if and only if $M, w \models p$. Arrow updates commute with negation because $M, w \models [U]\neg\varphi \Leftrightarrow (M * U), w \models \neg\varphi \Leftrightarrow (M * U), w \not\models \varphi \Leftrightarrow M, w \not\models [U]\varphi \Leftrightarrow M, w \models \neg[U]\varphi$. Arrow updates distribute over conjunctions because $M, w \models [U](\varphi \wedge \psi) \Leftrightarrow (M * U), w \models \varphi \wedge \psi \Leftrightarrow (M * U), w \models \varphi$ and $M, w \models \psi \Leftrightarrow M, w \models [U]\varphi$ and $M, w \models [U]\psi \Leftrightarrow M, w \models [U]\varphi \wedge [U]\psi$.

The last two statements require slightly more work to prove. We have $M, w \models [U]\Box_a\varphi \Leftrightarrow (M * U), w \models \Box_a\varphi \Leftrightarrow (M * U), w' \models \varphi$ for all w' that are a-accessible from w' in $M * U$. Note that we have $(M * U), w' \models \varphi \Leftrightarrow M, w' \models [U]\varphi$. So $M, w \models [U]\Box_a\varphi \Leftrightarrow M, w' \models [U]\varphi$ for all w' that are a-accessible from w in $M * U$. The trick is now to determine which worlds w' are a-accessible from w in $M * U$, so for which w' we must have $M, w' \models [U]\varphi$ in order to have $M, w \models [U]\Box_a\varphi$.

Consider any clause $(\psi, a, \chi) \in U$. If $M, w \models \psi$ then every a-successor w' of w that satisfies χ is an a-successor of w in $M * U$. The formula $[U]\varphi$ holds in all these worlds if and only if $M, w \models \psi \to \Box_a(\chi \to [U]\varphi)$. We then only have to repeat this for every $(\psi, a, \chi) \in U$: we have $M, w \models [U]\Box_a\varphi$ if and only if $M, w \models \bigwedge_{(\psi,a,\chi)\in U}(\psi \to \Box_a(\chi \to [U]\varphi))$.

Finally, in order to determine when we have $M, w \models [U][U']\varphi$ we must determine which arrows are retained if we apply $[U]$ and $[U']$ after each other. An arrow is retained by the first update $[U]$ if and only if it satisfies some clause $(\psi_1, a, \chi_1) \in U$. In order for this arrow to be retained by U' as well,

it must additionally satisfy some clause (ψ_2, a, χ_2). But it must satisfy this (ψ_2, a, χ_2) not in M but in $M * U$. Such an arrow must therefore start in a $\psi_1 \wedge [U]\psi_2$ world, and go to a $\chi_1 \wedge [U]\chi_2$ world. In that case it satisfies the clause $(\psi_1 \wedge [U]\psi_2, a, \chi_1 \wedge [U]\chi_2)$. We have such a combined clause for every $(\psi_1, a, \chi_1) \in U$ and every $(\psi_2, a, \chi_2) \in U'$, so $M, w \models [U][U']\varphi \Leftrightarrow M, w \models [U \times U']\varphi$ where $U \times U' = \{(\psi_1 \wedge [U]\psi_2, a, \chi_1 \wedge [U]\chi_2) \mid (\psi_1, a, \chi_1) \in U, (\psi_2, a, \chi_2) \in U'\}$. $\square$

In later sections we will also use the notions of *bisimulation* and *bisimulation contraction*. The following definitions are as usual.

Definition 4. Let $M_1 = (W_1, R_1, V_1)$ and $M_2 = (W_2, R_2, V_2)$ be models and let B be a relation on $W_1 \times W_2$. The relation B is a *bisimulation* on M_1 and M_2 if for all $(w_1, w_2) \in B$, we have

- for every $p \in P$, $w_1 \in V_1(p) \Leftrightarrow w_2 \in V_2(p)$,
- for every $a \in A$ and every $w_1' \in W_1$ such that $(w_1, w_1') \in R_1$, there is a $w_2' \in W_2$ such that $(w_2, w_2') \in R_2$ and $(w_1', w_2') \in B$ and
- for every $a \in A$ and every $w_2' \in W_2$ such that $(w_2, w_2') \in R_2$, there is a $w_1' \in W_1$ such that $(w_1, w_1') \in R_2$ and $(w_1', w_2') \in B$.

Two states $w_1 \in W_1$ and $w_2 \in W_2$ are *bisimilar*, denoted $w_1 \sim_{M_1, M_2} w_2$, if there is a bisimulation B on M_1 and M_2 such that $(w_1, w_2) \in B$.

We omit mentioning the models where this should not cause confusion, and write $w_1 \sim w_2$ if the states are bisimilar. The following is a standard result, see any textbook or handbook on modal logic for details.

Lemma 2. *Let $M_1 = (W_1, R_1, V_1)$ and $M_2 = (W_2, R_2, V_2)$ be models. The relation $\sim \subseteq W_1 \times W_2$ is an equivalence relation, and it is a bisimulation on M_1 and M_2. Furthermore, $\sim$ is the largest bisimulation on M_1 and M_2.*

Definition 5. Let $M = (W, R, V)$ be a model. The *bisimulation contraction* of M is the model $M_{BC} = (W_{BC}, R_{BC}, V_{BC})$ given by

- $W_{BC} := \{[w] \mid w \in W\}$, where $[w] := \{w' \in W \mid w \sim w'\}$,

- for every $a \in A$, $R_{BC}(a) = \{([w],[w']) \mid (w,w') \in R(a)\}$ and
- for every $p \in P$, $V_{BC}(p) = \{[w] \mid w \in V(p)\}$.

It as another standard result that, for every $w \in W$, we have $M, w \sim_{M,M_{BC}} M_{BC}, [w]$. Furthermore, using the Paige-Tarjan algorithm[9] [27], we can compute M_{BC} in polynomial time and linear space.

The relevance of bisimulation is that well-behaved modal logics tend to be invariant under bisimulation, i.e. if $w_1 \sim w_2$ then any formula ϕ that holds in one of the states also hold in the other. AAUL is no exception.

Lemma 3. *For every $\phi \in L_{AAUL}$ and every M_1, w_1 and M_2, w_2 such that $w_1 \sim w_2$, we have $M_1, w_1 \models \phi$ if and only if $M_2, w_2 \models \phi$.*

Proof. We first show that AUL is invariant under bisimulation. The proof is by induction on the construction of ϕ. The first clause of the definition of bisimilarity guarantees that w_1 and w_2 satisfy the same propositional variables. So if ϕ is atomic, we have $M_1, w_1 \models \phi \Leftrightarrow M_2, w_2 \models \phi$. Suppose then as induction hypothesis that ϕ is not atomic and that the lemma holds for all strict subformulas of ϕ.

We continue by case distinction on the main connective of ϕ. Most of the cases are as usual, so we do not discuss them in detail. The one relatively interesting case is $\phi = [U]\phi'$.

We first show that $\sim_{M_1,M_2}$ is also a bisimulation relation on $M_1 * U$ and $M_2 * U$. Take any $w_1 \in W_1$ and $w_2 \in W_2$ such that $w_1 \sim_{M_1,M_2} w_2$. The valuations of $M_1 * U$ and $M_2 * U$ are identical to that of M_1 and M_2, respectively, so w_1 and w_2 agree on propositional variables. Furthermore, take any $a \in A$ and any w_1' such that $(w_1, w_1') \in R_1^U(a)$. Then, in particular, $(w_1, w_1') \in R_1(a)$. Since $\sim_{M_1,M_2}$ is a bisimulation, there is a $(w_2, w_2') \in R_1(a)$ such that $w_1' \sim_{M_1,M_2} w_2'$.

The arrow from w_1 to w_1' is retained by the update U, so there is some clause $(\psi, a, \psi') \in U$ such that $M_1, w_1 \models \psi$ and $M_1, w_1' \models \psi'$. By the induction

[9] Technically, the algorithm described in [27] applies to single agent models only, but extending it to the multi-agent context is trivial.

hypothesis, and the facts that $w_1 \sim_{M_1,M_2} w_2$ and $w_1' \sim_{M_1,M_2} w_2'$, it follows that $M_2, w_2 \models \psi$ and $M_2, w_2' \models \psi'$. So the arrow from w_2 to w_2' matches the clause (ψ, a, ψ') is retained by U. It follows that $(w_2, w_2') \in R_2^U(a)$ and $(w_1', w_2') \in \sim_{M_1,M_2}$.

By similar reasoning, for every w_2' such that $(w_2, w_2') \in R_2 * U(a)$ there is some w_1' such that $(w_1, w_1') \in R_1 * U(a)$ and $w_1' \sim_{M_1,M_2} w_2'$. So $\sim_{M_1,M_2}$ is a bisimulation not only on M_1 and M_2, but also on $M_1 * U$ and $M_2 * U$. By the induction hypothesis it follows that $M_1 * U, w_1 \models \phi'$ if and only if $M_2 * U, w_2 \models \phi'$, and therefore $M_1, w_1 \models [U]\phi'$ if and only if $M_2, w_2 \models [U]\phi'$, which was to be shown.

Now that we have shown that AUL is invariant under bisimulation, we can show that AAUL is also invariant under bisimulation. The proof is again by induction and by case distinction on the main connective. The only new case is $\phi = [\updownarrow]\phi'$, so we omit the other cases.

Let $w_1 \sim w_2$. For every $U \in L_{AUL}$, we have $M_1, w_1 \models [U]\phi'$ if and only if $M_2, w_2 \models [U]\phi'$. It follows immediately that $M_1, w_1 \models [\updownarrow]\phi'$ if and only if $M_2, w_2 \models [\updownarrow]\phi'$. □

So bisimilar states are indistinguishable. In general it is not the case that every two non-bisimilar states are distinguishable, but for finite models it is the case.

Lemma 4. *Let $M_1 = (W_1, R_1, V_1)$ and $M_2 = (W_2, R_2, V_2)$ be finite models. For any $w_1 \in W_1$ and $w_2 \in W_2$ such that $w_1 \not\sim w_2$, there is a $\phi \in L_{AUL}$ such that $M_1, w_1 \models \phi$ and $M_2, w_2 \not\models \phi$.*

It is a standard result that the above lemma holds for basic modal logic (which we referred to as EL here). Since $L_{EL} \subseteq L_{AUL}$ it follows immediately that the lemma holds for AUL as well.

4. Expressivity

In this section we investigate the expressive power of the newly defined arbitrary arrow update logic relative to arbitrary public announcement logic

and a number of other well-known logics in (dynamic) epistemic logic.

Definition 6 (Expressivity)**.** Let two logical languages L_1 and L_2 and a class of structures C be given. If for every formula $\varphi \in L_1$ there is a $\psi \in L_2$ such that φ and ψ are equivalent on C, we say that L_2 is *at least as expressive* as L_1 on C, notation: $L_1 \preceq_C L_2$ (and $L_1 \not\preceq_C L_2$ if this is not the case). If the converse also holds, L_1 and L_2 are equally expressive on C, notation $L_1 \equiv_C L_2$. If the converse does not hold, L_2 is strictly more expressive than L_1, notation $L_1 \prec_C L_2$. When both $L_1 \not\preceq_C L_2$ and $L_2 \not\preceq_C L_1$ then L_1 and L_2 are expressively incomparable on C, notation: $L_1 \parallel_C L_2$. We omit the subscript C if C is the class of all Kripke models.

4.1. APAL and AAUL are incomparable in expressivity

In this subsection we prove that $L_{APAL} \parallel L_{AAUL}$. This proof will be significantly easier to understand if we first describe the general structure of the proof, without considering most of the technical details.

Let $\mathrm{Upd}_!$ and let $\mathrm{Upd}_\updownarrow$ be the set of updates quantified over by $\langle ! \rangle$ and $\langle \updownarrow \rangle$ respectively.[10] Now take any $\psi \in \mathrm{Upd}_!$. PAL and AUL are both equally expressive as EL [7, 22], so in particular PAL and AUL are equally expressive as each other. As such, there is a formula $\psi' \in L_{AUL}$ that is equivalent to ψ. Let $U_\psi := (\top, A, \psi')$, and note that $U_\psi \in \mathrm{Upd}_\updownarrow$. The public announcement $\langle \psi \rangle$ removes all worlds that do not satisfy ψ. The arrow update $\langle U_\psi \rangle$ does not remove any worlds, but does eliminate all arrows to worlds that do not satisfy ψ' (which is equivalent to ψ). Inaccessible worlds might as well not exist, so removing all arrows to a world has essentially the same effect as removing the world entirely: whenever $(M * \psi), w$ exists[11], it is bisimilar to $(M * U_\psi), w$.

On the other hand, for some $U \in \mathrm{Upd}_\updownarrow$ and M, w there is no $\psi \in \mathrm{Upd}_!$ that makes $(M * U), w$ and $(M * \psi), w$ bisimilar.[12] If we abuse notation by identifying

[10] So $\mathrm{Upd}_! = L_{PAL}$ and $\mathrm{Upd}_\updownarrow = \{U \in L_{AUL} \mid U \text{ is an arrow update}\}$.

[11] If $M, w \not\models \psi$ then w is removed by $\langle \psi \rangle$, so $(M * \psi), w$ doesn't exist.

[12] This follows from the fact that public announcements preserve S5, but arrow updates do not necessarily do so.

ψ with U_ψ, we therefore have $\mathrm{Upd}_! \subset \mathrm{Upd}_\updownarrow$. In other words, $\langle\updownarrow\rangle$ quantifies over a strictly larger set than $\langle!\rangle$. Crucially, this quantification is "all or nothing"; in AAUL we can quantify over $\mathrm{Upd}_\updownarrow$, by using $\langle\updownarrow\rangle$, but not over the smaller set $\mathrm{Upd}_!$. For suitably chosen φ we would expect to find three kinds of pointed models:

1. models M_1, w such that $M_1, w \not\models \langle X\rangle\varphi$ for all $X \in \mathrm{Upd}_\updownarrow$,
2. models M_2, w such that $M_2, w \not\models \langle X\rangle\varphi$ for all $X \in \mathrm{Upd}_!$ but $M_2, w \models \langle X\rangle\varphi$ for some $X \in \mathrm{Upd}_\updownarrow$,
3. models M_3, w such that $M_3, w \models \langle X\rangle\varphi$ for some $X \in \mathrm{Upd}_!$.

The AAUL formula $\langle\updownarrow\rangle\varphi$ distinguishes between M_1, w and M_2, w, but not between M_2, w and M_3, w. The APAL formula $\langle!\rangle\varphi$ on the other hand distinguishes between M_2, w and M_3, w, but not between M_1, w and M_2, w. At this point we cannot guarantee that there is no $\psi_1 \in L_{AAUL}$ that distinguishes between M_2, w and M_3, w, or that there is no $\psi_2 \in L_{APAL}$ that distinguishes between M_1, w and M_2, w. But, because AAUL cannot quantify over $\mathrm{Upd}_!$ and APAL cannot quantify over $\mathrm{Upd}_\updownarrow$, there is no reason to assume that such ψ_1 and/or ψ_2 exist. As such, we would expect there to be no APAL formula equivalent to $\langle\updownarrow\rangle\varphi$ (which implies that $L_{AAUL} \not\preceq L_{APAL}$) and no AAUL formula equivalent to $\langle!\rangle\varphi$ (which implies that $L_{APAL} \not\preceq L_{AAUL}$).

The method we use to show that $L_{APAL} \parallel L_{AAUL}$ is inspired by the considerations described above, but with one difference. Instead of three models M_1, M_2 and M_3, we use three sets $\{M_1^x \mid x \in P \setminus \{p\}\}$, $\{M_2^x \mid x \in P \setminus \{p\}\}$ and $\{M_3^x \mid x \in P \setminus \{p\}\}$.

Definition 7. For $x \in P \setminus \{p\}$ the model $M_1^x = (W_1, R_1, V_1^x)$ is given by

- $W_1 = \{w_1, w_2\}$,
- $R_1(a) = \{(w_1, w_2)\} \cup \{(w_1, w_1)\}$,
- $R_1(b) = \emptyset$ for all $b \neq a$,
- $V_1^x(p) = \{w_1\}$,

- $V_1^x(q) = \emptyset$ for all $q \neq p$.

The model M_1^x is shown in Figure 6. We use w_3 as an alias for w_1 and w_4 as an alias for w_2 in M_1^x, so $M_1^x, w_3 = M_1^x, w_1$ and $M_1^x, w_4 = M_1^x, w_2$.

Definition 8. For $x \in P \setminus \{p\}$, the model $M_2^x = (W_2, R_2, V_2^x)$ is given by

- $W_2 = \{w_1, w_2, w_3\}$,
- $R_1(a) = \{(w_1, w_2), (w_3, w_2), (w_1, w_3), (w_3, w_1)\} \cup \{(w_1, w_1), (w_3, w_3)\}$,
- $R_1(b) = \emptyset$ for all $b \neq a$,
- $V_2^x(p) = \{w_1, w_3\}$,
- $V_2^x(x) = \{w_3\}$,
- $V_2^x(q) = \emptyset$ for all $q \notin \{p, x\}$.

The model M_2^x is shown in Figure 7. We use w_4 as an alias for w_2 in M_2^x, so $M_2^x, w_4 = M_2^x, w_2$.

Definition 9. For $x \in P \setminus \{p\}$, the model $M_3^x = (W_3, R_3, V_3^x)$ is given by

- $W_3 = \{w_1, w_2, w_3, w_4\}$,
- $R_1(a) = \{(w_1, w_2), (w_3, w_4), (w_1, w_3), (w_3, w_1)\} \cup \{(w_1, w_1), (w_3, w_3)\}$,
- $R_1(b) = \emptyset$ for all $b \neq a$,
- $V_3^x(p) = \{w_1, w_3\}$,
- $V_3^x(x) = \{w_3, w_4\}$,
- $V_3^x(q) = \emptyset$ for all $q \notin \{p, x\}$.

The model M_3^x is shown in Figure 8

The reason for using w_3 as an alias for w_1 and w_4 as an alias for w_2 is that it allows us to succinctly point out the similarities between the models; there

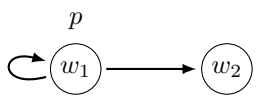


Figure 6: The model M_1^x.

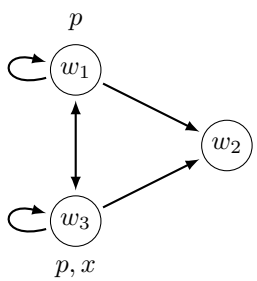


Figure 7: The model M_2^x.

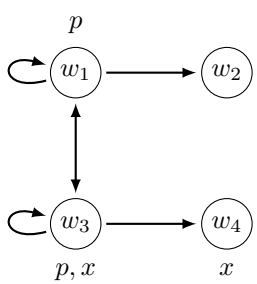


Figure 8: The model M_3^x.

is an arrow from w_i to w_j $(i, j \in \{1, 2, 3, 4\})$ in one of the models if and only if there is such an arrow in the other models.

Let $\phi_1 := \Box_a p \wedge \Diamond_a \neg \Box_a p$. We first show that $M_1^x, w_1 \not\models \langle \updownarrow \rangle \phi_1$, $M_2^x, w_1 \models \langle \updownarrow \rangle \phi_1$, $M_2^x, w_1 \not\models \langle ! \rangle \phi_1$ and $M_3^x, w_1 \models \langle ! \rangle \phi_1$ for all $x \in P \setminus \{p\}$. After that, we show that there is no APAL formula that distinguishes between M_1^x, w_1 and M_2^x, w_1 for all $x \in P \setminus \{p\}$, and that there is no AAUL formula that distinguishes between M_2^x, w_1 and M_3^x, w_1 for all $x \in P \setminus \{p\}$. This suffices to show that there is no APAL formula equivalent to $\langle \updownarrow \rangle \phi_1$ and no AAUL formula equivalent to $\langle ! \rangle \phi_1$, which implies that $L_{APAL} \parallel L_{AAUL}$.

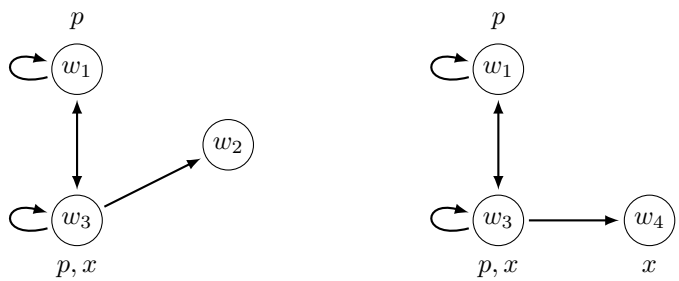
Lemma 5. *For every $x \in P \setminus \{p\}$ we have $M_1^x, w_1 \not\models \langle \updownarrow \rangle \phi_1$, $M_2^x, w_1 \models \langle \updownarrow \rangle \phi_1$, $M_2^x, w_1 \not\models \langle ! \rangle \phi_1$ and $M_3^x, w_1 \models \langle ! \rangle \phi_1$.*

Proof. Recall that $\phi_1 = \Box_a p \wedge \Diamond_a \neg \Box_a p$. Let $U_x := (p, a, p), (x \vee \neg p, a, \top)$ and $\psi_x := p \vee x$. Then $M_2^x * U_x$ and $M_3^x * \psi_x$ are as shown in Figure 9. We have $(M_2^x * U_x), w_1 \models \phi_1$ and $(M_3^x * \psi_x), w_1 \models \phi_1$, which implies that $M_2^x, w_1 \models \langle U_x \rangle \phi_1$ and $M_3^x, w_1 \models \langle \psi_x \rangle \phi_1$. Because $U_x \in L_{AUL}$ and $\psi_x \in L_{PAL}$ this, in turn, implies that $M_2^x, w_1 \models \langle \updownarrow \rangle \phi_1$ and $M_3^x, w_1 \models \langle ! \rangle \phi_1$.

Left to show is that $M_1^x, w_1 \not\models \langle \updownarrow \rangle \phi_1$ and $M_2^x, w_1 \not\models \langle ! \rangle \phi_1$. In order for any pointed model to satisfy ϕ_1 there must be at least two p-worlds in the model: one that satisfies $\Box_a p$ and one that satisfies $\neg \Box_a p$. The model M_1^x has only one p-world and arrow updates cannot add worlds. So for every arrow update U we have $M_1^x, w_1 \not\models \langle U \rangle \phi_1$. This implies that $M_1^x, w_1 \not\models \langle \updownarrow \rangle \phi_1$.

Now suppose towards a contradiction that there is some $\psi \in L_{PAL}$ such that $M_2^x, w_1 \models \langle \psi \rangle \phi_1$. In order for ϕ_1 to hold in $(M_2^x * \psi), w_1$ there must be at least two p-worlds in $M_2^x * \psi$. This means that both w_1 and w_3 must be retained by the update $\langle \psi \rangle$. Furthermore, w_3 must satisfy $\neg \Box_a p$ so w_2 must also be retained. But then $(M_2^x * \psi), w_1 \not\models \Box_a p$, contradicting the assumption that $M_2^x, w_1 \models \langle \psi \rangle \phi_1$. We therefore have $M_2^x, w_1 \not\models \langle \psi \rangle \phi_1$ for every $\psi \in L_{PAL}$, which implies that $M_2^x, w_1 \not\models \langle ! \rangle \phi_1$. $\Box$

It remains to be shown that APAL cannot distinguish between M_1^x, w_1 and M_2^x, w_1, and that AAUL cannot distinguish between M_2^x, w_1 and M_3^x, w_1. The



(a) The model $M_2^x * U_x$.

(b) The model $M_3^x * \psi_x$.

Figure 9: The updated models $M_2^x * U_x$ and $M_3^x * \psi_x$.

proofs of these claims are by induction, but there is one complication: due to the update modalities our induction hypothesis has to apply not only to M_i and M_j, but also to certain submodels of M_i and M_j. We call these submodels *APAL-equivalent* and *AAUL-equivalent*.

Definition 10. Let $N_1^x = (W_{N_1}, R_{N_1}, V_{N_1})$ and $N_2^x = (W_{N_2}, R_{N_2}, V_{N_2})$ be submodels of M_1^x and M_2^x respectively. We say that N_1^x and N_2^x are *APAL-equivalent* if

- N_1^x and N_2^x can be obtained from M_1^x and M_2^x respectively by a finite sequence of public announcements,
- for every $i \in \{1, 2, 3\}$ we have $w_i \in W_{N_1}$ if and only if $w_i \in W_{N_2}$.

Note that public announcements only remove arrows that go to or from a world that is removed. As a result, two APAL-equivalent models N_1^x and N_2^x also have the property that $(w_i, w_j) \in R_{N_1}$ if and only if $(w_i, w_j) \in R_{N_2}$ (for all $i, j \in \{1, 2, 3\}$).

Lemma 6. *Let $N_1^x = (W_{N_1}, R_{N_1}, V_{N_1})$ and $N_2^x = (W_{N_2}, R_{N_2}, V_{N_2})$ be submodels of M_1^x and M_2^x respectively that are APAL-equivalent, let $\varphi \in L_{APAL}$ and suppose $x \in P \setminus \{p\}$ is a propositional variable that does not occur in φ. Then for every $i \in \{1, 2, 3\}$, if $w_i \in W_{N_1}$ then φ does not distinguish between N_1^x, w_i and N_2^x, w_i.*

Proof. Fix any $i \in \{1, 2, 3\}$ such that $w_i \in W_{N_1}$ (and therefore also $w_i \in W_{N_2}$). We show by induction that φ does not distinguish between N_1^x, w_i and N_2^x, w_i.

As base case, suppose that $\varphi \in P$. The pointed models N_1^x, w_i and M_2^x, w_i agree on all propositional variables other than x. By assumption φ does not contain x, so it does not distinguish between those worlds.

As induction hypothesis, assume that $\varphi \notin P$ and that the lemma holds for all subformulas of φ. If a Boolean combination distinguishes between two worlds then so does at least one of the combined formulas, so we can assume without loss of generality that the main connective of φ is not Boolean. Furthermore, there are only arrows for agent a so we can assume without loss of generality that every $\Box_b$ operator has $a = b$. This means that φ is of the form $\Box_a\varphi'$, $[\varphi']\varphi''$ or $[!]\varphi'$ for some $\varphi', \varphi'' \in L_{APAL}$.

- Suppose $\varphi = \Box_a\varphi'$. The models N_1^x, w_i and N_2^x, w_i are APAL-equivalent, so a world w_j is accessible from w_i in N_1^x if and only if it is accessible in N_2^x. In order for φ to distinguish between N_1^x, w_i and N_2^x, w_i it is therefore necessary that φ' distinguishes between N_2^x, w_j and N_3^x, w_j for some w_j. That would contradict the induction hypothesis, so φ does not distinguish between N_1^x, w_i and N_2^x, w_i.

- Suppose $\varphi = [\varphi']\varphi''$. We claim that $N_1^x * \varphi'$ and $N_2^x * \varphi'$ are APAL-equivalent. Suppose towards a contradiction that they are not APAL-equivalent. Then there is some world w_j from N_1^x and N_2^x that is retained in $N_1^x * \varphi'$ but not in $N_2^x * \varphi'$, or retained in $N_2^x * \varphi'$ but not in $N_1^x * \varphi'$. That means φ' must distinguish between N_1^x, w_j and N_2^x, w_j, contradicting the induction hypothesis. We have arrived at a contradiction, so our assumption that $N_1^x * \varphi'$ and $N_2^x * \varphi'$ are not APAL-equivalent must be wrong.

  It then follows from the induction hypothesis that φ'' does not distinguish between $N_1^x * \varphi', w_i$ and $N_2^x * \varphi', w_i$, which implies that φ does not distinguish between N_1^x, w_i and N_2^x, w_i.

- Suppose $\varphi = [!]\varphi'$. Every world in N_1^x is uniquely identified by whether it satisfies p. Every world in N_2^x is uniquely identified by whether it satisfies p and x: w_1 satisfies $p \wedge \neg x$, w_2 satisfies $\neg p \wedge \neg x$ and w_3 satisfies $p \wedge x$. This means that for every formula $\psi \in L_{PAL}$ there is some formula $\psi' \in L_{PAL}$ such that, for every $j \in \{1, 2, 3\}$, we have $N_1^x, w_j \models \psi$ if and only if $N_2^x, w_j \models \psi'$. Likewise, for every formula $\psi' \in L_{PAL}$ there is some $\psi \in L_{PAL}$ such that, for every $j \in \{1, 2, 3\}$, $N_1^x, w_j \models \psi$ if and only if $N_2^x, w_j \models \psi'$.

  In either case, $N_1^x * \psi$ and $N_2^x * \psi'$ are APAL-equivalent. By the induction hypothesis this implies that φ' cannot distinguish between $N_1^x * \psi, w_i$ and $N_2^x * \psi', w_i$. As a result, there is a formula $\psi \in L_{PAL}$ such that $N_1^x, w_i \not\models [\psi]\varphi'$ if and only if there is a formula $\psi' \in L_{PAL}$ such that $N_2^x, w_i \not\models [\psi']\varphi'$. In other words, $N_1^x, w_i \not\models [!]\varphi'$ if and only if $N_2^x, w_i \not\models [!]\varphi'$.

In each of the three possible cases φ does not distinguish between N_1^x, w_i and N_2^x, w_i, completing the induction step and thereby the proof. □

The proof that AAUL cannot distinguish between N_2^x, w_1 and N_3^x, w_1 is very similar to the preceding proof, so we omit some of the details.

Definition 11. Let $N_2^x = (W_{N_2}, R_{N_2}, V_{N_2})$ and $N_3^x = (W_{N_3}, R_{N_3}, V_{N_3})$ be submodels of M_2^x and M_3^x respectively. We say that N_2^x and N_3^x are *AAUL-equivalent* if

- N_2^x and N_3^x can be obtained from M_2^x and M_3^x respectively by a finite sequence of arrow updates,
- for every $i, j \in \{1, 2, 3, 4\}$ we have $(w_i, w_j) \in R_{N_2}$ if and only if $(w_i, w_j) \in R_{N_3}$.

Lemma 7. *Let $N_2^x = (W_{N_2}, R_{N_2}, V_{N_2})$ and $N_3^x = (W_{N_2}, R_{N_2}, V_{N_2})$ be submodels of M_2^x and M_3^x respectively that are AAUL-equivalent, let $\varphi \in L_{AAUL}$ and suppose $x \in P \setminus \{p\}$ is a propositional variable that does not occur in φ. Then for every $i \in \{1, 2, 3, 4\}$, φ does not distinguish between N_2^x, w_i and N_3^x, w_i.*

Proof. Fix any $i \in \{1, 2, 3, 4\}$. We show by induction that φ does not distinguish between N_2^x, w_i and N_3^x, w_i. As base case, suppose that $\varphi \in P$. The models agree on all propositional variables other than x, so φ does not distinguish between them.

As induction hypothesis, assume that $\varphi \notin P$ and that the lemma holds for all subformulas of φ. We can assume without loss of generality that φ is of the form $\Box_a \varphi'$, $[U]\varphi'$ or $[\updownarrow]\varphi'$ for some $U \in L_{AAUL}$ and $\varphi' \in L_{AAUL}$.

- Suppose $\varphi = \Box_a \varphi'$. The models N_2^x and N_3^x are AAUL-equivalent so a world w_j is accessible from w_i in one of the models if and only if it is accessible in the other model. By the induction hypothesis φ' does not distinguish between N_2^x, w_j and N_3^x, w_j, so φ does not distinguish between N_2^x, w_i and N_3^x, w_i.

- Suppose $\varphi = [U]\varphi'$. We claim that $N_2^x * U$ and $N_3^x * U$ are AAUL-equivalent. In order for any arrow (w_j, w_k) to be retained in one of the models but not the other, there would have to be a clause $(\psi, a, \psi') \in U$ such that ψ distinguishes between N_2^x, w_j and N_3^x, w_j or ψ' distinguishes between N_2^x, w_k and N_3^x, w_k. This would contradict the induction hypothesis, so the models are AAUL-equivalent.

  It then follows from the induction hypothesis that φ' does not distinguish between $N_2^x * U, w_i$ and $N_3^x * U, w_i$, which implies that φ does not distinguish between N_2^x, w_i and N_3^x, w_i.

- Suppose $\varphi = [\updownarrow]\varphi'$. Every world in N_2^x can be uniquely identified by a combination of p and x, as can every world in N_3^x. This implies that for every $U \in L_{AUL}$ there is some $U' \in L_{AUL}$ such that $N_2^x * U$ and $N_3^x * U'$ are AAUL-equivalent, and that for every $U' \in L_{AUL}$ there is some $U \in L_{AUL}$ such that $N_2^x * U$ and $N_3^x * U'$ are AAUL-equivalent.

  By the induction hypotheses φ' cannot distinguish between $N_2^x * U, w_i$ and $N_3^x * U', w_i$ if $N_2^x * U$ and $N_3^x * U'$ are AAUL-equivalent. This means that $N_2^x, w_i \not\models [\updownarrow]\varphi'$ if and only if $N_3^x, w_i \not\models [\updownarrow]\varphi'$.

For each of the possible forms of φ we have shown that it does not distinguish between N_2^x, w_i and N_3^x, w_i. This completes the induction step and thereby the proof. $\square$

Using the three lemmas we can easily show that APAL and AAUL are incomparable in expressivity.

Theorem 1. $L_{APAL} \parallel L_{AAUL}$.

Proof. Recall that $\phi_1 = \Box_a p \wedge \Diamond_a \neg \Box_a p$. Suppose towards a contradiction that there is some $\psi \in L_{APAL}$ that is equivalent to $\langle \updownarrow \rangle \phi_1$. This ψ contains a finite number of propositional variables and P is infinite, so take an $x \in P \setminus \{p\}$ that does not occur in ψ.

By Lemma 5 we know that $\langle \updownarrow \rangle \phi_1$ distinguishes between M_1^x, w_1 and M_2^x, w_1. By Lemma 6 we know that ψ does not distinguish between M_1^x, w_1 and M_2^x, w_1. This contradicts the assumption that ψ is equivalent to $\langle \updownarrow \rangle \phi_1$. It follows that there is no $\psi \in L_{APAL}$ that is equivalent to $\langle \updownarrow \rangle \phi_1$ and therefore that $L_{AAUL} \not\preceq L_{APAL}$.

Now suppose towards a contradiction that there is some $\psi \in L_{AAUL}$ that is equivalent to $\langle ! \rangle \phi_1$. Again, take an $x \in P \setminus \{p\}$ that does not occur in ψ. By Lemma 5 we know that $\langle ! \rangle \phi_1$ distinguishes between M_2^x, w_1 and M_3^x, w_1 and by Lemma 7 we know that ψ does not. This contradicts our assumption, so there is no $\psi \in L_{AAUL}$ that is equivalent to $\langle ! \rangle \phi_1$. We therefore have $L_{APAL} \not\preceq L_{AAUL}$, which together with the previous conclusion $L_{AAUL} \not\preceq L_{APAL}$ shows that $L_{APAL} \parallel L_{AAUL}$. $\square$

Because $L_{EL} \subset L_{APAL}$ and $L_{EL} \subset L_{AAUL}$ we also get the following, rather unsurprising, corollary.

Corollary 1. *$L_{EL} \prec L_{APAL}$ and $L_{EL} \prec L_{AAUL}$.*

4.2. Expressivity on smaller classes of models

Above we chose a finite set A of agents and a countably infinite set P of propositional variables. Furthermore, we allowed all Kripke models. We consider these choices to be reasonable: if we model a real-life situation we expect

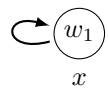


Figure 10: The one-world, one agent S5 model M_4^x.

the number of agents (represented by A) that are involved to be finite[13], whereas the number of potential facts (represented by P) might be infinite. Furthermore, the class of all Kripke models seems to be the "natural habitat" or arrow updates, since most of the smaller classes (such as S4, KD45 and S5) are not preserved under arrow updates.

Still, it is interesting to know the relative expressivity of APAL and AAUL if we use smaller A and P, or a smaller class of models. There are a lot of different combinations of A, P and a class of models. So we cannot feasibly give expressivity results for every one of them. Instead we only present a few salient results. Furthermore, for reasons of brevity, we only sketch most of the proofs.

First, let us consider the case where $A = \{a\}$ is a singleton, P is infinite and we use the class of S5 models.

Definition 12. Let $x \in P$. Consider the one-world one-agent model $M_4^x = (W_4, R_4, V_4^x)$ given by

- $W_4 = \{w_1\}$,
- $R_4(a) = W_4 \times W_4$,
- $V_4^x(x) = \{w_1\}$,
- $V_4^x(p) = \emptyset$ for $p \neq x$.

The model M_4^x is shown in Figure 10.

Definition 13. Let $x \in P$. Consider the two-world one-agent model $M_5^x = (W_5, R_5, V_5^x)$, given

[13]Note that the proofs given so far do not depend on A being finite. So we could safely allow an infinite set of agents.

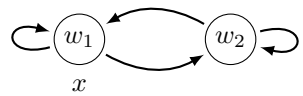


Figure 11: The two-world, one agent S5 model M_5^x.

- $W_5 = \{w_1, w_2\}$,
- $R_5(a) = W_5 \times W_5$,
- $V_5^x(x) = \{w_1\}$,
- $V_5^x(p) = \emptyset$ for $p \neq x$.

The model M_5^x is shown in Figure 11

Theorem 2. *If $A = \{a\}$ and P is countable infinite, then $L_{EL}(A, P) \prec_{S5} L_{AAUL}(A, P)$.*

Sketch of proof. No formula in the language of EL distinguishes M_4^x, w from M_5^x, w for all $x \in P$, since any epistemic formula only involves finitely many propositional variables.

If we execute the arrow update $(x, a, \neg x)$ in M_5^x, the result is a model where the only remaining arrow is from w_1 to w_2. A picture of the updated model is given in Figure 12. The AAUL formula $\langle\updownarrow\rangle\Diamond_a\Box_a\bot$ therefore holds in M_5^x, w_1. It clearly does not hold in M_4^x, w, so $\langle\updownarrow\rangle\Diamond_a\Box_a\bot$ can distinguish between M_4^x, w and M_5^x, w for all $x \in P$.

This implies that there is no single EL formula that is equivalent to the AAUL formula $\langle\updownarrow\rangle\Diamond_a\Box_a\bot$, so $L_{AAUL}(A, P) \not\preceq L_{EL}(A, P)$. We trivially have $L_{EL}(A, P) \preceq L_{AAUL}(A, P)$, so it follows that $L_{EL}(A, P) \prec_{S5} L_{AAUL}(A, P)$. $\Box$

Corollary 2. *If $A = \{a\}$ and P is countably infinite, then $L_{APAL}(A, P) \prec_{S5} L_{AAUL}(A, P)$.*

Proof. In [5, Proposition 3.12], it is proven that single-agent arbitrary announcement logic is equally expressive as epistemic logic over S5, irrespective of the size of P. Hence the corollary follows. $\Box$

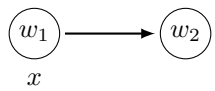


Figure 12: The updated model $M_5^x * \{(x, a, \neg x)\}$.

The difference in expressivity for one-agent S5 relies on the set of propositional variables being infinite. When this set is finite it is in fact the case that quantifying over arrow updates does not add any expressivity.

Theorem 3. *If $A = \{a\}$ and P is finite, then $L_{EL}(A, P) \equiv_{S5} L_{AAUL}(A, P)$.*

Proof. In single agent S5 with a finite number of atoms, we can use a finite set $\Phi \subset L_{EL}$ of characteristic formulas to identify each model up to bisimilarity. AAUL is invariant under bisimulation, so for each AAUL formula ψ and every $\chi \in \Phi$ we have either $\models_{S5} \chi \to \psi$ or $\models_{S5} \chi \to \neg\psi$. As a result, $\models_{S5} \psi \leftrightarrow \bigwedge_{\chi \in \Phi} \chi \to \delta_\chi$, where $\delta_\chi = \top$ if $\models_{S5} \chi \to \psi$ and $\delta_\chi = \bot$ if $\models_{S5} \chi \to \neg\psi$. $\square$

But now suppose that we have not one agent but two. We show that then arbitrary arrow updates add expressivity, even if P is a singleton. Consider the following set of models:

Definition 14. Let $m, n \in \mathbb{N} \setminus \{0\}$ be two positive integers. We now define the model $M_{mn} = (W_{mn}, R_{mn}, V_{mn})$ as follows:

- $W_{mn} = \{s_i \mid 0 \leq i \leq m\} \cup \{t_i \mid 0 \leq i \leq n\}$,
- $R_{mn}(a) = (\{(s_0, t_0)\} \cup \{(s_i, s_{i+1}) \mid i \text{ is odd}\} \cup \{(t_i, t_{i+1}) \mid i \text{ is odd}\})^*$,
- $R_{mn}(b) = (\{(s_i, s_{i+1}) \mid i \text{ is even}\} \cup \{(s_i, s_{i+1}) \mid i \text{ is even}\})^*$,
- $V_{mn}(p) = \{s_m, s_n\}$,

where * is a transitive reflexive closure operator. The model M_{mn} is shown in Figure 13.

Theorem 4. *If $A = \{a, b\}$ and $P = \{p\}$, then $L_{EL} \prec_{S5} L_{AAUL}$.*

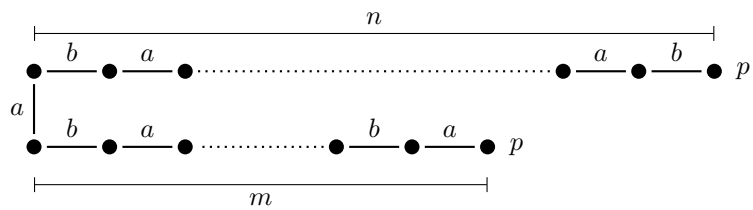


Figure 13: The model M_{mn}.

Sketch of proof. Let $\chi := \Diamond_a \Box_a \bot$, $\xi_0 := \neg p$ and $\xi_{k+1} = \Box_a \xi_k \wedge \Box_b \xi_k$ for every $k \in \mathbb{N} \setminus \{0\}$. So ξ_k means that there is no p world reachable in k or fewer steps.

Take any $m, n \in \mathbb{N} \setminus \{0\}$ such that $m \neq n$. If $m > n$, then $M_{mn}, s_0 \models \langle(\xi_n, a, \top)\rangle\chi$. Likewise, if $m < n$, then $M_{mn}, s_0 \models \langle(\neg\xi_m, a, \top)\rangle\chi$. So if $m \neq n$ we have $M_{mn}, s_0 \models \langle\updownarrow\rangle\chi$.

If $m = n$ on the other hand, then there is no way to distinguish between s_0 and t_0. This implies that $M_{mn}, s_0 \not\models \langle\updownarrow\rangle\chi$. So $\langle\updownarrow\rangle\chi$ distinguishes between the models with $m = n$ and the models with $m \neq n$.

Every $\psi \in L_{EL}$ is of finite modal depth d_ψ. If m and n are both greater than d_ψ, then ψ cannot distinguish between the cases $m = n$ and $m \neq n$. This implies that $L_{EL}(A, P) \not\preceq L_{AAUL}(A, P)$. $\Box$

APAL with two agents is more expressive than EL [5, Proposition 3.14], so Theorem 4 does not give us a counterpart to Corollary 2. We can, however, use a separate proof to show that, if $A = \{a, b\}$ and $P = \{p\}$, then $L_{AAUL}(A, P) \not\preceq_{S5} L_{APAL}(A, P)$.

Theorem 5. *If $A = \{a, b\}$ and $P = \{p\}$, then $L_{AAUL}(A, P) \not\preceq_{\mathcal{S}5} L_{APAL}(A, P)$.*

Sketch of proof. Recall that $\langle\updownarrow\rangle\chi \in L_{AAUL}$ distinguishes between the models M_{mn} with $m = n$ and those with $m \neq n$.

Now suppose there is a formula $\psi \in L_{APAL}$ that distinguishes those models where $m = n$ from those where $m \neq n$. This formula ψ is of depth d_ψ. Take m and n to be larger than d_ψ. It is clear that no epistemic formula will reach s_m or t_n, and after a public announcement that changes the model this remains the case, since public announcements preserve S5. Hence ψ will not be able

to distinguish the cases above d_ψ where $m = n$ from $m \neq n$. This implies that ψ is not equivalent to $\langle\updownarrow\rangle\chi$, since that formula does distinguish the models with $m = n$ from those with $m \neq n$. This is true for any $\psi \in L_{APAL}$, so $L_{AAUL}(A, P) \not\preceq_{S5} L_{APAL}(A, P)$. $\square$

The question whether $L_{APAL}(A, P) \prec_{S5} L_{APAL}(A, P)$ or $L_{APAL}(A, P) \parallel_{S5} L_{APAL}(A, P)$ if $A = \{a, b\}$ and $P = \{p\}$ remains open, although we suspect the latter to be the case.

4.3. Comparisons to other logics

In the preceding sections we compared L_{EL}, L_{APAL} and L_{AAUL} to each other. In this section we compare L_{AAUL} to three different logics.

From a technical perspective the comparisons made in this section are rather trivial. Our reason for presenting the results anyway is that they point to interesting differences between different types of quantification. For reasons of brevity we do not give full definitions of the logics considered here; instead we provide references to publications that do contain definitions.

First, let us compare L_{AAUL} to epistemic logic with common knowledge (ELC)[14]. In AAUL the operator $[\updownarrow]$ quantifies over an infinite number of updates. As a result, $[\updownarrow]\varphi$ could be seen as an infinite conjunction $\bigwedge_{U \in L_{AUL}} [U]\varphi$. In ELC the operator C_A likewise represents an infinite quantification; a formula $C_A\varphi$ can be seen as an infinite conjunction $\bigwedge_{i \in \mathbb{N}} E_A^i \varphi$ (where E_A is an "everybody knows" operator).

We can use the models M_{mn} to show that $L_{AAUL} \parallel_{S5} L_{ELC}$. As can be seen in Figure 13 at s_m and t_n either $\Box_a p$ is true or $\Box_b p$ is true. Using a common knowledge formula one can express what happens at the end, yet there is no formula in AAUL (or APAL for that matter) that is able to express this.

Theorem 6. $L_{AAUL} \parallel_{S5} L_{ELC}$.

[14] See for example [20].

Sketch of proof. Like L_{EL}, the language L_{ELC} cannot distinguish those pointed models M_{mn}, s_0 where $m = n$ from those where $m \neq n$.[15] This can be seen with model comparison games as they are for instance discussed in [16]. For any depth d, one can choose m and n sufficiently large, such that any C-move in such a game by spoiler can be matched by duplicator by choosing a world that is equivalent up to depth d. Hence ELC is not at least as expressive as AAUL.

Consider the formula $\hat{C}_{ab}\Box_a p$. This formula is true in all models M_{mn} where either m or n is odd and false in all models where both m and n are even. Yet, there is no formula in AAUL that distinguishes these. The quantifier in AAUL cannot distinguish between updates with formulas that depend on $\Box_a p$ being true at the final worlds or any other formula. $\Box$

Corollary 3. $L_{APAL} \parallel_{S5} L_{ELC}$.

Sketch of proof. The proof of Theorem 6 can be adapted to show that the APAL is not comparable to ELC by slightly adapting the valuation of p in the models M_{mn} to include s_0 and t_0. If $m \neq n$, there is some public announcement that detects the difference in length between the two sides. This announcement can then be used to remove one of s_1 and t_1 but not the other. If $m = n$, on the other hand, then s_0 and t_0 are bisimilar and so are s_1 and t_1. The formula $\langle ! \rangle(\Box_b p \wedge \Diamond_a \neg \Box_b p)$ therefore distinguishes the cases where $m = n$ from those where $m \neq n$. ELC still cannot distinguish between $m = n$ and $m \neq n$, so $L_{APAL} \not\preceq L_{ELC}$.

Like AAUL, APAL cannot distinguish the cases where m and n are even from those where they are not. ELC can do this, so $L_{ELC} \not\preceq L_{APAL}$, completing the proof. $\Box$

Let us consider two more logics, *refinement modal mogic* (RML) [12] and *arbitrary action model logic* (AAML) [19].

[15] L_{ELC} can distinguish the pointed models M_{mn}, s_0 where $m = n \bmod 2$ from those where $m \neq n \bmod 2$, but that is not sufficient to distinguish $m = n$ from $m \neq n$.

Corollary 4. $L_{RML} \prec_{\mathcal{S}5} L_{AAUL}$.

Proof. RML is equally expressive as EL (see [12, Proposition 36]). It therefore follows from $L_{EL} \prec_{S5} L_{AAUL}$ that $L_{RML} \prec_{\mathcal{S}5} L_{AAUL}$. □

Corollary 5. $L_{AAML} \prec_{\mathcal{S}5} L_{AAUL}$.

Proof. [19, Corollary IV.5] shows that AAML is equally expressive as EL. It follows that $L_{AAML} \prec_{\mathcal{S}5} L_{AAUL}$. □

An overview of the expressivity results discussed so far can be seen in Figure 14. The interesting thing about these different expressivity results is that they show that changing the scope of quantification can have wildly different effects. Let X and Y be two different operators that quantify over some sets S_X and S_Y. Then, in general, we would expect that logics using X to be incomparable to logics using Y (unless $S_X = S_Y$).

After all, if $S_X \not\subseteq S_Y$ and $S_Y \not\subseteq S_X$ then X and Y seem unrelated so we should expect logics using them to be incomparable. But if $S_X \subset S_Y$ then, by the reasoning presented in Section 4.1 we expect there to be worlds that can be distinguished by $X\varphi$ but not $Y\varphi$ as well as worlds that can be distinguished by $Y\varphi$ but not $X\varphi$.

Some of the logics studied and mentioned in this paper follow this expected pattern. The logics ELC, APAL and AAUL are indeed incomparable in expressivity. But, somewhat surprisingly, RML and AAML are equally expressive as EL and therefore less expressive than ELC, APAL and AAUL.

It therefore seems an interesting question for further research to ask why going from APAL to AAUL is so different to going from AAUL to AAML. Additionally, we could wonder whether there is any interesting set S larger than the set of arrow updates but smaller than the set of action models, with the property that quantification over S adds expressivity to EL.

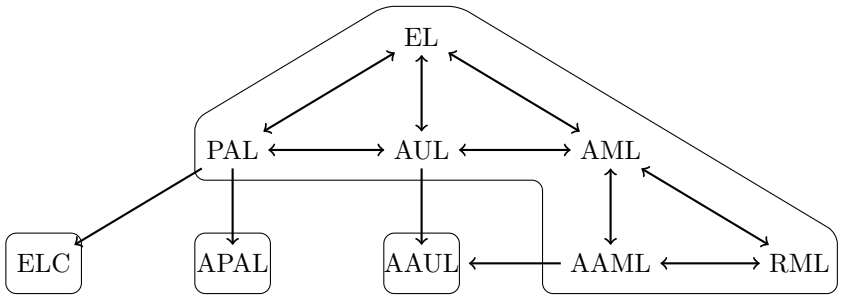


Figure 14: A comparison of the expressivity of several logics, when considered over the class of all Kripke models. An arrow from one logic to another means that the second logic is at least as expressive as the first. The "at least as expressive as" relation is transitive, so for reasons of clarity we omit some arrows that follow from transitivity. Borders around sets of logics indicate equivalence classes of equi-expressive logics.

5. Model Checking for AAUL

Here we show that the model checking problem for AAUL is PSPACE-complete. This is as expected, considering that the model checking problems for APAL and Group Announcement Logic are known to be PSPACE-complete as well [2]. Indeed, the proofs presented in this section are inspired by the ones given in [2]. Recall that, as shown in [25], the model checking for AUL is in PTIME. So the $[\updownarrow]$ operator significantly increases the complexity of model checking.

In order to show that the model checking problem for AAUL is PSPACE-complete we have to show that it is PSPACE-hard and that it is in PSPACE. We start by proving that it is PSPACE-hard.

5.1. Model Checking for AAUL is PSPACE hard

We show that the QBF-SAT problem can be reduced to the model checking problem for AAUL. Since QBF-SAT is known to be PSPACE-complete, this shows that AAUL model checking is PSPACE-hard.

Let us start by very briefly describing the QBF-SAT problem. Let $n \in \mathbb{N}$ be given, as well as $Q_1, \cdots, Q_n \in \{\forall, \exists\}$ and let $\Phi = \Phi(p_1, \cdots, p_n)$ be a Boolean formula containing n propositional variables. The QBF-SAT problem

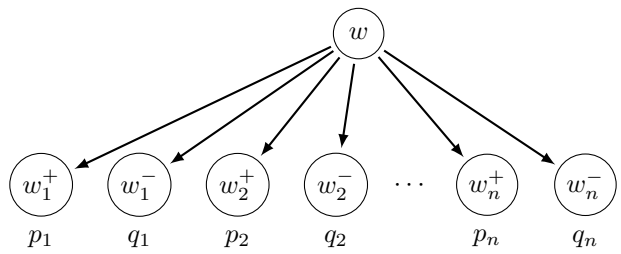


Figure 15: The model M_{Ha}^n used to show PSPACE-hardness.

for $n, Q_1, \cdots, Q_n$ and Φ is to determine whether $Q_1 p_1 \cdots Q_n p_n : \Phi(p_1, \cdots, p_n)$ is true.[16] A simple instance of the problem would be $\forall p_1 \exists p_2 : (p_2 \to p_1)$. The propositional variables are considered as Boolean variables here, so this instance could also be denoted $\forall p_1 \in \{\top, \bot\} \exists p_2 \in \{\top, \bot\} : (p_2 \to p_1)$, which happens to be true.

In order to reduce QBF-SAT to the model checking of AAUL, we need to create a corresponding model and an AAUL formula for each instance of QBF-SAT, with both this model and the formula having size polynomial in that of the instance. We start by constructing the model. The model, M_{Ha}^n, depends only on n, and is shown in Figure 15. The idea is that we will force a number of arbitrary arrow updates to choose between a world w_i^+ (corresponding to the choice $p_i = \top$) and a world w_i^- (corresponding to the choice $p_i = \bot$).

We will now define the formula corresponding to a QBF-SAT instance. First, in order to force the arbitrary arrow updates to choose one of the worlds, let us define a number of useful subformulas. For $1 \leq m \leq n$ let

$$\gamma_m := \bigwedge_{1 \leq i \leq m} ((\Diamond p_i \vee \Diamond q_i) \wedge \neg(\Diamond p_i \wedge \Diamond q_i)) \wedge \bigwedge_{m < j \leq n} (\Diamond p_j \wedge \Diamond q_j).$$

In other words, γ_m holds if and only if for every $1 \leq i \leq m$ the arrow to exactly one of w_i^+ and w_i^- is eliminated, while for every $m < j \leq n$ both w_j^+ and w_j^- are still reachable. This means that the values of all p_i with $i \leq m$ have

[16] We could equivalently ask whether $Q_1 p_1 \cdots Q_n p_n \Phi(p_1, \cdots, p_n)$ is satisfiable, the formula has no free variables so truth and satisfiability coincide.

been chosen, but the values of all p_j with $j > m$ have not. Now, consider the following, recursively defined, formula.

$$\begin{aligned}
\Psi_{n+1} &:= \Phi(\Diamond p_1, \cdots, \Diamond p_n) \\
\Psi_m &:= \begin{cases} [\updownarrow](\gamma_m \to \Psi_{m+1}) & \text{if } Q_m = \forall \\ \langle\updownarrow\rangle(\gamma_m \wedge \Psi_{m+1}) & \text{if } Q_m = \exists \end{cases} \\
&\quad \text{for } 1 \leq m \leq n \\
\Psi &:= \Psi_1
\end{aligned}$$

For our example formula $\forall p_1 \exists p_2 : (p_2 \to p_1)$ we obtain, writing $\Diamond p_i \leftrightarrow \neg\Diamond q_i$ for $(\Diamond p_i \vee \Diamond q_i) \wedge \neg(\Diamond p_i \wedge \Diamond q_i)$ the following formula Ψ:

$$\begin{aligned}
[\updownarrow] \quad & (((\Diamond p_1 \leftrightarrow \neg\Diamond q_1) \wedge (\Diamond p_2 \wedge \Diamond q_2)) \to \\
& \langle\updownarrow\rangle((\Diamond p_1 \leftrightarrow \neg\Diamond q_1) \wedge (\Diamond p_2 \leftrightarrow \neg\Diamond q_2) \wedge (\Diamond p_2 \to \Diamond p_1)))
\end{aligned}$$

We leave it to the reader to verify that Ψ holds in M^2_{Ha}, w.

Lemma 8. *$M^n_{Ha}, w \models \Psi$ iff $Q_1 p_1 \cdots Q_n p_n : \Phi(p_1, \cdots, p_n)$ is true.*

Proof. Recall that γ_m holds at w in any submodel of M^n_{Ha} if and only if for all $1 \leq i \leq m$ exactly one of w_i^+ and w_i^- remains reachable, and for every $m < i \leq n$ both w_i^+ and w_i^- remain reachable. This means we can interpret any submodel satisfying γ_m as a choice for the values of $p_1, \cdots, p_m$, where p_i takes value $\top$ if and only if the arrow to w_i^+ is retained (so if and only if $\Diamond p_i$ holds in w). The arbitrary updates $[\updownarrow]$ and $\langle\updownarrow\rangle$ can therefore be seen as universal and existential quantifiers for the choice of p_i. Finally, Ψ_{n+1} checks whether $Q(p_1, \cdots, p_n)$ holds for the chosen values of the p_i. □

Corollary 6. *The model checking problem for AAUL is PSPACE-hard.*

5.2. Model Checking for AAUL is in PSPACE

Left to show is that the model checking problem for AAUL is in PSPACE. We do this by presenting an algorithm MCheck(M, w, φ) that returns *true* if and only if $M, w \models \varphi$ and *false* if and only if $M, w \not\models \varphi$. MCheck(M, w, φ) works by

a case distinction on the main connective of φ. Most cases are exactly as one would expect; MCheck$(M, w, \varphi_1 \vee \varphi_2)$ returns *true* if either MCheck(M, w, φ_1) or MCheck(M, w, φ_1) does, and so on. For reasons of brevity we only consider the interesting cases, and omit the trivial ones. Let $M = (W, R, V)$ and $w \in W$.

The two cases that we consider in detail are $\varphi = [U]\chi$ and $\varphi = [\updownarrow]\chi$. We start with the case $\varphi = [U]\chi$, where $U = (\varphi_1, a_1, \psi_1), \cdots, (\varphi_n, a_n, \psi_n)$. In order to solve MCheck(M, w, φ) we simply have to call MCheck$(M * U, w, \chi)$; the difficult part is to compute $M * U$. We can do so as follows.

1. for every $w' \in W$ and $1 \leq i \leq n$, label w' with φ_i iff MCheck(M, w', φ_i) returns *true*, and with ψ_i iff MCheck(M, w', ψ_i) does.
2. for every $a \in A$, $(w_1, w_2) \in R(a)$ and $(\varphi_i, a, \psi_i) \in U$, if w_1 is labeled φ_i and w_2 is labeled ψ_i, then label (w_1, w_2) with "keep."
3. for every $a \in A$ and $(w_1, w_2) \in R(a)$, if (w_1, w_2) is not labeled "keep," then remove it.

The other non-trivial case is $\varphi = [\updownarrow]\chi$. We would like to solve this using "brute force," so for every U we would like to check whether $[U]\chi$ holds. Unfortunately, there are infinitely many different arrow updates, so we cannot check them all. But while there are infinitely many different updates U, there are only finitely many different updated models $M * U$. So instead of running MCheck$(M, w, [U]\chi)$ for every U, we run MCheck$(M * U, w, \chi)$ for all different $M * U$.

It is not the case that every submodel of M is of the form $M * U$ for some U, so we need to determine which submodels can be represented as $M * U$. In order to do this, we use the definitions and results about bisimilarity that were introduced in Section 3. In particular, we use the bisimulation contraction M_{BC} of M (see Definition 5).

For any state w, the pointed model M, w is bisimilar to $M_{BC}, [w]$ and AAUL is bisimulation invariant, as discussed in Section 3, so we are free to work with M_{BC} instead of M. In M_{BC} no two different worlds are bisimilar, which, because we are working with finite models, means that every two sets of worlds are

distinguishable. As such, every submodel (W_{BC}, R', V_{BC}) of M_{BC} is of the form $M_{BC} * U$ for some U. The $[\updownarrow]\chi$ case in $\mathrm{MCheck}(M, w, \varphi)$ is therefore as follows.

1. compute $M_{BC} = (W_{BC}, R_{BC}, V_{BC})$.
2. for every submodel $M' = (W_{BC}, R', V_{BC})$, if $\mathrm{MCheck}(M', [w], \chi)$ returns *false*, then return *false*
3. return *true*

We consider the correctness of $\mathrm{MCheck}(M, w, \varphi)$ to be immediately clear, but it remains to show that it requires at most polynomial space. So let us do some complexity analysis. In the $[U]\chi$ case we need to run $\mathrm{MCheck}(M, w', \varphi_i)$ and $\mathrm{MCheck}(M, w', \psi_i)$ for all $w' \in W$, but we can do those one at a time, so we need to keep only one in memory. We do need to keep $O(|U| \cdot |M|) \leq O(|\varphi| \cdot |M|)$ different labels in memory, as well as the submodel $M * U$ which is of size at most $O(|M|)$. The total space requirement for $\mathrm{MCheck}(M, w, [U]\chi)$ is therefore $O(|\varphi| \cdot |M|)$ plus the maximum space requirement of $\mathrm{MCheck}(M * U, w, \chi)$, $\mathrm{MCheck}(M, w', \varphi_i)$ and $\mathrm{MCheck}(M, w', \psi_i)$ for every i, w'.

In the $[\updownarrow]\chi$ case we first need to compute M_{BC}. This can be done in polynomial time and $O(|M|)$ space, by using the Paige-Tarjan algorithm [27]. If we use depth-first search we need to store only two additional models at a time, namely M_{BC} and M'. Both are of size at most $O(|M|)$. Finally, we need the space required to run $\mathrm{MCheck}(M', [w], \chi)$.

So the $[U]\chi$ and $[\updownarrow]\chi$ cases take at most $O(|\varphi| \cdot |M|)$ space, plus whatever is required to do the model checking for their subformulas. All other cases take less space. This means that every connective adds at most $O(|\varphi| \cdot |M|)$ to the space requirement. There are at most $O(|\varphi|)$ connectives in φ, so $\mathrm{MCheck}(M, w, \varphi)$ requires at most $O(|\varphi|^2 \cdot |M|)$ space. This means we have the following lemma.

Lemma 9. *Model checking for AAUL is in PSPACE.*

Corollary 7. *Model checking for AAUL is PSPACE-complete.*

6. Proof system

In this section we introduce an infinitary proof system for AAUL. This system is a conservative extension of the proof system for AUL given in [22]. The axiom and rule for the arbitrary arrow update are very similar to the axiom and rule for APAL given in [6]. The completeness proof we give for it, is very closely related to the proof system for APAL given in [6]. Before introducing the proof system, we need an auxiliary definition.

Definition 15. Let $x \notin P$ be a new atom. The set NF of formulas that are in *necessity form* is generated by the following normal form:

$$\xi(x) ::= x \mid \varphi \to \xi(x) \mid \Box_a \xi(x) \mid [U]\xi(x)$$

where $\varphi \in L_{AAUL}$ and $U \in L_{AAUL}$. Given a formula ψ and a formula $\xi(x) \in NF$ in necessity form, we use $\xi(\psi)$ to denote the result of replacing the unique occurrence of x in $\xi(x)$ by ψ.

Lemma 10. *If $\varphi = \xi([\updownarrow]\psi)$ for some $\xi(x) \in NF$ then this representation of φ is unique, i.e. if $\varphi = \xi'([\updownarrow]\psi')$ then $\xi'(x) = \xi(x)$ and $\psi' = \psi$.*

Proof. For given $\chi, U \in L_{AAUL}$, we can consider $\chi \to$, $\Box_a$ and $[U]$ to be unary operators. The symbol x can only occur inside the scope of such unary operators. These three operators do not include $[\updownarrow]$, so the $[\updownarrow]$ operators in $\xi([\updownarrow]\psi)$ and $\xi'([\updownarrow]\psi')$ must both be the outermost $[\updownarrow]$ operator. $\Box$

Now, we can consider the proof system $\mathcal{L}_{AAUL}$.

Definition 16. The proof system $\mathcal{L}_{AAUL}$ is given by the following eight axiom schemata and four rules.

(A1) All instances of propositional tautologies

(A2) $\Box_a(\varphi \to \psi) \to (\Box_a\varphi \to \Box_a\psi)$

(A3) $[U]p \leftrightarrow p$

(A4) $[U]\neg\varphi \leftrightarrow \neg[U]\varphi$

(A5) $[U](\varphi \wedge \psi) \leftrightarrow ([U]\varphi \wedge [U]\psi)$

(A6) $[U]\Box_a\varphi \leftrightarrow \bigwedge_{(\psi,a,\chi)\in U}(\psi \to \Box_a(\chi \to [U]\varphi))$

(A7) $[U][U']\varphi \leftrightarrow [U \times U']\varphi$ where $U \times U' =$
$\{(\psi_1 \wedge [U]\psi_2, a, \chi_1 \wedge [U]\chi_2) \mid (\psi_1, a, \chi_1) \in U, (\psi_2, a, \chi_2) \in U'\}$

(A8) $[\updownarrow]\varphi \to [U_0]\varphi$ where $U_0 \in L_{AUL}$

(R1) From $\varphi \to \psi$ and φ, infer ψ.

(R2) From φ, infer $\Box_a\varphi$.

(R3) From φ, infer $[U]\varphi$.

(R4) From $\{\xi([U_0]\varphi) \mid U_0 \in L_{AUL}\}$, infer $\xi([\updownarrow]\varphi)$, where $\xi(x) \in NF$.

A formula φ can be *derived* in $\mathcal{L}_{AAUL}$ if it is a member of the smallest set of formulas that contains all instances of (A1) – (A8) and that is closed under (R1) – (R4). If φ can be derived in $\mathcal{L}_{AAUL}$ we call φ a *theorem* (of $\mathcal{L}_{AAUL}$) and write $\vdash \varphi$.

Axioms (A1), (A2) and rules (R1) and (R2) together are the basic multi-agent modal system K. Axioms (A3), ..., (A7) are all so-called reduction axioms for the arrow update, i.e. going from left to right the number of arrow updates reduces ((A3) and (A7)) or the complexity of the formulas to which the arrow update is applied reduces ((A4), (A5) and (A6)). This means one can effectively translate any formula without arbitrary arrow updates to a provably equivalent formula of multi-agent modal logic, as was shown in [22]. Axioms (A8) and rule (R4) deal with arbitrary arrow updates, and given their presence in the proof system, one also needs (R3) for completeness. Note that while a rule "From φ, infer $[\updownarrow]\varphi$" is not included in the proof system, it is derivable. After all, if $\vdash \varphi$ then (R3) allows us to derive $[U]\varphi$ for all $U \in L_{AUL}$. Since $x \in NF$, this allows us to derive $[\updownarrow]\varphi$ by (R4). Before proving the soundness of $\mathcal{L}_{AAUL}$, let us consider one lemma.

Lemma 11. *Rule (R4) is truth preserving. That is, if $\xi(x) \in NF$, $\varphi \in L_{AAUL}$ and M, w are such that $M, w \models \xi([U_0]\varphi)$ for all $U_0 \in L_{AUL}$, then we have $M, w \models \xi([\updownarrow]\varphi)$.*

Proof. By induction on the construction of $\xi(x)$. As base case, suppose $\xi(x) = x$. If $M, w \models [U_0]\varphi$ for all $U_0 \in L_{AUL}$ it follows immediately from the semantics of AAUL that $M, w \models [\updownarrow]\varphi$. So (R4) is sound for $\xi(x) = x$.

Suppose then as induction hypothesis that $\xi(x) \neq x$ and that the lemma holds for every $\xi'(x)$ that precedes $\xi(x)$ in the recursive definition of NF. There are three possibilities for the form of $\xi(x)$.

The first possibility is that $\xi(x) = \psi \to \xi'(x)$. Fix any M, w and φ, and suppose that that $M, w \models \psi \to \xi'([U_0]\varphi)$ for all $U_0 \in L_{AUL}$. If $M, w \not\models \psi$ then, trivially, $M, w \models \psi \to \xi'([\updownarrow]\varphi)$. If, on the other hand, $M, w \models \psi$ then $M, w \models \xi'([U_0]\varphi)$ for all $U_0 \in L_{AUL}$ and therefore, by the induction hypothesis, $M, w \models \xi'([\updownarrow]\varphi)$. This implies that $M, w \models \psi \to \xi'([\updownarrow]\varphi)$. In either case, from $M, w \models \xi([U_0]\varphi)$ for all $U_0 \in L_{AUL}$ it follows that $M, w \models \xi([\updownarrow]\varphi)$.

The second possibility is that $\xi(x) = \Box_a \xi'(x)$. Fix any M, w and φ, and suppose that $M, w \models \Box_a \xi'([U_0]\varphi)$ for all $U_0 \in L_{AUL}$. Let w' be any world that is a-accessible from w in M. We have $M, w' \models \xi'([U_0]\varphi)$ for all $U_0 \in L_{AUL}$ and therefore, by the induction hypothesis, $M, w' \models \xi'([\updownarrow]\varphi)$. This holds for every a-successor of w, so $M, w \models \Box_a \xi'([\updownarrow]\varphi)$.

The third and final possibility is that $\xi(x) = [U]\xi'(x)$. Again, fix any M, w and φ, and suppose that $M, w \models [U]\xi'([U_0]\varphi)$ for all $U_0 \in L_{AUL}$. Then $(M * U), w \models \xi'([U_0]\varphi)$ for all $U_0 \in L_{AUL}$ and therefore, by the induction hypothesis, $(M * U), w \models \xi'([\updownarrow]\varphi)$. This, in turn, implies that $M, w \models [U]\xi'([\updownarrow]\varphi)$.

We have treated all possible forms of $\xi(x)$. This completes the induction step and thereby the proof. $\Box$

Theorem 7 (Soundness of $\mathcal{L}_{AAUL}$)**.** *Let $\varphi \in L_{AAUL}$. If $\vdash \varphi$, then $\models \varphi$.*

Proof. The soundness of the axioms (A1) – (A5) and (A8) follows immediately from the semantics of AAUL, as does the soundness of the rules (R1) – (R3).

The soundness of the non-straightforward axioms (A6) and (A7) follows from Lemma 1. The soundness of (R4) follows from Lemma 11. $\square$

6.1. Completeness of $\mathcal{L}_{AAUL}$: preliminaries

The completeness of $\vdash$ is, unfortunately but not unusually, harder to prove than the soundness. Before getting to the main proof we will need a number of definitions and lemmas. Firstly, we need definitions for the depth and size of formulas.

Definition 17. Let $\varphi \in L_{AAUL}$. The $[\updownarrow]$-depth $d(\varphi)$ of φ is given inductively by

$$\begin{array}{lcl}
d(p) & = & 0 \\
d(\neg\varphi) & = & d(\varphi) \\
d(\varphi_1 \wedge \varphi_2) & = & \max(d(\varphi_1), d(\varphi_2)) \\
d(\Box_a\varphi) & = & d(\varphi) \\
d(U) & = & \max\{d(\varphi_1), d(\varphi_2) \mid (\varphi_1, a, \varphi_2) \in U\} \\
d([U]\varphi) & = & \max(d(U), d(\varphi)) \\
d([\updownarrow]\varphi) & = & d(\varphi) + 1
\end{array}$$

The size $s(\varphi)$ of φ is a more complicated measure, given inductively by

$$\begin{array}{lcl}
s(p) & = & 1 \\
s(\neg\varphi) & = & s(\varphi) + 1 \\
s(\varphi_1 \wedge \varphi_2) & = & s(\varphi_1) + s(\varphi_2) + 1 \\
s(\Box_a\varphi) & = & s(\varphi) + 1 \\
s([\updownarrow]\varphi) & = & s(\varphi) + 1 \\
s([U]\varphi) & = & s(U)^{s(\varphi)} \\
s(U) & = & (|U| + 2)(9 + 2 \cdot \mathrm{smax}(U)) \\
 & & \text{where } |U| \text{ is the number of clauses in } U \text{ and} \\
 & & \mathrm{smax}(U) = \max\{s(\psi), s(\chi) \mid (\psi, a, \chi) \in U\}
\end{array}$$

We write $\varphi_1 <^s_d \varphi_2$ if either $d(\varphi_1) < d(\varphi_2)$ or $d(\varphi_1) = d(\varphi_2)$ and $s(\varphi_1) < s(\varphi_2)$.

The measure $d(\varphi)$ is simply the nesting depth of $[\updownarrow]$ in φ. The measure $s(\varphi)$ does not have such a simple description, it is designed purely to provide us with

a well-ordering we can do induction on. Specifically, we need it for the following lemma.

Lemma 12. *Let $\varphi, \psi, U, U' \in L_{AAUL}$. Then*

$$\begin{array}{lll} p & <^s_d & [U]p, \\ \neg[U]\varphi & <^s_d & [U]\neg\varphi, \\ [U]\varphi \wedge [U]\psi & <^s_d & [U](\varphi \wedge \psi), \\ \bigwedge_{(\psi,a,\chi)\in U}(\psi \to \Box_a(\chi \to [U]\varphi)) & <^s_d & [U]\Box_a\varphi \text{ and} \\ [U \times U']\varphi & <^s_d & [U][U']\varphi. \end{array}$$

Proof. In all five cases, the formulas on the left and right side of the inequality have the same $[\updownarrow]$-depth. It therefore suffices to show a difference in size. The first three cases are relatively easy to prove.

Every U contains at least one clause (see Definition 1), so $|U| \geq 1$ and $\mathrm{smax}(U) \geq 1$. We therefore have

$$s(U) = (|U| + 2)(9 + 2 \cdot \mathrm{smac}(U)) \geq 3 \cdot 11 = 33.$$

It follows that

$$s(p) = 1 < 33 \leq s(U)^1 = s([U]p),$$

and

$$s(\neg[U]\varphi) = s([U]\varphi) + 1 < 33 \cdot s([U]\varphi) \leq s([U]\neg\varphi).$$

Furthermore, for every $x_1 \geq 2$ and $x_2, x_3 \geq 1$, we have $x_1^{x_2} + x_1^{x_3} + 1 < x_1^{x_2} \cdot x_1^{x_3} \cdot x_1 = x_1^{x_2+x_3+1}$. This implies that

$$s([U]\varphi \wedge [U]\psi) = s(U)^{s(\varphi)} + s(U)^{s(\psi)} + 1 < s(U)^{s(\varphi)+s(\psi)+1} = s([U](\varphi \wedge \psi)).$$

Proving the inequality for the last two cases is simple but requires a lot of bookkeeping. Recall that $\psi_1 \to \psi_2$ is an abbreviation for $\neg(\psi_1 \wedge \neg\psi_2)$, so $s(\psi \to \Box_a(\chi \to [U]\varphi)) = s(\psi) + s(\chi) + 3 + 3 + 1 + s([U]\varphi)$. Furthermore, $\bigwedge$ represents a number of conjunction symbols equal to its number of conjuncts

minus one. We therefore have

$$s\left(\bigwedge_{(\psi,a,\chi)\in U}(\psi \to \Box_a(\chi \to [U]\varphi))\right) = \sum_{(\psi,a,\chi)\in U} s(\psi) + s(\chi) + 7 + s([U]\varphi) + 1$$
$$\leq |U|(8 + 2 \cdot \mathrm{smax}(U) + s([U]\varphi)).$$

Furthermore, for every $x_1, x_2 \geq 2$ we have $x_1 + x_2 \leq x_1 \cdot x_2$. So $2 \cdot \mathrm{smax}(U) + s([U]\varphi) \leq 2s([U]\varphi) \cdot \mathrm{smax}(U)$. Additionally, $8 < 9s([U]\varphi)$. As a result,

$$\begin{aligned}
|U|(8 + 2 \cdot \mathrm{smax}(U) + s([U]\varphi)) &< |U|(9s([U]\varphi) + 2s([U]\varphi) \cdot \mathrm{smax}(U)) \\
&= |U|(9 + 2 \cdot \mathrm{smax}(U))s([U]\varphi) \\
&< (|U| + 2)(9 + 2 \cdot \mathrm{smax}(U))s([U]\varphi) \\
&= s(U)s([U]\varphi) = s(U)s(U)^{s(\varphi)} \\
&= s(U)^{s(\varphi)+1} = s(U)^{s(\Box_a\varphi)} \\
&= s([U]\Box_a\varphi).
\end{aligned}$$

Left to show is that $[U \times U']\varphi <^s_d [U][U']\varphi$. Recall that $U \times U'$ is an abbreviation for

$$\{(\psi_1 \wedge [U]\psi_2, a, \chi_1 \wedge [U]\chi_2) \mid (\psi_1, a, \chi_1) \in U, (\psi_2, a, \chi_2) \in U'\}.$$

This gives us $|U \times U'| = |U| \cdot |U'|$. Furthermore, since $s(\psi_1 \wedge [U]\psi_2) = s(\psi_1) + s(U)^{s(\psi_2)} + 1$, we also have $\mathrm{smax}(U \times U') \leq \mathrm{smax}(U) + s(U)^{\mathrm{smax}(U')} + 1$. We now want to compare $s(U \times U')$ to $s(U)^{s(U')}$. On the one hand,

$$\begin{aligned}
s(U \times U') &= (|U \times U'| + 2)(9 + 2 \cdot \mathrm{smax}(U \times U')) \\
&\leq (|U| \cdot |U'| + 2)(9 + 2(\mathrm{smax}(U) + s(U)^{\mathrm{smax}(U')} + 1)).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
s(U)^{s(U')} &= s(U)^{(|U'|+2)(9+2\cdot\mathrm{smax}(U'))} \\
&\quad \text{(By the definition of } s(U')) \\
&\geq s(U)^{(|U'|+2)+(9+2\cdot\mathrm{smax}(U'))} \\
&\quad \text{(Because } x_1 \cdot x_2 \geq x_1 + x_2 \text{ for } x_1, x_2 \geq 2)
\end{aligned}$$

$$
\begin{aligned}
&= s(U)^{|U'|+2} s(U)^{9+2\cdot\mathrm{smax}(U')} \\
&= ((|U|+2)(9+2\cdot\mathrm{smax}(U)))^{|U'|+2} s(U)^{9+2\cdot\mathrm{smax}(U')} \\
&\quad \text{(By the definition of } s(U)) \\
&> (|U|+2)^{|U'|+2} s(U)^9 s(U)^{2\cdot\mathrm{smax}(U')} \\
&= (|U|+2)^{|U'|+2}((|U|+2)(9+2\cdot\mathrm{smax}(U)))^9 s(U)^{2\cdot\mathrm{smax}(U')} \\
&\quad \text{(By the definition of } s(U)) \\
&> (|U|+2)^{|U'|+2}(9+2\cdot\mathrm{smax}(U))^9 s(U)^{2\cdot\mathrm{smax}(U')} \\
&\geq (|U|+2)(|U'|+2)(9+2\mathrm{smax}(U))^9 s(U)^{2\cdot\mathrm{smax}(U')} \\
&\quad \text{(Because } x_1^{x_2} \geq x_1 \cdot x_2 \text{ for } x_1, x_2 \geq 2) \\
&> (|U|\cdot|U'|+2)(9+2\mathrm{smax}(U))^9 s(U)^{2\cdot\mathrm{smax}(U')} \\
&> (|U|\cdot|U'|+2)(9+2(\mathrm{smax}(U)+1+1)) s(U)^{\mathrm{smax}(U')} \\
&> (|U|\cdot|U'|+2)(9+2(\mathrm{smax}(U)+s(U)^{\mathrm{smax}(U')}+1)),
\end{aligned}
$$

Putting these inequalities together, we get $s(U \times U')^{s(\varphi)} < \left(s(U)^{s(U')}\right)^{s(\varphi)}$, and therefore

$$
s([U \times U']\varphi) = s(U \times U')^{s(\varphi)} < \left(s(U)^{s(U')}\right)^{s(\varphi)} \leq s(U)^{\left(s(U')^{s(\varphi)}\right)} = s([U][U']\varphi).
$$

□

The relevance of Lemma 12 is that, for each of (A3) – (A7), the formula on the right side of the equivalence is smaller than the formula on the left side.

6.2. Completeness of $\mathcal{L}_{AAUL}$: Lindenbaum lemma

We will now define theories, and work towards a Lindenbaum lemma, which states that every theory can be extended to a maximal consistent theory.

Definition 18. A set $\Phi \subseteq L_{AAUL}$ of formulas is a *theory* if it contains all theorems and is closed under rules (R1) and (R4).

We do not require Φ to be closed under (R2) and (R3) because, unlike (R1) and (R4), these rules preserve only validity, not truth. For example, $M, w \models \varphi$

does not guarantee $M, w \models [U]\varphi$, but $M, w \models \varphi \to \psi$ and $M, w \models \varphi$ do guarantee that $M, w \models \psi$.

Definition 19. A theory Φ is *consistent* if there is a $\varphi \in L_{AAUL}$ such that $\varphi \notin \Phi$. A theory Φ is *maximal* if for every formula $\varphi \in L_{AAUL}$ either $\varphi \in \Phi$ or $\neg\varphi \in \Phi$.

Lemma 13. *Fix any $\psi \in \Phi$. The following are equivalent:*

1. *Φ is inconsistent,*
2. *there is a φ such that $\varphi \in \Phi$ and $\neg\varphi \in \Phi$,*
3. *$\psi \wedge \neg\psi \in \Phi$.*

The proof is trivial and left to the reader. We also need some more notation to define sets of formulas.

Definition 20. Let Φ be a theory and $\varphi, U \in L_{AAUL}$. Then

$$\begin{aligned} \Phi + \varphi &:= \{\psi \mid \varphi \to \psi \in \Phi\} \\ \Box_a \Phi &:= \{\psi \mid \Box_a \psi \in \Phi\} \\ [U]\Phi &:= \{\psi \mid [U]\psi \in \Phi\} \end{aligned}$$

First, let us show that $\Phi + \varphi$ is an appropriate notation for $\{\psi \mid \varphi \to \psi \in \Phi\}$.

Lemma 14. *If Φ is a theory, then $\varphi \in \Phi + \varphi$ and $\Phi \subseteq \Phi + \varphi$.*

Proof. Firstly, we have $\vdash \varphi \to \varphi$ and therefore $\varphi \to \varphi \in \Phi$. This implies that $\varphi \in \Phi + \varphi$. Now, note that $\vdash \psi \to (\varphi \to \psi)$ and therefore $\psi \to (\varphi \to \psi) \in \Phi$. Since Φ is closed under (R1), this implies that if $\psi \in \Phi$, then $\varphi \to \psi \in \Phi$. We therefore have $\psi \in \Phi + \varphi$ for all $\psi \in \Phi$. $\Box$

Next, we need two relatively simple lemmas about theories.

Lemma 15. *If Φ is a theory, then so are $\Phi + \varphi$, $K_a\Phi$ and $[U]\Phi$.*

Proof. If $\vdash \psi$ then also $\vdash \varphi \to \psi$, $\vdash \Box_a\psi$ and $\vdash [U]\psi$. The set of theorems is therefore a subset of $\Phi + \varphi$, $\Box_a\Phi$ and $[U]\Phi$. It remains to be shown that the three sets are closed under (R1) and (R4).

Suppose $\psi \to \chi \in \Phi + \varphi$ and $\psi \in \Phi + \varphi$. Then, by definition, $\varphi \to (\psi \to \chi) \in \Phi$ and $\varphi \to \psi \in \Phi$. By (A1) and (R1) this implies that $\varphi \to \chi \in \Phi$ and therefore $\chi \in \Phi + \varphi$. So $\Phi + \varphi$ is closed under (R1).

Similarly, (A2) and (R1) guarantee that if $\Box_a(\psi \to \chi) \in \Phi$ and $\Box_a \psi \in \Phi$ then $\Box_a \chi \in \Phi$. Furthermore, (A4), (A5) and (R1) guarantee that if $[U](\psi \to \chi), [U]\psi \in \Phi$ then $[U]\chi \in \Phi$. The sets $\Box_a \Phi$ and $[U]\Phi$ are therefore also closed under (R1).

Now suppose that for some $\xi(x) \in NF$, we have $\{\xi([U]\psi) \mid [U] \in L_{AUL}\} \subseteq \Phi + \varphi$. By the definition of $\Phi + \varphi$, we have $\{\varphi \to \xi([U]\psi) \mid [U] \in L_{AUL}\} \subseteq \Phi$, which implies that $\varphi \to \xi([\updownarrow]\psi) \in \Phi$, since $\varphi \to \xi(x) \in NF$ and Φ is closed under (R4). Similarly, from $\Box_a \xi(x) \in NF$ and $[U]\xi(x) \in NF$ it follows that $\Box_a \Phi$ and $[U]\Phi$ are closed under (R4). $\Box$

Lemma 16. *$\Phi + \varphi$ is consistent if and only if $\neg\varphi \notin \Phi$.*

Proof. We prove that if $\neg\varphi \in \Phi$ then $\Phi + \varphi$ is inconsistent and that if $\Phi + \varphi$ is inconsistent then $\neg\varphi \in \Phi$.

Suppose $\neg\varphi \in \Phi$. Then, by Lemma 14 we have $\neg\varphi \in \Phi + \varphi$ and $\varphi \in \Phi + \varphi$. So $\Phi + \varphi$ is inconsistent.

Suppose then that $\Phi + \varphi$ is inconsistent. Then $\Phi + \varphi$ contains all AAUL formulas, so in particular $p \wedge \neg p \in \Phi + \varphi$. This implies that $\varphi \to (p \wedge \neg p) \in \Phi$. Since $(\varphi \to (p \wedge \neg p)) \to \neg\varphi$ is a propositional tautology and Φ is closed under modus ponens this implies that $\neg\varphi \in \Phi$. $\Box$

We now have all we need to prove our Lindenbaum lemma.

Lemma 17 (Lindenbaum lemma)**.** *Every consistent theory can be extended to a maximal consistent theory.*

Proof. Let Φ be a consistent theory. The set of all AAUL formulas is countably infinite, so we can enumerate it as $\{\varphi_0, \varphi_1, \cdots\}$. Define the sequence Φ_n of theories inductively as follows.

$$
\begin{array}{lcll}
\Phi_0 & = & \Phi & \\
\Phi_{n+1} & = & \Phi_n + \varphi_n & \text{if } \neg\varphi_n \notin \Phi_n \\
\Phi_{n+1} & = & \Phi_n & \text{if } \neg\varphi_n \in \Phi_n \text{ and } \varphi_n \text{ is not of the form } \xi([\updownarrow]\psi) \\
\Phi_{n+1} & = & \Phi_n + \varphi_j & \text{if } \neg\varphi_n \in \Phi_n \text{ and } \varphi_n \text{ is of the form } \xi([\updownarrow]\psi)
\end{array}
$$

where φ_j is the lowest numbered formula that is of the form $\varphi_j = \neg\xi([U]\psi)$ with $U \in L_{AUL}$, and such that $\neg\neg\xi([U]\psi) \notin \Phi_n$.

First, let us show that such φ_j is well defined. The first important observation here is that, by Lemma 10, the representation $\xi([\updownarrow]\psi)$ is unique. The second important observation is that (assuming that it is defined) each Φ_{n+1} is a consistent theory, since they are of the form $\Phi_n + \psi$ with $\neg\psi \notin \Phi_n$. So if $\neg\xi([\updownarrow]\psi) \in \Phi_n$ then there must be some $\xi([U]\psi) \notin \Phi_n$, as otherwise closure under (R4) would imply that $\xi([\updownarrow]\psi) \in \Phi_n$. If $\xi([U]\psi) \notin \Phi_n$ then also $\neg\neg\xi([U]\psi) \notin \Phi_n$. The lowest numbered formula with this property is φ_j.

Now let $\Psi = \bigcup_{n=0}^{\infty} \Phi_n$. We claim that Ψ is a maximal consistent theory that contains Φ. To this end, first note that $\{\Phi_n\}$ is an increasing sequence: $\Phi_n \subseteq \Phi_{n+1}$ for all $n \in \mathbb{N}$. Now, consider the following.

1. Ψ contains the theory Φ, so it contains all theorems.
2. Take any $\varphi_n \in L_{AAUL}$. We have either $\neg\varphi_n \in \Phi_n \subseteq \Psi$ or $\neg\varphi_n \notin \Phi_n$ and therefore $\varphi_n \in \Phi_n + \varphi_n = \Phi_{n+1} \subseteq \Psi$.
3. If $\varphi \to \psi \in \Psi$ and $\varphi \in \Psi$ then there is some $n \in \mathbb{N}$ such that $\varphi \to \psi \in \Phi_n$ and $\varphi \in \Phi_n$. This implies that $\psi \in \Phi_n$, and therefore also $\psi \in \Psi$.
4. If $\varphi \in \Psi$ then $\neg\varphi \notin \Psi$. By contradiction: suppose $\varphi, \neg\varphi \in \Psi$. Then there is an $n \in \mathbb{N}$ such that $\varphi, \neg\varphi \in \Phi_n$. That contradicts Φ_n being a consistent theory.
5. Ψ is closed under (R4). Proof: suppose φ_n is of the right form to be a conclusion of (R4), so $\varphi_n = \xi([\updownarrow]\psi)$. If $\neg\varphi_n \notin \Phi_n$ then $\varphi_n \in \Phi_{n+1} \subseteq \Psi$, so Ψ is closed with respect to this instance of (R4). Suppose then that $\neg\varphi_n \in \Phi_n$. Then $\varphi_j = \neg\xi([U]\psi) \in \Phi_{n+1} \subseteq \Psi$. By point 4, this implies that $\xi([U]\psi) \notin \Psi$, so one of the premises of (R4) is not satisfied. Again, Ψ is closed with respect to this instance of (R4).

From points 1, 3 and 5 it follows that Ψ is a theory. From point 1 it follows that Ψ is an extension of Φ. From 4 it follows that Ψ is consistent. Finally, from 2 it follows that Ψ is maximal. $\square$

6.3. Completeness of $\mathcal{L}_{AAUL}$: truth lemma

We can now define the canonical model, and prove a truth lemma for this model.

Definition 21 (Canonical model)**.** The canonical model $M_c = (W_c, R_c, V_c)$ is given as follows:

$$\begin{aligned} W_c &= \{\Phi \mid \Phi \text{ is a maximal consistent theory}\} \\ R_c(a) &= \{(\Phi, \Psi) \mid \Box_a \Phi \subseteq \Psi\} \\ V_c(p) &= \{\Phi \mid p \in \Psi\} \end{aligned}$$

Before considering the truth lemma, let us consider two more small lemmas.

Lemma 18. *Let Φ be a theory. If $\Box_a \varphi \notin \Phi$, then there is a maximal consistent theory Ψ such that $\Box_a \Phi \subseteq \Psi$ and $\varphi \notin \Psi$.*

Proof. By assumption, $\Box_a \varphi \notin \Phi$ and therefore $\varphi \notin \Box_a \Phi$. This implies that $\neg\neg\varphi \notin \Box_a \Phi$, since $\Box_a \Phi$ contains the tautology $\neg\neg\varphi \to \varphi$ and is closed under (R1). As such, $\Box_a \Phi + \neg\varphi$ is a consistent theory, which can be extended to a maximal consistent theory Ψ. This Ψ contains $\neg\varphi$ and is consistent, so in particular $\varphi \notin \Psi$. $\square$

Lemma 19. *Let Φ be a maximal consistent theory. Then $[U]\Phi$ is also a maximal consistent theory.*

Proof. We know from Lemma 15 that $[U]\Phi$ is a theory. Suppose towards a contradiction that $[U]\Phi$ is inconsistent. Then $p, \neg p \in [U]\Phi$ and therefore $[U]p, [U]\neg p \in \Phi$. But then, using (A4), we have $\neg[U]p \in \Phi$. So Φ is inconsistent, contradicting our assumptions. The theory $[U]\Phi$ must therefore be consistent.

Suppose then, towards a contradiction, for some φ we have $\varphi \notin [U]\Phi$ and $\neg\varphi \notin [U]\Phi$. Then $[U]\varphi \notin \Phi$ and $[U]\neg\varphi \notin \Phi$. By (A4), this implies that

$\neg[U]\varphi \notin \Phi$. But then Φ is not complete, contradicting out assumptions. The theory $[U]\Phi$ must therefore be complete. $\square$

Now, finally, we arrive at the truth lemma.

Lemma 20 (Truth lemma)**.** *For every maximal consistent theory Φ and every $\varphi \in L_{AAUL}$, we have $\varphi \in \Phi$ if and only if $M_c, \Phi \models \varphi$.*

Proof. The proof is by induction on $<^s_d$. As base case, suppose $d(\varphi) = 0$ and $s(\varphi) = 1$. Then $\varphi = p$ for some $p \in P$, so it follows immediately from the definition of $V_c(p)$ that $M_c, \Phi \models \varphi$ if and only if $p \in \Phi$.

Suppose then as induction hypothesis that $d(\varphi) > 0$ or $s(\varphi) > 1$, and that the lemma holds for all ψ with $\psi <^s_d \varphi$. The proof continues with a case distinction. Note that for every strict subformula ψ of φ we have $\psi <^s_d \varphi$.

Case 1. Suppose $\varphi = \neg\psi$. By the induction hypothesis, $\psi \notin \Phi \Leftrightarrow M_c, \Phi \not\models \psi$. By the semantics of AAUL we have $M_c, \Phi \not\models \psi \Leftrightarrow M_c, \Phi \models \neg\psi$. By maximality and consistency of Φ we have $\psi \notin \Phi \Leftrightarrow \neg\psi \in \Phi$. The three equivalences together show that $\varphi \in \Phi \Leftrightarrow M_c, \Phi \models \varphi$.

Case 2. Suppose $\varphi = \psi_1 \wedge \psi_2$. By the induction hypothesis we have $M_c, \Phi \models \psi_i \Leftrightarrow \psi_i \in \Phi$, $i \in \{1, 2\}$. By the semantics of AAUL we have $M_c, \Phi \models \psi_1 \wedge \psi_2 \Leftrightarrow M_c, \Phi \models \psi_1$ and $M_c, \Phi \models \psi_2$. Finally, because Φ is a maximal consistent theory we have $\psi_1 \wedge \psi_2 \in \Phi \Leftrightarrow \psi_1 \in \Phi$ and $\psi_2 \in \Phi$. Together, these equivalences show that $\varphi \in \Phi \Leftrightarrow M_c, \Phi \models \varphi$.

Case 3. Suppose $\varphi = \Box_a\psi$. We have $M_c, \Phi \models \Box_a\psi \Leftrightarrow M_c, \Psi \models \psi$ for all Ψ such that $\Box_a\Phi \subseteq \Psi$. By the induction hypothesis, the latter is equivalent to $\psi \in \Psi$ for all Ψ such that $\Box_a\Phi \subseteq \Psi$.

If $\Box_a\psi \in \Phi$ then $\psi \in \Psi$ for all Ψ such that $\Box_a\Phi \subseteq \Psi$. So $\Box_a\psi \in \Phi \Rightarrow M_c, \Phi \models \Box_a\psi$. Furthermore, by Lemma 18, if $\Box_a\psi \notin \Phi$ then there is some maximal consistent theory Ψ such that $\psi \notin \Psi$ and $\Box_a\Phi \subseteq \Psi$. By contraposition, this implies that if $\psi \in \Psi$ for all Ψ such that $\Box_a\Phi \subseteq \Psi$ then $\Box_a\psi \in \Phi$. So $M_x, \Phi \models \Box_a\psi \Rightarrow \Box_a\psi \in \Phi$.

Case 4. Suppose φ is of the form $[U]p, [U]\neg\psi, [U](\psi_1 \wedge \psi_2)$, $[U]\Box_a\psi$ or $[U][U']\psi$. Then φ occurs on the left side of one of the axioms (A3) – (A7). Let φ' be the corresponding formula on the right side.

The set Φ is a theory, so it contains (A3) – (A7) and is closed under modus ponens. So $\varphi \in \Phi \Leftrightarrow \varphi' \in \Phi$. Furthermore, by the semantics of AAUL (see Lemma 1), we have $M_c, \Phi \models \varphi \Leftrightarrow M_c, \Phi \models \varphi'$. Finally, $\varphi' <^s_d \varphi$ (see Lemma 12), so by the induction hypothesis $\varphi' \in \Phi \Leftrightarrow M_c, \Phi \models \varphi'$. Together, these three equivalences imply that $\varphi \in \Phi \Leftrightarrow M_c, \Phi \models \varphi$.

Case 4. Suppose $\varphi = [U][\updownarrow]\psi$. We treat the two directions of the bi-implication separately. Firstly, suppose that $[U][\updownarrow]\psi \in \Phi$. Observe that $[U]([\updownarrow]\psi \to [U_0]\psi) \in \Phi$ and $[U]([\updownarrow]\psi \to [U_0]\psi) \to ([U][\updownarrow]\psi \to [U][U_0]\psi)$ for every $U_0 \in L_{AUL}$, since both formulas are derivable in $\mathcal{L}_{AAUL}$. Furthermore, Φ is closed under (R1), so $[U][U_0]\psi \in \Phi$.

The update U_0, being an element of L_{AUL}, does not contain any $[\updownarrow]$ operators. As such, the $[\updownarrow]$-depth of $[U][U_0]\psi$ is strictly lower than that of $[U][\updownarrow]\psi$. We therefore have $[U][U_0]\psi <^s_d [U][\updownarrow]\psi$, so by the induction hypothesis our assumption that $[U][U_0]\psi \in \Phi$ yields the conclusion that $M_c, \Phi \models [U][U_0]\psi$. By the semantics of AAUL, the latter is equivalent to $M_c * U, \Phi \models [U_0]\psi$. Note that this holds for all $U_0 \in L_{AUL}$, so $M_c * U, \Phi \models [\updownarrow]\psi$ and therefore $M_c, \Phi \models [U][\updownarrow]\psi$. We have shown that $[U][\updownarrow]\psi \in \Phi \Rightarrow M_c, \Phi \models [U][\updownarrow]\psi$.

Suppose then that $M_c, \Phi \models [U][\updownarrow]\psi$. By the induction hypothesis, this implies that $[U][U_0]\psi \in \Phi$ for all $U_0 \in L_{AUL}$. Taking $\xi = [U]x \in NF$ we have $\{\xi([U_0]\psi) \mid U_0 \in L_{AUL}\} \subseteq \Phi$ which, since Φ is closed under (R4), gives us $\xi([\updownarrow]\psi) = [U][\updownarrow]\psi \in \Phi$. We have now shown that $M_c, \Phi \models [U][\updownarrow]\psi \Rightarrow [U][\updownarrow]\psi \in \Phi$. Together with our previous conclusion, this shows that $[U][\updownarrow]\psi \in \Phi \Leftrightarrow M_c, \Phi \models [U][\updownarrow]\psi$.

Case 5. Suppose $\varphi = [\updownarrow]\psi$. We have $[\updownarrow]\psi \in \Phi \Leftrightarrow ([U_0]\psi \in \Phi$ for all $U_0 \in L_{AUL})$; where $\Rightarrow$ is due to (A8) and $\Leftarrow$ is due to (R4). Furthermore,

since $[U_0]\psi <^s_d [\updownarrow]\psi$, so we can use the induction hypothesis to obtain $[U_0]\psi \in \Phi \Leftrightarrow M_c, \Phi \models [U_0]\psi$. Finally, by the semantics of AAUL, we have $(M_c, \Phi \models [U_0]\psi$ for all $U_0 \in L_{AUL}) \Leftrightarrow M_c, \Phi \models [\updownarrow]\psi$. Together, these equivalences imply that $[\updownarrow]\psi \in \Phi \Leftrightarrow M_c, \Phi \models [\updownarrow]\psi$.

Cases 1 – 5 are exhaustive, and in each case $\varphi \in \Phi \Leftrightarrow M_c, \Phi \models \varphi$. This completes the induction step and thereby the proof. □

6.4. $\mathcal{L}_{AAUL}$ is sound and complete for $\models$

The hard parts of the proof are done, now we can quickly prove that $\mathcal{L}_{AAUL}$ is complete.

Theorem 8 (Completeness of $\mathcal{L}_{AAUL}$)**.** *For all $\varphi \in L_{AAUL}$, if $\models \varphi$ then $\vdash \varphi$.*

Proof. By contraposition. Suppose $\nvdash \varphi$. Let Φ be the set of theorems, and note that we have $\varphi \notin \Phi$ and, since Φ is a theory, $\neg\neg\varphi \notin \Phi$. This means $\Phi + \neg\varphi$ is a consistent theory, so there is a maximal consistent theory Ψ that contains $\Phi + \neg\varphi$. We have $\varphi \notin \Psi$ and therefore, by Lemma 20, $M_c, \Psi \not\models \varphi$. As such, $\not\models \varphi$. □

Together with the soundness Theorem 7 this shows that the proof system $\mathcal{L}_{AAUL}$ is sound and complete for arbitrary arrow update logic.

If a finitary axiomatization of a logic exists, it follows that the set of validities of that logic is recursively enumerable. The axiomatization of AAUL is infinitary, however, so no such conclusion can be drawn. In fact, it is not currently known whether the set of validities of AAUL is RE, and therefore whether the satisfiability problem for AAUL is co-RE. It was shown in in [17] that the satisfiability problem of AAUL, like that of APAL [18], can encode the tiling problem. So while we do not know whether the satisfiability problem of AAUL is co-RE, we do know that is is not RE. In particular, this means that the satisfiability problem of AAUL is undecidable.

7. Conclusion

In this paper we introduced arbitrary arrow update logic where one can quantify over arrow updates. We investigated its expressivity relative to other logics in the family of dynamic epistemic logics, including epistemic logic with common knowledge. For a finite set of agents and a countably infinite set of propositional variables we managed to completely chart the expressivity landscape over the class of all Kripke models. For one agent and countably many propositional variable we also completely charted the landscape, mostly because in S5 all related systems boil down to epistemic logic. For two agents and one propositional variable, there is only one question remaining and that is one half of the relative expressivity of APAL and AAUL, and we conjecture that the logics have non-comparable expressivity.

We also showed that the model checking problem for arbitrary arrow update logic is PSPACE-complete, and we introduced a sound and complete infinitary axiomatization for arbitrary arrow update logic.

As far as future research is concerned there are other arbitrary variants of dynamic modal logics to consider and investigate their relative expressivity to AAUL, moreover we can further develop variants of APAL present in the literature and investigate what happens if we replace public announcements by arrow updates.

Another interesting question for future research is whether we can characterize for which dynamic operators "arbitrary version" are incomparable in expressivity. As mentioned in Section 4.3, we would generally expect any two logics using different "arbitrary operators" to be incomparable in expressivity. Yet the logics RML and AAML turn out to be only as expressive as basic epistemic logic, and therefore less expressive than the logics APAL and AAUL. It would be interesting to know exactly why RML and AAML deviate from the expected pattern.

Finally, we could add more temporal connectives to AAUL, and study their interaction with the $[\updownarrow]$ operator. In particular, if we add CTL-connectives like

AG and AE we could use AAUL (with the normative interpretation) to study concepts like *liveness*, *fairness* and *safety*.

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