Irrationality and transcendence of infinite continued fractions of square roots

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Abstract

We give conditions on a sequence of positive integers $\{a_n\}_{n=1}^{\infty}$ sufficient to ensure that the number defined by the continued fraction expansion $[0; \sqrt{a_1}, \sqrt{a_2}, \dots]$ is either irrational or transcendental.

1 Introduction

As is well known, every infinite simple continued fraction expansion is an irrational number. We know that the Siegel's constant $[0; 1, 2, 3, \dots]$ is transcendental and its irrationality measure has been found [5]. The authors however do not know how to prove the irrationality of $[\sqrt{0}; \sqrt{1}, \sqrt{2}, \dots]$. Also the irrationality measure of the Napier's constant $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots] = [2; \overline{1, 3\lambda + 2, 1}]_{\lambda=0}^{\infty}$ has been found [2] but the authors do not know how to prove the irrationality of

$$[\sqrt{2}; 1, \sqrt{2}, 1, 1, 2, 1, 1, \sqrt{6}, 1, 1, \sqrt{8}, \dots] = [\sqrt{2}; \overline{1, \sqrt{3\lambda + 2}, 1}]_{\lambda=0}^{\infty}.$$

The bar here indicates that the expression beneath it is repeated with the next value of $\lambda = 0, 1, 2, 3, \dots,$ successively and then concatenated to form the sequences of partial quotients in the continued fraction expansion. Special challenges like those we have just mentioned apart it is possible to say something about questions of this type in general.

Adapting a method of Erdős [1] and Liouville [10] we prove the following three theorems.

Theorem 1. *Let k be a positive integer and let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that*

$$\limsup_{n \to \infty} a_n^{\frac{1}{k 2^{n-1} \prod_{j=1}^{n-2} k(2^j + 1)}} = \infty. \tag{1}$$

Then the continued fraction $[0; \sqrt{a_1}, \sqrt{a_2}, \dots]$ is not the root of a polynomial of degree at most k with the rational coefficients.

Theorem 2. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that*

$$\limsup_{n \to \infty} a_n^{\frac{1}{2^{n-1} \prod_{j=1}^{n-2} (2^j + 1)}} = \infty. \tag{2}$$

Then the continued fraction $[0; \sqrt{a_1}, \sqrt{a_2}, \dots]$ is irrational.

Theorem 3. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of positive integers such that*

$$\limsup_{n\to\infty} a_n^{2^{-n^2}} = \infty.$$

Then the continued fraction $[0; \sqrt{a_1}, \sqrt{a_2}, \dots]$ *is transcendental.*

The background to these results is as follows. In 1975 Erdős [1] proved that if we suppose $\{a_n\}_{n=1}^{\infty}$ to be a non-decreasing sequence of positive integers such that $\lim_{n\to\infty} a_n^{\frac{1}{2^n}} = \infty$ then the number $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is irrational. Later in 1991 Hančl [3] proved that if $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive real numbers such that $a_n \le 2^{\frac{1}{n^2}2^n}$ holds for any positive integer n, then there exists a sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers such that the number $\sum_{n=1}^{\infty} \frac{1}{c_n a_n}$ is rational. Subsequently Šustek [16] found a new irrationality measure for such a number. Later Rucki [13] established a criteria for the sums of reciprocals of natural numbers to be irrational. Recently Hančl and Kolouch [4] give a criterion for infinite products of rational numbers to be irrational. A nice review of these results can be found in [9]. See also [6].

Having considered the irrationality and transcendence of real numbers defined as series it is natural to investigate the irrationality and transcendence of real numbers defined via continued fraction expansions. Here we begin by observing that if s and m are non-negative integers and $a_0, \dots, a_s, c_1, \dots, c_m$ are positive algebraic numbers greater than 1, then the number

$$[a_0; a_1, \dots, a_s, \overline{c_1, \dots, c_m}]$$

is algebraic. Here the bar means that the expression underneath is repeated. The results of this paper are a complement to this observation. Note that our results are of a quite general character. We do not for instance require that the elements of $\{a_n\}_{n=1}^{\infty}$ be approximable by the elements of a finite union of power sequences or be associated with any differential equation as with the method of K. Mahler for which see K. Nishioka' s book [12]. Other relevant information on continued fractions can be found in the book of Karpenkov [8]. Throughout the paper $\mathbb{Z}^+$ and $\mathbb{Z}$ will denote the set of all positive integers and integers, respectively.

2 Notation and preliminary results

Let $A_n \geq 1$ for all $n \in \mathbb{Z}^+$. For the (not necessary simple) continued fraction $A = [0; A_1, A_2, \cdots]$ set $\frac{p_n}{q_n} = [0; A_1, A_2, \cdots, A_n]$. We have

$$p_0 = 0, \quad q_0 = 1, \quad p_1 = 1, \quad q_1 = A_1, \quad p_{n+2} = A_{n+2}p_{n+1} + p_n,$$

$$\begin{aligned} q_{n+2} &= A_{n+2}q_{n+1} + q_n, \quad q_{n+1}p_n - p_{n+1}q_n = (-1)^{n+1}, \\ \left| A - \frac{p_n}{q_n} \right| &= \frac{1}{q_n^2([A_{n+1}; A_{n+2}, \ldots] + [0; A_n, A_{n-1}, \ldots, A_1])} \\ &= \frac{1}{q_n^2([A_{n+1}; A_{n+2}, \ldots] + \frac{q_{n-1}}{q_n})} \leq \frac{1}{A_{n+1}(\prod_{j=1}^n A_j)^2} \end{aligned} \tag{3}$$

for all $n \in \mathbb{Z}^+$. All of this can be found in the book of Schmidt [14] pages 7-10 for instance.

Let α be an algebraic number with minimal polynomial $P(x) = \sum_{j=0}^d a_j x^j$ and conjugates $\alpha = \alpha_1, \alpha_2 \ldots, \alpha_d$. Then the Mahler measure $M(\alpha)$ of the number α is defined to be $M(\alpha) :=\mid a_d \mid \prod_{j=1}^d \max(1, \mid \alpha_j \mid)$. Set $H(\alpha) = M(\alpha)^{\frac{1}{d}}$. Now we have the following lemma.

Lemma 4. *Let n be a positive integer and let $\beta_1, \ldots, \beta_n$ be algebraic numbers. Then*

$$H(\sum_{j=1}^n \beta_j) \leq n \prod_{j=1}^n H(\beta_j) \tag{4}$$

and

$$\deg(\sum_{j=1}^n \beta_j) \leq \prod_{j=1}^n \deg(\beta_j). \tag{5}$$

Moreover if α is an algebraic number with $\alpha \neq 0$ then we have

$$H(\alpha^{-1}) = H(\alpha) \tag{6}$$

and

$$\deg(\alpha) = \deg(\alpha^{-1}). \tag{7}$$

For the proofs of (4) and (6) see Waldschmidt [17], Property 3.3, pages 75 and 77 together with Lemma 3.10, on page 79. See also Stewart [15]. Proofs of (5) and (7) can be found in Isaacs [7].

We also need the following theorem [11].

Theorem 5. *Let α and β be different algebraic numbers of degree A and B, respectively. Then*

$$\mid \alpha - \beta \mid \geq \frac{1}{2^{AB} M(\alpha)^B M(\beta)^A}. \tag{8}$$

3 Proof of Main results

Proof. (Proof of Theorem 1.) Set $\gamma_N = [0; \sqrt{a_1}, \sqrt{a_2}, \dots \sqrt{a_N}]$. From this and Lemma 4 we obtain that

$$\deg(\gamma_N) = \deg([0; \sqrt{a_1}, \sqrt{a_2}, \dots \sqrt{a_N}])$$

$$\leq 2 \deg([0; \sqrt{a_2}, \sqrt{a_3}, \dots \sqrt{a_N}]) \leq \dots \leq 2^N$$

and

$$M(\gamma_N) = H(\gamma_N)^{\deg(\gamma_N)} \leq H(\gamma_N)^{2^N}$$

$$\leq (2H(\sqrt{a_1})H([0; \sqrt{a_2}, \sqrt{a_3}, \dots \sqrt{a_N}]))^{2^N} \leq \dots \leq (2^{N-1} \prod_{n=1}^{N} H(\sqrt{a_n}))^{2^N}$$

$$\leq (2^{N-1} \prod_{n=1}^{N} \sqrt{a_n})^{2^N}.$$

This and Theorem 5 yield that

$$\gamma(N) =\mid \gamma - \gamma_N \mid \geq \frac{1}{2^{\deg(\gamma)\deg(\gamma_N)} M(\gamma)^{\deg(\gamma_N)} M(\gamma_N)^{\deg(\gamma)}}$$

$$\geq \frac{1}{(2^k M(\gamma))^{2^N} M(\gamma_N)}$$

$$\geq \frac{1}{(K 2^N \prod_{n=1}^{N} \sqrt{a_n})^{2^N}} \geq \frac{1}{(K^2 2^{2N} \prod_{n=1}^{N} a_n)^{2^{N-1}}},$$

where $K = K(k)$ does not depend on N. Hence for all sufficiently large N we have that

$$\gamma(N)((K2^N)^{2^N} (\prod_{n=1}^{N} a_n)^{2^{N-1}} \geq 1. \tag{9}$$

From (3) we now obtain for all large N that

$$\gamma(N) \leq \frac{1}{\sqrt{a_{N+1}} \prod_{n=1}^{N} a_n}. \tag{10}$$

Inequalities (9) and (10) together imply that for all sufficiently large N that

$$\frac{((K2^N)^{2^N} (\prod_{n=1}^{N} a_n)^{2^{N-1}}}{\sqrt{a_{N+1}} \prod_{n=1}^{N} a_n} \geq 1. \tag{11}$$

Hence for all sufficiently large N we have that

$$\frac{2^{N^2 2^N} (\prod_{n=1}^{N} a_n)^{2^N}}{a_{N+1}} \geq 1. \tag{12}$$

Now from (1) we obtain for infinitely many N that

$$a_{N+1}^{\frac{1}{k2^N \prod_{j=1}^{N-1}(k2^j+1)}} \geq (1 + \frac{1}{(N+1)^2}) \max_{n=1,\dots N} a_n^{\frac{1}{k2^{n-1} \prod_{j=1}^{n-2}(k2^j+1)}}. \tag{13}$$

This is because otherwise there would exist N_0 such that for every $N \geq N_0$ we have

$$a_{N+1}^{\frac{1}{k2^N \prod_{j=1}^{N-1}(k2^j+1)}} < (1 + \frac{1}{(N+1)^2}) \max_{n=1,\dots N} a_n^{\frac{1}{k2^{n-1} \prod_{j=1}^{n-2}(k2^j+1)}}$$

$$< (1 + \frac{1}{(N+1)^2})(1 + \frac{1}{N^2}) \max_{n=1,\dots N-1} a_n^{\frac{1}{k2^{n-1} \prod_{j=1}^{n-2}(k2^j+1)}}$$

$$< \cdots < \prod_{n=N_0+1}^{N+1}(1+\frac{1}{n^2}) \max_{n=1,\ldots N_0} a_n^{\frac{1}{k2^{n-1}\prod_{j=1}^{n-2}(k2^j+1)}}$$

$$< \prod_{n=1}^{\infty}(1+\frac{1}{n^2}) \max_{n=1,\ldots N_0} a_n^{\frac{1}{k2^{n-1}\prod_{j=1}^{n-2}(k2^j+1)}} = const.$$

This contradicts (1). From (13) we obtain that for infinitely many N that

$$a_{N+1} \geq ((1+\frac{1}{(N+1)^2}) \max_{n=1,\ldots N} a_n^{\frac{1}{k2^{n-1}\prod_{j=1}^{n-2}(k2^j+1)}})^{(k2^N\prod_{j=1}^{N-1}(k2^j+1))}$$

$$= (1+\frac{1}{(N+1)^2})^{(k2^N\prod_{j=1}^{N-1}(k2^j+1))}(\max_{n=1,\ldots N} a_n^{\frac{1}{k2^{n-1}\prod_{j=1}^{n-2}(k2^j+1)}})^{(k2^N\prod_{j=1}^{N-1}(k2^j+1))}$$

$$> 2^{3^N}(\max_{n=1,\ldots N} a_n^{\frac{1}{k2^{n-1}\prod_{j=1}^{n-2}(k2^j+1)}})^{(k2^N\prod_{j=1}^{N-1}(k2^j+1))}$$

$$> 2^{3^N}(\max_{n=1,\ldots N} a_n^{\frac{1}{k2^{n-1}\prod_{j=1}^{n-2}(k2^j+1)}})^{(k2^{N-1}\prod_{j=1}^{N-2}(k2^j+1)+k2^{N-2}\prod_{j=1}^{N-3}(k2^j+1)+\cdots+k2)2^N}$$

$$\geq 2^{3^N}(\prod_{j=1}^{N} a_j)^{2^N}.$$

This contradicts (12). □

Theorems 2 and 3 are an immediate consequence of Theorem 1.

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