HAAS-MOLNAR CONTINUED FRACTIONS AND METRIC DIOPHANTINE APPROXIMATION

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Abstract. Haas-Molnar maps are a family of maps of the unit interval introduced by A. Haas and D. Molnar. They include the regular continued fraction map and A. Renyi's backward continued fraction map as important special cases. As shown by Haas and Molnar, it is possible to extend the theory of metric diophantine approximation already well developed for the Gauss continued fraction map, to the class of Haas-Molnar maps. In particular for a real number x, if $\left(\frac{p_n}{q_n}\right)_{n\geq 1}$ denotes its sequence of regular continued fraction convergents, set $\theta_n(x) = q_n^2\left|x - \frac{p_n}{q_n}\right|$ $(n = 1, 2 \dots)$. The metric behaviour of the Cesàro averages of the sequence $(\theta_n(x))_{n\geq 1}$ has been studied by a number of authors. Haas and Molnar have extended this study to the analogues of the sequence $(\theta_n(x))_{n\geq 1}$ for the Haas-Molnar family of continued fraction expansions. In this paper we extend the study of $(\theta_{k_n}(x))_{n\geq 1}$ for certain sequences $(k_n)_{n\geq 1}$, initiated by the second named author, to Haas-Molnar maps.

In memory of A.A. Karatsuba.

1. Introduction

For $x \in [0, 1]$, let

$$x = [0; v_1, v_2, v_3, \cdots] := \cfrac{1}{v_1 + \cfrac{1}{v_2 + \cfrac{1}{v_3 + \ddots}}} \tag{1}$$

with $(v_i)_{i\geq 1} \subset \mathbb{N}$ denoting the partial quotients of the regular continued fraction expansion of x related by $v_1(x) = [1/x]$ and

$$v_n(x) = v_{n-1}(G(x)), \qquad (n = 2, 3, \dots)$$

where G denotes the Gauss map

$$G(x) = \{\tfrac{1}{x}\} = \tfrac{1}{x} - [\tfrac{1}{x}].$$

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Here for real y we have used $\{y\}$ to denote the fractional part of y and used $[y]$ to denote the integer part of y (the biggest integer less than or equal to y). The truncates

$$\frac{p_n}{q_n} := [0; v_1, v_2, \cdots, v_n] := \cfrac{1}{v_1 + \cfrac{1}{v_2 + \cfrac{1}{v_3 + \ddots \cfrac{1}{v_n}}}}$$

with $(p_n, q_n) = 1$ and $q_n > 0$ are called the convergents of the continued fraction expansion of x. The convergents tend to x as $n \to \infty$. Now define

$$\theta_n(x) = q_n^2 \left| x - \frac{p_n}{q_n} \right|$$

to measure the distance between x and its n^{th} convergent $(n = 1, 2, \dots)$. It is well known [L] that $\theta_n(x) \in [0, 1]$. Now let

$$F(z) = \begin{cases} \frac{z}{\log 2} & z \in [0, \frac{1}{2}] \\ \frac{1}{\log 2}(1 - z + \log(2z)) & z \in [\frac{1}{2}, 1]. \end{cases}$$

H. Lenstra conjectured that for almost all $x \in [0, 1]$

$$\lim_{n\to\infty} \frac{1}{n} \#\{j : 1 \le j \le n, \theta_j(x) \le z\} = F(z).$$

This was proved in the early 1980's by W. Bosma, H. Jager and F. Wiedijk in [BJW], using a method based on the Birkhoff's pointwise ergodic theorem and a representation of the natural extention of the Gauss map. A more refined question of ongoing interest is the distribution of $\theta_{k_j}(x)$ for suitable subsequence $\{k_j : j \in \mathbb{N}\}$ of $\mathbb{N}$. Following established practice, we will say that a sequence of natural numbers $(k_j)_{j\ge 1}$ is L^p good- universal if given any dynamical system (X, β, μ, T) with space X, a σ-algebra of its subsets β , a β-measurable map $T : X \to X$, a T-invariant probability μ on β, and $f \in L^p(X, \beta, \mu)$, then the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f(T^{k_j} x)$$

exists μ almost everywhere. See Section 7 for some examples of such sequences.

The requirement that μ be a probability is not strictly part of the concept of good universality, but it is convenient for our considerations in this paper.

Recall that we say that a sequence of real numbers $(x_n)_{n=1}^{\infty}$ is *uniformly distributed modulo* 1 if, for each interval $I \subseteq [0, 1)$, we have

$$\lim_{n\to\infty} \frac{1}{n} \#\{1 \le j \le n \colon \{x_j\} \in I\} = |I|,$$

where $|I|$ denotes the length of I.

Call a sequence $(k_j)_{j=1}^{\infty}$ of natural numbers p -good if it is L^p good-universal for a fixed $p \in [0, \infty]$ and $(k_j\alpha)_{j=1}^{\infty}$ is uniformly distributed modulo 1 for any irrational number α. We call $(k_j)_{j=1}^{\infty}$ good if it is p-good for some $p \geq 1$. See [N4]. In 1998 [N4] the second author proved the following result.

Suppose $0 < z < 1$. Then

$$\lim_{n\to\infty} \frac{1}{n} \#\{j : 1 \leq j \leq n, \theta_{k_j}(x) \leq z\} = F(z)$$

for almost all $x \in [0, 1]$ with respect to Lebesgue measure, for any good sequence.

See [N4] for a number of other applications of this idea. Later D. Hensley [H2] used a transfer operator approach to show the condition in [N4] on $(k_j)_{j\geq 1}$ can be replaced by the assumption that $(k_j)_{j\geq 1}$ is strictly increasing.

Other continued fraction like families which show similar behaviour to the map G have been studied. See for example [BKS, DKW, Na, GH, RW]. In 2003 A. Haas and D. Molnar [HM1] introduced a family of piecewise Möbius transformations $T_u(x) : [0, 1] \to [0, 1]$ which include the Gauss map $G(x) = T_{(1,0)}(x)$ as a special case. The role of the subscript u will be explained later. In this paper we will extend the class of investigations initiated in [N4] for the map $G(x) = T_{(1,0)}$ to the family of maps T_u. This gives a symbolic representation of $x \in [0, 1]$ as $x = [v_1, v_2, v_3, \cdots]_u$ for $(v_j)_{j\geq 1} \subset \mathbb{N}$ analogous to the regular continued fraction expansion.

We now describe the Haas-Molnar family. See [HM1] for details of proofs.

To a 2×2 real entry matrix $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ associate the Möbius transformation

$$C(x) = \frac{ax + b}{cx + d}.$$

Matrix multiplication respects composition of the corresponding Möbius transformations. Let

$$\Psi_{+,0} = (0, \infty) \times \{0\}, \Psi_{-,0} = (-\infty, -1) \times \{0\},$$

and

$$\Psi_{+,1} = (1, \infty) \times \{1\}, \Psi_{-,1} = (-\infty, 0) \times \{1\}.$$

The Haas-Molnar family will be parametrized by $u = (k, m) \in \Psi = \Psi_0 \cup \Psi_1 \subset \mathbb{R} \times \{0, 1\}$, in which

$$\Psi_0 = \Psi_{+,0} \cup \Psi_{-,0}$$

and

$$\Psi_1 = \Psi_{+,1} \cup \Psi_{-,1}.$$

This restriction to matrices from Ψ is required because in other cases the associated dynamical systems are not known to have the ergodic properties required to complete this investigation [Ha]. Consider the family of matrices

$$A_{(k,0)} = \begin{pmatrix} \frac{k}{\sqrt{|k|}} & \frac{-k}{\sqrt{|k|}} \\ \frac{-1}{\sqrt{|k|}} & 0 \end{pmatrix}, \tag{2}$$

and

$$A_{(k,1)} = \begin{pmatrix} \frac{k}{\sqrt{|k|}} & 0 \\ \frac{-1}{\sqrt{|k|}} & \frac{1}{\sqrt{|k|}} \end{pmatrix}. \tag{3}$$

In the form of Möbius transformations these are,

$$A_{(k,0)}(x) = \frac{k(1-x)}{x} \tag{4}$$

and

$$A_{(k,1)}(x) = \frac{kx}{1-x}, \tag{5}$$

respectively. Now define the piecewise Möbius transformation (the Haas-Molnar family) $T_u : J \to J$ for $u \in \Psi$ on the unit interval J as

$$T_u(x) = \{A_u(x)\}. \tag{6}$$

Notice that $T_{(1,0)}$ and $T_{(1,1)}$ are the Gauss and Renyi [GH] transformations respectively. It turns out that $T_{(k,0)}$ and $T_{(k,1)}$ share some common properties with $T_{(1,0)}$, and $T_{(1,1)}$ respectively. We will therefore call the map T_u *Gauss-like* when $u \in \Psi_0$ and *Renyi-like* when $u \in \Psi_1$.

Let

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{7}$$

For $u \in \Psi$, let V_u denote the set of u-digits. That is

$$V_u = \{l \in \mathbb{Z} : \text{ there exists } x \in (0,1) \text{ with } [A_u(x)] = l\},$$

which coincides with the set of non-negative integers if $k > 0$ and with the set of negative integers if $k < 0$. For a finite sequence of u-digits $a_1, a_2, \cdots, a_n$, set

$$C_n = B^{-a_n} A_u B^{-a_{n-1}} A_u \cdots B^{-a_1} A_u. \tag{8}$$

We now define the *u-continued fraction* expansion of x to be

$$x = \lim_{n \to \infty} [a_1, \cdots, a_n]_u,$$

where

(9)

$$[a_1, a_2, \cdots, a_n]_u = A_u^{-1} B^{a_1} A_u^{-1} B^{a_2} \ldots A_u^{-1} B^{a_n} A_u^{-1}(\infty) = C_n^{-1} A_u^{-1}(\infty) \in [0,1].$$

We similarly define the u-truncates

$$\frac{p_n}{q_n} := [v_1, v_2, v_3, \cdots, v_n]_u,$$

$(n = 1, 2, \dots)$ with $q_n > 0$. In the case $u \in \Psi_0$ for instance this leads to the representation

$$[v_1, v_2, \cdots]_u = \cfrac{k}{v_1 + k + \cfrac{k}{v_2 + k + \cfrac{k}{v_3 + k + \ddots}}}. \tag{10}$$

See Lemma 2.9 for a justification of this. We now define

$$\theta_{n,u}(x) = q_n^2 \left| x - \frac{p_n}{q_n} \right|$$

to measure the rate at which the sequence $\left(\frac{p_n}{q_n}\right)_{n \geq 1}$ converges to x. What is different from the case of the regular continued fraction expansion is that now $\theta_{n,u}(x)$ is either in $[0, \frac{1}{|k|}]$ or in $[0, \frac{1}{|k|-1}]$ depending on whether $(-1)^m k > 0$ or < 0 respectively, instead of $[0, 1]$ where $u = (k, m)$ is the subscript. We will show that $\theta_{n,u}(x)$ still possesses a distribution on $[0, \frac{1}{|k|}]$ as $n \to \infty$ on a full measure set of $x \in [0, 1]$ for each element of $u \in \Psi_{+,0}$.

Now for $u = (k, m) \in \Psi$, define

$$c_u = \frac{sgn(k)}{\log |\frac{k+1-m}{k-m}|}.$$

In particular, for each $k \geq 0$, let $c_k := c_{k,0} = \frac{1}{\log(\frac{k+1}{k})}$. It was shown in [Ha] that the measure μ_u on $[0, 1]$ with density $\frac{d\mu_u}{dx} = c_u(x + k)^{-1}$ if $u \in \Psi_0$ and $\frac{d\mu_u}{dx} = c_u(x + k - 1)^{-1}$ if $u \in \Psi_1$, is T_u-invariant and T_u is ergodic with respect to μ_u. For $k \in (0, \infty)$, define

$$F_k(z) = \begin{cases} c_k z & \text{if } z \in [0, \frac{1}{k+1}] \\ 1 - c_k (kz - 1 - \log(kz)) & \text{if } z \in [\frac{1}{k+1}, \frac{1}{k}] \end{cases}$$

We will show that

Theorem 1.1. *For a strictly increasing sequence of positive integers $(k_j)_{j \geq 1}$, $u \in \Psi_{+,0}$ and $0 < z < \frac{1}{k}$, we have*

$$\lim_{n \to \infty} \frac{1}{n} \#\{j : 1 \leq j \leq n, \theta_{k_j,u}(x) \leq z\} = F_k(z)$$

for almost all $x \in [0, 1]$ with respect to Lebesgue measure.

Remark 1. *The theorem generalises* [HM2, Theorem 4] *in the case where $u \in \Psi_{+,0}$ to subsequences. This theorem also generalises* [H2, Theorem 1.1] *and* [N4, Theorem 3.1] *to Haas-Molnar maps.*

Apply Lemma 5.1 on T_u we get the following straightforwardly.

Theorem 1.2. *Suppose $(k_j)_{j\geq 0}$ is p-good for $p \geq 1$. Also suppose*

$$\int_{[0,1]} |F(a_1(x))|^p d\mu_u(x) < \infty.$$

For non-negative real numbers $c_1, \dots, c_n$ set

$$M_F(c_1, \dots, c_n) = F^{-1}\left[\frac{F(c_1) + \dots F(c_n)}{n}\right].$$

Then

$$\lim_{n\to\infty} M_F(a_{k_1}(x), \dots, a_{k_n}(x)) = F^{-1}\left[\int_{[0,1]} F(a_1(x))d\mu_u(x)\right],$$

for almost all $x \in [0,1]$.

Theorem 1.2 has a couple of immediate corollaries.

Corollary 1.2.1. *Suppose $(k_j)_{j\geq 0}$ is a p-good sequence of natural numbers for $p \geq 1$ and $u \in \Psi$. Then if $x = [a_1, a_2, \dots]_u$, we have*

$$\lim_{n\to\infty} \frac{1}{n}(a_{k_1}(x) + \dots + a_{k_n}(x)) = sgn(k)\infty,$$

for almost all $x \in [0,1]$.

Corollary 1.2.2. *Suppose $(k_j)_{j\geq 0}$ is a p-good sequence of natural numbers $p \geq 1$ and $u \in \Psi$. Then if $x = [a_1, a_2, \dots]_u$, we set $\alpha_i = a_i + 1$ when $a_i \geq 0$ and $\alpha_i = |a_i|$ if $a_i < 0$, we have*

$$\lim_{n\to\infty} (\alpha_{k_1}(x). \dots .\alpha_{k_n}(x))^{\frac{1}{n}} = \prod_{i=1}^{\infty}\left(\frac{(i+|k|)^2}{(i+|k|)^2 - 1}\right)^{c_u \log i},$$

for almost all $x \in [0,1]$.

Now we introduce some notations and definitions from [HM2]. Define the functions

$$F_u(x,y) = sgn(k)\left(\frac{1}{x-y}, \frac{xy}{k(x-y)}\right)$$

if $u \in \Psi_0$ and

$$F_u(x,y) = sgn(k)\left(\frac{1}{x-y}, \frac{(1-x)(1-y)}{k(x-y)}\right)$$

if $u \in \Psi_1$. Set

$$J_u = \begin{cases} (-\infty, -k] & \text{if } u = (k,m) \in \Psi_{+,0} \\ [-k, \infty) & \text{if } u \in \Psi_{-,0} \\ (-\infty, 1-k] & \text{if } u \in \Psi_{+,1} \\ [1-k, \infty) & \text{if } u \in \Psi_{-,1} \end{cases} \tag{11}$$

and

$$\Omega_u = F_u(I \times J_u).$$

In $\mathbb{R}^2$ set $O = (0,0)$, $P_u = (\frac{1}{|k|}, 0)$, $R_u = (0, \frac{1}{|k|})$ and set $Q_u = (\frac{1}{|k|+1}, \frac{1}{|k|+1})$ if $(-1)^m k > 0$ and $Q_u = (\frac{1}{|k|-1}, \frac{1}{|k|-1})$ if $(-1)^m k < 0$. Let Ω_u^* denote the quadrilateral with vertices O, P_u, Q_u and R_u in the (w, z) plane, from which the line segment from O to R_u has been removed. For $|k| < 1$ let Λ_u be bounded by the lines $\overline{P_u Q_u}$ and $\overline{Q_u R_u}$ and the hyperbola $z = \frac{1}{4|k|w}$. Let $\Lambda_u = \emptyset$ when $|k| \geq 1$. Define $\Omega_u = \Omega_u^* \cup \Lambda_u$ and

$$\lambda_u(w,z) = \begin{cases} \frac{|k|c_u}{\sqrt{1+(-1)^{m+1}4kzw}} & \text{if } (w,z) \in \Omega_u^* \\ \frac{2|k|c_u}{\sqrt{1+(-1)^{m+1}4kzw}} & \text{if } (w,z) \in \Lambda_u. \end{cases}$$

For any Borel set $D \subseteq \mathbb{R}$ set

$$\lambda_u(D) = \int_D \lambda_u(w,z)d\mu,$$

where μ is Lebesgue measure on $\mathbb{R}^2$. We prove the following extentions to Theorems 2, and 5 of [HM2], replacing $\mathbb{N}$ by good subsequences $(k_j)_{j\geq 1}$.

Theorem 1.3. *Suppose $u \in \Psi$ and $(k_j)_{j\geq 1}$ is good. Then for any Borel set $D \subseteq \mathbb{R}$ we have*

$$\lim_{n\to\infty} \frac{1}{n} \#\{1 \leq j \leq n : (\theta_{k_j}(x), \theta_{k_j+1}(x)) \in D\} = \lambda_u(D),$$

for almost all $x \in [0,1]$.

Note this means for any bounded uniformly continuous function f that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(\theta_{k_j}(x), \theta_{k_j+1}(x)) = \int_{\Omega_u} f d\lambda_u,$$

for almost all $x \in [0,1]$.

Corollary 1.3.1. *Suppose $u \in \Psi$ and $(k_j)_{j\geq 1}$ is good. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} |\theta_{k_j}(x) - \theta_{k_j+1}(x)| = c_u|k| \int_{\Omega_u} \frac{|w-z|}{\sqrt{1-4kzw}} dzdw.$$

for almost all $x \in [0,1]$.

The limit in Corollary 1.3.1 for $|k| \geq 1$ is calculated explicitly in Theorem 3 of [HM2].

Corollary 1.3.2. *Suppose $u \in \Psi$ and $(k_j)_{j\geq 1}$ is good, then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\theta_{k_j}(x)=$$
$$\begin{cases} 2\left((k^2+|k|)\log\left(\frac{|k|+1}{|k|}\right)\right)^{-1} & \text{if } (-1)^m k>0 \\ 2\left((k^2-|k|)\log\left(\frac{|k|}{|k|-1}\right)\right)^{-1} & \text{if } (-1)^m k<0, \end{cases}$$

for almost all $x\in[0,1]$.

In light of Theorem 1.1, it is natural to ask whether all the results in this section are true for arbitrary strictly increasing sequences of integers $(k_j)_{j\geq 1}$. Moving average versions of all the results in this section except Theorem 1.1 appear in Section 6 .

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2. Properties of The Haas-Molnar family

Recall that the partial quotients $[a_1,a_2,\cdots,a_n]_u$ of the *u-continued fraction* expansion are defined in (9). One can check that $T_u([a_1,a_2,\cdots,a_n]_u)=[a_2,a_3,\cdots,a_n]_u$. If we write

$$C_n^{-1}A_u^{-1}=\begin{pmatrix} p_n & r_n \\ q_n & s_n \end{pmatrix}, \tag{12}$$

then $[a_1,a_2,\cdots,a_n]_u=C_n^{-1}A_u^{-1}(\infty)=\frac{p_n}{q_n}$. We first state some results taken directly from section 4 and 5 of [HM1] as following.

Lemma 2.1. *For* $u\in\Psi$ *and* $x=[a_1,a_2,\cdots]_u$, $T_u(x)=[a_2,a_3,\cdots]_u$.

Lemma 2.2. *For* $u\in\Psi_{+,0}$ *and* $n\in\mathbb{N}$, *we have* $\theta_{n,u}(x)\in[0,\frac{1}{k}]$.

Lemma 2.3. *For* $u\in\Psi_{+,0}$ *and* $n\in\mathbb{N}$, *we have* $C_n(\infty)=\frac{-\sqrt{k}q_n}{q_{n-1}}$.

Lemma 2.4. *For* $u\in\Psi_{+,0}$ *and* $n\in\mathbb{N}$, *we have* $s_n=kq_n+\sqrt{k}q_{n-1}, r_n=kp_n+\sqrt{k}p_{n-1}$.

Lemma 2.5. *For* $u\in\Psi_{+,0}$ *and* $n\in\mathbb{N}$, *we have* $p_nq_{n-1}-q_np_{n-1}=\frac{(-1)^{n-1}}{\sqrt{k}}$.

Lemma 2.6. *For* $u\in\Psi_{+,0}, x=[a_1,a_2,\cdots]_u$ *and* $n\in\mathbb{N}$, *we have*

$$0<\frac{q_{n-1}}{q_n}=\frac{1}{\sqrt{k}}[a_n,\cdots,a_1]_u<\frac{1}{\sqrt{k}}.$$

Remark 2. *This lemma is a corollary of* [HM1, Proposition 6]. *Note that if* $a_1=0$, *then* $\frac{q_0}{q_1}=\frac{1}{\sqrt{k}}$, *so this estimation is best possible for every* $x\in(0,1)$.

When $u\in\Psi_{+,0}$ the following identity will be useful,

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ \frac{1}{\sqrt{k}} & 0 \end{pmatrix}\begin{pmatrix} \frac{a_1+k}{\sqrt{k}} & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} \frac{a_n+k}{\sqrt{k}} & 1 \\ 1 & 0 \end{pmatrix}, n\geq 1,$$

or equivalently

$$\left\{ \begin{array}{l} q_{n+1} = \frac{1}{\sqrt{k}}(a_{n+1}+k)q_n + q_{n-1} \\ p_{n+1} = \frac{1}{\sqrt{k}}(a_{n+1}+k)p_n + p_{n-1}. \end{array} \right. \tag{13}$$

In particular $p_0 = 0$ and $q_0 = \frac{1}{\sqrt{k}}$. In order to prove our Theorem 1.1, we still need the following results.

Lemma 2.7. *For $u \in \Psi_{+,0}$ and $n \in \mathbb{N}$, we have*

$$x = \frac{p_n + \frac{T_u^n(x)}{\sqrt{k}} p_{n-1}}{q_n + \frac{T_u^n(x)}{\sqrt{k}} q_{n-1}} = \frac{p_n\sqrt{k} + p_{n-1}T_u^n(x)}{q_n\sqrt{k} + q_{n-1}T_u^n(x)}.$$

Proof. We have

$$\begin{aligned} T_u^n(x) &= [a_{n+1}, a_{n+2}, \cdots]_u(\infty) \\ &= \lim_{N\to\infty} A_u^{-1}B^{a_{n+1}}A_u^{-1}B^{a_{n+2}}\cdots B^{a_N}A_u^{-1}(\infty). \end{aligned}$$

Thus

$$\begin{aligned} x = [a_1, a_2, \cdots]_u &= A_u^{-1}B^{a_1}A_u^{-1}B^{a_2}\cdots A_u^{-1}B^{a_n}A_u^{-1}B^{a_{n+1}}\cdots(\infty) \\ &= C_n^{-1}(T_u^n(x)) = C_n^{-1}A_u^{-1}A_u(T_u^n(x)) \\ &= \begin{pmatrix} p_n & r_n \\ q_n & s_n \end{pmatrix} \begin{pmatrix} \sqrt{k} & -\sqrt{k} \\ -\frac{1}{\sqrt{k}} & 0 \end{pmatrix} (T_u^n(x)) \\ &= \begin{pmatrix} p_n\sqrt{k} - \frac{r_n}{\sqrt{k}} & -p_n\sqrt{k} \\ q_n\sqrt{k} - \frac{s_n}{\sqrt{k}} & -q_n\sqrt{k} \end{pmatrix} (T_u^n(x)). \end{aligned}$$

The last equality is obtained by substituting s_n and r_n via Lemma 2.4. Converting this into the language of Möbius transformations one gets the equations in the lemma.

□

Lemma 2.8. *For $u \in \Psi_{+,0}$ and $n \in \mathbb{N}$, we have*

$$\theta_{n,u}(x) = \frac{\frac{T_u^n(x)}{k}}{1 + \frac{q_{n-1}}{q_n}\frac{T_u^n(x)}{\sqrt{k}}}.$$

Proof. By Lemma 2.7,

$$\begin{aligned} \theta_{n,u}(x) &= |q_n||q_n x - p_n| \\ &= q_n^2 \left| x - \frac{p_n}{q_n} \right| \end{aligned}$$

$$= q_n^2 \left| \frac{p_n\sqrt{k} + p_{n-1}T_u^n(x)}{q_n\sqrt{k} + q_{n-1}T_u^n(x)} - \frac{p_n}{q_n} \right|$$
$$= q_n^2 \left| \frac{q_n p_{n-1}T_u^n(x) - p_n q_{n-1}T_u^n(x)}{q_n(q_n\sqrt{k} + q_{n-1}T_u^n(x))} \right|.$$

By Lemma 2.5, $|p_n q_{n-1} - q_n p_{n-1}| = \frac{1}{\sqrt{k}}$, then the lemma follows. □

If $u \in \Psi_{+,0}$, we have

$$\begin{aligned}[v_1, v_2, \cdots]_u &= \lim_{n\to\infty} [v_1, v_2, \cdots, v_n]_u \\ &= \lim_{n\to\infty} \frac{p_n}{q_n} \\ &= \lim_{n\to\infty} C_n^{-1} A_u^{-1}(\infty) \\ &= \lim_{n\to\infty} C_n^{-1}(0).\end{aligned}$$

In order to have a clearer idea about what these notations represent and to compare with the classical expression, we show that

Lemma 2.9. *For $u \in \Psi_{+,0}$, $[v_1, v_2, \cdots]_u$ is given by (10).*

Proof. As $[v_1, v_2, \cdots]_u = \lim_{n\to\infty} \frac{p_n}{q_n}$, in order to prove the lemma we only need to show that

$$\frac{p_n}{q_n} = \cfrac{k}{v_1 + k + \cfrac{k}{v_2 + k + \cfrac{k}{v_3 + k + \cfrac{k}{\ddots + \cfrac{k}{v_n + k}}}}}. \tag{14}$$

We do this by induction. The first step is obvious. Now assume the truth of the equation for $n-1$. We have

$$\begin{aligned}\frac{p_n}{q_n} = C_n^{-1}(0) &= A_u^{-1}B^{v_1}(A_u^{-1}B^{v_2}\cdots A_u^{-1}B^{v_n}(0)) \\ &= A_u^{-1}(v_1 + A_u^{-1}B^{v_2}\cdots A_u^{-1}B^{v_n}(0)) \\ &= \frac{k}{k + v_1 + A_u^{-1}B^{v_2}\cdots A_u^{-1}B^{v_n}(0)} \\ &= \frac{k}{k + v_1 + [v_2, v_3, \cdots, v_n]_u},\end{aligned}$$

and by assumption,

$$[v_2, v_3, \cdots, v_n]_u = \cfrac{k}{v_2 + k + \cfrac{k}{v_3 + k + \cfrac{k}{v_4 + k + \cfrac{k}{\ddots + \cfrac{k}{v_n + k}}}}},$$

so (14) holds.

□

3. The transfer operator L_u based on the Haas-Molnar family

The *transfer operator* (sometimes called the Ruelle operator) encodes information about the associated dynamical system. While we can extend the transfer operator to more general frameworks [R1, M], in this paper we confine attention to Haas-Molnar continued fraction maps.

We now recall some notions from [H1] adapted to our purposes. For a function $f : (0,1) \to \mathbb{C}$ of bounded variation, let V_f be its total variation. Set

$$f(a^+) := \lim_{x \to a^+} f(x), \ f(a^-) := \lim_{x \to a^-} f(x).$$

We call a function *normal* if $f(a) = \dfrac{f(a^+) + f(a^-)}{2}$ for all $a \in (0,1)$. Let

$$\Sigma = \{f : (0,1) \to \mathbb{C}, \ f \text{ is normal with } V_f < \infty\}.$$

For $f \in \Sigma$, let

$$\|f\| = |f(0^+)| + V_f.$$

Then $(\Sigma, \| \, \|)$ is a Banach space [H1]. The *transfer operator* $L_u : \Sigma \to \Sigma$ for $T_u, u \in \Psi_{+,0}$ is defined by

$$(L_u f)(t) = \sum_{y \in T_u^{-1}(t)} f(y) \frac{y^2}{k} = \sum_{j=0}^{\infty} f\left(\frac{k}{j+t+k}\right) \frac{k}{(j+t+k)^2}$$

for $f \in \Sigma$ and $k \in (0, \infty)$. Throughout the rest of this section and the next we will, without further comment, assume $u \in \Psi_{+,0}$,

When $k = 1$, the operator $L_{(1,0)}$ is the famous Gauss-Kuzmin-Wirsing operator [B]. The importance of the operator L_u is that if $\mathcal{X}$ is a random variable on I with density function $f(t)$, then $T_u^n \mathcal{X}$ has density $(L_u^n f)(t)$ [R2]. The operator L_u has simple dominant eigenvalue 1 by reasoning similar to that used for the Gauss-Kuzmin-Wirsing operator in [A, V, MR1, MR2]. Its eigenfunction plays an important role in our investigation.

Lemma 3.1. *The operator $L_u : \Sigma \to \Sigma$ has eigenfunction*

$$g(t) = \frac{c_k}{k+t}$$

corresponding to the dominant eigenvalue 1 *for* $k \in (0, \infty)$.

Proof.

$$\begin{aligned}(L_u g)(t) &= \sum_{j=0}^{\infty} \frac{c_k}{k + \frac{k}{j+t+k}} \frac{k}{(j+t+k)^2} \\ &= c_k \sum_{j=0}^{\infty} \frac{1}{(j+t+k)(j+t+k+1)} \\ &= c_k \sum_{j=0}^{\infty} (\frac{1}{j+t+k} - \frac{1}{j+t+k+1}) \\ &= \frac{c_k}{k+t}.\end{aligned}$$

□

Remark 3. *After we finish the proof, we will see that this is a special case of* [Ha, Corollary 1, 2]. *Invariant measures for a large class of* $T_{m,k}$ *maps are indicated in* [Ha].

Now apply the classical theory of probability to the sequence of random variables $(T_u^n \mathcal{X})_{n \geq 1}$ here. Let V_r be the r-fold Cartesian product of the positive integers. For $v = (v_1, v_2, \cdots, v_r) \in V_r$, set

$$[v] := [v_1, v_2, \cdots, v_r]_u = \frac{p_r}{q_r},$$
$$\{v\} := \frac{q_{r-1}}{q_r},$$
$$|v| := q_r$$

and

$$[v+t] := \cfrac{k}{v_1 + k + \cfrac{k}{v_2 + k + \cfrac{k}{v_3 + k + \cfrac{k}{\ddots + \cfrac{k}{v_r + k + t}}}}}$$
$$= \frac{p_r + \frac{t}{\sqrt{k}} p_{r-1}}{q_r + \frac{t}{\sqrt{k}} q_{r-1}}.$$

By the definition of L_u, we have

$$(L_u^r f)(t) = \sum_{v \in V_r} |v|^{-2} (\sqrt{k} + \{v\} t)^{-2} f([v+t]). \tag{15}$$

Now for any function $f : I \to I$ of bounded variation, let $||f||$ be its total variation on $I = [0, 1]$. For an event E, denote by $Prob\{E\}$ to be the probability of it. By a method similar to [H1, Lemma 6], we can obtain the following result.

Lemma 3.2. *For any positive function f of bounded variation $||f||$ on I, there exists a constant $\kappa_u \in (0, 1)$ depending on u such that*

$$(L_u^r f)(t) = \left(\int_0^1 f(x)dx\right) g(t) + O(\kappa_u^r ||f||)$$
$$= \frac{c_k}{k+t} + O(\kappa_u^r ||f||).$$

Proof. First, let $\mathbb{X}_\theta(x) = 1$ if $0 < x < \theta$ and 0 if $\theta < x < 1$. Then any function f of bounded variation can be expressed as

$$f(t) = \int_{\theta=0}^1 \mathbb{X}_\theta(t) d\mu(\theta)$$

for a finite complex Borel measure $d\mu$ on $(0, 1)$. It is easy to show that

$$(L_u f)(t) = \int_{\theta=0}^1 (L_u \mathbb{X}_\theta)(t) d\mu(\theta), \tag{16}$$

$$(L_u)^r (1 + \theta t)^{-2} = \frac{c_k}{(1+\theta)(k+t)} + O(\kappa_u^r) \tag{17}$$

by Babenko and Jurév's method [B] [BY]. Now for $\theta \in [0, 1]$, let

$$\theta = [v_1(\theta), v_2(\theta), \cdots, v_r(\theta) + \xi_r(\theta)]_u$$

for some $\xi_r(\theta) \in (0, 1)$. Then by induction,

$$(L_u)^r \mathbb{X}_\theta(t) = \sum_{j_1 > v_1(\theta)} (\frac{j_1 + k}{k})^{-2} L_u^{r-1} \left(\sqrt{k} + \frac{\sqrt{k}}{j_1 + k} t\right)^{-2}$$
$$+ \sum_{j_2 < v_2(\theta)} |(v_1(\theta), j_2)|^{-2} L_u^{r-2} (\sqrt{k} + \{(v_1(\theta), j_2)\} t)^{-2} + \cdots$$
$$+ \sum_{j_r \gtrless v_r(\theta)} |(v_1(\theta), v_2(\theta), \cdots, v_{r-1}(\theta), j_r)|^{-2}$$
$$\times (\sqrt{k} + \{(v_1(\theta), v_2(\theta), \cdots, v_{r-1}(\theta), j_r)\} t)^{-2}$$
$$+ |(v_1(\theta), v_2(\theta), \cdots, v_r(\theta))|^{-2} (\sqrt{k} + \{(v_1(\theta), v_2(\theta), \cdots, v_r(\theta))\} t)^{-2}$$
$$\times \mathbb{X}_\theta([v_1(\theta), v_2(\theta), \cdots, v_r(\theta) + t])$$
$$= g(t) C(r, \theta) + O(\kappa_u^r),$$

in which the symbol $\gtrless$ is $>$ for odd r and $<$ for even r.

As

$$\theta = \int_0^1 \mathbb{X}_\theta(t)dt = \int_0^1 L_u^r \mathbb{X}_\theta(t)dt,$$

we have

$$C(r,\theta) = \theta + O(\kappa_u^r).$$

Thus

$$(L_u^r f)(t) = \Big(\int_0^1 \theta d\mu(\theta)\Big)g(t) + O(\kappa_u^r ||f||)$$
$$= \Big(\int_0^1 f(x)dx\Big)g(t) + O(\kappa_u^r ||f||).$$

□

Remark 4. *The result is also true as an application of* [I-TM, (1.7)].

Now let $T_u^r(x) = t$. By Lemma 2.7,

$$x = \frac{p_r + \frac{t}{\sqrt{k}} p_{r-1}}{q_r + \frac{t}{\sqrt{k}} q_{r-1}}.$$

Thus, with $x = [v+t] = [v_1, \cdots, v_{r-1}, v_r + t]_u$, we have

$$|dx/dt| = \frac{1}{k}|v|^{-2}(1 + \{v\}t/\sqrt{k})^{-2}.$$

Let $\mathcal{X}(x)$ denote $T_u : [0,1] \to [0,1]$ from now on, and continue letting f denote the density function of $\mathcal{X}(x)$ on $[0,1]$. That is, if one defines $\psi(y) = \mu\{x \in (0,1) \mid \mathcal{X}(x) \le y\}$ with μ being Lebesgue measure, we have $f = \psi'$. So the probability $P_{r,z,f} = Prob\{\{v\} \le z\}$ is given by

$$P_{r,z,f} = \int_{t=0}^1 \sum_{v \in V_r, \{v\} \le z} \frac{1}{k}|v|^{-2}(1 + \{v\}t/\sqrt{k})^{-2} f([v+t])dt.$$

So the conditional density of $T_u^r(x)$ given that $\{v(\mathcal{X}, r)\} \le z$ is

$$g_{r,z,f}(t) := \sum_{v \in V_r, \{v\} \le z} \frac{1}{k}|v|^{-2}(1 + \{v\}t/\sqrt{k})^{-2} f([v+t])/P_{r,z,f}. \tag{18}$$

Now let

$$h_{r,z,f}(t) = \sum_{v \in V_r, \{v\} \le z} \frac{1}{k}|v|^{-2}(1 + \{v\}t/\sqrt{k})^{-2} f([v+t]).$$

We can show that

Lemma 3.3. *For any density function f of bounded variation $||f||$ on J, $t \in (0,1)$ and $z \in (0, \frac{1}{\sqrt{k}})$, the function $h_{r,z,f}(t)$ has bounded variation and*

$$h_{r,z,f}(t) = \frac{c_k z}{\sqrt{k}+zt} + O(r\kappa_u^r ||f||). \tag{19}$$

Proof. This is proved in the same way as [H2, Theorem 2.1]. Now by (13), we have $\frac{q_r}{q_{r+1}} = \frac{1}{(v_{r+1}+k)/\sqrt{k}+q_{r-1}/q_r} \le z$, which implies $v_{r+1} \ge [\sqrt{k}/z] - k$. Then the inductive step $r+1$ goes as

$$h_{r,z,f}(t) = \sum_{v\in V_{r+1}, \{v\}\le z} \frac{1}{k}|v|^{-2}(1+\{v\}t/\sqrt{k})^{-2} f([v+t])$$

$$= \sum_{n=[\frac{\sqrt{k}}{z}]-k+1}^{\infty} n^{-2} \sum_{v\in V_r} |v|^{-2}(\sqrt{k}+\{v\}t)^{-2}(\sqrt{k}+\frac{t}{\frac{n+k}{\sqrt{k}}+\{v\}})^{-2} f([v,n+t])$$

$$- \sum_{n=[\frac{\sqrt{k}}{z}]-k+1}^{\infty} n^{-2} \sum_{v\in V_r, \{v\}\le \langle \frac{\sqrt{k}}{z}\rangle} |v|^{-2}(\sqrt{k}+\{v\}t)^{-2}(\sqrt{k}+\frac{t}{\frac{n+k}{\sqrt{k}}+\{v\}})^{-2} f([v,n+t])$$

$$= \frac{c_k z}{\sqrt{k}+zt} + O((r+1)\kappa_u^{r+1}||f||)$$

□

We can now use (18) to compute the conditional density for $T_u^r \mathcal{X}$ under the condition $\{v\} \le z$ by virtue of (19).

Corollary 3.4. *We have*

$$g_{r,z,f}(t) = \frac{z}{(\sqrt{k}+tz)(\log(\sqrt{k}+z) - \log\sqrt{k})} + O(r\kappa_u^r ||f||).$$

As the first step to prove our Theorem 1.1, we show the following theorem.

Theorem 3.5. *The probability $Q_{r,f} = Prob\{\theta_{r,u}(\mathcal{X}) \le z, z \in [0, \frac{1}{k}]\}$ for $\mathcal{X}$ with initial density f satisfies*

$$Q_{r,f}(z) = F_k(z) + O(\kappa_u^r ||f||). \tag{20}$$

Proof. Define the function $I(E) = 1$ if the event E is true otherwise 0, then by applying the change of variable $x = [v+t]$ and Lemma 2.8,

$$Q_{r,f}(z) = \int_{x=0}^{1} f(x) I\left(\theta_{r,u} \le z\right) dx$$

$$= \int_{t=0}^{1} \sum_{v\in V_r} \frac{1}{k}|v|^{-2}(1+\{v\}t/\sqrt{k})^{-2} f([v+t]) I\left(\theta_{r,u} \le z\right) dt$$

$$= \int_{t=0}^{1} \sum_{v \in V_r} \frac{1}{k} |v|^{-2} \left(1 + \{v\} t/\sqrt{k}\right)^{-2} f([v+t]) I \left(\frac{\frac{t}{k}}{1 + \{v\} \frac{t}{\sqrt{k}}} \leq z \right) dt$$

$$= \int_{t=0}^{1} \sum_{v \in V_r} \frac{1}{k} |v|^{-2} (1+\{v\} t/\sqrt{k})^{-2} f([v+t]) \left(1 - I \left(\{v\} \leq \frac{1}{z\sqrt{k}} - \frac{\sqrt{k}}{t} \right) \right) dt.$$

Remember that $\{v\} \in (0, \frac{1}{\sqrt{k}})$. We set

$$\left(\tfrac{1}{z\sqrt{k}} - \tfrac{\sqrt{k}}{t}\right)^* := \begin{cases} 0 & \text{if } \frac{1}{z\sqrt{k}} - \frac{\sqrt{k}}{t} < 0 \\ \frac{1}{z\sqrt{k}} - \frac{\sqrt{k}}{t} & \text{if } \frac{1}{z\sqrt{k}} - \frac{\sqrt{k}}{t} \in [0, \frac{1}{\sqrt{k}}] \\ \frac{1}{\sqrt{k}} & \text{if } \frac{1}{z\sqrt{k}} - \frac{\sqrt{k}}{t} > \frac{1}{\sqrt{k}} \end{cases}$$

Now replace z by $\left(\frac{1}{z\sqrt{k}} - \frac{\sqrt{k}}{t} \right)^*$ in (19) and do the integration by suitably chosen integral limit. When $z \in (0, \frac{1}{k+1})$ (the point $\frac{1}{k+1}$ is obtained by comparing one of the integral limit $\frac{kz}{1-z}$ with 1), one gets

$$Q_{r,f}(z) = 1 - c_k \left(\int_{kz}^{\frac{kz}{1-z}} (\frac{1}{t} - \frac{zk}{t^2}) dt + \int_{\frac{kz}{1-z}}^{1} \frac{1}{k+t} dt \right) + O(\kappa_u^r ||f||)$$
$$= c_k z + O(\kappa_u^r ||f||).$$

When $z \in (\frac{1}{k+1}, \frac{1}{k})$, one gets

$$Q_{r,f}(z) = 1 - c_k \int_{kz}^{1} (\frac{1}{t} - \frac{zk}{t^2}) dt + O(\kappa_u^r ||f||) =$$
$$1 - c_k (kz - 1 - \log(kz)) + O(\kappa_u^r ||f||).$$

So we have proved

$$Q_{r,f}(z) = F_k(z) + O(\kappa_u^r ||f||). \tag{21}$$

□

Now by Theorem 3.5 we can see that our Theorem 1.1 holds for sequences $\{k_j\}_{j=0}^{\infty}$ with $\inf\{|k_{j-1} - k_j|\}$ large enough. In the next section we will deal with the general case by a trick from [H2].

4. Proof of Theorem 1.1

We follow the method in [H2] in this section.

Suppose $\{k_j\}_{j=0}^{\infty}$ is a strictly increasing sequence of positive integers. Without loss of generality we assume that $k_0 > N_0$ for N_0 large enough. Consider the first n random variables $T_u^{k_j}(\mathcal{X}), v(\mathcal{X}, k_j)$ and $\theta_{u,k_j}(\mathcal{X})$ for integers $j \in [0, n-1]$. We then let $n \to \infty$ to prove our result.

Note that for this length-n sequence, $\inf\{k_{j+1} - k_j\}$ may not be large enough, so to make use of (20), we split the n-sequence $\{k_j\}_{j=1}^n$ into $n^{\frac{3}{4}}$ subsequences $\{k_{j_1,j_2}\}, 0 \le j_1 < n^{\frac{3}{4}}, 0 \le j_2 < n^{\frac{1}{4}}$ such that each sequence satisfies

$$|k_{j_1,j_2+1} - k_{j_1,j_2}| > n^{\frac{3}{4}}.$$

By choosing the number of sequences $n^{\frac{3}{4}} > n^{\frac{1}{4}}$, the number of entries in each subsequences, we can control the norm of the density function $\parallel g_{k_{j_1,j_2},z,f} \parallel$ at each step. Now for some fixed z choose a positive number γ small enough to ensure that $\gamma < \frac{1}{3}F_k(z)$. For fixed j_1 define

$$R_{z,j_1} = Prob\{ \text{ More than } n^{\frac{1}{4}}(F_k(z) + 2\gamma) \text{ entries in}\{\theta_{k_{j_1,j_2}}\} \text{ satisfy } \theta_{k_{j_1,j_2}} > z, 0 \le j_2 < n^{\frac{1}{4}}\}.$$

It is shown in [H2] that

Lemma 4.1. *There exists $C > 0$ such that $R_{z,j_1} \le O(e^{-C\gamma^2 n^{\frac{1}{4}}})$.*

Using this lemma, among all the other $n^{\frac{3}{4}}$ subsequences, there will be $O\left(n^{\frac{3}{4}} e^{-C\gamma^2 n^{\frac{1}{4}}}\right)$ subsequences with the property that there are more than $n^{\frac{1}{4}}(F_k(z) + 2\gamma)$ entries in $\{\theta_{k_{j_1,j_2}}\}$ satisfying $\theta_{k_{j_1,j_2}} < z$. Note that

$$\lim_{n\to\infty} n^{\frac{3}{4}} e^{-C\gamma^2 n^{\frac{1}{4}}} = 0.$$

By a similar argument one can show that among all the $n^{\frac{3}{4}}$ subsequences, there will be $O(n^{\frac{3}{4}} e^{-C\gamma^2 n^{\frac{1}{4}}})$ ones satisfying the property that, there are more than $n^{\frac{1}{4}}(1 - F_k(z) + 2\gamma)$ entries in $\{\theta_{k_{j_1,j_2}}\}$ such that $\theta_{k_{j_1,j_2}} > z$. As γ can be made arbitrarily small, the probability $\frac{1}{n}\#\{j : 1 \le j \le n, \theta_{k_j,u}(x) \le z\}$ must tend to $F_k(z)$ as $n \to \infty$ by equation (20).

5. Distribution of the orbits of the natural extention of T_u

The results in this section are based on the following lemma taken from [N2]. An extensive list of good universal sequences is contained in the final section.

Lemma 5.1. *Suppose the dynamical system (X, β, μ, T) is weak mixing and that the sequence of natural numbers $(k_j)_{j\ge1}$ is L^p good universal and $(k_j\alpha)_{j\ge1}$ is uniformly distributed modulo one for each irrational α. Then if $f \in L^p(X, \beta, \mu)$ we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(T^{k_j}(x)) = \int_X f d\mu,$$

almost everywhere with respect to μ.

We now introduce some definitions and results from [HM1].

Let $I = [0,1)$ and for $D \subseteq I \times J_u$ set

$$\rho_u(D) = \int\int_D \rho_u(x,y)dxdy,$$

where $\rho_u(x,y) = c_u(x-y)^{-2}$. We now define $\tilde{T}_u : I \times J_u \to I \times J_u$ by

$$\tilde{T}_u(x,y) = (T_u(x), A_u(y) - [A_u(x)]).$$

One can check [HM1, Thm. 1] that $\tilde{T}_u$ is the natural extention of T_u which preserves the measure ρ_u. Then as T_u is weak mixing then so is $\tilde{T}_u$. See [CFS p. 241] for details of this. Now fix $u \in \Psi$, given $x \in (0,1)$, set $(x_n, y_n) = \tilde{T}_u^n(x,\infty)$ $(n = 1,2,\dots)$. In terms of $\tilde{T}_u$ we have $\theta_n = |x_n - y_n|^{-1}$ [HM1, Theorem 2] and $(\theta_n, \theta_{n+1}) = F_\mu(x_n, y_n)$ [HM2, Prop 1].. We need the following Lemma from [HM1, Lemma 1].

Lemma 5.2. *Given an u-irrational x and $\epsilon > 0$ then there is N such that if $n > N$ and $y, y' \in J_u \cup \{\infty\}$ we have $|\tilde{T}_u(x,y) - \tilde{T}_u(x,y')| < \epsilon$.*

In light of Lemma 5.1 we have the following analogue of Theorem 3 from [HM1].

Theorem 5.3. *The points $\tilde{T}_u^n(x,y)$ for the integers $n \geq 0$ are distributed in the interior of $I \times J_u$ with density function $\rho_u(x,y)$ for all $y \in J_u \cup \{\infty\}$. In other words, for $(k_j)_{j\geq 1}$ as in Lemma 5.1 and any Borel set $D \subset I \times J_u$ having boundary of Lebesgue measure 0, and for almost all x,*

$$\lim_{n\to\infty} \frac{1}{n}\#\{1 \leq j \leq n : \tilde{T}_u^{k_j}(x,y) \in D\} = \rho_u(D)$$

for all $y \in J_u \cup \{\infty\}$. Furthermore for any uniformly continuous function $f \in L^1(\rho_u)$ and almost all $x \in I$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} f(\tilde{T}_u^{k_j}(x,y)) = \int_{I\times J_u} f d\rho_u$$

for all $y \in J_u \cup \{\infty\}$.

Proof. Let D be a Borel set with measure zero boundary contained in $I \times J_u$. Consider the ϵ-collar C_ϵ of D, consisting of the ϵ boundary of D. Set $D_\epsilon^+ = D \cup C_\epsilon$ and $D_\epsilon^- = D - C_\epsilon$. Then we have $D_\epsilon^- \subset D \subset D_\epsilon^+$. Suppose we are given $x \in (0,1)$, some $y \in J_u \cup \{\infty\}$ and $\epsilon > 0$ together with $N^- \in \mathbb{N}$ such that if $n > N^-$, then $\tilde{T}_u^n(x,y) \in D$. Then as a consequence of Lemma 5.2, we have $\tilde{T}_u^n(x,y) \in D_\epsilon^-$ for all y for large n. Similarly suppose we are given $x \in (0,1)$, some $y \in J_u \cup \{\infty\}$ and $\epsilon > 0$ together with $N^+ \in \mathbb{N}$ such that if $n > N^+$, then $\tilde{T}_u^n(x,y) \in D$. Then as a consequence of Lemma 5.2, we have $\tilde{T}_u^n(x,y) \in D_\epsilon^+$ for all y for large n.

If $N^+ \in \mathbb{N}$ can be chosen such that if for all $n > N^+$ and some $y \in J_u \cup \{\infty\}$ we have $\tilde{T}_u^n(x,y) \in D_\epsilon^+$ then $\tilde{T}_u^n(x,y) \in D_\epsilon^+$ for all y for larger n. Fix $y* \in J_u \cup \{\infty\}$. By the above, for any $(x,y) \in I \times J_u$, for which the first and last limits exist

$$\lim_{n\to\infty} \frac{1}{n} \#\{1 \le j \le n : \tilde{T}_u^{k_j}(x,y) \in D_\epsilon^-\}$$

$$\le \liminf_{n\to\infty} \frac{1}{n} \#\{1 \le j \le n : \tilde{T}_u^{k_j}(x,y*) \in D\}$$

$$\le \limsup_{n\to\infty} \frac{1}{n} \#\{1 \le j \le n : \tilde{T}_u^{k_j}(x,y*) \in D\}$$

$$\le \lim_{n\to\infty} \frac{1}{n} \#\{1 \le j \le n : \tilde{T}_u^{k_j}(x,y) \in D_\epsilon^+\}.$$

Using the weak mixing of the natural extention of T_u and Lemma 5.1, the first and last limits exist for almost all (x,y) in $I \times J_u$ and are $\rho_u(D_\epsilon^-)$ and $\rho_u(D_\epsilon^+)$ respectively. Since D has a boundary of measure zero, the difference between $\rho_u(D_\epsilon^-)$ and $\rho_u(D_\epsilon^+)$ can be made as small as you wish by choosing ϵ small enough. Since $y* \in J_u \cup \{\infty\}$ is arbitrary it follows that for almost all x

$$\lim_{n\to\infty} \frac{1}{n} \#\{1 \le j \le n : \tilde{T}_u^{k_j}(x,y*) \in D\} = \rho_u(D),$$

proving that $(\tilde{T}_u^n(x,y))_{n\ge 1}$ is distributed according to the density ρ_u.

We now consider the second assertion of the Theorem. Suppose

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(\tilde{T}_u^{k_j}(x,y)) = L.$$

Then for $y' \ne y$

$$\left| \frac{1}{n} \sum_{j=1}^{n} f(\tilde{T}_u^{k_j}(x,y')) - L \right|$$

$$< \left| \frac{1}{n} \sum_{j=1}^{n} f(\tilde{T}_u^{k_j}(x,y')) - f(\tilde{T}_u^{k_j}(x,y')) \right| + \left| \frac{1}{n} \sum_{j=1}^{n} f(\tilde{T}_u^{k_j}(x,y)) - L \right|.$$

The first term on the right can be made as small as we like for large n, by Lemma 5.2 and the uniform continuity of f. The second term on the right is small by hypothesis. Thus the limit with y' instead of y converges to the limit L as well. Since f is integrable by Lemma 5.1 the weak mixing of $\tilde{T}_u$ shows that the limit L exists for almost all x and equals $\int_{I\times J_u} f d\rho_u$.

□

As in [HM2], let Γ_u denote the square in the (x, y) plane with vertices $(0, 1-k)$, $(1-k, 1+k)$, $(k+1, 2)$ and $(0, 2)$ if $m = 1$ and $-1 < k < 0$. Observe that $\sigma_u(x, y) = (-y, -x)$ is an order two self map of Γ_u.

We also need the following Lemma from ([HM2, Corollary 1]):

Lemma 5.4. *If $|k| > 1$, then F_u is a homeomorphism from $I \times J_u$ to Ω_u. If $|k| < 1$ then $F_u : \Gamma_u \to \Lambda_u$ is two to one, with $F_u(x, y) = F_u(\sigma_u(x, y))$, where $\sigma_u(x, y) = (-y, -x)$ and $F_u : I \times J_u \backslash \Gamma_u \to \Omega_u \backslash \Lambda_u$ is a homeomorphism, and in particular for any $u \in \Psi$ and for u-irrational x the approximating pairs belong to Ω_u.*

One can induce a Bernoulli and hence weak mixing shift on Ω_u. Suppose $|k| > 1$. Then $F_u : I \times J_u \to \Omega_u$ is a homeomorphism so we can define the map $\tilde{S}_u : \Omega_u \to \Omega_u$ by $\tilde{S}_u = F_u \circ \tilde{T}_u \circ F_u^{-1}$. Because of the way we have defined the measure λ_u, the map F_u is an isomorphism of probability spaces and $\tilde{T}_u$ is a measure preserving transformation of Ω_u. Moreover, the dynamical system $(I \times J_u, \tilde{T}_u, \rho_u)$ is isomorphic via this conjugacy to the dynamical system $(\Omega_u, \tilde{S}_u, \lambda)$. Hence we have the following lemma.

Lemma 5.5. *The dynamical system $(\Omega_u, \tilde{S}_u, \lambda_u)$ is Bernoulli.*

In light of Theorem 5.3 we now have

Theorem 5.6. *For $u \in \Psi$ and good $(k_j)_{j\geq 1}$ the sequence $(\theta_{k_j}, \theta_{k_j+1})_{j\geq 1}$ is distributed in Ω_u according to the density $\lambda_u(w, z)$. In other words, given any Borel set $D \subset \Omega_u$ with a boundary of measure zero we have for almost all x that*

$$\lim_{n\to\infty} \frac{1}{n} \# \left\{1 \leq j \leq n : (\theta_{k_j}(x), \theta_{k_j+1}(x)) \in D\right\} = \lambda_u(D).$$

Furthermore for any uniformly continuous function $f \in L^1(\lambda_u)$ and almost all $x \in [0, 1]$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f((\theta_{k_j}(x), \theta_{k_j+1}(x)) = \int_{I\times J_u} f d\lambda_u$$

for all $y \in J_u \cup \{\infty\}$.

Applying Theorem 5.6 with $f(w, z) = |w - z|$ we get Corollary 1.3.1 . Now let I_u be $(0, \frac{1}{|k|})$ if $(-1)^m k > 0$ and let it be $(0, \frac{1}{|k|-1})$ if $(-1)^m k < 0$. Let $\phi : \Omega_u \to I_u$ denote the projection $\phi(w, z) = w$. Define the measure β_u on I_u by $\beta_u(D) = \lambda_u(\phi^{-1}(D))$. We can see for almost every x, that

$$\begin{aligned} \beta_u(D) &= \lim_{n\to\infty} \frac{1}{n} \#\{1 \leq j \leq n : (\theta_{k_j}(x), \theta_{k_{j+1}}(x)) \in \phi^{-1}(D)\} \\ &= \lim_{n\to\infty} \frac{1}{n} \#\{1 \leq j \leq n : \theta_{k_j}(x) \in D\}. \end{aligned}$$

Corollary 1.3.2 now follows from [HM2, Theorem 4].

6. Moving Averages

We begin by introducing some notations. Let Z be a collection of points in $\mathbb{Z} \times \mathbb{N}$, and let

$$Z^h = \{(m,n) \in Z \colon n \geq h\},$$
$$Z^h_\alpha = \{(x,y) \in \mathbb{Z}^2 \colon |x-m| < \alpha(y-n) \text{ for some } (m,n) \in Z^h\},$$
$$Z^h_\alpha(k) = \{x \colon (x,k) \in Z^h_\alpha\} \qquad (k \in \mathbb{N}).$$

Geometrically, we can think of Z^1_α as the lattice points contained in the union of all solid cones with aperture α and vertex contained in $Z^1 = Z$. We say that a sequence of pairs of natural numbers $((d_n, e_n))_{n=1}^\infty$ is *Stolz* if there exists a collection of points Z in $\mathbb{Z} \times \mathbb{N}$ and a function $h = h(t)$ tending to infinity with t such that $((d_n, e_n))_{n=t}^\infty \in Z^{h(t)}$, and if there exist h_0, α_0, and $c > 0$ such that, for all $k \in \mathbb{N}$, we have the cardinality $\# Z^{h_0}_{\alpha_0}(k) \leq ck$. This technical condition is interesting because of the following lemma which is an implication of Theorem 1 part (a) from [BJR], deduced for instance using the same method as that used to prove Theorem 1.14 from [W]. We refer the reader to Section 1, for the definition of mathematical terms used but not defined in this section. The proofs of results in this section are also foregone as they are essentially the same as those of results in Section 1 with the role of subsequence ergodic theorems replaced by the moving average theorem.

Lemma 6.1. *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system. Suppose that $((d_n, e_n))_{n=1}^\infty$ is a Stolz sequence. Then, for any $f \in L^1(X, \mathcal{B}, \mu)$, the limit*

$$\lim_{n\to\infty} \frac{1}{e_n} \sum_{j=1}^{e_n} f(T^{d_n+j-1}\alpha)$$

exists μ-almost everywhere.

Note that if we set

$$M_f(\alpha) = \lim_{n\to\infty} \frac{1}{e_n} \sum_{j=1}^{e_n} f(T^{d_n+j-1}\alpha) \qquad \text{and} \qquad M_{n,f}(\alpha) = \frac{1}{e_n} \sum_{j=1}^{e_n} f(T^{d_n+j-1}\alpha)$$

and observe that

$$M_{n,f}(T\alpha) - M_{n,f}(\alpha) = \frac{1}{e_n}(f(T^{d_n+e_n}\alpha) - f(T^{d_n}\alpha)),$$

then we can see that $M_f(T\alpha) = M_f(\alpha)$ μ-a.e. A standard fact in ergodic theory is that if $(X, \mathcal{B}, \mu, T)$ is ergodic and if $M_f(T\alpha) = M_f(\alpha)$ μ-a.e., then $M_f(\alpha) = \int_X f \, d\mu$ μ-a.e., [CFS, p 14]. We have the following lemma.

Lemma 6.2. *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system. Suppose that $((d_n, e_n))_{n=1}^{\infty}$ is a Stolz sequence. Then, for any $f \in L^1(X, \mathcal{B}, \mu)$,*

$$\lim_{n\to\infty} \frac{1}{e_n} \sum_{j=1}^{e_n} f(T^{d_n+j-1}\alpha) = \int_X f\, d\mu$$

μ-almost everywhere.

Theorem 6.3. *Suppose $((d_n, e_n))_{n\geq 0}$ is a Stolz sequence. Suppose $F : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is continuous and increasing. Let μ_u denote the measure on $[0, 1]$ defined in the introduction. Suppose*

$$\int_{[0,1]} |F(a_1(x))|^p d\mu_u(x) < \infty.$$

For a finite number of non-negative reals $c_1, \ldots, c_n$ set

$$M_F(c_1, \ldots, c_n) = F^{-1}\left[\frac{F(c_1) + \ldots F(c_n)}{n}\right].$$

Then, for almost all x

$$\lim_{n\to\infty} M_F(a_{d_n}(x), \ldots, a_{d_n+e_n-1}(x)) = F^{-1}\left[\int_{[0,1]} F(a_1(x))d\mu_u(x)\right].$$

In the rest of this section $((d_n, e_n))_{n\geq 0}$ is a Stolz sequence and $u \in \Psi$. Theorem 6.3 has a couple of immediate corollaries.

Corollary 6.4. *For almost every $x = [a_1, a_2, \ldots]_u$, we have*

$$\lim_{n\to\infty} \frac{1}{n}(a_{d_n}(x) + \ldots + a_{d_n+e_n-1}(x)) = sgn(k)\infty.$$

Corollary 6.5. *If $x = [a_1, a_2, \ldots]_u$, we set $\alpha_i = a_i + 1$ when $a_i \geq 0$ and $\alpha_i = |a_i|$ if $a_i < 0$. Then for almost all x we have*

$$\lim_{n\to\infty} (\alpha_{d_n}(x). \ldots .\alpha_{d_n+e_n-1)}(x))^{\frac{1}{n}} = \prod_{i=1}^{\infty} \left(\frac{(i + |k|)^2}{(i + |k|)^2 - 1}\right)^{c_u \log i}.$$

Theorem 6.6. *For any Borel set with a zero measure boundary*

$$\lim_{n\to\infty} \frac{1}{e_n} \#\{1 \leq j \leq e_n : (\theta_{d_n+j-1}, \theta_{d_n+j}) \in D\} = \lambda_u(D).$$

Corollary 6.7. *For almost all $x \in [0, 1]$*

$$\lim_{n\to\infty} \frac{1}{e_n} \sum_{j=1}^{e_n} |\theta_{a_n+j-1}(x) - \theta_{a_n+j}(x)| = c_u|k| \int_{\Omega_u} \frac{|w - z|}{\sqrt{1 - 4kzw}} dzdw.$$

Corollary 6.8. *For almost every $x \in [0, 1]$*

$$\lim_{n\to\infty}\frac{1}{e_n}\sum_{j=1}^{e_n}\theta_{d_n+j-1}(x) =$$
$$\begin{cases} 2\left((k^2+|k|)\log\left(\frac{|k|+1}{|k|}\right)\right)^{-1} & \textit{if } (-1)^m k>0 \\ 2\left((k^2-|k|)\log\left(\frac{|k|}{|k|-1}\right)\right)^{-1} & \textit{if } (-1)^m k<0. \end{cases}$$

7. List of known good universal sequences

In this section we give some examples of L^p-good universal sequences for some $p \geq 1$.

1. *The natural numbers:* The sequence $(n)_{n=1}^{\infty}$ is L^1-good universal. This is Birkhoff's pointwise ergodic theorem.
2. *Polynomial like sequences:* If $\phi(x)$ is a polynomial such that $\phi(\mathbb{N}) \subseteq \mathbb{N}$ and $p>1$, then $(\phi(n))_{n=1}^{\infty}$ and $(\phi(p_n))_{n=1}^{\infty}$, where p_n is the nth prime, are L^p-good universal sequences. See [Bo1], [Bo2] and [N1], respectively.
3. *Condition* H*:* Sequences $(a_n)_{n=1}^{\infty}$ that are both L^p-good universal and Hartman uniformly distributed can be constructed as follows. Denote by $[x]$ the integer part of a real number x. Set $a_n=[\tau(n)]$ $(n=1,2,\dots)$, where $\tau:[1,\infty)\to[1,\infty)$ is a differentiable function whose derivative increases with its argument. Let Ω_m denote the cardinality of the set $\{n\colon a_n\leq m\}$, and suppose, for some function $\varphi:[1,\infty)\to[1,\infty)$ increasing to infinity as its argument does, that we set

$$\varrho(m)=\sup_{\{z\}\in\left[\frac{1}{\varphi(m)},\frac{1}{2}\right)}\left|\sum_{n\colon a_n\leq m} e(za_n)\right|,$$

where $e(x)=e^{2\pi ix}$ for a real x. Suppose also, for some decreasing function $\rho:[1,\infty)\to[1,\infty)$ and some positive constant $\omega>0$, that

$$\frac{\varrho(m)+\Omega_{[\varphi(m)]}+\frac{m}{\varphi(m)}}{\Omega_m}\leq\omega\rho(m).$$

Then if we have

$$\sum_{n=1}^{\infty}\rho(\theta^n)<\infty$$

for all $\theta>0$, we say that $(a_n)_{n=1}^{\infty}$ satisfies condition H, see [N3].

Sequences satisfying condition H are both Hartman uniformly distributed and L^p-good universal. Specific sequences of integers that satisfy condition H include $a_n=[\tau(n)]$ $(n=1,2,\dots)$ where:

I. $\tau(n)=n^{\gamma}$ if $\gamma>1$ and $\gamma\notin\mathbb{N}$.
II. $\tau(n)=e^{\log^{\gamma} n}$ for $\gamma\in(1,\frac{3}{2})$.

III. $\tau(n) = b_k n^k + \cdots + b_1 n + b_0$ for $b_k, \ldots, b_1$ not all rational multiplies of the same real number.

IV. *Hardy fields:* By a Hardy field, we mean a closed subfield (under differentiation) of the ring of germs at $+\infty$ of continuous real-valued functions with addition and multiplication taken to be pointwise. Let $\mathcal{H}$ denote the union of all Hardy fields. Conditions for $(a_n)_{n=1}^{\infty} = ([\psi(n)])_{n=1}^{\infty}$, where $\psi \in \mathcal{H}$ to satisfy condition H are given by the hypotheses of Theorems 3.4, 3.5 and 3.8. in [BKQW]. Note the term ergodic is used in this paper in place of the older term Hartman uniformly distributed.

4. *A random example:* Suppose that $S = (b_n)_{n=1}^{\infty}$ is a strictly increasing sequence of natural numbers. By identifying S with its characteristic function χ_S, we may view it as a point in $\Lambda = \{0,1\}^{\mathbb{N}}$, the set of maps from $\mathbb{N}$ to $\{0,1\}$. We may endow Λ with a probability measure by viewing it as a Cartesian product $\Lambda = \prod_{n=1}^{\infty} X_n$, where, for each natural number n, we have $X_n = \{0,1\}$ and specify the probability ν_n on X_n by $\nu_n(\{1\}) = \omega_n$ with $0 \leq \omega_n \leq 1$ and $\nu_n(\{0\}) = 1 - \omega_n$ such that $\lim_{n\to\infty} \omega_n n = \infty$. The desired probability measure on Λ is the corresponding product measure $\nu = \prod_{n=1}^{\infty} \nu_n$. The underlying σ-algebra $\mathcal{A}$ is that generated by the cylinders
$$\{(\Delta_n)_{n=1}^{\infty} \in \Lambda \colon \Delta_{n_1} = \alpha_{n_1}, \ldots, \Delta_{n_k} = \alpha_{n_k}\}$$
for all possible choices of $n_1, \ldots, n_k$ and $\alpha_{n_1}, \ldots, \alpha_{n_k}$. Then almost every point $(a_n)_{n=1}^{\infty}$ in Λ, with respect to the measure ν, is Hartman uniformly distributed. See Proposition 8.2 (i) in [Bo1] for the details of this. Again Hartman uniformly distributed sequences are called ergodic sequences in this paper.

5. *Block sequences:* Suppose that $(a_n)_{n=1}^{\infty} = \bigcup_{n=1}^{\infty}[d_n, e_n]$ is ordered by absolute value for disjoint $([d_n, e_n])_{n=1}^{\infty}$ with $d_{n-1} = O(e_n)$ as n tends to infinity. Note that this allows the possibility that $(a_n)_{n=1}^{\infty}$ is zero density. This example is an immediate consequence of Tempelman's semigroup ergodic theorem. See page 218 of [T]. Being a group average ergodic theorem this pointwise limit must be invariant. which ensures that the block sequence must be Hartman uniformly distributed. The proof of this, which we don't need in this paper and is hence forgone is a simple exercise in spectral theory.

6. *Random perturbation of good sequences:* Suppose that $(a_n)_{n=1}^{\infty}$ is an L^p-good universal sequence which is also Hartman uniformly distributed. Let $\theta = (\theta_n)_{n=1}^{\infty}$ be a sequence of $\mathbb{N}$-valued

independent, identically distributed random variables with basic probability space $(Y, \mathcal{A}, \mathcal{P})$, and a $\mathcal{P}$-complete σ-field $\mathcal{A}$. Let $\mathbb{E}$ denote expectation with respect to the basic probability space $(Y, \mathcal{A}, \mathcal{P})$. Assume that there exist $0 < \alpha < 1$ and $\beta > 1/\alpha$ such that

$$a_n = O(e^{n^\alpha}) \qquad \text{and} \qquad \mathbb{E} \log_+^\beta |\theta_1| < \infty.$$

Then $(a_n + \theta_n(\omega))_{n=1}^\infty$ is both L^p-good universal and Hartman uniformly distributed [NW].

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