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Large-n_f Contributions to the Four-Loop Splitting Functions in QCD

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Abstract

We have computed the fourth-order n_f^2 contributions to all three non-singlet quark-quark splitting functions and their four n_f^3 flavour-singlet counterparts for the evolution of the parton distributions of hadrons in perturbative QCD with n_f effectively massless quark flavours. The analytic form of these functions is presented in both Mellin N-space and momentum-fraction x-space; the large-x and small-x limits are discussed. Our results agree with all available predictions derived from lower-order information. The large-x limit of the quark-quark cases provides the complete n_f^2 part of the four-loop cusp anomalous dimension which agrees with two recent partial computations.

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1 Introduction

In the past years the next-to-next-to-leading order (NNLO) corrections in perturbative QCD have been determined for many high-energy processes, see Refs. [1–8] for some recent calculations. For processes with initial-state protons, NNLO analyses require parton distributions evolved with the three-loop splitting functions [9, 10]. In some cases also the next-to-next-to-next-to-leading order (N^3LO) corrections are important, e.g., for quantities with a slow convergence of the perturbation series or for cases where a very high accuracy is required. An example of the former is Higgs production at proton-proton colliders [11, 12]. An example of the latter is the determination of the strong coupling constant α_s from the structure functions F_2 and F_3 in lepton-nucleon deep-inelastic scattering (DIS), see Ref. [13], for which the N^3LO coefficient functions have been obtained in Refs. [14, 15]. In principle N^3LO analyses of these processes require the four-loop splitting functions, although estimates of these functions via, for example, Padé approximants can be sufficient in some cases such as for DIS at large Bjorken-x.

At present a direct computation of the four-loop splitting functions $P_{\rm ik}^{(3)}(x)$ appears to be too difficult. Work on low-integer Mellin moments of these functions started ten years ago [16]; until recently only the $N = 2$ and $N = 4$ moments had been obtained of the quark+antiquark non-singlet splitting function $P_{\rm ns}^{(3)+}$ together with the $N = 3$ result for its quark–antiquark counterpart $P_{\rm ns}^{(3)-}$ [17–19]. Using FORCER [20, 21], a four-loop generalization of the well-known MINCER program [22, 23] for the parametric reduction of self-energy integrals, it is now possible to derive more moments in the same manner as in Refs. [24–26] at the third order in α_s. So far the moments up to $N = 6$ and $N = 4$ have been computed, respectively, for the non-singlet and singlet cases [27, 28], and computations up to $N = 8$ are feasible. Further conceptual and/or computational developments are required, however, in order to obtain sufficient information for the construction of approximate x-space expressions analogous to those at three loops in Ref. [29].

The situation is far more favourable for the contributions to the functions $P_{\rm ik}^{(3)}(x)$ which are leading (in the singlet case) or leading and sub-leading (in the non-singlet case) in the number n_f of effectively massless quark flavours. Here the harder four-loop diagram topologies do not contribute, and FORCER calculations above $N = 20$, and in some cases above $N = 40$, are possible. If suitably combined with information and expectations on the structure of these contributions in terms of harmonic sums [30, 31], these fixed-N results turn out to be sufficient to find and validate the analytic dependence of these parts of the four-loop splitting functions on N, and hence on x in terms of harmonic polylogarithms [32], by LLL-based techniques [33–35]. This approach has been used before, e.g. in Refs. [36, 37] for the three-loop transversity and helicity-difference splitting functions, and may be applicable to other four-loop quantities in the future. The present results include the n_f^2 part of the four-loop cusp anomalous dimension also obtained in Refs. [38–40].

The remainder of this article is organized as follows: in Section 2 we set up our notations and briefly discuss the diagram calculations and the LLL analyses of the resulting integer-N moments. The analytic results for the n_f^3 parts of $P_{\rm ik}^{(3)}$ and the n_f^2 parts of $P_{\rm ns}^{(3)}$ in N- and x-space are presented and discussed in Sections 3 and 4. We summarize our results in Section 5.

2 Notations and calculations

The renormalization-group evolution equations for the dependence of the parton momentum distributions $f_{\rm a} = u,\, \bar{u},\, d,\, \bar{d},\, \ldots,\, g$ of hadrons on the mass factorization scale μ_f,

$$\frac{d}{d\ln\mu_f^2}\, f_{\rm a}\left(x,\mu_f^2\right) \;=\; \int_x^1 \frac{dy}{y}\, P_{\rm ab}(y,\alpha_{\rm s})\, f_{\rm b}\left(\frac{x}{y},\mu_f^2\right) \,, \tag{2.1}$$

form a system of $2n_f{+}1$ coupled integro-differential equations. These equations can be turned into ordinary differential equations by a Mellin transformation,

$$f_{\rm a}\left(N,\mu_f^2\right) \;=\; \int_0^1 dx\, x^{N-1}\, f_{\rm a}\left(x,\mu_f^2\right) \,, \tag{2.2}$$

and decomposed into $2n_f{-}1$ scalar (non-singlet) equations for the combinations

$$q_{\rm ik}^{\pm} \;=\; q_{\rm i} \pm \bar{q}_{\rm i} - (q_{\rm k} \pm \bar{q}_{\rm k}) \,, \quad q_{\rm v} \;=\; \sum_{\rm i=1}^{n_f} (q_{\rm i} - \bar{q}_{\rm i}) \tag{2.3}$$

of quark distributions and the $2{\times}2$ flavour-singlet quark-gluon system

$$\frac{d}{d\ln\mu_f^2} \begin{pmatrix} q_{\rm s} \\ g \end{pmatrix} \;=\; \begin{pmatrix} P_{\rm qq} & P_{\rm qg} \\ P_{\rm gq} & P_{\rm gg} \end{pmatrix} \otimes \begin{pmatrix} q_{\rm s} \\ g \end{pmatrix} \,, \quad q_s \;=\; \sum_{\rm i=1}^{n_f} (q_{\rm i} + \bar{q}_{\rm i}) \tag{2.4}$$

by using the general properties of QCD such as $P_{{\rm gq}_{\rm i}} = P_{{\rm g}\bar{\rm q}_{\rm i}} = P_{\rm gq}$. Note that $P_{\rm qg} = 2n_f P_{{\rm q}_{\rm i}{\rm g}}$.

The splitting functions in Eqs. (2.1) admit an expansion in powers of $\alpha_{\rm s}$ which we write as

$$P_{\rm ij}(x,\alpha_{\rm s}) \;=\; \sum_{n=0} a_{\rm s}^{n+1} P_{\rm ij}^{(n)}(x) \quad \text{with} \quad a_{\rm s} \;=\; \alpha_{\rm s}(\mu_f^2)/(4\pi) \,, \tag{2.5}$$

i.e., we identify (without loss of information) the mass-factorization and the coupling-constant renormalization scales. The difference between the splitting functions $P_{\rm ns}^{+}$ and $P_{\rm ns}^{-}$ for the first two non-singlet combinations in Eq. (2.4) and the pure-singlet quark-quark splitting function

$$P_{\rm ps} \;=\; P_{\rm qq} - P_{\rm ns}^{+} \tag{2.6}$$

starts at the second order in $\alpha_{\rm s}$, the remaining difference

$$P_{\rm ns}^{\rm s} \;=\; P_{\rm ns}^{\rm v} - P_{\rm ns}^{-} \tag{2.7}$$

at the third order in $\alpha_{\rm s}$. To order $\alpha_{\rm s}^4$ the latter quantity is proportional to the cubic group invariant $d^{abc}d_{abc}/n_c$, while the other splitting functions can be expressed in terms of $C_F = 4/3$ and $C_A = n_c = 3$ in QCD and quartic group invariants; the latter do not occur with the powers of n_f that are considered in this article. The even-N or odd-N moments of the splitting functions are related to the anomalous dimensions $\gamma(N)$ of twist-2 spin-N operators in the light-cone operator product expansion (OPE), see, e.g., Refs. [41,42]; we use the standard convention $\gamma^{(n)}(N) \;=\; -P^{(n)}(N)$.

Our calculation of the four-loop splitting functions proceeds along the lines of Refs. [24–26]. The partonic DIS structure functions are mapped by the optical theorem to forward amplitudes

$$\text{probe}\,(q) + \text{parton}\,(p) \;\longrightarrow\; \text{probe}\,(q) + \text{parton}\,(p) \tag{2.8}$$

with $p^2 = 0$ and $q^2 = -Q^2 < 0$. Via a dispersion relation their coefficients of $(2p\cdot q/Q^2)^N$ then provide, depending on the structure function under consideration, the even-N or odd-N moments of the unfactorized partonic structure functions. These quantities are calculated in dimensional regularization with $D = 4-2\varepsilon$, and the n-loop splitting functions can be extracted from the coefficients of $\varepsilon^{-1}\alpha_{\rm s}^n$. For the even-$N$ determination of the splitting functions $P_{\rm ns}^{+}$ and $P_{\rm ik}$ in Eq. (2.4) we use the photon and the Higgs boson in the heavy-top limit as the probes. The splitting functions $P_{\rm ns}^{-}$ and $P_{\rm ns}^{\rm v}$ are determined from the odd-N vector−axial-vector interference structure function F_3.

The projection on the $N^{\rm th}$ power in the parton momentum p leads to self-energy integrals that can be solved by the FORCER program. The complexity of these integrals increases by four if N is increased by two. Together with the steep increase of the number of integrals with N, see the discussion of the harmonic projection in Ref. [23], this limits the number of moments that can be calculated. So far high values of N cannot be reached for the top-level 4-loop diagram topologies.

The raw diagram databases provided by QGRAF [43] are heavily manipulated by (T)FORM [44–46] programs to provide the best possible starting point for the main integral computations. As discussed in Ref. [47], one important step is the identification of ℓ-loop self-energy insertions, which reduces many n-loop diagrams to fewer $(n-\ell)$-loop diagrams in which one or more propagators have a non-integer power. For the large-n_f contributions under consideration in the article, genuine four-loop diagrams remain after this step only in the calculation of the $C_A n_f^3$ part of $P_{\rm qg}^{(3)}$, and these diagrams have a rather simple topology: in the notation of MINCER they are generalizations of the Y3 and O1 three-loop topologies. The hardest diagrams occur in the $C_A C_F n_f^2$ and $n_f^2\, d^{abc}d_{abc}/n_c$ non-singlet cases: these are three-loop BE topologies with a one-loop gluon propagator, see Fig. 2; the highest N calculated here for any of these is $N = 27$.

As far as they are known from fixed-order calculations [9, 10] and all-order resummations of leading large-n_f terms [48–50], the even-N or odd-N moments of the splitting functions (i.e. the anomalous dimensions) can be expressed in terms of simple denominators, $D_a^k = (N+a)^{-k}$ and harmonic sums [30, 31] with argument N which are recursively defined by

$$S_{\pm m}(N) \;=\; \sum_{i=1}^{N} \frac{(\pm 1)^i}{i^m} \tag{2.9}$$

and

$$S_{\pm m_1, m_2, \ldots, m_d}(N) \;=\; \sum_{i=1}^{N} \frac{(\pm 1)^i}{i^{m_1}}\, S_{m_2,\ldots,m_d}(i)\;. \tag{2.10}$$

The weight w of the harmonic sums is defined by the sum of the absolute values of the indices m_d. Sums up to $w = 2n-1$ occur in the n-loop anomalous dimensions, but no sums with an index -1. For terms with D_a^k and/or coefficients that include values ζ_m of the Riemann ζ-function (with $m \geq 3$, ζ_2 does not occur in these functions), the maximal weight of the sums is reduced by $k+m$.

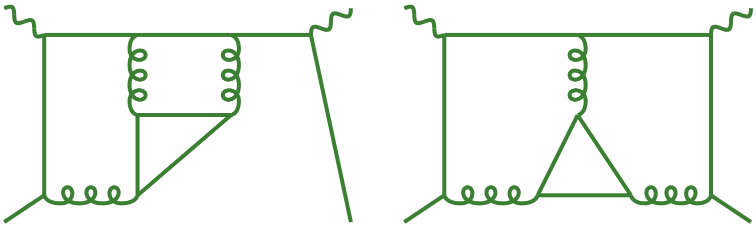

Figure 1: The three-loop gauge-boson–quark forward scattering diagrams with MINCER topology BE that contribute to $C_A C_F n_f$ part of the three-loop splitting functions for the quark$\pm$antiquark flavour differences in Eq. (2.3). The same diagrams, but with a one-loop insertion in one of the gluon lines, form the hardest part of the corresponding calculation of the four-loop $C_A C_F n_f^2$ contribution.

It is, of course, possible that other structures occur in the n-loop anomalous dimensions at $n \geq 4$ – already the three-loop DIS coefficient functions include terms where special combinations of sums are multiplied by low positive powers of N [14, 15]. However, one may expect this to happen at $n = 4$ only in the terms with low powers of n_f which receive contributions from generically new diagram topologies. Disregarding new structures and terms with $\zeta_{m\geq 3}$ which are much easier to fix from low-N results, a general ansatz for the n-loop anomalous dimensions then is

$$\gamma^{(n)}(N) \;=\; \sum_{w=0}^{2n+1} c_{00w}\, S_w(N) + \sum_{a} \sum_{k=1}^{2n+1} \sum_{w=0}^{2n+1-k} c_{akw}\, D_a^k\, S_w(N) \;, \tag{2.11}$$

where $S_w(N)$ is a shorthand for all harmonic sums with weight w and $S_0(N) \equiv 1$. The terms with c_{00w} only occur in the quark-quark and gluon-gluon splitting functions and are restricted by the known large-N structure of these functions [51–53]. In all cases the range of the sums is reduced for large-n_f contributions in a manner that can be inferred from the results at $n \leq 3$ and from the prime-factor decompositions of the denominators of the calculated moments.

Even so, Eq. (2.11) usually includes far too many coefficients for a direct determination from as many calculated moments. These coefficients, however, are integer modulo some predictable powers of $1/3$ at $n \leq 2$ [9, 10] and in Refs. [48–50]. Hence the systems of equations can by turned into Diophantine systems which require far fewer equations than unknowns. Given the present limitations of the calculation of diagrams with BE topology, this is still not sufficient for the n_f^2 contributions to the four-loop non-singlet splitting functions. However, these functions include additional structures that facilitate solving these equations with the calculable moments.

The crucial point for the determination of the n_f^2 parts of $\gamma_{\rm ns}^{(3)\pm}(N)$, already presented in [27], is to write its colour-factor decomposition in two ways,

$$\begin{aligned} \gamma_{\rm ns}^{(3)\pm}(N)\Big|_{n_f^2} \;&=\; C_F n_f^2 \left\{ C_F\, 2A^{(3)}(N) \qquad + (C_A - 2C_F) B_{\pm}^{(3)}(N) \right\} \\ &=\; C_F n_f^2 \left\{ C_F \left(2A^{(3)}(N) - 2B_{\pm}^{(3)}(N) \right) + C_A B_{\pm}^{(3)}(N) \right\} . \end{aligned} \tag{2.12}$$

$A^{(3)}(N)$ is the large-n_c result; it is the same for the even-N (+) and odd-N (−) cases and should include only non-alternating harmonic sums, i.e., only positive indices in Eqs. (2.9) and (2.10).

Once $A^{(3)}(N)$ is known, it is possible to determine $B_+^{(3)}(N)$ and $B_-^{(3)}(N)$ from the C_F parts in the second line of Eq. (2.12) which require only two-loop diagrams with one two-loop or two one-loop insertions. The corresponding three-loop coefficient, defined as in Eq. (2.12) but with n_f^1, reads

$$\begin{aligned} A^{(2)}(N) \;=\;& 8/3\,(-2S_{1,3}-4S_{2,2}-6S_{3,1}+6S_4+20/3\,(S_{1,2}+S_{2,1})-(11-\eta)\,S_3) \\ &+\left(-1331/27-256/9\eta+64/9\eta^2+8\eta^3+256/9D_1^2-16\zeta_3\right)S_1 \\ &+\left(1246/27-32/9\eta+16/3\eta^2-32/3D_1^2\right)S_2\;-\;17/2+323/54\eta \\ &-248/27\eta^2+8/9\eta^3-4\eta^4+2686/27D_1^2+152/9D_1^3+(12+8\eta)\,\zeta_3\;. \end{aligned} \qquad (2.13)$$

As below, the argument N of the sums is suppressed for brevity. η is defined in Eq. (2.15) below.

We have computed the even and odd moments up to $N=22$ for the determination and validation of $A^{(3)}(N)$, and the even-N or odd-N moments up to $N=42$ for $B_+^{(3)}(N)$ and $B_-^{(3)}(N)$. The Diophantine systems have been solved using the LLL-based program in Refs. [34, 35] at $N\le 18$ for $A^{(3)}(N)$ with 55 unknowns and at $N\le 40$ for $B_\pm^{(3)}(N)$ with 115 unknowns.

For the determination of the n_f^2 part of $\gamma_{\rm ns}^{(3)\rm s}(N)$ only the odd moments at $N\le 25$ were available; the result at $N=27$ was obtained afterwards and used as a check. As mentioned below Eq. (2.7), the function $\gamma_{\rm ns}^{\rm s}(N)$ only starts at order α_s^3. This 'leading order' $d^{abc}d_{abc}/n_c$ contribution reads

$$\begin{aligned} \gamma_{\rm ns}^{(2)\rm s}(N) \;=\;& 16n_f\,d^{abc}d_{abc}/n_c\Big\{(S_{-2,1}-S_{1,-2})(-4\eta-8\eta^2)-S_1S_{-2}(32\nu-20\eta-8\eta^2) \\ &+S_{-2}\,(32\nu-36\eta-28\eta^2-8\eta^3)+S_1\,(-32\nu+26\eta+56\eta^2+46\eta^3+12\eta^4) \\ &+S_3\,(-2\eta-4\eta^2)+32\nu-32\eta-60\eta^2-92\eta^3-44\eta^4-8\eta^5\Big\} \end{aligned} \qquad (2.14)$$

where the result has been rendered more compact by using the abbreviations

$$\eta\;\equiv\;\{N(N+1)\}^{-1}\;=\;D_0D_1\;,\quad \nu\;\equiv\;\{(N-1)(N+2)\}^{-1}\;=\;D_{-1}D_2\;. \qquad (2.15)$$

As the overall leading-order quantity $P_{\rm qq}^{(0)}$, the splitting function corresponding to Eq. (2.14) is the same for the present initial-state and the final-state (fragmentation distributions) evolution, cf. Refs. [54–56], and invariant under the x-space transformation $f(x)\to xf(1/x)$. The (combinations of) harmonic sums in Eq. (2.14) are 'reciprocity respecting' (RR), i.e., their Mellin inverses are invariant under the above transformation. The same holds for the combinations of denominators in Eq. (2.15). Except for S_1^2 and S_1^3 – products of RR sums lead to higher weight RR sums – all reciprocity-respecting sums to weight three occur in Eq. (2.14). The list of RR function to this weight has been given in Ref. [36] with a slightly different basis choice at $w=3$.

Like the overall NLO anomalous dimensions $\gamma_{\rm ns}^{(1)\pm}(N)$, the next-to-leading order $d^{abc}d_{abc}/n_c$ contribution $\gamma_{\rm ns}^{(3)\rm s}(N)$ is not reciprocity-respecting. However, and this is the crucial point, its RR-breaking part can be calculated from Eq. (2.14) according to the conjecture of Ref. [53]. For the n_f^2 contribution addressed here it is given by $-\frac{2}{3}n_f\frac{d}{dN}\gamma_{\rm ns}^{(2)\rm s}(N)$, where the differentiation can be carried out, for example, via the asymptotic expansion of the sums, see also Ref. [57]. That leaves an unknown reciprocity-respecting generalization of the form (2.14) with additional $w=4$ sums which can be chosen as

$$S_1^4\,,\ S_1S_3\,,\ S_{3,1}-S_{1,3}\,,\ S_{-2}^2 \tag{2.16}$$

and

$$S_{-4}\,,\ S_1^2S_{-2}\,,\ S_1(S_{-2,1}-S_{1,-2})\,,\ S_{-3,1}+S_{1,-3}-2S_{1,-2,1}\ . \tag{2.17}$$

Including also ν^2 terms, one arrives at a trial function with 79 coefficients, of which as many as 15 can be eliminated by imposing the existence of the first moment and the correct values (zero) for its ζ-function contributions, and 9 can be assumed to vanish (all contributions with S_1^3 and S_1^4). The remaining 56 coefficients have then been found using the 12 odd moments with $3 \le N \le 25$.

The correctness of the solution has been verified by the (non-ζ) value of the first moment and the result at $N = 27$. It is possible, though, to judge 'by inspection' whether a solution returned by the Diophantine equation solver [34, 35] is correct. For example, the above solution is returned as

```
A short solution is b[45]
= 160 372 816 -185 -494 238 52 -64 620 -616 308 112 0 -196 256 12 0 -30
208 -282 160 92 -136 96 64 4 0 16 -32 40 -64 0 0 -8 0 22 -32 2 0 24 -40
24 -4 24 -24 8 0 0 16 0 -16 12 4 0 0 0
```

where the numbers, ordered by overall weight and the weight of the sums (the details are not relevant here), are the remaining coefficients c_{akw} in Eq. (2.11) times $3/32$. The factor 3 ensures that the effective coefficients are integer, the factor $1/32$ removes some overall powers of 2 introduced by our choice for the expansion parameter $a_{\rm s}$ in Eq. (2.5).

A pattern such as the one above for the about 30 coefficients of the highest-weight functions, with larger and more random coefficients at the left (low-weight) end, is a hallmark of a correct solution. In fact, correct and incorrect solutions were correctly identified by inspection in all present calculations as well as in the preparation of Ref. [37].

Of the n_f^3 contributions to the singlet splitting functions in Eq. (2.4), only the case of $P_{\rm qg}^{(3)}$ is critical. Unlike the other three cases this function is suppressed by only two powers of n_f relative to the lowest-n_f term, recall the remark below Eq. (2.4), and includes contributions from sums up to weight four instead of weight three. Hence a considerably larger basis set is required in Eq. (2.11). At the same time the fixed-N calculations are harder for $P_{\rm qg}^{(3)}$ than for the other three cases, in particular for the $C_A n_f^3$ contribution, as already indicated on p. 4.

Yet, using reasonable assumptions based on the three-loop splitting function, we managed to find suitable functional forms with 101 unknown coefficients for the $C_F n_f^3$ part (with only positive-index sums but overall weight up to six) and 115 unknown coefficients for the $C_A n_f^3$ part (including alternating sums but an overall weight of five), which we were able to determine from the even moments $2 \le N \le 40$ in the former and $2 \le N \le 44$ in the latter case. Several higher moments were employed for the validation of the $C_F n_f^3$ result and the $C_A n_f^3$ coefficients were checked using $N = 46$. Some of the four-loop and three-loop $C_A n_f^3$ diagrams at $N > 40$ were calculated using an alternative approach for generalized Y and O MINCER topologies that avoids the harmonic projection [23]. This approach may be reported on later in a more general context.

3 Results in N-space

In this section we present the analytic expressions for the n_f^2 and n_f^3 contributions to the three non-singlet anomalous dimensions and the n_f^3 parts of their four flavour-singlet counterparts in the $\overline{\mathrm{MS}}$ scheme. As in Eqs. (2.13) and (2.14) above, all harmonic sums (2.9) and (2.10) have the argument N which is suppressed in the formulae for brevity.

The results for $\gamma_{\rm ns}^{(3)\pm}$ are presented in terms of the decomposition (2.12). The large-n_c part

$$
\begin{aligned}
A^{(3)}(N) \;=\; \tfrac{16}{27}\Big\{ & -12\,S_{1,3,1} + 6\,S_{1,4} - 12\,S_{2,3} - 24\,S_{3,2} - 30\,S_{4,1} + 36\,S_5 + 20\,S_{1,3} \\
& + 40\,S_{2,2} + 6\,S_{3,1}\Big(10+\eta\Big) - 3/2\,S_4\Big(53+2\eta\Big) - 38/3\,S_{1,2} - 38/3\,S_{2,1} \\
& + 1/3\,S_3\Big(287 - 12\eta + 18\eta^2 - 36\,D_1^2\Big) - 1/12\,S_2\Big(416\eta - 12\eta^2 - 144\eta^3 \\
& - 768\,D_1^2 + (1259 + 216\,\zeta_3)\Big) + 1/48\,S_1\Big(3392\eta - 3656\eta^2 + 432\eta^3 \\
& + 720\eta^4 - 3392\,D_1^2 - 576\,D_1^3 - 1728\,D_1^4 + (2119 + 2880\,\zeta_3 - 1296\,\zeta_4)\Big) \\
& + 1/96\Big(944\eta^3 - 864\eta^5 - 7088\,D_1^3 - 2736\,D_1^4 - 1728\,D_1^5 + 9\,(127 - 264\,\zeta_3 \\
& + 216\,\zeta_4) - 24\,(1705 + 72\,\zeta_3)\,D_1^2 - 2\,(2275 - 432\,\zeta_3)\,\eta^2 + (20681 - 2880\,\zeta_3 \\
& + 1296\,\zeta_4)\,\eta\Big)\Big\}
\end{aligned}
\tag{3.1}
$$

is the same for these two cases, while the contributions with the $1/n_c$-suppressed 'non-planar' colour factor $(C_A - 2C_F)$ are valid at even N for $B_+^{(3)}$ and odd N for $B_-^{(3)}$. These functions read

$$
\begin{aligned}
B_+^{(3)}(N) \;=\; \tfrac{32}{27}\Big\{ & -9\,S_{-5} - 12\,S_{-4,1} - 6\,S_{-3,-2} - 12\,S_{-3,1,1} + 6\,S_{1,-4} + 12\,S_{1,-3,1} \\
& + 12\,S_{1,-2,-2} + 24\,S_{1,-2,1,1} - 6\,S_{1,3,1} + 24\,S_{1,4} + 6\,S_{2,-3} + 12\,S_{2,-2,1} + 9\,S_{2,3} \\
& + 6\,S_{3,-2} - 3\,S_{3,2} - 6\,S_{4,1} + 9\,S_5 + S_{-4}\Big(20 - 3\eta\Big) + 2\,S_{-3,1}\Big(10 - 3\eta\Big) \\
& - 6\,S_{-2,-2}\,\eta - 12\,S_{-2,1,1}\,\eta - 20\,S_{1,-3} - 40\,S_{1,-2,1} - 30\,S_{1,3} - 20\,S_{2,-2} \\
& + S_{3,1}\Big(10 + 3\eta\Big) - 1/2\,S_4\Big(73 + 24\eta\Big) - 1/3\,S_{-3}\Big(19 - 30\eta + 9\eta^2 - 18\,D_1^2\Big) \\
& + 2\,S_{-2,1}\Big(10\eta - 3\eta^2 + 6\,D_1^2\Big) + 38/3\,S_{1,-2} + 1/12\,S_3\Big(619 + 180\eta \\
& - 54\eta^2 + 108\,D_1^2\Big) + 1/3\,S_{-2}\Big(8\eta + 39\eta^2 - 96\,D_1^2\Big) + 6\,S_{1,1}\Big(2\eta^2 + \eta^3\Big) \\
& + 1/48\,S_2\Big(144\eta^2 + 72\eta^3 - (1585 + 864\,\zeta_3)\Big) + 1/96\,S_1\Big(1584\eta - 3672\eta^2 \\
& + 720\eta^3 + 864\eta^4 - 1728\,D_1^2 - 1728\,D_1^3 - 2592\,D_1^4 + (923 + 5760\,\zeta_3 \\
& - 2592\,\zeta_4)\Big) - 1/192\Big(1392\eta^3 - 1584\eta^4 + 3168\,D_1^4 - 3\,(193 - 1584\,\zeta_3 \\
& + 1296\,\zeta_4) + 2\,(2447 - 864\,\zeta_3)\,\eta^2 + 4\,(7561 + 864\,\zeta_3)\,D_1^2 - (15077 - 5760\,\zeta_3 \\
& + 2592\,\zeta_4)\,\eta\Big)\Big\}
\end{aligned}
\tag{3.2}
$$

and

$$
\begin{aligned}
B_{-}^{(3)}(N) \;=\; \frac{32}{27}\Big\{ & -9\,S_{-5} - 12\,S_{-4,1} - 6\,S_{-3,-2} - 12\,S_{-3,1,1} + 6\,S_{1,-4} + 12\,S_{1,-3,1} \\
& + 12\,S_{1,-2,-2} + 24\,S_{1,-2,1,1} - 6\,S_{1,3,1} + 24\,S_{1,4} + 6\,S_{2,-3} + 12\,S_{2,-2,1} + 9\,S_{2,3} \\
& + 6\,S_{3,-2} - 3\,S_{3,2} - 6\,S_{4,1} + 9\,S_5 + S_{-4}\Big(20 - 3\eta\Big) + 2\,S_{-3,1}\Big(10 - 3\eta\Big) \\
& - 6\,S_{-2,-2}\,\eta - 12\,S_{-2,1,1}\,\eta - 20\,S_{1,-3} - 40\,S_{1,-2,1} - 30\,S_{1,3} - 20\,S_{2,-2} \\
& + S_{3,1}\Big(10 + 3\eta\Big) - 1/2\,S_4\Big(73 + 24\eta\Big) - 1/3\,S_{-3}\Big(19 - 30\eta + 9\eta^2 - 18\,D_1^2\Big) \\
& + 2\,S_{-2,1}\Big(10\eta - 3\eta^2 + 6\,D_1^2\Big) + 38/3\,S_{1,-2} + 1/12\,S_3\Big(619 + 180\eta - 54\eta^2 \\
& + 108\,D_1^2\Big) + 1/3\,S_{-2}\Big(8\eta + 3\eta^2 - 18\eta^3 - 96\,D_1^2\Big) - 6\,S_{1,1}\Big(2\eta^2 + \eta^3\Big) \\
& + 1/48\,S_2\Big(144\eta^2 + 72\eta^3 - (1585 + 864\,\zeta_3)\Big) - 1/96\,S_1\Big(432\eta - 1032\eta^2 \\
& + 240\eta^3 + 288\eta^4 - 576\,D_1^2 - 576\,D_1^3 - 864\,D_1^4 - (923 + 5760\,\zeta_3 - 2592\,\zeta_4)\Big) \\
& + 1/192\Big(7280\eta^3 - 336\eta^4 - 1728\eta^5 - 11136\,D_1^3 - 18144\,D_1^4 + 4608\,D_1^5 \\
& + 3\,(193 - 1584\,\zeta_3 + 1296\,\zeta_4) - 18\,(583 - 96\,\zeta_3)\,\eta^2 - 4\,(10489 + 864\,\zeta_3)\,D_1^2 \\
& + (25541 - 5760\,\zeta_3 + 2592\,\zeta_4)\,\eta\Big)\Big\}\,. \qquad (3.3)
\end{aligned}
$$

As for the complete corresponding three-loop quantities in Ref. [10], the difference between the odd-N result (3.3) and the even-N result (3.2) is much simpler than those expressions and given by

$$
\begin{aligned}
\delta B^{(3)}(N) \;=\; \frac{32}{27}\Big\{ & -6\,S_{-2}\Big(2\eta^2 + \eta^3\Big) - 12\,S_{1,1}\Big(2\eta^2 + \eta^3\Big) - S_1\Big(21\eta - 49\eta^2 \\
& + 10\eta^3 + 12\eta^4 - 24\,D_1^2 - 24\,D_1^3 - 36\,D_1^4\Big) + 1/6\Big(327\eta - 175\eta^2 + 271\eta^3 \\
& - 60\eta^4 - 54\eta^5 - 366\,D_1^2 - 348\,D_1^3 - 468\,D_1^4 + 144\,D_1^5\Big)\Big\}\,. \qquad (3.4)
\end{aligned}
$$

Finally the additional $n_f^2 a_s^4$ contribution to the evolution of the valence distribution, see Eq. (2.7), is

$$
\begin{aligned}
\gamma_{\rm ns}^{(3)\rm s}\big|_{n_f^2 d^{abc}d_{abc}/n_c}(N) \;=\; \frac{64}{3}\Big\{ & 2\Big[S_{-4} + 2\,S_{-3,1} + 2\,S_{1,-3} - 4\,S_{1,-2,1} - S_{1,3}\Big]\Big(8\nu - 5\eta \\
& - 2\eta^2\Big) - 8\Big[2\,S_{-2,-2} + 4\,S_{-2,1,1} - S_{-2,2}\Big]\Big(2\nu - \eta\Big) - 4\Big[4\,S_{1,1,-2} - S_{2,-2} \\
& + S_{3,1}\Big]\Big(4\nu - 3\eta - 2\eta^2\Big) + 2\,S_4\Big(16\nu - 11\eta - 6\eta^2\Big) - 2/3\,S_{-3}\Big(128\nu - 87\eta \\
& - 21\eta^2 + 6\eta^3 - 6\,D_1^2 + 24\,D_1^3 + 16\,D_2^2\Big) + 4/3\,S_{-2,1}\Big(88\nu - 57\eta - 21\eta^2 - 6\eta^3 \\
& - 12\,D_1^2 + 8\,D_2^2\Big) + 8/3\,S_{1,-2}\Big(44\nu - 42\eta - 21\eta^2 + 3\,D_1^2 + 12\,D_1^3 + 4\,D_2^2\Big) \\
& + S_3\Big(16\nu - 9\eta - 7\eta^2 - 6\eta^3 - 6\,D_1^2 - 8\,D_1^3\Big) - 1/3\,S_{-2}\Big(304\nu - 273\eta \\
& - 312\eta^2 - 84\eta^3 - 84\,D_1^2 + 24\,D_1^3 - 72\,D_1^4 + 32\,D_2^2\Big) - \Big[4\,S_{1,1} - S_2\Big]\Big(16\nu
\end{aligned}
$$

$$
\begin{aligned}
&- 13\eta - 28\eta^2 - 23\eta^3 - 6\eta^4\Big) + 1/6\, S_1 \Big(608\nu - 855\eta - 984\eta^2 - 972\eta^3 \\
&- 144\eta^4 + 24\eta^5 + 300 D_1^2 + 456 D_1^3 + 36 D_1^4 + 288 D_1^5 + 64 D_2^2\Big) \\
&- 2/3 \Big(104\nu + 96\eta - 261\eta^2 - 252\eta^3 - 54\eta^4 + 36\eta^5 + 12\eta^6 - 216 D_1^2 \\
&- 168 D_1^3 - 162 D_1^4 + 24 D_1^5 - 60 D_1^6 + 16 D_2^2\Big)\Big\}\,. \qquad (3.5)
\end{aligned}
$$

The leading large-n_f contribution is the same for the three types of non-singlet quark distributions in Eq. (2.3). It has been obtained to all orders in α_s in Ref. [48]. Our results agree with the corresponding fourth-order coefficient which in our notation reads, for even and odd N,

$$
\begin{aligned}
\gamma_{\rm ns}^{(3)}|_{n_f^3}(N) \;=\; & \frac{16}{81} C_F \Big\{ 6 S_4 - 10 S_3 - 2 S_2 - S_1 (2 - 12\zeta_3) + 131/16 \\
& - 9\zeta_3 - \eta\,(20 + 6\zeta_3) + 15\eta^2 - \eta^3 - 3\eta^4 + 24 D_1^2 + 6 D_1^4 \Big\}\,. \qquad (3.6)
\end{aligned}
$$

The new functions (3.1) – (3.5) are illustrated in Fig. 2. The results at non-integer values of N have been calculated by a numerical Mellin transformation of the x-space expressions in the next section; for the analytic continuation in N of the harmonic sums to weight five see also Ref. [57].

Up to terms suppressed by two powers of $1/N$, also the large-N behaviour of the three non-singlet anomalous dimension is the same with

$$
\gamma_{\rm ns}^{(n-1)}(N) \;=\; A_n (\ln N + \gamma_e) - B_n + C_n \,\frac{\ln N + \gamma_e}{N} - D_n + O\Big(N^{-2} \ln^{\ell} N\Big) \qquad (3.7)
$$

where γ_e is the Euler-Mascheroni constant. The coefficients A_n are relevant beyond the evolution of the parton distributions, since they are identical to the n-loop cusp anomalous dimensions [51]. The result at three loops can be found in Eq. (3.11) of Ref. [9], its n_f part was derived before in Refs. [58, 59]. Our new results (3.1) and (3.2) specify the n_f^2 coefficient of A_4. Together with the long-known n_f^3 result [48, 60] given by the large-N limit of Eq. (3.6) we obtain

$$
\begin{aligned}
A_4\Big|_{n_f^{a>1}} \;=\; & C_F C_A n_f^2 \left(\frac{923}{81} - \frac{608}{81}\zeta_2 + \frac{2240}{27}\zeta_3 - \frac{112}{3}\zeta_4\right) \\
& + C_F^2 n_f^2 \left(\frac{2392}{81} - \frac{640}{9}\zeta_3 + 32\zeta_4\right) - C_F n_f^3 \left(\frac{32}{81} - \frac{64}{27}\zeta_3\right). \qquad (3.8)
\end{aligned}
$$

The large-n_c limit of this result has also been derived in Ref. [38], and the $C_F^2 n_f^2$ part in Refs. [39, 40]. Hence all n_f^2 contributions in Eq. (3.8) are covered by two independent determinations. Since this result involves several coefficients in (3.1) and (3.2), this agreement can also be viewed as another verification of our determination of the all-N n_f^2 expressions for $\gamma_{\rm ns}^{(3)\pm}$. The n_f^2 part of the coefficient C_4 in Eq. (3.7) is found to be

$$
C_4\Big|_{n_f^2} \;=\; \frac{1216}{81}\, C_F^2 n_f^2 \;=\; \Big[(A_2)^2 + A_1 A_3\Big]_{n_f^2} \qquad (3.9)
$$

as conjectured in Ref. [53] – the first verification of this conjecture by a fourth-order calculation.

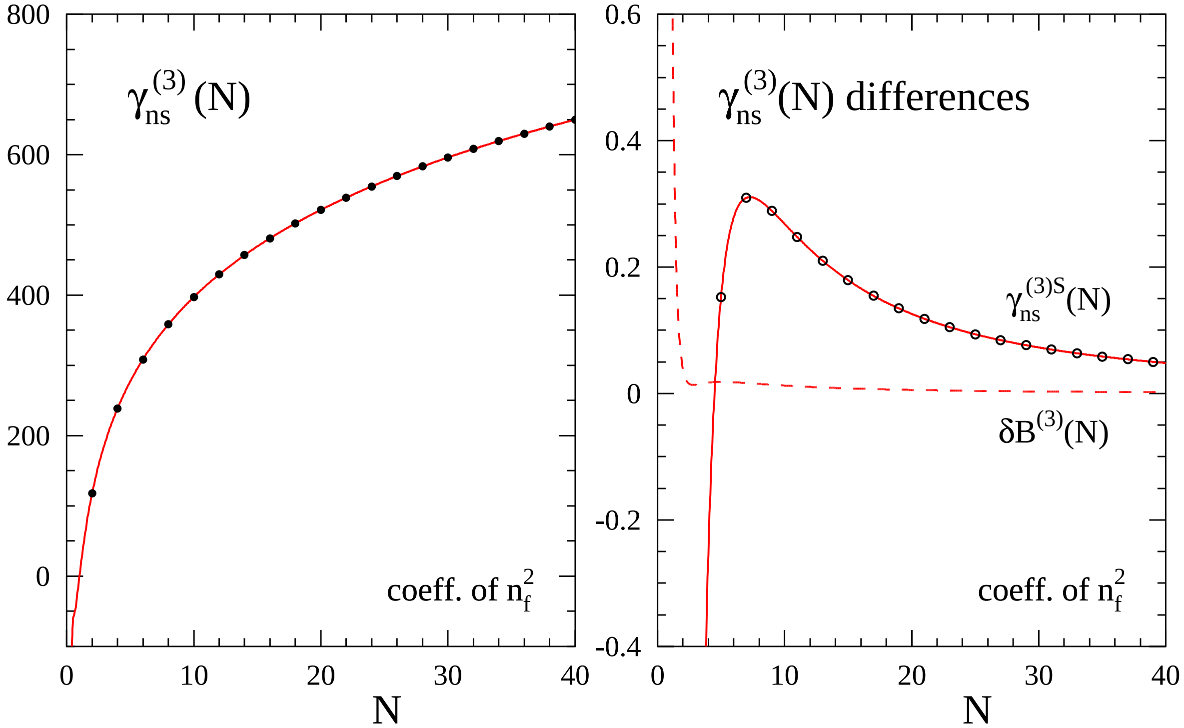

Figure 2: The n_f^2 parts of the anomalous dimensions $\gamma_{\rm ns}^{(3)+}(N)$ (left) and $\gamma_{\rm ns}^{(3)s}(N) = \gamma_{\rm ns}^{(3)v}(N) - \gamma_{\rm ns}^{(3)-}(N)$ (right). Their even-N (left) and odd-N (odd) moments computed using FORCER [20,21] are shown together with the numerical all-N curves. Also shown on the right, where we focus on $\gamma_{\rm ns}^{(3)s}$ at $N > 4$, is the difference $\delta B^{(3)}(N) = \gamma_{\rm ns}^{(3)-}(N) - \gamma_{\rm ns}^{(3)+}(N)$. Note the normalization of our expansion parameter $a_{\rm s}$ in Eq. (2.5).

We now turn to the leading large-n_f anomalous dimensions for the even-N flavour-singlet evolution (2.4), starting with the pure singlet contribution (2.6):

$$
\begin{aligned}
\gamma_{\rm ps}^{(3)}\big|_{n_f^3}(N) \;=\; & C_F \Big\{ -64/27\, S_{1,1,1} \Big(3D_0 - 6D_0^2 - 3D_1 - 6D_1^2 - 4D_2 + 4D_{-1}\Big) \\
& + 64/27\, S_{1,1} \Big(11D_0 - 13D_0^2 + 6D_0^3 - 17D_1 - 4D_1^2 + 12D_1^3 + 2D_2 + 8D_2^2 \\
& + 4D_{-1}\Big) - 32/81\, S_1 \Big(94D_0 - 98D_0^2 + 87D_0^3 - 18D_0^4 - 226D_1 + 100D_1^2 \\
& + 111D_1^3 - 90D_1^4 + 128D_2 + 88D_2^2 - 48D_2^3 + 4D_{-1}\Big) + 16/81 \Big(146D_0^3 \\
& - 87D_0^4 + 18D_0^5 - 54D_1^3 - 309D_1^4 + 198D_1^5 + 72D_2^2 - 176D_2^3 + 96D_2^4 \\
& - 4(1-18\zeta_3)D_{-1} + 2(26+27\zeta_3)D_0 - 2(59+54\zeta_3)D_0^2 \\
& + 4(91-18\zeta_3)D_2 - 2(206+27\zeta_3)D_1 + 2(215-54\zeta_3)D_1^2\Big)\Big\}\,. \qquad (3.10)
\end{aligned}
$$

As expected from the lower orders, the highest-weight sums in the four-loop off-diagonal contributions are proportional to the leading-order structures

$$
p_{\rm qg} \;=\; D_0 - 2D_1 + 2D_2 \quad \text{and} \quad p_{\rm gq} \;=\; 2D_{-1} - 2D_0 + D_1\,. \qquad (3.11)
$$

Using these abbreviations, the fourth-order leading-n_f parts of the gluon-quark and quark-gluon anomalous dimensions are given by

$$
\begin{aligned}
\gamma_{\rm qg}^{(3)}\big|_{n_f^3}(N) \; = \; & C_F \Bigg\{ 32/27 \Big[3\, S_4 - S_{1,1,1,1}\Big] p_{\rm qg} - 32/81\, S_{1,1,1} \Big(71\, D_0 - 30\, D_0^2 + 18\, D_0^3 \\
& - 115\, D_1 - 36\, D_1^3 + 42\, D_2 + 24\, D_2^2 - 8\, D_{-1}\Big) + 32/81 \Big[S_{1,2} + S_{2,1}\Big] \Big(81\, D_0 \\
& - 27\, D_0^2 + 18\, D_0^3 - 135\, D_1 - 36\, D_1^3 + 62\, D_2 + 24\, D_2^2 - 8\, D_{-1}\Big) \\
& + 32/81\, S_3 \Big(71\, D_0 - 27\, D_0^2 + 18\, D_0^3 - 109\, D_1 - 36\, D_1^3 + 36\, D_2 + 24\, D_2^2 - 8\, D_{-1}\Big) \\
& - 16/243\, S_{1,1} \Big(416\, D_0 - 102\, D_0^2 - 72\, D_0^3 - 1633\, D_1 + 90\, D_1^2 - 288\, D_1^3 - 216\, D_1^4 \\
& + 1174\, D_2 + 648\, D_2^2 + 288\, D_2^3 + 72\, D_{-1}\Big) - 32/243\, S_2 \Big(976\, D_0 - 891\, D_0^2 \\
& + 360\, D_0^3 - 216\, D_0^4 + 88\, D_1 - 459\, D_1^2 - 72\, D_1^3 + 540\, D_1^4 - 1101\, D_2 - 852\, D_2^2 \\
& - 432\, D_2^3 + 68\, D_{-1}\Big) - 16/729\, S_1 \Big(8634\, D_0^2 - 6822\, D_0^3 + 2430\, D_0^4 - 1620\, D_0^5 \\
& + 1125\, D_1^2 - 2070\, D_1^3 - 3456\, D_1^4 + 3240\, D_1^5 - 1812\, D_2^2 - 2448\, D_2^3 - 1728\, D_2^4 \\
& + 352\, D_{-1} + 24\,(427 + 27\,\zeta_3)\, D_1 - (763 + 648\,\zeta_3)\, D_2 - 12\,(802 + 27\,\zeta_3)\, D_0\Big) \\
& + 4/729 \Big(17370\, D_0^4 - 15012\, D_0^5 - 25992\, D_1^4 + 49464\, D_1^5 - 28512\, D_1^6 - 5280\, D_2^3 \\
& - 3456\, D_2^4 + 13824\, D_2^5 + 128\,(31 + 27\,\zeta_3)\, D_{-1} - 6\,(281 - 9936\,\zeta_3)\, D_1 \\
& + 72\,(635 - 18\,\zeta_3)\, D_1^2 - 54\,(835 + 144\,\zeta_3)\, D_0^3 + 24\,(959 - 432\,\zeta_3)\, D_2^2 \\
& - 6\,(1621 - 2592\,\zeta_3)\, D_1^3 + 24\,(1988 + 459\,\zeta_3)\, D_0^2 - 9\,(7037 + 3852\,\zeta_3)\, D_0 \\
& + 2\,(31649 - 14688\,\zeta_3)\, D_2\Big)\Bigg\} \\
& + C_A \Bigg\{ 32/27 \Big[4\, S_{-4} + S_{1,1,1,1} - S_{1,1,2} + S_{1,2,1} - S_{1,3} + S_{2,1,1} - S_{2,2} + S_{3,1} \\
& + 3\, S_4\Big] p_{\rm qg} - 128/81\, S_{-3} \Big(5\, D_0 - 7\, D_1 + 7\, D_2\Big) + 64/81 \Big[-S_{1,1,1} + S_{1,2} \\
& - S_{2,1}\Big] \Big(5\, D_0 - 10\, D_1 + 3\, D_1^2 + 10\, D_2 - 3\, D_2^2\Big) - 64/81\, S_3 \Big(5\, D_0 - 4\, D_1 \\
& - 3\, D_1^2 + 4\, D_2 + 3\, D_2^2\Big) + 16/243\, S_{-2} \Big(38\, D_0 - 10\, D_1 + 9\, D_1^2 + 28\, D_2\Big) \\
& - 4/243\, S_{1,1} \Big(316\, D_0 - 45\, D_0^2 + 144\, D_0^3 - 641\, D_1 - 354\, D_1^2 + 349\, D_2 \\
& + 792\, D_2^2 - 288\, D_2^3 - 104\, D_{-1}\Big) - 4/243\, S_2 \Big(468\, D_0 - 45\, D_0^2 + 144\, D_0^3 \\
& - 1659\, D_1 + 912\, D_1^2 - 576\, D_1^3 + 1277\, D_2 - 168\, D_2^2 + 288\, D_2^3 - 104\, D_{-1}\Big) \\
& - 2/729\, S_1 \Big(6354\, D_0^2 - 3258\, D_0^3 + 3456\, D_0^4 + 5298\, D_1^2 + 648\, D_1^3 - 5184\, D_1^4 \\
& + 15408\, D_2^2 + 16992\, D_2^3 - 3456\, D_2^4 - 128\, D_{-1} - 6\,(1895 + 864\,\zeta_3)\, D_1 \\
& - 3\,(2863 - 864\,\zeta_3)\, D_0 + (17447 + 5184\,\zeta_3)\, D_2\Big) + 2/243 \Big(554\, D_0^3 + 696\, D_0^4
\end{aligned}
$$

$$
\begin{aligned}
& + 432\, D_0^5 + 8508\, D_1^3 - 6816\, D_1^4 + 3168\, D_1^5 + 2720\, D_2^3 - 4608\, D_2^4 + 2304\, D_2^5 \\
& - 192\,(2 - 3\zeta_3)\, D_{-1} + 6\,(125 + 288\zeta_3)\, D_1 - 3\,(269 + 912\zeta_3)\, D_2 \\
& + 2\,(643 - 432\zeta_3)\, D_0^2 + 8\,(653 - 216\zeta_3)\, D_2^2 - (655 - 432\zeta_3)\, D_0 \\
& - 2\,(2399 + 864\zeta_3)\, D_1^2 \Big) \Big\}
\end{aligned}
\tag{3.12}
$$

and

$$
\begin{aligned}
\gamma^{(3)}_{\rm gq}\big|_{n_f^3}(N) \;=\; & C_F \Big\{ -64/27\, S_{1,1,1}\, p_{\rm gq} + 64/81\, S_{1,1} \Big(8\, p_{\rm gq} - 3\, D_1^2 \Big) \\
& - 64/81\, S_1 \Big(4\, p_{\rm gq} - 8\, D_1^2 + 3\, D_1^3 \Big) - 64/81 \Big(6\, p_{\rm gq}\,\zeta_3 + 4\, D_1^2 - 8\, D_1^3 + 3\, D_1^4 \Big) \Big\} \; .
\end{aligned}
\tag{3.13}
$$

Finally the corresponding contribution to the gluon-gluon anomalous dimension reads

$$
\begin{aligned}
\gamma^{(3)}_{\rm gg}\big|_{n_f^3}(N) \;=\; & C_F \Big\{ 64/27 \Big(3\, D_0 - 6\, D_0^2 - 3\, D_1 - 6\, D_1^2 - 4\, D_2 + 4\, D_{-1} \Big) \Big[S_{1,1,1} - S_{1,2} \\
& - S_{2,1} + S_3/2 \Big] + 64/81\, S_{1,1} \Big(57\, D_0 + 21\, D_0^2 + 18\, D_0^3 - 39\, D_1 + 12\, D_1^2 + 20\, D_2 \\
& - 38\, D_{-1} \Big) - 32/81\, S_2 \Big(42\, D_0 + 69\, D_0^2 + 18\, D_0^3 - 42\, D_1 + 69\, D_1^2 - 18\, D_1^3 \\
& + 70\, D_2 - 70\, D_{-1} \Big) - 32/243\, S_1 \Big(429\, D_0 + 276\, D_0^2 + 207\, D_0^3 + 54\, D_0^4 - 33\, D_1 \\
& - 30\, D_1^2 + 135\, D_1^3 - 54\, D_1^4 - 26\, D_2 - 370\, D_{-1} \Big) - 2/243 \Big(77 - 3360\, D_0^3 \\
& - 1656\, D_0^4 - 432\, D_0^5 - 3840\, D_1^3 + 3816\, D_1^4 - 1296\, D_1^5 - 1296\,(3 + \zeta_3)\, D_1 \\
& - 432\,(11 - 3\zeta_3)\, D_0 + 96\,(43 - 18\zeta_3)\, D_2 + 96\,(47 + 18\zeta_3)\, D_{-1} \\
& - 24\,(179 + 108\zeta_3)\, D_0^2 + 24\,(193 - 108\zeta_3)\, D_1^2 \Big) \Big\} \\
+ & C_A \Big\{ 4/81 \Big[-2\, S_{1,1} + S_2 \Big] \Big(33\, D_0 + 48\, D_0^2 - 33\, D_1 + 48\, D_1^2 + 52\, D_2 - 52\, D_{-1} \Big) \\
& + 4/243\, S_1 \Big(480\, D_0 + 456\, D_0^2 + 144\, D_0^3 - 480\, D_1 + 456\, D_1^2 - 144\, D_1^3 + 527\, D_2 \\
& - 527\, D_{-1} - 24\,(1 - 6\zeta_3) \Big) - 1/243 \Big(5 + 1380\, D_0^2 + 912\, D_0^3 + 288\, D_0^4 \\
& + 1380\, D_1^2 - 912\, D_1^3 + 288\, D_1^4 + 6\,(229 - 96\zeta_3)\, D_0 - 6\,(229 - 96\zeta_3)\, D_1 \\
& + 4\,(331 - 144\zeta_3)\, D_2 - 4\,(331 - 144\zeta_3)\, D_{-1} \Big) \Big\} \; .
\end{aligned}
\tag{3.14}
$$

The C_A part of Eq. (3.14), which is a non-singlet-type quantity and hence could be written in a more compact manner in terms of the quantities in Eq. (2.15), has been obtained already in Ref. [50]. Its leading large-N coefficient is related to that in Eq. (3.8) by the 'Casimir scaling' C_A/C_F. Moreover two linear combinations of Eq. (3.13) with Eq. (3.10) and the C_F part of Eq. (3.14) were derived in Ref. [49, 50]; our results agree also with those findings. Eq. (3.12) is entirely new.

The results (3.6), (3.10) and (3.12) – (3.14) and their continuations to non-integer N are illustrated in Figs. 3 and 4 for the normalization specified for their x-space counterparts in Eq. (2.5).

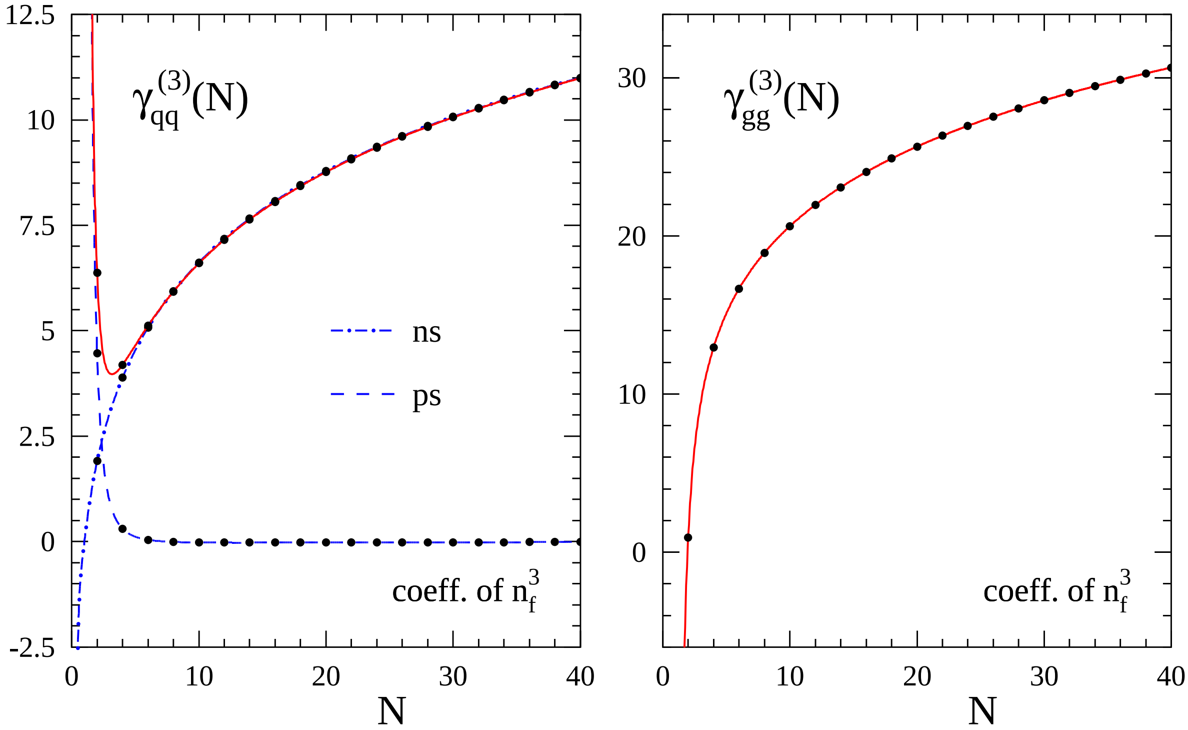

Figure 3: The n_f^3 parts of the 'diagonal' quark-quark and gluon-gluon four-loop anomalous dimensions. The analytically calculated even-N moments are shown together with their continuation calculated via a numerical Mellin transformation of the corresponding x-space expressions using the program of Ref. [61]. For the quark-quark case the non-singlet and pure-singlet contributions are displayed separately.

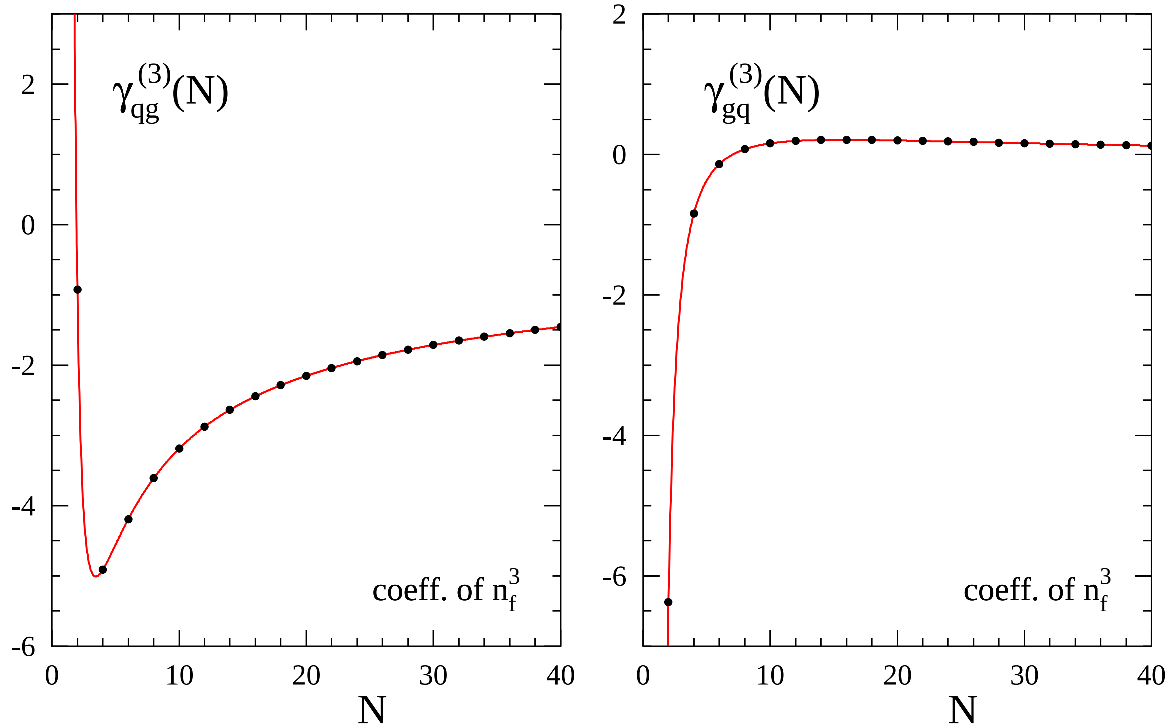

Figure 4: As Figure 3, but for the 'off-diagonal' gluon-quark and quark-gluon anomalous dimensions.

4 Results in x-space

The four-loop splitting functions $P_{\rm ik}^{(3)}(x)$ are obtained from the above N-space results by an inverse Mellin transformation which expresses these functions in terms of harmonic polylogarithms. This transformation can be performed by a completely algebraic procedure [32, 62] based on the fact that harmonic sums occur as coefficients of the Taylor expansion of harmonic polylogarithms.

Before we present our results, we recall the basic definitions [32]: The lowest-weight ($w = 1$) functions $H_m(x)$ are given by

$$H_0(x) \;=\; \ln x\;, \qquad H_{\pm 1}(x) \;=\; \mp \ln(1 \mp x)\;. \tag{4.1}$$

The higher-weight ($w \geq 2$) functions are recursively defined as

$$H_{m_1,...,m_w}(x) \;=\; \begin{cases} \dfrac{1}{w!}\ln^w x\;, & \text{if } m_1,...,m_w = 0,\ldots,0 \\[2mm] \displaystyle\int_0^x dz\, f_{m_1}(z)\, H_{m_2,...,m_w}(z)\;, & \text{otherwise} \end{cases} \tag{4.2}$$

with

$$f_0(x) \;=\; \frac{1}{x}\;, \qquad f_{\pm 1}(x) \;=\; \frac{1}{1 \mp x}\;. \tag{4.3}$$

For chains of indices ‘zero’ we employ the abbreviated notation

$$H_{\underbrace{0,\ldots,0}_{m},\pm 1,\underbrace{0,\ldots,0}_{n},\pm 1,\ldots}(x) \;=\; H_{\pm(m+1),\pm(n+1),\ldots}(x) \tag{4.4}$$

and suppress the argument x in all results below.

The splitting functions for the quark$\pm$antiquark flavour differences in Eq. (2.3) are expressed in a decomposition analogous to the first line of Eq. (2.12),

$$P_{\rm ns}^{(3)\pm}(x)\big|_{n_f^2} \;=\; C_F n_f^2 \left\{ 2C_F \widetilde{A}^{(3)} \;+\; (C_A - 2C_F)\widetilde{B}_{\pm}^{(3)} \right\} \tag{4.5}$$

with

$$\begin{aligned}
\widetilde{A}^{(3)}(x) \;=\; & -\frac{16}{9}\Big\{ p_{\rm qq}(x)\Big(6\,{\rm H}_{0,0,0,0} - {\rm H}_{1,0,0,0} - 2\,{\rm H}_{1,3} + 2\,{\rm H}_{2,0,0} + 4\,{\rm H}_{3,0} + 5\,{\rm H}_{4} \\
& + \frac{53}{4}\,{\rm H}_{0,0,0} + \frac{10}{3}\,{\rm H}_{1,0,0} + \frac{20}{3}\,{\rm H}_{2,0} + 10\,{\rm H}_{3} + \frac{287}{18}\,{\rm H}_{0,0} - 5\,{\rm H}_{0,0}\,\zeta_2 + \frac{19}{9}\,{\rm H}_{1,0} \\
& + 2\,{\rm H}_{1,0}\,\zeta_2 + \frac{19}{9}\,{\rm H}_{2} + \frac{1259}{72}\,{\rm H}_{0} + 6\,{\rm H}_{0}\,\zeta_3 - 10\,{\rm H}_{0}\,\zeta_2 + 2\,{\rm H}_{1}\,\zeta_3 - \frac{19}{9}\,\zeta_2 + \frac{40}{3}\,\zeta_3 \\
& - \frac{7}{2}\,\zeta_4 + \frac{2119}{288}\Big) + (1-x)\Big(2\,{\rm H}_{1,0,0} + \frac{29}{3}\,{\rm H}_{1,0} + \frac{23}{3}\,{\rm H}_{1}\Big) + x\Big(-3\,{\rm H}_{0,0,0,0} \\
& - \frac{37}{4}\,{\rm H}_{0,0,0} - 3\,{\rm H}_{2,0} - 3\,{\rm H}_{3} - \frac{188}{9}\,{\rm H}_{0,0} - \frac{35}{6}\,{\rm H}_{2} - \frac{2539}{48}\,{\rm H}_{0} + 3\,{\rm H}_{0}\,\zeta_2 + \frac{35}{6}\,\zeta_2 \\
& - 9\,\zeta_3 - \frac{5729}{72}\Big) - 3\,{\rm H}_{0,0,0,0} + \frac{7}{4}\,{\rm H}_{0,0,0} + 5\,{\rm H}_{2,0} + 7\,{\rm H}_{3} + \frac{232}{9}\,{\rm H}_{0,0} + \frac{27}{2}\,{\rm H}_{2} \\
& + \frac{8911}{144}\,{\rm H}_{0} - 7\,{\rm H}_{0}\,\zeta_2 - \frac{27}{2}\,\zeta_2 + 9\,\zeta_3 + \frac{5729}{72} + \delta(1-x)\Big(-\frac{127}{32} + \frac{1259}{36}\,\zeta_2 \\
& - \frac{233}{12}\,\zeta_3 - \frac{323}{12}\,\zeta_4 + 10\,\zeta_3\,\zeta_2 + 6\,\zeta_5\Big)\Big\}
\end{aligned} \tag{4.6}$$

and

$$
\begin{aligned}
\widetilde{B}_{+}^{(3)}(x) \;=\; -\frac{32}{9}\Big\{ & p_{\rm qq}(x)\Big(-\mathrm{H}_{-3,0} + \frac{3}{2}\mathrm{H}_{0,0,0,0} - 2\mathrm{H}_{1,-2,0} - 4\mathrm{H}_{1,0,0,0} - \mathrm{H}_{1,3} \\
& - \frac{3}{2}\mathrm{H}_{2,0,0} + \frac{1}{2}\mathrm{H}_{3,0} + \mathrm{H}_{4} + \frac{73}{12}\mathrm{H}_{0,0,0} - 5\mathrm{H}_{1,0,0} + \frac{5}{3}\mathrm{H}_{3} + \frac{619}{72}\mathrm{H}_{0,0} - 2\mathrm{H}_{0,0}\zeta_2 \\
& + \frac{1585}{288}\mathrm{H}_{0} + \frac{7}{2}\mathrm{H}_{0}\zeta_3 - \frac{10}{3}\mathrm{H}_{0}\zeta_2 - 2\mathrm{H}_{1}\zeta_3 - \frac{19}{18}\zeta_2 + 15\zeta_3 - 6\zeta_4 + \frac{923}{576}\Big) \\
& + \frac{1}{2}p_{\rm qq}(-x)\Big(2\mathrm{H}_{-3,0} + 2\mathrm{H}_{-2,0,0} - 4\mathrm{H}_{-2,2} + 2\mathrm{H}_{-1,0,0,0} + 8\mathrm{H}_{-1,2,1} - 4\mathrm{H}_{-1,3} \\
& - 3\mathrm{H}_{0,0,0,0} - 4\mathrm{H}_{3,1} + 4\mathrm{H}_{4} + \frac{20}{3}\mathrm{H}_{-2,0} + \frac{20}{3}\mathrm{H}_{-1,0,0} - \frac{40}{3}\mathrm{H}_{-1,2} - \frac{20}{3}\mathrm{H}_{0,0,0} \\
& + \frac{20}{3}\mathrm{H}_{3} + 4\mathrm{H}_{-2}\zeta_2 + \frac{38}{9}\mathrm{H}_{-1,0} + 2\mathrm{H}_{-1,0}\zeta_2 - \frac{19}{9}\mathrm{H}_{0,0} - 2\mathrm{H}_{0,0}\zeta_2 \\
& - 4\mathrm{H}_{-1}\zeta_3 + \frac{40}{3}\mathrm{H}_{-1}\zeta_2 - \mathrm{H}_{0}\zeta_3 - \frac{10}{3}\mathrm{H}_{0}\zeta_2 + \frac{19}{9}\zeta_2 - 10\zeta_3 + \frac{1}{2}\zeta_4\Big) \\
& + (1-x)\Big(-3\mathrm{H}_{1,0,0} + \mathrm{H}_{1,0} - 4\mathrm{H}_{1,1} + 14\mathrm{H}_{1}\Big) + (1+x)\Big(2\mathrm{H}_{-1,0,0} - 4\mathrm{H}_{-1,2} \\
& + \frac{1}{2}\mathrm{H}_{2,0} - 2\mathrm{H}_{2,1} + \frac{17}{3}\mathrm{H}_{-1,0} + 4\mathrm{H}_{-1}\zeta_2\Big) + x\Big(3\mathrm{H}_{-2,0} + \frac{7}{2}\mathrm{H}_{3} - \frac{211}{24}\mathrm{H}_{0,0} \\
& + \frac{31}{4}\mathrm{H}_{2} - \frac{139}{4}\mathrm{H}_{0} - \frac{1}{2}\mathrm{H}_{0}\zeta_2 - \frac{25}{12}\zeta_2 - 10\zeta_3 - \frac{1187}{48}\Big) + \mathrm{H}_{-2,0} + \frac{9}{2}\mathrm{H}_{3} \\
& + \frac{47}{8}\mathrm{H}_{0,0} + \frac{39}{4}\mathrm{H}_{2} + \frac{83}{4}\mathrm{H}_{0} - \frac{9}{2}\mathrm{H}_{0}\zeta_2 - \frac{39}{4}\zeta_2 + 6\zeta_3 + \frac{1187}{48} \\
& + \delta(1-x)\Big(-\frac{193}{192} + \frac{1585}{144}\zeta_2 - 10\zeta_3 - \frac{5}{2}\zeta_4 + \frac{15}{2}\zeta_3\zeta_2 - \frac{11}{4}\zeta_5\Big)\Big\}\,, \qquad (4.7)
\end{aligned}
$$

where we have used the abbreviation

$$
p_{\rm qq}(x) \;=\; 2\,(1-x)^{-1} - 1 - x\,. \qquad (4.8)
$$

All divergences for $x \to 1$ are to be read as plus-distributions. The second contribution to $P_{\rm ns}^{(3)-}(x)$ in Eq. (4.5) can be expressed via

$$
\widetilde{B}_{-}^{(3)}(x) \;=\; \widetilde{B}_{+}^{(3)}(x) + \delta\widetilde{B}^{(3)}(x) \qquad (4.9)
$$

and

$$
\begin{aligned}
\delta\widetilde{B}^{(3)}(x) \;=\; -\frac{32}{9}\Big\{ & p_{\rm qq}(-x)\Big(2\mathrm{H}_{-3,0} + 2\mathrm{H}_{-2,0,0} - 4\mathrm{H}_{-2,2} + 2\mathrm{H}_{-1,0,0,0} \\
& + 8\mathrm{H}_{-1,2,1} - 4\mathrm{H}_{-1,3} - 3\mathrm{H}_{0,0,0,0} - 4\mathrm{H}_{3,1} + 4\mathrm{H}_{4} + \frac{20}{3}\mathrm{H}_{-2,0} + \frac{20}{3}\mathrm{H}_{-1,0,0} \\
& - \frac{40}{3}\mathrm{H}_{-1,2} - \frac{20}{3}\mathrm{H}_{0,0,0} + \frac{20}{3}\mathrm{H}_{3} + 4\mathrm{H}_{-2}\zeta_2 + \frac{38}{9}\mathrm{H}_{-1,0} + 2\mathrm{H}_{-1,0}\zeta_2 \\
& - \frac{19}{9}\mathrm{H}_{0,0} - 2\mathrm{H}_{0,0}z2 + \frac{40}{3}\mathrm{H}_{-1}\zeta_2 - 4\mathrm{H}_{-1}\zeta_3 - \frac{10}{3}\mathrm{H}_{0}\zeta_2 - H(0)\,\zeta_3 \\
& + \frac{19}{9}\zeta_2 - 10\zeta_3 + \frac{1}{2}\zeta_4\Big) - (1-x)\Big(+8\mathrm{H}_{1,1} - \frac{61}{3}\mathrm{H}_{1} + \frac{277}{18}\Big) \\
& + (1+x)\Big(4\mathrm{H}_{-2,0} + 4\mathrm{H}_{-1,0,0} - 8\mathrm{H}_{-1,2} - 5\mathrm{H}_{0,0,0} - 4\mathrm{H}_{2,1} + 6\mathrm{H}_{3} - \frac{29}{2}\mathrm{H}_{0,0} \\
& + \frac{46}{3}\mathrm{H}_{-1,0} + \frac{41}{3}\mathrm{H}_{2} + 8\mathrm{H}_{-1}\zeta_2 - \frac{151}{9}\mathrm{H}_{0} - 3\mathrm{H}_{0}\zeta_2 - 4\zeta_3\Big) - (4+8x)\,\zeta_2\Big\}\,. \qquad (4.10)
\end{aligned}
$$

The inverse Mellin transform of Eq. (3.5), up to the conventional minus sign between the anomalous dimensions and splitting functions, is given by the rather lengthy expression

$$
\begin{aligned}
P_{\mathrm{ns}}^{(3)\mathrm{s}}\big|_{n_f^2\, d^{abc}d_{abc}/n_c} \;=\; & \frac{128}{3}\bigg\{ \Big(\frac{1}{x}-x^2\Big)\bigg(-\frac{16}{3}\,\mathrm{H}_{1,-2,0} + \frac{16}{3}\,\mathrm{H}_{1,0,0,0} + \frac{8}{3}\,\mathrm{H}_{1,1,0,0} + \frac{8}{3}\,\mathrm{H}_{1,3} \\
& - \frac{20}{3}\,\mathrm{H}_{1,0}\,\zeta_2 - \frac{16}{3}\,\mathrm{H}_{1,1}\,\zeta_2 - \frac{44}{3}\,\mathrm{H}_{1}\,\zeta_3 + \frac{40}{9}\,\mathrm{H}_{1}\,\zeta_2\bigg) + \Big(\frac{1}{x}+x^2\Big)\bigg(-\frac{8}{3}\,\mathrm{H}_{-1,-2,0} \\
& + \frac{32}{3}\,\mathrm{H}_{-1,-1,-1,0} - \frac{16}{3}\,\mathrm{H}_{-1,-1,0,0} - \frac{32}{3}\,\mathrm{H}_{-1,-1,2} - \frac{8}{3}\,\mathrm{H}_{-1,0,0,0} + \frac{8}{3}\,\mathrm{H}_{-1,2,0} \\
& + \frac{32}{3}\,\mathrm{H}_{-1,2,1} + \frac{16}{3}\,\mathrm{H}_{-1,3} + \frac{80}{9}\,\mathrm{H}_{-1,-1,0} - \frac{80}{9}\,\mathrm{H}_{-1,0,0} - \frac{80}{9}\,\mathrm{H}_{-1,2} + 16\,\mathrm{H}_{-1,-1}\,\zeta_2 \\
& - \frac{28}{3}\,\mathrm{H}_{-1,0}\,\zeta_2 - \frac{56}{3}\,\mathrm{H}_{-1}\,\zeta_3 + \frac{40}{3}\,\mathrm{H}_{-1}\,\zeta_2\bigg) + (1-x)\bigg(2\,\mathrm{H}_{-3,0,0} - 4\,\mathrm{H}_{-3,2} \\
& + 4\,\mathrm{H}_{-2,-2,0} - 16\,\mathrm{H}_{-2,-1,-1,0} + 4\,\mathrm{H}_{-2,-1,0,0} + 8\,\mathrm{H}_{-2,-1,2} + 2\,\mathrm{H}_{-2,0,0,0} - 4\,\mathrm{H}_{-2,3} \\
& + 13\,\mathrm{H}_{-2,0,0} + 8\,\mathrm{H}_{1,-2,0} + \mathrm{H}_{1,0,0,0} - \mathrm{H}_{1,1,0,0} + 2\,\mathrm{H}_{1,3} + 4\,\mathrm{H}_{-3}\,\zeta_2 - 16\,\mathrm{H}_{-2,-1}\,\zeta_2 \\
& + 6\,\mathrm{H}_{-2,0}\,\zeta_2 + \frac{77}{6}\,\mathrm{H}_{1,0,0} + 14\,\mathrm{H}_{-2}\,\zeta_3 + \frac{91}{6}\,\mathrm{H}_{1,0} + \mathrm{H}_{1,0}\,\zeta_2 + \frac{182}{3}\,\mathrm{H}_{1,1} - 4\,\mathrm{H}_{1,1}\,\zeta_2 \\
& - \frac{131}{36}\,\mathrm{H}_{1} + 10\,\mathrm{H}_{1}\,\zeta_3 + \frac{16}{3}\,\mathrm{H}_{1}\,\zeta_2\bigg) + (1+x)\bigg(6\,\mathrm{H}_{2,0,0,0} + 2\,\mathrm{H}_{2,1,0,0} + 4\,\mathrm{H}_{2,3} \\
& + 3\,\mathrm{H}_{3,0,0} + 3\,\mathrm{H}_{4,0} + 12\,\mathrm{H}_{4,1} + 2\,\mathrm{H}_{-1,-2,0} - 8\,\mathrm{H}_{-1,-1,-1,0} - 2\,\mathrm{H}_{-1,-1,0,0} \\
& - 4\,\mathrm{H}_{-1,-1,2} - \mathrm{H}_{-1,0,0,0} + 4\,\mathrm{H}_{-1,2,0} + 16\,\mathrm{H}_{-1,2,1} + 2\,\mathrm{H}_{-1,3} - \frac{32}{3}\,\mathrm{H}_{-1,-1,0} \\
& + \frac{7}{3}\,\mathrm{H}_{-1,0,0} - \frac{70}{3}\,\mathrm{H}_{-1,2} + \frac{41}{6}\,\mathrm{H}_{2,0} - 6\,\mathrm{H}_{2,0}\,\zeta_2 + \frac{82}{3}\,\mathrm{H}_{2,1} - 8\,\mathrm{H}_{2,1}\,\zeta_2 + \frac{155}{18}\,\mathrm{H}_{-1,0} \\
& - 5\,\mathrm{H}_{-1,0}\,\zeta_2 - 8\,\mathrm{H}_{2}\,\zeta_3 - 7\,\mathrm{H}_{-1}\,\zeta_3 + 18\,\mathrm{H}_{-1}\,\zeta_2\bigg) + x\bigg(-6\,\mathrm{H}_{5} - 16\,\mathrm{H}_{-3,0} \\
& + 32\,\mathrm{H}_{-2,-1,0} - 6\,\mathrm{H}_{-2,2} + \frac{7}{2}\,\mathrm{H}_{2,0,0} - \frac{9}{2}\,\mathrm{H}_{3,0} - 18\,\mathrm{H}_{3,1} - 15\,\mathrm{H}_{4} + \frac{64}{3}\,\mathrm{H}_{-2,0} \\
& - 12\,\mathrm{H}_{0,0,0} + 6\,\mathrm{H}_{0,0,0}\,\zeta_2 + \frac{85}{3}\,\mathrm{H}_{3} + 22\,\mathrm{H}_{-2}\,\zeta_2 - \frac{250}{9}\,\mathrm{H}_{0,0} + 15\,\mathrm{H}_{0,0}\,\zeta_2 - \frac{382}{9}\,\mathrm{H}_{2} \\
& + 16\,\mathrm{H}_{2}\,\zeta_2 - \frac{41}{4}\,\mathrm{H}_{0} + \frac{25}{2}\,\mathrm{H}_{0}\,\zeta_4 - \frac{85}{3}\,\mathrm{H}_{0}\,\zeta_2 + 14\,\zeta_3\,\zeta_2 + \frac{382}{9}\,\zeta_2 - 10\,\zeta_3 - \frac{275}{4}\,\zeta_4 \\
& + \frac{850}{9}\bigg) + x^2\bigg(\frac{8}{3}\,\mathrm{H}_{-3,0} - \frac{16}{3}\,\mathrm{H}_{-2,-1,0} + \frac{16}{3}\,\mathrm{H}_{-2,2} + \frac{8}{3}\,\mathrm{H}_{0,0,0,0} - \frac{8}{3}\,\mathrm{H}_{2,0,0} \\
& - \frac{8}{3}\,\mathrm{H}_{3,0} - \frac{32}{3}\,\mathrm{H}_{3,1} - \frac{8}{3}\,\mathrm{H}_{4} - \frac{80}{9}\,\mathrm{H}_{-2,0} + \frac{80}{9}\,\mathrm{H}_{0,0,0} + \frac{80}{9}\,\mathrm{H}_{3} - 8\,\mathrm{H}_{-2}\,\zeta_2 \\
& + \frac{16}{3}\,\mathrm{H}_{0,0}\,\zeta_2 + \frac{8}{3}\,\mathrm{H}_{2}\,\zeta_2 + 20\,\mathrm{H}_{0}\,\zeta_3 - \frac{160}{9}\,\mathrm{H}_{0}\,\zeta_2 - \frac{200}{9}\,\zeta_3 + \frac{19}{3}\,\zeta_4\bigg) - 4\,\mathrm{H}_{0,0,0,0,0} \\
& - 2\,\mathrm{H}_{5} + 14\,\mathrm{H}_{-3,0} - 28\,\mathrm{H}_{-2,-1,0} + 2\,\mathrm{H}_{-2,2} - 12\,\mathrm{H}_{0,0,0,0} + \frac{11}{2}\,\mathrm{H}_{2,0,0} + \frac{1}{2}\,\mathrm{H}_{3,0} \\
& + 2\,\mathrm{H}_{3,1} - 22\,\mathrm{H}_{4} - 2\,\mathrm{H}_{-2,0} - \frac{26}{3}\,\mathrm{H}_{0,0,0} + 2\,\mathrm{H}_{0,0,0}\,\zeta_2 - 5\,\mathrm{H}_{3} - 16\,\mathrm{H}_{-2}\,\zeta_2 \\
& - \frac{400}{9}\,\mathrm{H}_{0,0} - 12\,\mathrm{H}_{0,0}\,\zeta_3 + 36\,\mathrm{H}_{0,0}\,\zeta_2 - \frac{109}{9}\,\mathrm{H}_{2} + 14\,\mathrm{H}_{2}\,\zeta_2 - \frac{725}{9}\,\mathrm{H}_{0} - \frac{5}{2}\,\mathrm{H}_{0}\,\zeta_4 \\
& + 32\,\mathrm{H}_{0}\,\zeta_3 + 3\,\mathrm{H}_{0}\,\zeta_2 + 30\,\zeta_3\,\zeta_2 + \frac{373}{18}\,\zeta_2 - \frac{125}{3}\,\zeta_3 - 13\,\zeta_4 - 38\,\zeta_5 - \frac{850}{9}\bigg\}
\end{aligned}
\tag{4.11}
$$

where our normalization of the colour factor is $d^{abc}d_{abc}/n_c = 5/18$ in QCD; for use with third-order results note the discussion below Eq. (30) in Ref. [63]. Finally the common leading large-n_f contribution to the N^{3}LO evolution of all three types of quark distributions in Eq. (2.3) reads

$$P_{\rm ns}^{(3)}(x)\Big|_{n_f^3} = \frac{32}{9}\,C_F\,\Big\{ p_{\rm qq}(x)\Big(-\frac{1}{6}\,{\rm H}_{0,0,0} - \frac{5}{18}\,{\rm H}_{0,0} + \frac{1}{18}\,{\rm H}_0 + \frac{1}{3}\,\zeta_3 - \frac{1}{18}\Big) + x\Big(\frac{1}{3}\,{\rm H}_{0,0} + \frac{13}{18}\,{\rm H}_0 + \frac{1}{6}\Big) - \frac{1}{3}\,{\rm H}_{0,0} - \frac{13}{18}\,{\rm H}_0 - \frac{1}{6} + \delta(1-x)\Big(-\frac{131}{288} + \frac{1}{9}\,\zeta_2 + \frac{19}{18}\,\zeta_3 - \frac{1}{3}\,\zeta_4\Big)\Big\}\,. \tag{4.12}$$

Also the large-x limit is the same for the three non-singlet splitting functions. It is given by

$$P_{\rm ns}^{(n-1)\pm,{\rm v}}(x) = \frac{A_n}{(1-x)_+} + B_n\,\delta(1-x) + C_n\,\ln(1-x) + D_n + O\Big((1-x)\ln^\ell(1-x)\Big) \tag{4.13}$$

in terms of the same constants as in Eq. (3.7), i.e., the $n_f^{a>1}$ contributions to A_4 and C_4 have been given in Eqs. (3.8) and (3.9). The coefficients B_4 can be read of from Eqs. (4.6), (4.7) and (4.12). The difference between $P_{\rm ns}^{-}$ and $P_{\rm ns}^{+}$ and the splitting function (2.7) are suppressed by two powers of $(1-x)$ with respect to the leading term in Eq. (4.13).

The non-singlet splitting functions include double-logarithmic small-x contributions up to $\ln^{2\ell}x$ at N$^\ell$LO. The coefficients of these leading-logarithmic (LL) parts of $P_{\rm ns}^{\pm}$ have long been known to all orders [64, 65]; the presence of a $\ln^4 x$ term in $P_{\rm ns}^{\rm s}$ at NNLO had not been predicted before the three-loop calculation in Ref. [9]. Contributions where k powers of (C_A, C_F) in the colour factor are replaced by n_f^k are suppressed by k powers of $\ln x$ relative to the overall leading logarithms. Hence we expect terms up to $\ln^4 x$ and $\ln^5 x$, respectively, in $P_{\rm ns}^{(3)\pm}$ and $P_{\rm ns}^{(3){\rm s}}$ at n_f^2. Indeed we find

$$P_{\rm ns}^{(3)+}|_{n_f^2} = \ln^4 x\Big(\frac{4}{9}\,C_F^2\Big) + \ln^3 x\Big(\frac{152}{27}\,C_F^2 + \frac{44}{27}\,C_F C_A\Big) + \ln^2 x\Big(\frac{16}{81}\,[134+9\,\zeta_2]\,C_F^2 + \frac{4}{27}\,[161-36\,\zeta_2]\,C_F C_A\Big) + \ln x\Big(\frac{8}{81}\,[967+72\,\zeta_2]\,C_F^2 + \frac{1}{81}\,[7561-2736\,\zeta_2+864\,\zeta_3]\,C_F C_A\Big)\,, \tag{4.14}$$

$$P_{\rm ns}^{(3)-}|_{n_f^2} = \ln^4 x\Big(\frac{4}{9}\,C_F C_A - \frac{4}{9}\,C_F^2\Big) + \ln^3 x\Big(\frac{692}{81}\,C_F C_A - \frac{664}{81}\,C_F^2\Big) + \ln^2 x\Big(\frac{4}{81}\,[1081-36\,\zeta_2]\,C_F C_A - \frac{16}{27}\,[55+9\,\zeta_2]\,C_F^2\Big) + \ln x\Big(\frac{1}{27}\,[4131-304\,\zeta_2+384\,\zeta_3]\,C_F C_A - \frac{8}{81}\,[241+384\,\zeta_2+72\,\zeta_3]\,C_F^2\Big) \tag{4.15}$$

and

$$P_{\rm ns}^{(3){\rm s}}|_{n_f^2 d^{abc}d_{abc}/n_c} = -\frac{64}{45}\,\ln^5 x - \frac{64}{3}\,\ln^4 x - \frac{128}{9}\,(3-\zeta_2)\,\ln^3 x - \frac{256}{3}\,(14-9\,\zeta_2+3\,\zeta_3)\,\ln^2 x - \frac{64}{3}\,(138+26\,\zeta_2-64\,\zeta_3+5\,\zeta_4)\,\ln x \tag{4.16}$$

up to constants and terms vanishing for $x \to 0$. The corresponding limit of Eq. (4.12) reads

$$P_{\rm ns}^{(3)}|_{C_F n_f^3} = -\frac{8}{81}\,\ln^3 x - \frac{88}{81}\,\ln^2 x - \frac{64}{27}\,\ln x + O(1)\,. \tag{4.17}$$

The analytic structure of the LL resummations is very different for $P_{\rm ns}^{+}$ and $P_{\rm ns}^{-}$ with [64,65]

$$P_{\rm ns,LL}^{+}(N,a_{\rm s}) \;=\; \frac{N}{2}\left\{1-\left(1-\frac{8a_{\rm s}C_F}{N^2}\right)^{-1/2}\right\} \tag{4.18}$$

and

$$P_{\rm ns,LL}^{-}(N,a_{\rm s}) \;=\; \frac{N}{2}\left\{1-\left(1-\frac{8a_{\rm s}C_F}{N^2}\left[1-\frac{8a_{\rm s}n_c}{N}\frac{d}{dN}\ln\left(e^{z^2/4}D_{-1/[2n_c^2]}(z)\right)\right]\right)^{-1/2}\right\} \tag{4.19}$$

where $z=N(2a_{\rm s}N_c)^{-1/2}$, and $D_p(z)$ denotes a parabolic cylinder function [66]. The expansion of Eq. (4.19) in powers of $a_{\rm s}$ is an asymptotic expansion, in contrast to Eq. (4.18). The difference between the two expansions vanishes in the large-n_c limit.

An extension of these resummations to next-to-leading logarithmic (NLL) accuracy and beyond is known so far only for the former case — for the $x^{2n}\ln^{\ell}x$ terms at $n\geq 0$; for the $x^{2n+1}\ln^{\ell}x$ terms the roles of $P_{\rm ns}^{+}$ and $P_{\rm ns}^{-}$ are interchanged in this respect. A determination of the N$^{\ell}$LL terms on the basis of N$^{\ell}$LO information is possible from the D-dimensional structure of the unfactorized expressions, analogous to the case of the final-state splitting functions and coefficient functions in semi-inclusive annihilation [67,68]. The first term in Eq. (4.15), an overall NNLL contribution, agrees with the result in Eq. (4.6) of Ref. [69] after $\alpha_{\rm s}$-expansion and Mellin inversion. For details and results on the singlet cases and coefficient functions see Ref. [70].

A generalization of the equation underlying Eq. (4.18) to all powers of $\ln x$, i.e., the terms with $1/N^{a>1}$ in the expansion about $N=0$, has been suggested in Ref. [71] as

$$P_{\rm ns}^{+}(N,a_{\rm s})\,\left(P_{\rm ns}^{+}(N,a_{\rm s})-N+\beta(a_{\rm s})/a_{\rm s}\right) \;=\; O(1)$$
$$\text{up to terms with}\quad \zeta_2(C_A-2C_F)\,, \tag{4.20}$$

where $\beta(a_{\rm s})=-\beta_0\,a_{\rm s}^2-\beta_1\,a_{\rm s}^3-\ldots$ with $\beta_0=11/3\,C_A-2/3\,n_f$ is the beta function of QCD; the terms including β_2 [72,73] enter the four-loop evaluation of Eq. (4.20). This evaluation indeed reproduces Eq. (4.14) except for the $\zeta_2(C_A-2C_F)$ contributions — note that there are typos in Eq. (25) and (26) of Ref. [71] — as well as the corresponding terms up to overall NNLL accuracy resulting from Eq. (4.6) of Ref. [69].

The n_f^2 contributions to the three non-singlet splitting functions are illustrated In Figs. 5 and 6 on linear and logarithmic scales in x. The latter have been extended to $x=10^{-6}$ in order to include the onset of the steep small-x rise of all these functions. The difference (4.10) between the n_f^2 parts of $P_{\rm ns}^{(3)-}$ and $P_{\rm ns}^{(3)+}$ is numerically irrelevant except at very small x. The $n_f^2\,d^{abc}d_{abc}/n_c$ difference between $P_{\rm ns}^{(3)\rm v}$ and $P_{\rm ns}^{(3)-}$, on the other hand, is non-negligible up to $x\simeq 0.5$.

At asymptotically small values of x, the behaviour of these functions is given by their respective leading $\ln^4 x$ and $\ln^5 x$ logarithms in Eqs. (4.14) – (4.16). As shown in the figures, though, the onset of the resulting steep rise towards $x=0$ is delayed to $x\approx 10^{-5}$ by the effect of the non-leading logarithms. In fact, even at the lowest x-values shown here a relevant approximation for $P_{\rm ns}^{+}$ and $P_{\rm ns}^{-}$ is obtained only if all $\ln x$ terms are taken into account. The situation is more favourable for $P_{\rm ns}^{\rm s}$ but, unlike for the three-loop contribution [9] to this function, also here the leading logarithmic result is totally different from the actual function at all physically sensible values of x.

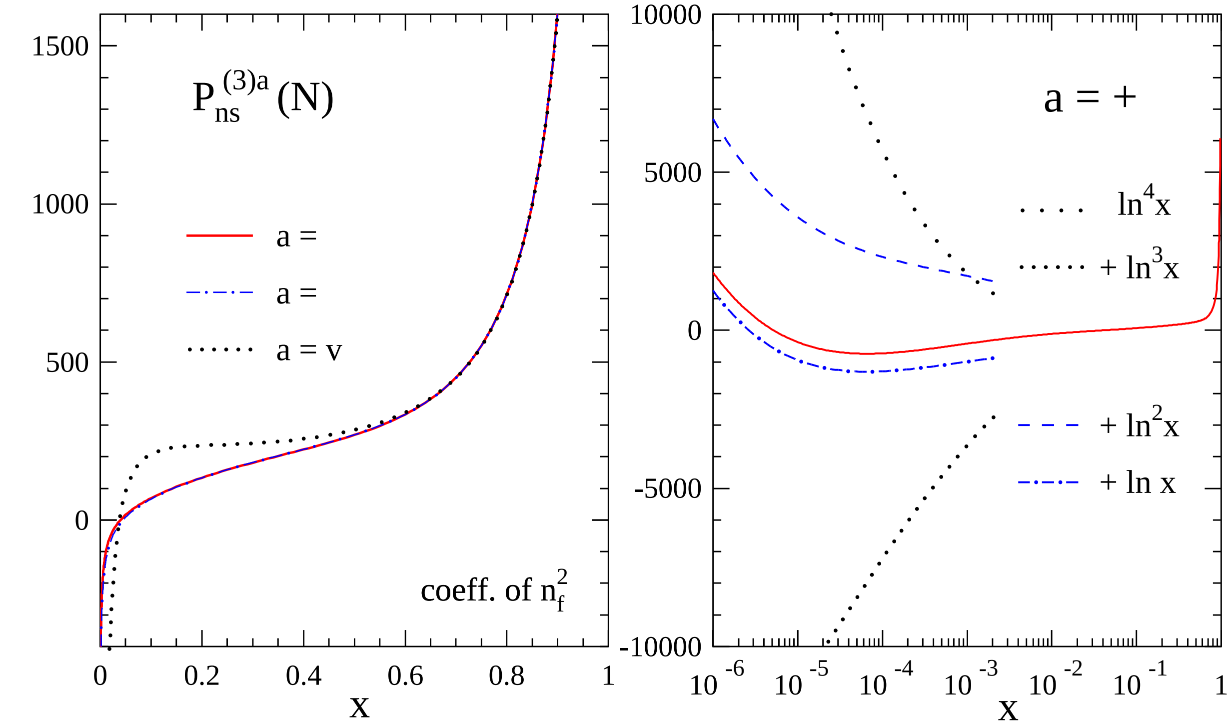

Figure 5: Left: the n_f^2 parts of the four-loop splitting functions for the evolution of the combinations (2.3) of quark and anti-quark distributions given by Eqs. (4.5) – Eqs. (4.11). Right: the small-x behaviour of this contribution to $P_{\rm ns}^{+}$, compared to its successive approximations by the small-x logarithms in Eq. (4.14).

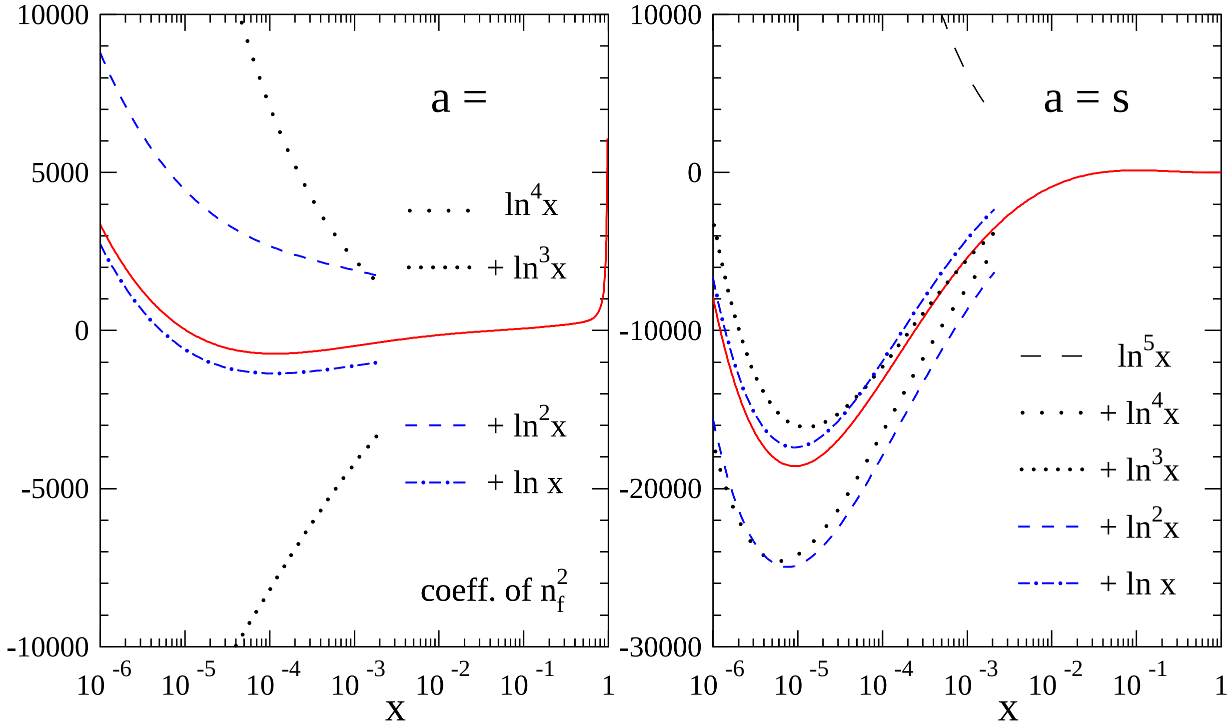

Figure 6: As the right panel of Figure 5, but for the splitting functions $P_{\rm ns}^{-}$ (left) and $P_{\rm ns}^{\rm s}$ (right). Due to Eq. (2.5) all numbers have to be divided by $(4\pi)^4 \simeq 25000$ for an expansion in powers of $\alpha_{\rm s}$.

The x-space splitting functions corresponding to the flavour-singlet anomalous dimensions in Eqs. (3.10) – (3.14) are given by

$$
\begin{aligned}
P^{(3)}_{\rm ps}(x)\Big|_{n_f^3} \;=\; & \frac{32}{9}\,{\color{blue}C_F}\,\Big\{\frac{1}{x}\Big(\frac{8}{3}\,\mathrm{H}_{1,1,1}+\frac{4}{9}\,\mathrm{H}_1-4\,\zeta_3+\frac{2}{3}\Big)+(1+x)\Big(-\mathrm{H}_{0,0,0,0}+4\,\mathrm{H}_{2,1,1} \\
& -4\,\mathrm{H}_{3,1}+2\,\mathrm{H}_4-\frac{29}{6}\,\mathrm{H}_{0,0,0}+\frac{29}{3}\,\mathrm{H}_3-\frac{73}{9}\,\mathrm{H}_{0,0}-2\,\mathrm{H}_{0,0}\,\zeta_2-4\,\mathrm{H}_0\,\zeta_3-\frac{29}{3}\,\mathrm{H}_0\,\zeta_2 \\
& -5\,\zeta_4\Big)+x\Big(-2\,\mathrm{H}_{1,1,1}-\frac{14}{3}\,\mathrm{H}_{2,1}+14\,\mathrm{H}_{1,1}+\frac{2}{9}\,\mathrm{H}_2-\frac{11}{9}\,\mathrm{H}_0-\frac{166}{9}\,\mathrm{H}_1-\frac{2}{9}\,\zeta_2 \\
& -2\,\zeta_3+\frac{38}{9}\Big)+x^2\Big(-\frac{8}{3}\,\mathrm{H}_{1,1,1}+\frac{8}{3}\,\mathrm{H}_{2,1}-4\,\mathrm{H}_{1,1}-\frac{76}{9}\,\mathrm{H}_2+\frac{64}{9}\,\mathrm{H}_0+\frac{68}{9}\,\mathrm{H}_1 \\
& +\frac{76}{9}\,\zeta_2+\frac{4}{3}\,\zeta_3-\frac{14}{9}\Big)+2\,\mathrm{H}_{1,1,1}-\frac{26}{3}\,\mathrm{H}_{2,1}-10\,\mathrm{H}_{1,1}+\frac{98}{9}\,\mathrm{H}_2-\frac{59}{9}\,\mathrm{H}_0+\frac{94}{9}\,\mathrm{H}_1 \\
& -\frac{98}{9}\,\zeta_2-4\,\zeta_3-\frac{10}{3}\Big\}\;, \qquad (4.21)
\end{aligned}
$$

$$
\begin{aligned}
P^{(3)}_{\rm qg}(x)\Big|_{n_f^3} \;=\; & \frac{32}{9}\,{\color{blue}C_F}\,\Big\{\frac{1}{x}\Big(\frac{8}{9}\,\mathrm{H}_{1,0,0}-\frac{8}{9}\,\mathrm{H}_{1,1,0}-\frac{8}{9}\,\mathrm{H}_{1,1,1}-\frac{8}{9}\,\mathrm{H}_{1,2}-\frac{92}{27}\,\mathrm{H}_{1,0}+\frac{4}{9}\,\mathrm{H}_{1,1} \\
& +\frac{284}{81}\,\mathrm{H}_1+\frac{8}{9}\,\mathrm{H}_1\,\zeta_2-\frac{16}{3}\,\zeta_3+\frac{136}{81}\Big)+(1-2x)\Big(-2\,\mathrm{H}_{3,0,0}+2\,\mathrm{H}_{3,1,0}+2\,\mathrm{H}_{3,1,1} \\
& +2\,\mathrm{H}_{3,2}-8\,\mathrm{H}_{4,0}-10\,\mathrm{H}_5+\mathrm{H}_{1,0,0,0}+\frac{1}{3}\,\mathrm{H}_{1,1,1,1}+10\,\mathrm{H}_{0,0,0}\,\zeta_2-2\,\mathrm{H}_3\,\zeta_2 \\
& +6\,\mathrm{H}_{0,0}\,\zeta_3-4\,\mathrm{H}_0\,\zeta_4-2\,\mathrm{H}_1\,\zeta_3-2\,\zeta_3\,\zeta_2-2\,\zeta_5\Big)+x\Big(-\frac{163}{3}\,\mathrm{H}_{0,0,0,0}-\frac{2}{3}\,\mathrm{H}_{2,1,1} \\
& +\frac{8}{3}\,\mathrm{H}_{3,0}-\frac{16}{3}\,\mathrm{H}_{3,1}+16\,\mathrm{H}_4-\frac{538}{9}\,\mathrm{H}_{0,0,0}+\frac{109}{9}\,\mathrm{H}_{1,0,0}-15\,\mathrm{H}_{1,1,0}-\frac{121}{9}\,\mathrm{H}_{1,1,1} \\
& -15\,\mathrm{H}_{1,2}-32\,\mathrm{H}_{2,0}-\frac{130}{9}\,\mathrm{H}_{2,1}-\frac{265}{9}\,\mathrm{H}_3+\frac{341}{36}\,\mathrm{H}_{0,0}-16\,\mathrm{H}_{0,0}\,\zeta_2-\frac{346}{27}\,\mathrm{H}_{1,0} \\
& -\frac{2029}{54}\,\mathrm{H}_{1,1}-\frac{1262}{27}\,\mathrm{H}_2+\frac{1426}{27}\,\mathrm{H}_0-\frac{10}{3}\,\mathrm{H}_0\,\zeta_3+\frac{265}{9}\,\mathrm{H}_0\,\zeta_2+\frac{323}{27}\,\mathrm{H}_1+15\,\mathrm{H}_1\,\zeta_2 \\
& +\frac{1262}{27}\,\zeta_2-\frac{973}{9}\,\zeta_3-6\,\zeta_4-\frac{31627}{324}\Big)+x^2\Big(10\,\mathrm{H}_{0,0,0,0}+2\,\mathrm{H}_{1,0,0,0}+\frac{2}{3}\,\mathrm{H}_{1,1,1,1} \\
& +\frac{8}{3}\,\mathrm{H}_{2,0,0}-\frac{8}{3}\,\mathrm{H}_{2,1,0}-2\,\mathrm{H}_{2,1,1}-\frac{8}{3}\,\mathrm{H}_{2,2}+\frac{40}{3}\,\mathrm{H}_{3,0}+\frac{8}{3}\,\mathrm{H}_{3,1}+\frac{40}{3}\,\mathrm{H}_4 \\
& -\frac{128}{3}\,\mathrm{H}_{0,0,0}-4\,\mathrm{H}_{1,0,0}+\frac{62}{9}\,\mathrm{H}_{1,1,0}+\frac{16}{3}\,\mathrm{H}_{1,1,1}+\frac{62}{9}\,\mathrm{H}_{1,2}-\frac{82}{3}\,\mathrm{H}_{2,0}-\frac{28}{3}\,\mathrm{H}_{2,1} \\
& -\frac{158}{9}\,\mathrm{H}_3+\frac{359}{9}\,\mathrm{H}_{0,0}-\frac{40}{3}\,\mathrm{H}_{0,0}\,\zeta_2+\frac{151}{3}\,\mathrm{H}_{1,0}+\frac{785}{27}\,\mathrm{H}_{1,1}+\frac{547}{27}\,\mathrm{H}_2+\frac{8}{3}\,\mathrm{H}_2\,\zeta_2 \\
& +\frac{9425}{162}\,\mathrm{H}_0-\frac{28}{3}\,\mathrm{H}_0\,\zeta_3+\frac{158}{9}\,\mathrm{H}_0\,\zeta_2+\frac{7547}{162}\,\mathrm{H}_1-4\,\mathrm{H}_1\,\zeta_3-\frac{62}{9}\,\mathrm{H}_1\,\zeta_2-\frac{547}{27}\,\zeta_2 \\
& +\frac{122}{9}\,\zeta_3+12\,\zeta_4+\frac{821}{324}\Big)+\frac{139}{6}\,\mathrm{H}_{0,0,0,0}-3\,\mathrm{H}_{2,0,0}+3\,\mathrm{H}_{2,1,0}+\frac{10}{3}\,\mathrm{H}_{2,1,1}+3\,\mathrm{H}_{2,2} \\
& -\frac{40}{3}\,\mathrm{H}_{3,0}-\frac{4}{3}\,\mathrm{H}_{3,1}-15\,\mathrm{H}_4+\frac{965}{36}\,\mathrm{H}_{0,0,0}-\frac{71}{9}\,\mathrm{H}_{1,0,0}+9\,\mathrm{H}_{1,1,0}+\frac{71}{9}\,\mathrm{H}_{1,1,1} \\
& +9\,\mathrm{H}_{1,2}-33\,\mathrm{H}_{2,0}+\frac{17}{9}\,\mathrm{H}_{2,1}-\frac{379}{9}\,\mathrm{H}_3+\frac{2473}{36}\,\mathrm{H}_{0,0}+15\,\mathrm{H}_{0,0}\,\zeta_2-\frac{952}{27}\,\mathrm{H}_{1,0} \\
& +\frac{232}{27}\,\mathrm{H}_{1,1}-\frac{1415}{27}\,\mathrm{H}_2-3\,\mathrm{H}_2\,\zeta_2+\frac{2104}{27}\,\mathrm{H}_0+\frac{20}{3}\,\mathrm{H}_0\,\zeta_3+\frac{379}{9}\,\mathrm{H}_0\,\zeta_2-\frac{1640}{27}\,\mathrm{H}_1 \\
& -9\,\mathrm{H}_1\,\zeta_2+\frac{1415}{27}\,\zeta_2+\frac{499}{18}\,\zeta_3-10\,\zeta_4+\frac{58277}{648}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{32}{9}\,C_A\Big\{\frac{1}{x}\Big(\frac{13}{27}\,\mathrm{H}_{1,0}-\frac{13}{27}\,\mathrm{H}_{1,1}-\frac{47}{81}\,\mathrm{H}_{1}-\frac{4}{3}\,\zeta_3-\frac{14}{81}\Big)+(1-2x)\Big(\mathrm{H}_{1,0,0,0}\\
&+\frac{1}{3}\,\mathrm{H}_{1,1,0,0}-\frac{1}{3}\,\mathrm{H}_{1,1,1,0}-\frac{1}{3}\,\mathrm{H}_{1,1,1,1}+\frac{1}{3}\,\mathrm{H}_{1,1,2}+\frac{1}{3}\,\mathrm{H}_{1,2,0}+\frac{1}{3}\,\mathrm{H}_{1,2,1}-\frac{1}{3}\,\mathrm{H}_{1,3}\\
&-\frac{2}{3}\,\mathrm{H}_{3,0}+\frac{2}{3}\,\mathrm{H}_{3,1}-\frac{8}{3}\,\mathrm{H}_{4}-\frac{181}{72}\,\mathrm{H}_{3}+\frac{8}{3}\,\mathrm{H}_{0,0}\,\zeta_2+\frac{1}{3}\,\mathrm{H}_{1,0}\,\zeta_2-\frac{1}{3}\,\mathrm{H}_{1,1}\,\zeta_2\\
&+\frac{8}{3}\,\mathrm{H}_{1}\,\zeta_3\Big)+x\Big(-\frac{8}{3}\,\mathrm{H}_{-1,0,0,0}+\frac{2}{3}\,\mathrm{H}_{0,0,0,0}-\frac{1}{6}\,\mathrm{H}_{-2,0}-\frac{28}{9}\,\mathrm{H}_{-1,0,0}-8\,\mathrm{H}_{0,0,0}\\
&-\frac{14}{9}\,\mathrm{H}_{1,0,0}-\frac{14}{9}\,\mathrm{H}_{1,1,0}-\frac{14}{9}\,\mathrm{H}_{1,1,1}+\frac{14}{9}\,\mathrm{H}_{1,2}+\frac{4}{3}\,\mathrm{H}_{2,0}-\frac{5}{4}\,\mathrm{H}_{2,1}-\frac{5}{27}\,\mathrm{H}_{-1,0}\\
&-\frac{493}{36}\,\mathrm{H}_{0,0}+\frac{131}{24}\,\mathrm{H}_{1,0}-\frac{1121}{216}\,\mathrm{H}_{1,1}-\frac{29}{6}\,\mathrm{H}_{2}-\frac{2449}{216}\,\mathrm{H}_{0}-\frac{28}{3}\,\mathrm{H}_{0}\,\zeta_3-\frac{187}{36}\,\mathrm{H}_{0}\,\zeta_2\\
&-\frac{67}{6}\,\mathrm{H}_{1}-\frac{14}{9}\,\mathrm{H}_{1}\,\zeta_2+\frac{251}{54}\,\zeta_2-\frac{169}{36}\,\zeta_3-\frac{17}{3}\,\zeta_4-\frac{35987}{1296}\Big)+x^2\Big(-\frac{8}{3}\,\mathrm{H}_{-1,0,0,0}\\
&+2\,\mathrm{H}_{1,0,0,0}+\frac{2}{3}\,\mathrm{H}_{1,1,0,0}-\frac{2}{3}\,\mathrm{H}_{1,1,1,0}-\frac{2}{3}\,\mathrm{H}_{1,1,1,1}+\frac{2}{3}\,\mathrm{H}_{1,1,2}+\frac{2}{3}\,\mathrm{H}_{1,2,0}+\frac{2}{3}\,\mathrm{H}_{1,2,1}\\
&-\frac{2}{3}\,\mathrm{H}_{1,3}-\frac{28}{9}\,\mathrm{H}_{-1,0,0}+\frac{25}{3}\,\mathrm{H}_{0,0,0}+\frac{14}{9}\,\mathrm{H}_{1,0,0}+\frac{14}{9}\,\mathrm{H}_{1,1,0}+\frac{14}{9}\,\mathrm{H}_{1,1,1}-\frac{14}{9}\,\mathrm{H}_{1,2}\\
&+\frac{13}{9}\,\mathrm{H}_{2,0}-\frac{13}{9}\,\mathrm{H}_{2,1}+\frac{65}{9}\,\mathrm{H}_{3}-\frac{14}{27}\,\mathrm{H}_{-1,0}-\frac{3293}{216}\,\mathrm{H}_{0,0}-\frac{797}{216}\,\mathrm{H}_{1,0}+\frac{2}{3}\,\mathrm{H}_{1,0}\,\zeta_2\\
&+\frac{829}{216}\,\mathrm{H}_{1,1}-\frac{2}{3}\,\mathrm{H}_{1,1}\,\zeta_2-\frac{2387}{216}\,\mathrm{H}_{2}+\frac{8861}{1296}\,\mathrm{H}_{0}-\frac{65}{9}\,\mathrm{H}_{0}\,\zeta_2+\frac{20549}{1296}\,\mathrm{H}_{1}+\frac{16}{3}\,\mathrm{H}_{1}\,\zeta_3\\
&+\frac{14}{9}\,\mathrm{H}_{1}\,\zeta_2+\frac{2387}{216}\,\zeta_2+\frac{67}{9}\,\zeta_3+\frac{18079}{648}\Big)-\frac{4}{3}\,\mathrm{H}_{-1,0,0,0}-\mathrm{H}_{0,0,0,0}-\frac{20}{9}\,\mathrm{H}_{-1,0,0}\\
&+\frac{29}{18}\,\mathrm{H}_{0,0,0}+\frac{10}{9}\,\mathrm{H}_{1,0,0}+\frac{10}{9}\,\mathrm{H}_{1,1,0}+\frac{10}{9}\,\mathrm{H}_{1,1,1}-\frac{10}{9}\,\mathrm{H}_{1,2}-\frac{5}{24}\,\mathrm{H}_{2,0}+\frac{5}{24}\,\mathrm{H}_{2,1}\\
&-\frac{19}{27}\,\mathrm{H}_{-1,0}-\frac{277}{216}\,\mathrm{H}_{0,0}-\frac{13}{6}\,\mathrm{H}_{1,0}+\frac{79}{54}\,\mathrm{H}_{1,1}-\frac{353}{72}\,\mathrm{H}_{2}+\frac{539}{216}\,\mathrm{H}_{0}-\frac{4}{3}\,\mathrm{H}_{0}\,\zeta_3\\
&+\frac{181}{72}\,\mathrm{H}_{0}\,\zeta_2-\frac{295}{48}\,\mathrm{H}_{1}+\frac{10}{9}\,\mathrm{H}_{1}\,\zeta_2+\frac{353}{72}\,\zeta_2+\frac{8}{9}\,\zeta_3+\frac{1}{2}\,\zeta_4+\frac{3341}{1296}\Big\}\,, \qquad (4.22)
\end{aligned}
$$

$$
\begin{aligned}
P^{(3)}_{\mathrm{gq}}(x)\Big|_{n_f^3} &= \frac{32}{9}\,C_F\Big\{\frac{1}{x}\Big(\frac{4}{3}\,\mathrm{H}_{1,1,1}-\frac{20}{9}\,\mathrm{H}_{1,1}-\frac{4}{9}\,\mathrm{H}_{1}+\frac{8}{3}\,\zeta_3-\frac{4}{9}\Big)+x\Big(\frac{2}{3}\,\mathrm{H}_{1,1,1}\\
&-\frac{16}{9}\,\mathrm{H}_{1,1}+\frac{8}{9}\,\mathrm{H}_{1}+\frac{4}{3}\,\zeta_3\Big)-\frac{4}{3}\,\mathrm{H}_{1,1,1}+\frac{20}{9}\,\mathrm{H}_{1,1}+\frac{4}{9}\,\mathrm{H}_{1}-\frac{8}{3}\,\zeta_3+\frac{4}{9}\Big\} \qquad (4.23)
\end{aligned}
$$

and

$$
\begin{aligned}
P^{(3)}_{\mathrm{gg}}(x)\Big|_{n_f^3} &= \frac{32}{9}\,C_F\Big\{(\frac{1}{x}-x^2)\Big(-\frac{4}{3}\,\mathrm{H}_{1,0,0}-\frac{8}{3}\,\mathrm{H}_{1,1,0}-\frac{8}{3}\,\mathrm{H}_{1,1,1}-\frac{8}{3}\,\mathrm{H}_{1,2}+\frac{46}{9}\,\mathrm{H}_{1,0}\\
&+\frac{8}{3}\,\mathrm{H}_{1}\,\zeta_2+4\zeta_3\Big)+\frac{1}{x}\Big(\frac{52}{9}\,\mathrm{H}_{1,1}-\frac{142}{27}\,\mathrm{H}_{1}-\frac{34}{27}\Big)+(1-x)\Big(-\mathrm{H}_{1,0,0}-2\,\mathrm{H}_{1,1,0}\\
&-2\,\mathrm{H}_{1,1,1}-2\,\mathrm{H}_{1,2}-2\,\mathrm{H}_{1,0}+2\,\mathrm{H}_{1}\,\zeta_2\Big)+(1+x)\Big(-\mathrm{H}_{0,0,0,0}-2\,\mathrm{H}_{2,0,0}-4\,\mathrm{H}_{2,1,0}\\
&-4\,\mathrm{H}_{2,1,1}-4\,\mathrm{H}_{2,2}-2\,\mathrm{H}_{3,0}-4\,\mathrm{H}_{3,1}-2\,\mathrm{H}_{4}+\frac{14}{3}\,\mathrm{H}_{2,1}+2\,\mathrm{H}_{0,0}\,\zeta_2+4\,\mathrm{H}_{2}\,\zeta_2\\
&+8\,\mathrm{H}_{0}\,\zeta_3-5\,\zeta_4\Big)+x\Big(\frac{29}{6}\,\mathrm{H}_{0,0,0}+\frac{29}{3}\,\mathrm{H}_{2,0}+\frac{29}{3}\,\mathrm{H}_{3}-\frac{40}{9}\,\mathrm{H}_{0,0}+6\,\mathrm{H}_{1,1}+\frac{64}{9}\,\mathrm{H}_{2}\\
&-\frac{11}{2}\,\mathrm{H}_{0}-\frac{29}{3}\,\mathrm{H}_{0}\,\zeta_2+\frac{5}{9}\,\mathrm{H}_{1}-\frac{64}{9}\,\zeta_2+2\,\zeta_3-\frac{43}{27}\Big)+x^2\Big(\frac{4}{3}\,\mathrm{H}_{0,0,0}+\frac{8}{3}\,\mathrm{H}_{2,0}\\
&+\frac{8}{3}\,\mathrm{H}_{2,1}+\frac{8}{3}\,\mathrm{H}_{3}-\frac{46}{9}\,\mathrm{H}_{0,0}-\frac{16}{9}\,\mathrm{H}_{1,1}-\frac{16}{9}\,\mathrm{H}_{2}-\frac{74}{27}\,\mathrm{H}_{0}-\frac{8}{3}\,\mathrm{H}_{0}\,\zeta_2-\frac{74}{27}\,\mathrm{H}_{1}
\end{aligned}
$$

$$
\begin{aligned}
&+ \tfrac{16}{9}\zeta_2 + \tfrac{58}{27}\Big) + \tfrac{23}{6}\,\mathrm{H}_{0,0,0} + \tfrac{23}{3}\,\mathrm{H}_{2,0} + \tfrac{23}{3}\,\mathrm{H}_3 - \tfrac{58}{9}\,\mathrm{H}_{0,0} - 10\,\mathrm{H}_{1,1} - \tfrac{68}{9}\,\mathrm{H}_2 \\
&+ \tfrac{7}{2}\,\mathrm{H}_0 - \tfrac{23}{3}\,\mathrm{H}_0\,\zeta_2 + \tfrac{67}{9}\,\mathrm{H}_1 + \tfrac{68}{9}\zeta_2 + 6\zeta_3 + \tfrac{19}{27} + \tfrac{77}{432}\,\delta(1-x)\Big\} \\
&+ \tfrac{32}{9}\,C_A\,\Big\{p_{\rm gg}(x)\Big(\tfrac{2}{3}\zeta_3 - \tfrac{1}{9}\Big) + (\tfrac{1}{x} - x^2)\Big(-\tfrac{13}{18}\,\mathrm{H}_{1,0} - \tfrac{13}{9}\,\mathrm{H}_{1,1} + \tfrac{215}{216}\,\mathrm{H}_1 + \tfrac{8}{27}\Big) \\
&+ (1-x)\Big(\tfrac{11}{24}\,\mathrm{H}_{1,0} + \tfrac{11}{12}\,\mathrm{H}_{1,1} - \tfrac{7}{9}\,\mathrm{H}_1\Big) + (1+x)\Big(-\tfrac{1}{3}\,\mathrm{H}_{0,0,0} - \tfrac{2}{3}\,\mathrm{H}_{2,0} - \tfrac{4}{3}\,\mathrm{H}_{2,1} \\
&- \tfrac{2}{3}\,\mathrm{H}_3 + \tfrac{2}{3}\,\mathrm{H}_0\,\zeta_2\Big) + x\Big(\tfrac{43}{72}\,\mathrm{H}_{0,0} + \tfrac{43}{36}\,\mathrm{H}_2 - \tfrac{7}{72}\,\mathrm{H}_0 - \tfrac{43}{36}\zeta_2 + \tfrac{2}{3}\zeta_3 + \tfrac{103}{432}\Big) \\
&+ x^2\Big(\tfrac{13}{18}\,\mathrm{H}_{0,0} + \tfrac{13}{9}\,\mathrm{H}_2 - \tfrac{215}{216}\,\mathrm{H}_0 - \tfrac{13}{9}\zeta_2\Big) + \tfrac{19}{18}\,\mathrm{H}_{0,0} + \tfrac{19}{9}\,\mathrm{H}_2 - \tfrac{7}{8}\,\mathrm{H}_0 - \tfrac{19}{9}\zeta_2 \\
&+ \tfrac{2}{3}\zeta_3 - \tfrac{103}{432} + \tfrac{5}{864}\,\delta(1-x)\Big\}
\end{aligned}
\tag{4.24}
$$

with

$$
p_{\rm gg}(x) \;=\; (1-x)^{-1} + x^{-1} - 2 + x - x^2 \;. \tag{4.25}
$$

The pure-singlet splitting function $P_{\rm ps}(x)$ is suppressed by two powers of $(1-x)$ in the limit $x \to 1$, hence the large-x limit of $P_{\rm qq}(x)$ is given by Eq. (4.13). The same functional form holds for the large-x expansion of $P_{\rm gg}(x)$. The n_f^3 contribution to $A_{4,\rm g}$ is related to Eq. (3.8) for $A_4 \equiv A_{4,\rm q}$ by the Casimir scaling C_A/C_F. The n_f^3 part of $B_{4,\rm g}$ can be readily read off from Eq. (4.24). As for the quark case in Eq. (3.9), non-vanishing contributions to $C_{4,\rm g}$ occur only for $n_f^{a<3}$.

Unlike these diagonal quantities, the off-diagonal entries $P_{\rm qg}$ and $P_{\rm gq}$ in Eq. (2.4) show a double-logarithmic large-x enhancement, i.e., terms up to $\ln^{2n}(1-x)$ contribute to $P^{(n)}_{\rm qg}(x)$ and $P^{(n)}_{\rm qg}(x)$. The highest three of these have been deduced at order α_s^4 from the large-x behaviour of physical evolution kernels of DIS structure functions in Ref. [74] and verified and resummed to all orders in Ref. [75]; a closed form of the next-to-next-next-to-leading logarithmic (N^{3}LL) terms has been obtained in Ref. [76]. The large-x enhanced contributions to Eqs. (4.22) and (4.23) read

$$
\begin{aligned}
P^{(3)}_{\rm qg}\big|_{n_f^3} \;=\;& \ln^4(1-x)\,\tfrac{4}{81}\,(C_F - C_A) + \ln^3(1-x)\,\tfrac{160}{243}\,(C_F - C_A) \\
&- \ln^2(1-x)\Big(\tfrac{16}{243}\,(10 - 9\zeta_2)\,C_A - \tfrac{232}{243}\,C_F\Big) \\
&+ \ln(1-x)\Big(\tfrac{32}{243}\,(55 + 30\zeta_2 - 36\zeta_3)\,C_A - \tfrac{16}{243}\,(71 - 108\zeta_3)\,C_F\Big) \;+\; O(1)
\end{aligned}
\tag{4.26}
$$

and

$$
P^{(3)}_{\rm gq}\big|_{n_f^3} \;=\; -\tfrac{32}{81}\ln^3(1-x)\,C_F - \tfrac{256}{81}\ln^2(1-x)\,C_F - \tfrac{256}{81}\ln(1-x)\,C_F \;+\; O(1)\;. \tag{4.27}
$$

The coefficient of $\ln^4(1-x)$ in Eq. (4.26), and the lack of a $\ln^4(1-x)$ contributions in Eq. (4.27), agree with the results of Refs. [74]. The same holds for the power-suppressed $(1-x)^a\ln^4(1-x)$, terms at all $a \geq 1$ resulting from Eqs. (4.22) and the corresponding $(1-x)^a\ln^3(1-x)$ coefficients of the large-x expansions of Eqs. (4.21) and (4.24), as given by the last lines of Eqs. (5.15) and (5.19) of Ref. [74] together with the relation (5.20) between the pure-singlet and gluon-gluon results.

Like their non-singlet counterparts, the singlet splitting functions receive a double-logarithmic small-x enhancement of the form $\alpha_s^n \ln^\ell x$ with $0 \le \ell \le 2n$. However, the small-x behaviour in the singlet case is dominated by additional single-logarithmic $x^{-1} \ln^\ell x$ terms, see Refs. [77–80]. In the present $\alpha_s^4 n_f^3$ cases, only non-logarithmic x^{-1} terms occur and the small-x expansions read

$$\begin{aligned} P_{\rm ps}^{(3)}\big|_{n_f^3} \;=\;& \frac{1}{x}\,\frac{64}{27}\,(1-6\zeta_3)\,C_F \;-\; \ln^4 x\,\frac{4}{27}\,C_F \;-\; \ln^3 x\,\frac{232}{81}\,C_F \\ & -\; \ln^2 x\,\frac{16}{81}\,(73+18\zeta_2)\,C_F \;-\; \ln x\,\frac{32}{81}\,(59+87\zeta_2+36\zeta_3)\,C_F \;+\; O(1)\;, \end{aligned} \tag{4.28}$$

$$\begin{aligned} P_{\rm qg}^{(3)}\big|_{n_f^3} \;=\;& \frac{1}{x}\left\{\frac{256}{729}\,(17-54\zeta_3)\,C_F \,-\, \frac{64}{729}\,(7+54\zeta_3)\,C_A\right\} \\ & +\; \ln^4 x\left\{\frac{278}{81}\,C_F \,-\, \frac{4}{27}\,C_A\right\} \;+\; \ln^3 x\left\{\frac{20}{243}\,(193+72\zeta_2)\,C_F \,+\, \frac{232}{243}\,C_A\right\} \\ & +\; \ln^2 x\left\{\frac{4}{27}\,(835+180\zeta_2+72\zeta_3)\,C_F \,-\, \frac{2}{243}\,(277-576\zeta_2)\,C_A\right\} \\ & +\; \ln x\left\{\frac{32}{243}\,(1988+1137\zeta_2+180\zeta_3-108\zeta_4)\,C_F \right. \\ & \qquad \left. +\; \frac{4}{243}\,(643+543\zeta_2-288\zeta_3)\,C_A\right\} \;+\; O(1)\;, \end{aligned} \tag{4.29}$$

$$P_{\rm gq}^{(3)}\big|_{n_f^3} \;=\; -\frac{1}{x}\,\frac{128}{81}\,(1-6\zeta_3)\,C_F \;+\; O(1)\;, \tag{4.30}$$

$$\begin{aligned} P_{\rm gg}^{(3)}\big|_{n_f^3} \;=\;& \frac{1}{x}\left\{\frac{32}{243}\,(5+18\zeta_3)\,C_A \,-\, \frac{64}{243}\,(17-54\zeta_3)\,C_F\right\} \;-\; \ln^4 x\,\frac{4}{27}\,C_F \\ & +\; \ln^3 x\left\{\frac{184}{81}\,C_F \,-\, \frac{16}{81}\,C_A\right\} \;+\; \ln^2 x\left\{\frac{152}{81}\,C_A \,-\, \frac{32}{81}\,(35-9\zeta_2)\,C_F\right\} \\ & +\; \ln x\left\{\frac{16}{81}\,(179-138\zeta_2+144\zeta_3)\,C_F \,-\, \frac{4}{81}\,(115-48\zeta_2)\,C_A\right\} \;+\; O(1)\;. \end{aligned} \tag{4.31}$$

The coefficients of $\ln^4 x$ in these results agree with the results of the double-logarithmic small-x resummation [70]. The pattern in Eq. (4.30), no small-x logarithms, is the same as for the $C_F n_f^2$ contribution to $P_{\rm gq}$ at order α_s^3.

The x^{-1} terms in Eqs. (4.29) and (4.31) show an interesting feature in the large-n_c limit $C_F \to \frac{1}{2}n_c$: the resulting coefficients of $x^{-1} n_c$ for $P_{\rm qg}^{(3)}$ and $P_{\rm gg}^{(3)}$ are identical to those of $x^{-1} C_F$ for $P_{\rm ps}^{(3)}$ and $P_{\rm gq}^{(3)}$ in Eqs. (4.28) and (4.30), respectively. For the QCD values of the colour factors, the ratio between the x^{-1} coefficients is 2.11 for the upper-row splitting functions and 2.09 for their lower-row counterparts; hence these ratios are between their overall large-n_c limit of 2 and the Casimir-scaling value of 9/4.

The leading large-n_f 4-loop contributions for the splitting function $P_{\rm qq}(x) = P_{\rm ns}(x) + P_{\rm ps}(x)$ given by Eq. (4.12) and (4.21), and those for $P_{\rm qg}(x)$, $P_{\rm gq}(x)$ and $P_{\rm gg}(x)$ given by Eqs. (4.22) – (4.24) are illustrated at $x < 1$ in Figs. 7 and 8. All functions have been multiplied by $x(1-x)$, hence their small-x and large-x limits are constants in the figures. For these $a_s^4 n_f^3$ coefficients, the pure-singlet contribution to $P_{\rm qq}$ remains relevant up to rather large values of x. The importance of the $\ln^\ell x$ small-x terms is largest for $P_{\rm qg}$ and $P_{\rm gg}$.

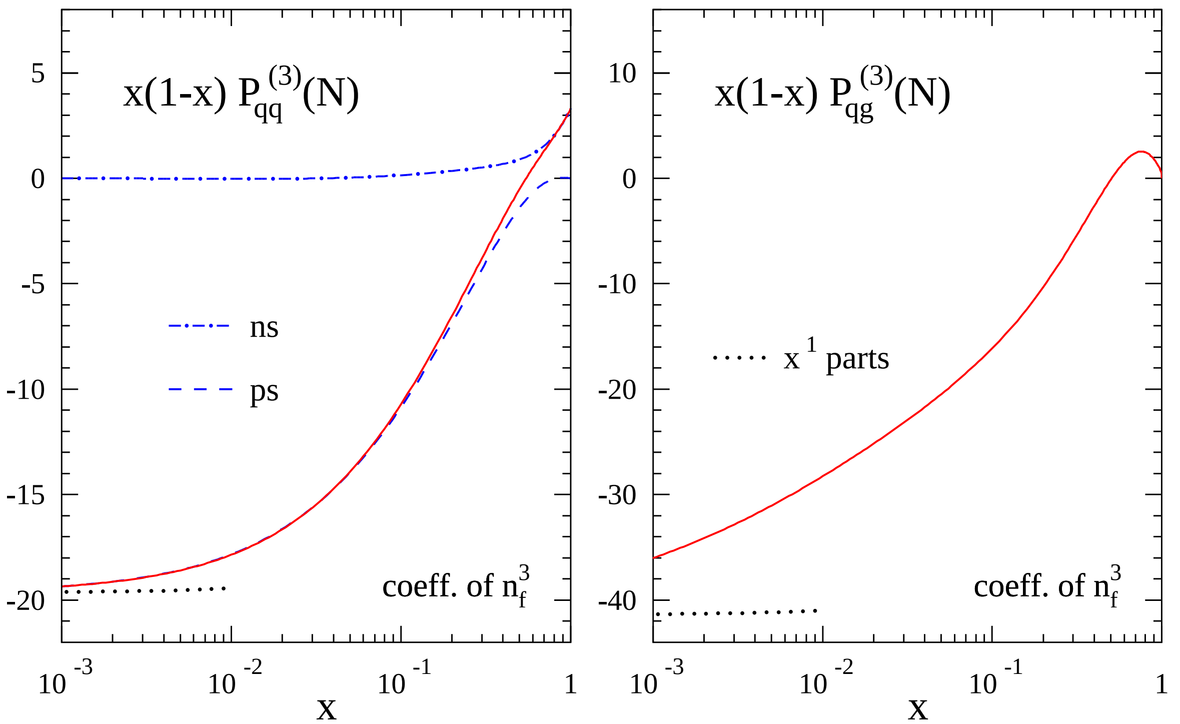

Figure 7: The n_f^3 parts of the 'upper row' quark-quark and gluon-quark four-loop splitting functions in the $\overline{\text{MS}}$ scheme, multiplied by $x(1-x)$ for display purposes, together with their x^{-1} leading small-x terms at $x < 10^{-2}$. For the quark-quark case also the the non-singlet and pure-singlet contributions are shown.

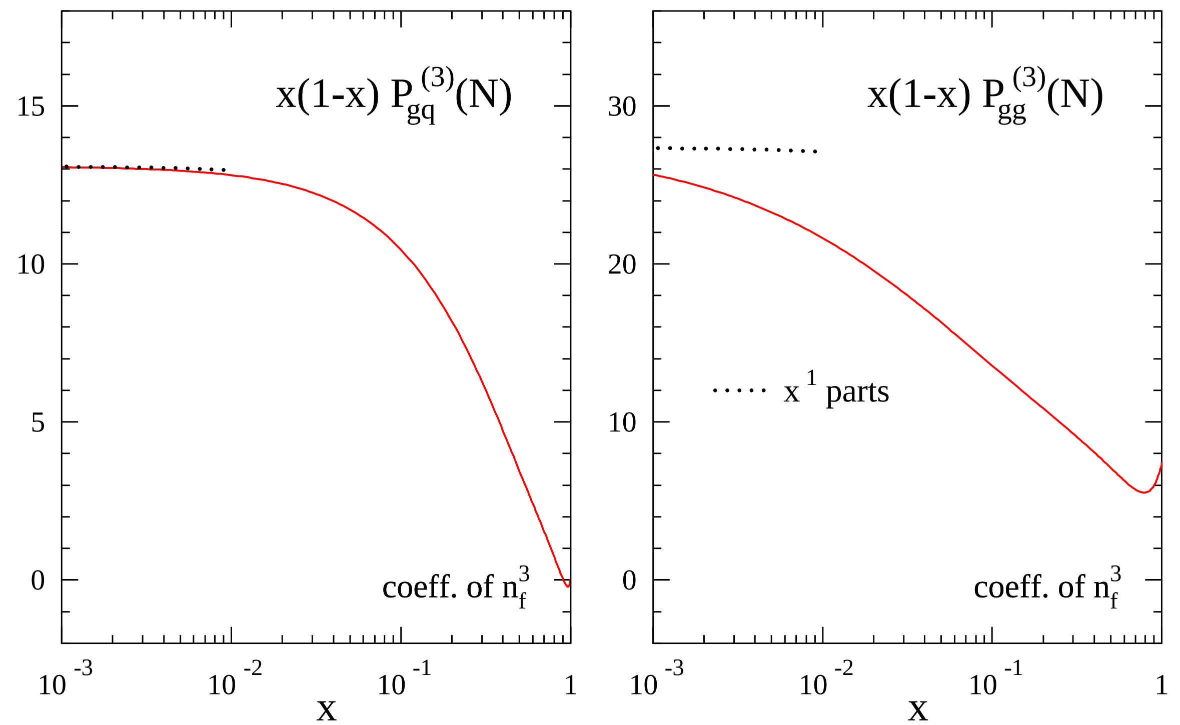

Figure 8: As Figure 7, but for the 'lower row' quark-gluon and gluon-gluon splitting functions at N^{3}LO.

5 Summary

As a first step towards the determination of the N^{3}LO splitting functions $P^{(3)}_{\rm ab}(x)$ in perturbative QCD beyond the leading large-n_f results of Refs. [48–50], we have derived the complete n_f^2 parts of the four-loop non-singlet quark-quark splitting functions and all n_f^3 contributions to their flavour-singlet counterparts in the $\overline{\rm MS}$ scheme. These results have been obtained by analytically computing a fairly large number of Mellin moments N in the approach of Refs. [24–26] – made possible by the development of the FORCER program [20, 21] for the computation of massless four-loop self-energy integrals – and a subsequent determination of the all-N and all-x expressions using the number-theoretical results and tools of Refs. [33–35], a method that has been applied already to three-loop splitting functions in Refs. [36, 37].

Of course this is not a mathematically rigorous procedure, but given the computation of extra 'guard moments' and the additional and non-trivial agreement with previous partial results and structural conjectures summarized below, the chance that our results are not correct is virtually zero. More rigorous approaches would include methods like those used in Refs. [9,10] or Ref. [81], but they require far more resources than are available to us.

Our results agree with Refs. [48–50], with the pioneering low-N non-singlet computations of Refs. [17–19], and with the recent determinations of n_f contributions to the four-loop cusp anomalous dimension [38–40] which appear in our results as the coefficient of $\ln N$ at large N or $1/(1-x)_+$ in the large-x expansion. We also agree with the prediction of Ref. [53] for the coefficient of $\ln(1-x)$ in the non-singlet cases and, in the small region of overlap, with the resummations of highest three small-x and large-x double logarithms in Refs. [69, 70, 74, 75]. Most interestingly our results are in agreement with the remarkably simple (if incomplete – the $\zeta_2(C_A - 2C_F)$ contributions are excluded) generalization of the leading-log small-x resummation [64, 65] for the quark+antiquark non-singlet splitting function $P^+_{\rm ns}$ to all powers of $\ln x$ proposed in Ref. [71].

By themselves the present results are not phenomenologically useful. We hope, though, that it will be possible to complement them in the near future by approximate expressions of the remaining (and numerically more important) contributions to the functions $P^{(3)}_{\rm ab}(x)$, analogous to those employed at NNLO [29] before the results [9, 10] became available, and hence facilitate improved N^{3}LO analyses of DIS and hard processes at colliders. One may also hope that the present results will provide useful additional 'data' for future studies of the structure of the perturbation series for the splitting functions which, in turn, may lead to more explicit four-loop calculations and results.

FORM [44–46] files of our N-space expressions in terms of harmonic sums [30, 31] and their x-space counterparts in terms of harmonic polylogarithms [32] can be obtained from the preprint server `http://arXiv.org` by downloading the source of this article. Furthermore they are available from the authors upon request.

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