The Complexity of the Empire Colouring Problem for Linear Forests

Andrew R. A. McGrae, Michele Zito[1]

Department of Computer Science
University of Liverpool

Abstract

Let r and s be fixed positive integers. Assume that the n vertices of a planar graph are partitioned into blocks (or *empires*) each containing exactly r vertices. The (s, r)-*colouring* problem (s-COL$_r$) asks for a colouring of the vertices of the graph that uses at most s colours, never assigns the same colour to adjacent vertices in different empires and, conversely, assigns the same colour to all vertices in the same empire, disregarding adjacencies. For $r = 1$ the problem coincides with the classical vertex colouring problem on planar graphs. The generalization for $r \geq 2$ was defined by Percy Heawood in 1890 in the same paper in which he refuted a previous "proof" of the famous Four Colour Theorem.

In a recent paper we have shown that if $s \geq 3$, s-COL$_r$ is NP-hard for linear forests if $s < r$. Furthermore, the hardness extends to $s < 6r - 3$ (resp. $s < 7$) when $r \geq 3$ (resp. for $r = 2$) for arbitrary planar graphs. For trees, our argument entails a nice dichotomy: s-COL$_r$ is NP-hard for $s \in \{3, \ldots, 2r - 1\}$ and solvable in polynomial time for any other positive value of s. In this paper we argue that linear forests don't make the problem any easier, even for small values of r. We prove that the s-COL$_r$ problem is NP-hard for linear forests even if $r = 2$ and $s = 3$, or $r = 3$ and $s \in \{3, 4\}$.

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Email addresses: `A.McGrae@liverpool.ac.uk` (Andrew R. A. McGrae), `M.Zito@liverpool.ac.uk` (Michele Zito)

[1]Corresponding author. Full postal address: Department of Computer Science, University of Liverpool, Ashton Building, Ashton Street, Liverpool L69 3BX, United Kingdom

1. Introduction

Let r and s be fixed positive integers. Assume that a partition is defined on the n vertices of a planar graph G. In this paper we usually call the blocks of such partition the *empires* of G and we will assume that each block contains exactly r vertices. The graph G along with a partition of this type will be referred to as an *r-empire graph*. The *(s, r)-colouring* problem (s-COL$_r$) asks for a colouring of the vertices of G that uses at most s colours, never assigns the same colour to adjacent vertices in different empires and, conversely, assigns the same colour to all vertices in the same empire, disregarding adjacencies. We call such colourings (s, r)-colourings of G. For $r = 1$, the problem coincides with the classical vertex colouring problem on planar graphs and an $(s, 1)$-colouring is just a proper vertex colouring of the the host graph. The generalization for $r \geq 2$ was defined by Heawood (1890) in the same paper in which he refuted a previous "proof" of the famous Four Colour Theorem. It has since been shown (Jackson and Ringel, 1983) that $6r$ colours are always sufficient and in some cases necessary to solve this problem.

McGrae (2010) studied the (s, r)-colouring problem on trees. He proved (also see McGrae and Zito (2008)), that $2r$ colours suffice and are sometimes needed to colour a collection of empires defined in an arbitrary tree (in fact the result generalizes to arbitrary forests). He also looked at the proportion of (s, r)-colourable trees on n vertices. He showed that, as n tends to infinity, for each r there exist positive integers s_1 and s_2 with $s_1 < s_2$, depending on r, such that almost no r-empire tree can be coloured with at most s_1 colours and, conversely, s_2 colours suffice with (at least) constant positive probability. Later on Cooper et al. (2009) improved on this showing that, as n tends to infinity, the minimum value s for which a random tree is (s, r)-colourable is concentrated in a very short interval with high probability.

Although this investigation considerably expanded the state of knowledge on s-COL$_r$, it failed to shed light on its computational complexity. Clearly the (s, r)-colouring problem can be solved in polynomial time if one has a sufficiently large palette. However the picture is less clear when s is smaller than, say, r times the maximum average degree of a graph in the class of interest. McGrae and Zito (2011) explored questions of this type. They proved that s-COL$_r$ is NP-hard for linear forests (i.e. forests whose components are paths) if $3 \leq s < r$ and solvable in polynomial time for $s < 3$ or $s \geq 2r$. Furthermore, the hardness extends to $s < 6r - 3$ (resp. $s < 7$) when $r \geq 3$ (resp. $r = 2$) on arbitrary planar graphs. Finally, for trees, the authors uncovered a nice dichotomy: s-COL$_r$ is NP-hard for $s \in \{3, \ldots, 2r-1\}$ and solvable in polynomial time for any other positive value of s (the same algorithm that provides a polynomial solution for trees works for linear forests). The hardness proofs mentioned above hinge on the fact that the connectivity within empires has no effect on the graph colourability. Essentially, to solve an instance of s-COL$_r$ it suffices to be able to colour with at most s distinct colours (in such a way that no two distinct vertices connected by an edge receive the same colour) the *reduced graph* of the input graph G. This is a (multi)graph obtained by contracting each empire to a distinct pseudo-vertex and adding an edge between a pair of pseudo-vertices u and v for each edge connecting two vertices in G, one belonging to the empire represented by u, the other one to that represented by v. In particular such reduction to the classical vertex colouring problem implies that s-COL$_r$ can be solved in polynomial time for $s = 2$, as checking whether the reduced graph of an r-empire graph is bipartite is easy.

The results above left the possibility that, for any positive integer s, s-COL$_r$ might be solvable in polynomial time when $r < 4$ if the input instance graph is a linear forest. In this paper we show that this is not the case. We prove that, for $r = 2$, the problem is not any easier than in the case where the input is a tree (thus one may argue that connectivity does not make the empire colouring problem any harder). Furthermore, we show that, for $r = 3$, the problem is NP-hard for $s \in \{3, 4\}$. In what follows LFOREST denotes the collection of linear forests.

The rest of the paper is organized as follows. In Section 2 we introduce a few computational problems whose complexity will be related to that of s-COL$_r$. We then (Section 3) discuss a couple of gadgets that will be used in the proofs of our main results. Then Section 4 is devoted to our NP-hardness result for $r \in \{2, 3\}$ and $s = 3$, whereas Section 5 deals with the case $r = 3$, $s = 4$. The last section provides a summary of our results and a brief discussion of open problems and relevant issues.

2. Decision Problems

In this section we define a few computational problems that will be used in our main reductions. Unless otherwise stated we follow Diestel (1999) for all our graph-theoretic notations.

Let k and s be positive integers greater than two. In what follows k-SAT denotes the well known (e.g., Garey and Johnson, 1979; Karp, 1972) NP-complete problem of checking the satisfiability of a k-CNF boolean formula. Similarly s-COL denotes the problem of deciding whether the vertices of a graph G can be coloured using at most s distinct colours in such a way that no edge of G connects two vertices of the same colour. Also, if Π is a decision problem and $\mathcal{I}$ is a particular set of instances for it, then $\Pi(\mathcal{I})$ will denote the restriction of Π to instances belonging to $\mathcal{I}$. If Π_1 and Π_2 are decision problems, then $\Pi_1 \leq_p \Pi_2$ will denote the fact that Π_1 is polynomial-time reducible to Π_2. The following result (Garey et al., 1976) shows the NP-hardness of 3-COL. The proof of Theorem 3 in Section 4 is reminiscent of the reduction that proves this result.

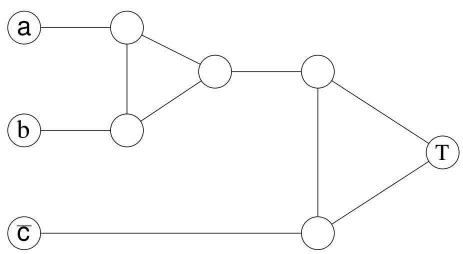


Figure 1: The clause gadget in the reduction of Theorem 1 for the clause $\mathsf{a} \vee \mathsf{b} \vee \bar{\mathsf{c}}$. The five "special" vertices for this gadget are the five ones without a label.

Theorem 1. 3-SAT $\leq_p$ 3-COL.

Proof. Given a formula ϕ of m clauses on variables $\mathsf{a}_1, \ldots, \mathsf{a}_n$ we construct a graph $G_\phi = (V, E)$ having a vertex for each literal on the variables $\mathsf{a}_1, \ldots, \mathsf{a}_n$, five "special" vertices for each clause of ϕ and three additional vertices labelled T, F, and X. The sets $\{T, F, X\}$ as well as $\{a_i, \overline{a_i}, X\}$ for $i = 1, \ldots, n$ span a triangle. Each clause of ϕ is represented by a gadget like the one in Figure 1.

To complete the reduction one needs to prove that ϕ is satisfiable if and only if G_ϕ is 3-colourable. This is a consequence of the following two claims:

1. In any 3-colouring of the subgraph of G_ϕ induced by the set $\{T, F, X\} \cup \{\mathsf{a}_1, \ldots, \mathsf{a}_n\} \cup \{\bar{\mathsf{a}}_1, \ldots, \bar{\mathsf{a}}_n\}$, exactly one of a_i and $\bar{\mathsf{a}}_i$ is coloured with the

same colour as vertex T and the other one is coloured with the same colour as vertex F.

Hence, for each choice of colour for T, F, and X, there are exactly 2^n possible ways of completing the colouring of the subgraph of G_ϕ induced by the set $\{T, F, X\} \cup \{\mathsf{a}_1, \ldots, \mathsf{a}_n\} \cup \{\bar{\mathsf{a}}_1, \ldots, \bar{\mathsf{a}}_n\}$.

2. For each clause gadget if the vertices corresponding to variables are coloured like vertices T or F then the gadget is 3-colourable if and only if at least one of the variable vertices is coloured like vertex T.

To believe the first claim notice that T, F and all vertices corresponding to literals of ϕ are adjacent to vertex X. Hence none of them can be coloured like X in any colouring of G_ϕ. Also, T and F (resp. a_i and $\bar{\mathsf{a}}_i$, for each $i \in \{1, \ldots, n\}$) must have a different colour because they are adjacent to each other.

As to the second claim, if all three variables corresponding to literals in the gadget in Figure 1 are coloured like vertex F then, in particular, the white vertex at the bottom of the picture must be coloured like vertex X and that implies that its other special neighbour must be coloured like vertex F. This in turn entails that all remaining special vertices are adjacent to a vertex coloured like vertex F and therefore they cannot be coloured using just the colours of vertex T and X. Conversely, if at least one of the literal vertices is coloured like vertex T then all special vertices can be coloured with the three available colours.

The proof of the Lemma follows. $\square$

Let s and k be positive integers with $s > \max(2, k)$. Also, let n and m be positive integers. An (s, k)-*formula graph* (see McGrae and Zito, 2011) is an undirected graph $\Phi = (V, E)$ such that $V(\Phi) = \mathcal{T} \cup \mathcal{C} \cup \mathcal{A}$ where $\mathcal{T} = \{T, F, X^1, \ldots, X^{s-2}\}$, $\mathcal{C}$ contains m groups of vertices $\{c^{1,1}, \ldots, c^{1,s-1}\}$, $\{c^{2,1}, \ldots c^{2,s-1}\}$, $\ldots, \{c^{m,1}, \ldots, c^{m,s-1}\}$ and $\mathcal{A}$ is a set of $2n$ vertices paired up in some recognizable way. In particular, in what follows we will denote the elements of $\mathcal{A}$ by $a_1, \ldots, a_n, \overline{a_1}, \ldots, \overline{a_n}$, and we will say that for each $i \in \{1, \ldots, n\}$, a_i and $\overline{a_i}$ are a *pair of complementary vertices*. Set $\mathcal{T}$ spans a complete graph; for each pair of complementary vertices a and $\overline{a}$, $\{a, \overline{a}, X^j\}$ spans a complete graph for each $j \in \{1, \ldots, s-2\}$; for each $i \in \{1, \ldots, m\}$, $\{T, c^{i,1}, \ldots, c^{i,s-1}\}$ spans a complete graph and if $j \in \{1, \ldots, k\}$ then there is a single edge connecting $c^{i,j}$ to some vertex in $\mathcal{A}$, else if $j \geq k+1$ then $\{c^{i,j}, F\} \in E(\Phi)$. Figure 2 gives a simple example of a $(5, 3)$-formula graph with $m = 1$ and $n = 3$.

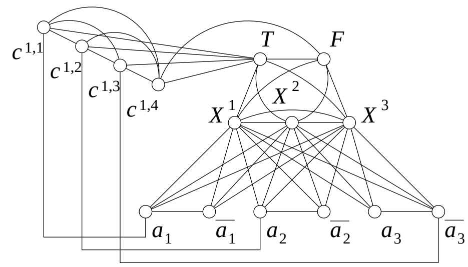


Figure 2: A small formula graph

Let $\mathrm{FG}(s,k)$ denote the class of all (s,k)-formula graphs. McGrae and Zito (2011) proved the following result.

Theorem 2. *Let s be an integer with $s \geq 3$. Then k-SAT $\leq_p$ s-COL(FG(s,k)) for any integer $k < s$.*

Theorem 2 implies in particular that s-COL(FG(s,k)) is NP-hard for any $k \geq 3$ and $s > k$. In Section 5 we will use 4-COL(FG$(4,3)$) to show the NP-hardness of 4-COL$_3$.

3. Gadgets

Before presenting our main results we introduce a number of useful gadgets.

Clique Gadgets. Let r and s be positive integers with $s < 2r$. In what follows the *clique gadget* $B_{r,s}$ is an r-empire graph satisfying the following properties.

- **B0** The graph $B_{r,s}$ has $r(s+1)$ vertices partitioned into $s+1$ empires of size r.
- **B1** The graph $B_{r,s}$ is a linear forest.
- **B2** The reduced graph of $B_{r,s}$ contains a spanning copy of K_{s+1}. Hence $B_{r,s}$ admits an $(s+1,r)$-colouring and cannot be coloured with fewer colours.

Figure 3 describes the clique gadgets that will be used in this paper.

Connectivity Gadgets. For positive integers r, s and m with $r,s \geq 2$ and $m \geq r$, the *connectivity gadget*, denoted by $A_{r,s,m}$, is an r-empire graph satisfying the following conditions:

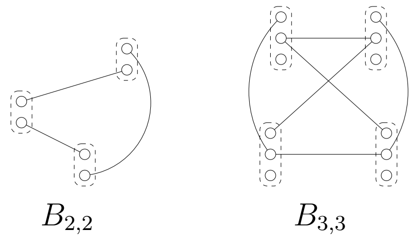


Figure 3: The clique gadgets used in this paper. Empires are denoted by sets of vertices enclosed by a dashed line.

A0 The graph $A_{r,s,m}$ contains no more than $2srm$ vertices split into empires of size r.

A1 The graph $A_{r,s,m}$ is a linear forest.

A2 The graph $A_{r,s,m}$ contains a set of m isolated vertices and such vertices must be given the same colour in any (s,r)-colouring of $A_{r,s,m}$. These vertices define the so called *monochromatic set* of the gadget and will collectively be denoted by Z. The elements of such a set will generically denoted by z.

Figure 4 shows the reduced graphs of two elements of the families of connectivity gadgets that will be used in this paper. The white vertices in each case span disjoint copies of a complete graph. These are used to constrain the colour that can be assigned to the other vertices. Details of the construction process as well as a proof that the relevant gadgets satisfy properties **A0**, **A1**, and **A2** are provided in the following Lemma.

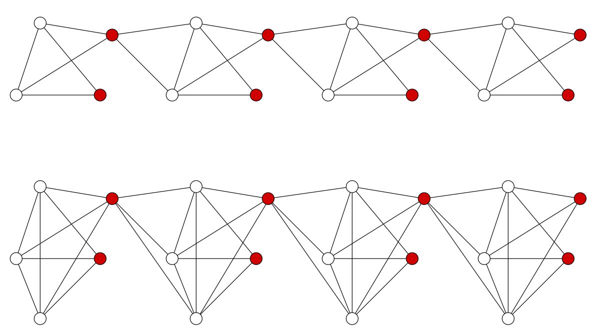
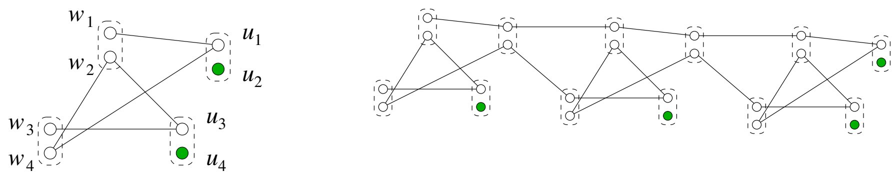


Figure 4: The reduced graphs of the connectivity gadgets: $A_{2,3,5}$ ($A_{3,3,11}$, $A_{3,3,12}$, and $A_{3,3,13}$) at the top, and $A_{3,4,5}$ in the bottom picture.



$A_{2,3,2}$ $\qquad\qquad$ $A_{2,3,4}$

Figure 5: Small connectivity gadgets used for $s = 3$. The dark vertices form the monochromatic set of the gadget.

Lemma 1. *Let m, r and s be positive integers, with $r \in \{2, 3\}$ and $s = 3$ or $r = 3$ and $s = 4$, and $m \geq r$. Then there exists an r-empire graph $A_{r,s,m}$ satisfying properties* **A0**, **A1**, *and* **A2**. *Furthermore $A_{r,s,m}$ can be constructed in time polynomial in r, s and m.*

Proof. In all cases the graph $A_{r,s,m}$ will be built up two distinct sets of vertices. There will be a *colour constraining set* $W_{r,s,m}$, and an independent set of vertices $U_{r,s,m}$.

We describe a construction for $A_{r,3,m}$ first. For $r = 2$ we provide an inductive contruction. In particular, the empires defined on the set $U_{2,3,m}$ in the resulting graph will either have all vertices of degree two, or be formed by a vertex of degree two, and one isolated vertex which belongs to the gadget's monochromatic set. The graph $A_{2,3,2}$ is represented on the left-hand side of Figure 5. Sets $W_{2,3,2}$ and $U_{2,3,2}$ are defined as follows:

$$\begin{aligned} W_{2,3,2} &= \{w_1, w_2, w_3, w_4\}, \\ U_{2,3,2} &= \{u_1, u_2, u_3, u_4\}. \end{aligned}$$

Blocks $\mathbf{w}_1 \equiv \{w_1, w_2\}$, $\mathbf{w}_2 \equiv \{w_3, w_4\}$, $\{u_1, u_2\}$ and $\{u_3, u_4\}$ define the empires of $A_{2,3,2}$. Thus condition **A0** is obvious. Note that $A_{2,3,2}$ is in fact just a simple path connecting all vertices except u_2 and u_4, and therefore it also satisfies condition **A1**. As to **A2**, the monochromatic set of $A_{2,3,2}$ contains the two isolated vertices: u_2 and u_4. Note that, because of the edge $\{w_2, w_4\}$, empires $\mathbf{w}_1$, and $\mathbf{w}_2$ are adjacent and therefore they must receive different colours in any $(3, 2)$-colouring of $A_{2,3,2}$. The remaining empires form an independent set (in the reduced graph of $A_{2,3,2}$) and hence they can be coloured with a single additional colour. Notice that they must be coloured with one additional colour, as empires $\{u_1, u_2\}$ and $\{u_3, u_4\}$ are both adjacent to each of $\mathbf{w_1}$ and $\mathbf{w_2}$. Thus **A2** is verified for $A_{2,3,2}$.

For $m > 2$, assume that the graph $A_{2,3,m-1} = (W_{2,3,m-1} \cup U_{2,3,m-1}, E_{m-1})$ has been built already, with $W_{2,3,m-1} \equiv \{w_1, \ldots, w_{4(m-2)}\}$ and $U_{2,3,m-1} \equiv \{u_1, \ldots, u_{4(m-2)}\}$ and each empire in $U_{2,3,m-1}$ either having all vertices of degree two or consisting of a vertex of degree two and a vertex of degree zero belonging to the monochromatic set of $A_{2,3,m-1}$. Let $\{u, u'\}$ be an arbitrary empire in $U_{2,3,m-1}$ such that the degree of u is two whereas u' is part of the monochromatic set of $A_{2,3,m-1}$. The graph $A_{2,3,m}$ will have vertex set defined by the union of the following sets:

$$\begin{aligned} W_{2,3,m} &= W_{2,3,m-1} \cup \{w_{4(m-2)+i} : 1 \leq i \leq 4\}, \\ U_{2,3,m} &= U_{2,3,m-1} \cup \{u_{4(m-2)+i} : 1 \leq i \leq 4\}, \end{aligned}$$

and edge set formed by the union of the edges of $A_{2,3,m-1}$, with one of the edges incident with u, say $\{w, u\}$, replaced by the edge $\{w, u'\}$, and the edges

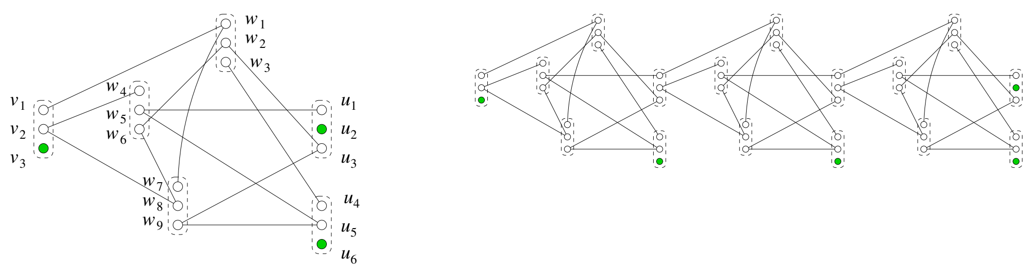
$$\begin{aligned} &\{u, w_{4m-7}\}, \{u', w_{4m-5}\}, \\ &\{w_{4m-6}, w_{4m-4}\}, \\ &\{w_{4m-7}, u_{4m-7}\}, \{w_{4m-4}, u_{4m-7}\}, \{w_{4m-6}, u_{4m-5}\}, \{w_{4m-5}, u_{4m-5}\}. \end{aligned}$$

The graph $A_{2,3,4}$ (with vertex labels omitted for clarity) is shown on the right-hand side of Figure 5. Thus $A_{2,3,m}$ has eight more vertices than $A_{2,3,m-1}$ and its monochromatic set contains one more element than that of $A_{2,3,m-1}$ (vertex u' is not part of the monochromatic set any more, but u_{4m-6} and u_{4m-4} are added to Z). The empires of $A_{2,3,m}$ are those of $A_{2,3,m-1}$ plus four new ones: $\mathbf{w}_{2m-3} \equiv \{w_{4m-7}, w_{4m-6}\}$, $\mathbf{w}_{2m-2} \equiv \{w_{4m-5}, w_{4m-4}\}$, $\{u_{4m-7}, u_{4m-6}\}$ and $\{u_{4m-5}, u_{4m-4}\}$. Condition **A0** follows. Furthermore the graph induced by the newly introduced vertices is isomorphic to $A_{2,3,2}$ and the "re-wiring" of the edges incident to $\{u, u'\}$ implies that $A_{2,3,m}$ is a simple path. Hence **A1** holds for $A_{2,3,m}$. The following properties are an immediate consequence of our construction:

1. The empires defined on $U_{2,3,m}$ form an independent set in the reduced graph of $A_{2,3,m}$, and each of them consists either entirely of vertices of degree two or of a vertex of degree two and a vertex of degree zero.
2. For each integer $x \in \{2, \ldots, m\}$, empires $\mathbf{w}_{2x-3}$ and $\mathbf{w}_{2x-2}$ are adjacent, they are not adjacent to any other empire defined over $W_{2,3,m}$, and they are both adjacent to exactly three empires defined over $U_{2,3,m}$.

Condition **A2** is satisfied as, we claim, all vertices in the set $U_{2,3,m}$ must be given the same colour in any $(3,2)$-colouring of $A_{2,3,m}$. Assume by contradiction that this claim does not hold. This implies that there exists a $(3,2)$-colouring of $A_{2,3,m}$ such that for some $x \in \{2,\ldots,m\}$ empires $\mathbf{w}_{2x-3}$ and $\mathbf{w}_{2x-2}$ are adjacent to three empires which are coloured with at least two different colours. But then there is no way to complete the colouring of $\mathbf{w}_{2x-3}$ and $\mathbf{w}_{2x-2}$ using only one more colour.

The construction of the connectivity gadgets for the case $s = 3$ is completed by noticing that, when $r = 3$, for $m \geq 3$, gadget $A_{3,3,m}$ can be obtained from $A_{2,3,\lceil (m+2)/3 \rceil}$ by adding an isolated vertex to each empire. Conditions **A0** and **A1** are clearly satisfied. Condition **A2** is also true provided we define the monochromatic set of $A_{3,3,3}$ as a set of three isolated elements of $U_{3,3,3}$ and that of $A_{3,3,m}$ by extending the one in $A_{2,3,\lceil (m+2)/3 \rceil}$.



$A_{3,4,3}$ $\qquad\qquad$ $A_{3,4,5}$

Figure 6: Small connectivity gadgets for $s = 4$.

The construction of $A_{3,4,m}$ is similar to that of $A_{2,3,m}$. In particular, for each $m \geq 3$, the empires defined on the set $U_{3,4,m}$ will either have all vertices of degree two or a vertex of degree two, a vertex of degree one, and a vertex of degree zero. The graph $A_{3,4,3}$ is represented on the left-hand side of Figure 6. Sets $W_{3,4,3}$ and $U_{3,4,3}$ are defined as follows:

$$\begin{aligned} W_{3,4,3} &= \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9\}, \\ U_{3,4,3} &= \{v_1, v_2, v_3, u_1, u_2, u_3, u_4, u_5, u_6\}. \end{aligned}$$

The empires of $A_{3,4,3}$ are $\mathbf{w}_i \equiv \{w_{3(i-1)+j} : 1 \leq j \leq 3\}$ for $i \in \{1,2,3\}$, and blocks $\{v_1, v_2, v_3\}$, $\{u_1, u_2, u_3\}$ and $\{u_4, u_5, u_6\}$. Condition **A0** is clearly satisfied. By construction $A_{3,4,3}$ is just a collection of paths connecting all vertices except v_3, u_2 and u_6, and therefore it satisfies condition **A1**.

As for **A2**, note that the monochromatic set of $A_{3,4,3}$ contains the three isolated vertices v_3, u_2 and u_6. Also, empires $\mathbf{w}_1, \mathbf{w}_2$, and $\mathbf{w}_3$ are all adjacent and therefore they must receive different colours in any $(4,3)$-colouring of $A_{3,4,3}$. The remaining empires form an independent set (in the reduced graph of $A_{3,4,3}$) and hence they can be coloured with only one additional colour. Thus **A2** is verified for $A_{3,4,3}$.

As in the case $s = 3$, for $m > 3$, $A_{3,4,m}$ is defined by concatenating a copy of $A_{3,4,m-1}$ with a copy of $A_{3,4,3}$. Let $\{u, u', u''\}$ be a particular empire in $U_{3,4,m-1}$ such that the degree of u' is one, that of u'' is two, whereas u is part of the monochromatic set of $A_{3,4,m-1}$. The graph $A_{3,4,m}$ will have vertex set defined by the union of the following sets:

$$\begin{aligned} W_{3,4,m} &= W_{3,4,m-1} \cup \{w_{9(m-3)+i} : 1 \leq i \leq 9\}, \\ U_{3,4,m} &= U_{3,4,m-1} \cup \{u_{6(m-3)+i} : 1 \leq i \leq 6\} \end{aligned}$$

(here $W_{3,4,m-1} \equiv \{w_1, \ldots, w_{9(m-3)}\}$ and $U_{3,4,m-1} \equiv \{u_1, \ldots, u_{6(m-3)}\}$). The edge set of $A_{3,4,m}$ is formed by the union of the edges of $A_{3,4,m-1}$, and the edges

$$\begin{gathered} \{u', w_{9m-26}\}, \{u, w_{9m-23}\}, \{u, w_{9m-19}\}, \\ \{w_{9m-26}, w_{9m-20}\}, \{w_{9m-25}, w_{9m-21}\}, \{w_{9m-21}, w_{9m-19}\}, \\ \{w_{9m-22}, u_{6m-17}\}, \{w_{9m-25}, u_{6m-15}\}, \{w_{9m-18}, u_{6m-15}\}, \\ \{w_{9m-24}, u_{6m-14}\}, \{w_{9m-22}, u_{6m-13}\}, \{w_{9m-18}, u_{6m-13}\}. \end{gathered}$$

Note that each of the empires defined on $U_{3,4,m}$ consists either entirely of vertices of degree two or of a vertex of degree two, a vertex of degree one, and a vertex of degree zero. The graph $A_{3,4,5}$ is shown in Figure 6. In general $A_{3,4,m}$ has fifteen more vertices than $A_{3,4,m-1}$ and its monochromatic set contains one more element than that of $A_{3,4,m-1}$. The empires of $A_{3,4,m}$ are those of $A_{3,4,m-1}$ and five new empires: $\mathbf{w}_{3m-8}$, $\mathbf{w}_{3m-7}$, $\mathbf{w}_{3m-6}$ defined on the new elements of $W_{3,4,m}$, and $\{u_{6m-17}, u_{6m-16}, u_{6m-15}\}$, $\{u_{6m-14}, u_{6m-13}, u_{6m-12}\}$, Condition **A0** follows. The reader can easily verify that the resulting graph also satisfies **A1**.

The empires defined on $U_{3,4,m}$ form an independent set in the reduced graph of $A_{2,3,m}$. Furthermore, for each integer $x \in \{3, \ldots, m\}$, empires $\mathbf{w}_{3x-8}$, $\mathbf{w}_{3x-7}$ and $\mathbf{w}_{3x-6}$ are all adjacent, they are not adjacent to any other empire defined over $W_{3,4,m}$, and they are both adjacent to exactly three empires defined over $U_{3,4,m}$. Condition **A2** can be readily verified arguing as in the case $s = 3$. □

Given an r-empire graph G, and an empire $\mathbf{v}$ in $V(G)$, the *r-degree of* $\mathbf{v}$ is simply the degree of vertex $\mathbf{v}$ in the reduced graph of G. Gadgets $A_{r,s,m}$ will be used to replace particular empires with high r-degree by an array of vertices of degree one or two, chosen among the monochromatic vertices of the gadget. Let m be an integer at least as large as the r-degree of $\mathbf{v}$. The *linearization of* $\mathbf{v}$ *in* G is the r-empire graph obtained by replacing $\mathbf{v}$ with a copy of $A_{r,s,m}$ attaching each edge incident with some element of $\mathbf{v}$ to an element of the monochromatic set of $A_{r,s,m}$. We will often denote the monochromatic set of the connectivity gadget $A_{r,s,m}$ that linearize $\mathbf{v}$ as $Z(\mathbf{v})$ and say that the chosen elements of such set *simulate* the connectivity of the empire $\mathbf{v}$.

4. Hardness $s = 3$

We are now ready to describe our main results. We start from the case $s = 3$.

Theorem 3. *Let* $r \in \{2, 3\}$. *Then* 3-SAT $\leq_p$ 3-COL$_r$(LFOREST).

Proof. The proof construction is reminiscent of that used in Theorem 1. Given an instance ϕ of 3-SAT we can produce a linear forest $P(\phi)$ and a partition of $V(P(\phi))$ into empires of size r such that $P(\phi)$ admits a $(3, r)$-colouring if and only if ϕ is satisfiable. The key property of $P(\phi)$ is that it is closely related to the graph G_ϕ used in the proof of Theorem 1.

$P(\phi)$ consists of one *truth gadget*, one *variable gadget* for each variable used in ϕ, and one *clause gadget* for each clause in ϕ. To define the truth gadget, we start by adding $r - 2$ distinct isolated vertices to each empire in $B_{2,2}$. The empires in the resulting graph (which we denote by $B_{2,2}^{+r}$) will be labelled $\mathbf{T}$, $\mathbf{F}$ and $\mathbf{X}$. If ϕ uses n different variables and m clauses, we

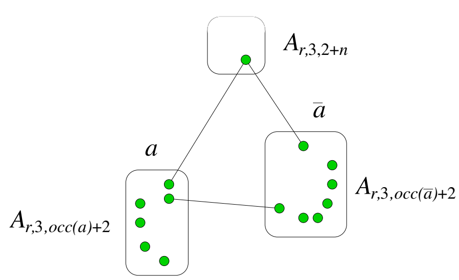


Figure 7: The shape of a variable gadget for $s = 3$.

linearize $\mathbf{T}$ and $\mathbf{X}$ in $B_{2,2}^{+r}$, using one copy of $A_{r,3,2+2m}$, and one copy of $A_{r,3,2+n}$ respectively (as both $\mathbf{T}$ and $\mathbf{X}$ have r-degree equals to two with respect to the graph $B_{2,2}^{+r}$). This completes the definition of the truth gadget. Since $\mathbf{T}$, $\mathbf{F}$ and $\mathbf{X}$ are all pairwise adjacent (in the reduced graph of $B_{2,2}^{+r}$) and the linearization preserves colour constraints (because of property **A2**), the vertices of the truth gadget simulating the three empires of $B_{2,2}^{+r}$ must have different colours in any $(3, r)$-colouring of the truth gadget. Without loss of generality we call these colours TRUE, FALSE and OTHER respectively.

For each variable a in ϕ, $P(\phi)$ contains a variable gadget. Let $\mathrm{occ}(\cdot)$ be a function taking as input a literal of ϕ and returning the number of occurrences of its argument in the given formula. The variable gadget for a is defined as the graph formed by the two connectivity gadgets $A_{r,3,\mathrm{occ}(\mathsf{a})+2}$ and $A_{r,3,\mathrm{occ}(\bar{\mathsf{a}})+2}$, along with a single monochromatic vertex $z_{\mathbf{X}}$ in $A_{r,3,2+n}$ (a distinct monochromatic vertex is used for each variable of ϕ). The edges in the variable gadgets will be those of $A_{r,3,\mathrm{occ}(\mathsf{a})+2}$ and $A_{r,3,\mathrm{occ}(\bar{\mathsf{a}})+2}$ plus three further edges: $\{z_{\mathbf{X}}, z_{\mathbf{a}}\}$, $\{z_{\mathbf{X}}, z_{\bar{\mathbf{a}}}\}$, and $\{z'_{\mathbf{a}}, z'_{\bar{\mathbf{a}}}\}$. Here $z_{\mathbf{a}}$ and $z'_{\mathbf{a}}$ (resp. $z_{\bar{\mathbf{a}}}$ and $z'_{\bar{\mathbf{a}}}$) are in the monochromatic sets of the gadgets used to linearize $\mathbf{a}$ (resp. $\bar{\mathbf{a}}$). Figure 7 gives a schematic view of the variable gadget for an arbitrary variable a. Since $\mathbf{X}$ has colour OTHER, there are only two possible colourings for $\mathbf{a}$ and $\bar{\mathbf{a}}$ — either $\mathbf{a}$ is TRUE and $\bar{\mathbf{a}}$ is FALSE, or $\mathbf{a}$ is FALSE and $\bar{\mathbf{a}}$ is TRUE (this corresponds to claim 1. in the proof of Theorem 1).

Finally, for each clause in ϕ, $P(\phi)$ contains a gadget like the one depicted in Figure 8. This is connected to the rest of the graph via four connectivity gadgets. More specifically, the two vertices labelled T_1 and T_2 (in the Figure) are two monochromatic vertices in $A_{r,3,2+2m}$ (a distinct pair of such monochromatic vertices for each case clause gadget). Also, vertices labelled a, b and $\bar{c}$ in the Figure belong to the monochromatic set of three connectivity gadgets of the form $A_{r,3,\mathrm{occ}(\ell)+2}$ used to replace a literal ℓ in ϕ ($\ell = a$, b, and $\bar{c}$ in the given example). The careful reader will notice the similarity between the clause gadget used in this proof and the one used in the proof of Theorem 1. Reasoning as in the proof of claim 2. in Theorem 1 it can be shown that the clause gadget admits a $(3, r)$-colouring if and only if at least one of the empires corresponding to a literal is coloured like empire $\mathbf{T}$.

The fact that $P(\phi)$ is $(3, r)$-colourable if and only if ϕ is satisfiable follows from the argument above and the fact that the graph obtained from $P(\phi)$ by shrinking each connectivity gadget first and then each remaining empire in $P(\phi)$ to a distinct (pseudo-)vertex coincides with the graph G_ϕ used in the proof of Theorem 1. $\square$

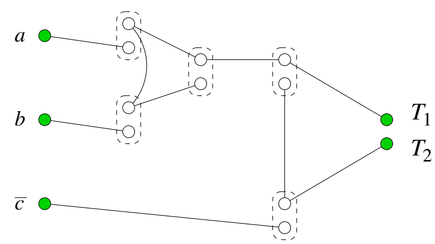


Figure 8: The clause gadget for the clause $(\mathsf{a} \vee \mathsf{b} \vee \overline{\mathsf{c}})$, for $r = 2$. Each dashed curve encloses a pair of vertices belonging to the same empire. The vertices labelled T_1 and T_2 are in $Z(\mathbf{T})$, while vertices labelled a, b and $\overline{c}$ are in $Z(\mathbf{a})$, $Z(\mathbf{b})$ and $Z(\overline{\mathbf{c}})$ respectively.

5. Hardness $s = 4$

For $s = 4$, it is convenient to show the NP-hardness of the relevant empire colouring problem using a reduction from 4-COL(FG(4, 3)), rather than directly from 3-SAT.

Theorem 4. 4-COL(FG$(4, 3)$) $\leq_p$ 4-COL$_3$(LFOREST).

Proof. Let Φ be a $(4, 3)$-formula graph defined on a set of complementary vertices $\mathcal{A} = \{a_1, \ldots, a_n, \overline{a_1}, \ldots, \overline{a_n}\}$ and with set $\mathcal{C}$ containing m groups of three vertices. A few simple replacement rules enable us to define a forest of paths $P(\Phi)$ and a partition of $V(P(\Phi))$ into empires of size three such that Φ is 4-colourable if and only if $P(\Phi)$ is $(4, 3)$-colourable.

First, the complete graph on $\mathcal{T} = \{T, F, X^1, X^2\}$ is replaced by four empires of size three labelled $\mathbf{T}$, $\mathbf{F}$, $\mathbf{X}^1$ and $\mathbf{X}^2$ spanning a copy of $B_{3,3}$. Moreover we linearize $\mathbf{T}$, $\mathbf{X}^1$ and $\mathbf{X}^2$ in $B_{3,3}$ using a copy of $A_{3,4,3+3m}$, $A_{3,4,3+2n}$ and $A_{3,4,3+n}$, respectively. Three monochromatic vertices in each connectivity gadget are used in the linearization. The $2n$ and n monochromatic vertices in the connectivity gadgets replacing $\mathbf{X}^1$ and $\mathbf{X}^2$ will be needed in the definition of the subgraph of $P(\Phi)$ replacing the subgraph of Φ induced by $\mathcal{A}$, X^1 and X^2, while the $3m$ monochromatic vertices in the connectivity gadget replacing $\mathbf{T}$ are needed to replace the complete graphs induced in Φ by $\{T, c^{i,1}, c^{i,2}, c^{i,3}\}$, for each $i \in \{1, \ldots, m\}$. Details will follow. By **A1**, the graph resulting from such linearization is a collection of paths and isolated vertices. Furthermore, because of **A2**, all vertices simulating $\mathbf{T}$ (resp. $\mathbf{X}^1$, or $\mathbf{X}^2$) must be given the same colour in any 4-colouring of the resulting 3-empire graph.

Next, for each pair of complementary vertices $a, \overline{a} \in \mathcal{A}$, we define two empires, $\mathbf{a} = \{a_1, a_2, a_3\}$ and $\overline{\mathbf{a}} = \{\overline{a}_1, \overline{a}_2, \overline{a}_3\}$, and we replace the cycle $X^1, a, X^2, \overline{a}$ in Φ with a path $z_{\mathbf{X}^1}, a_1, z_{\mathbf{X}^2}, \overline{a}_1, z'_{\mathbf{X}^1}$ (distinct cycles replaced by paths using distinct vertices $z_{\mathbf{X}^1}, z'_{\mathbf{X}^1} \in Z(\mathbf{X}^1)$ and $z_{\mathbf{X}^2} \in Z(\mathbf{X}^2)$). We also replace the edge $\{a, \overline{a}\}$ with $\{a_2, \overline{a}_2\}$. As a result of these replacements, by the properties of the connectivity gadgets, empires $\mathbf{a}$ and $\overline{\mathbf{a}}$ in $P(\Phi)$ cannot take any of the colours assigned to the vertices in either $Z(\mathbf{X}^1)$ or $Z(\mathbf{X}^2)$ in any $(4, 3)$-colouring of $P(\Phi)$. To finish dealing with complementary vertices we linearize $\mathbf{a}$ (resp. $\overline{\mathbf{a}}$) using a copy of $A_{3,4,\mathrm{occ}(a)+2}$ (resp. $A_{3,4,\mathrm{occ}(\overline{a})+2}$).

Finally, for each $i \in \{1, \ldots, m\}$, the clique on $\{T, c^{i,1}, c^{i,2}, c^{i,3}\}$ is replaced by a copy of $B_{3,3}$ on the empires $\mathbf{c}^{i,1}, \mathbf{c}^{i,2}, \mathbf{c}^{i,3}$ and three vertices from $Z(\mathbf{T})$ (for different values of i three different monochromatic vertices are used). Also, for each i, j and vertex $\ell \in \mathcal{A}$ such that $\{c^{i,j}, \ell\} \in E(\Phi)$ the edge is replaced by one connecting $c_3^{i,j}$, the currently unused vertex from $\mathbf{c}^{i,j}$, and a vertex from $Z(\ell)$.

To see the correctness of the reduction, notice that when we first shrink each connectivity gadget and then each remaining empire of $P(\Phi)$ to a distinct (pseudo-)vertex, the graph obtained coincides with the initial formula graph. □

6. Conclusions

The results in this paper prove that linear forests are no easier than trees with respect to the (s, r)-colouring problem, even for small values of r. Thus perhaps long distance connections, rather than connectivity, are really instrumental to making the colouring problem hard.

Somewhat disappointingly, the constructions presented in this paper do not cover the case $r = 3$ and $s = 5$. More generally, for arbitrary values of r it seems difficult to prove hardness results when s is very close to $2r - 1$.

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