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New developments in special geometry

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ABSTRACT

We review recent developments in special geometry, emphasizing the role of real coordinates. In the first part we discuss the para-complex geometry of vector and hypermultiplets in rigid Euclidean $N = 2$ supersymmetry. In the second part we study the variational principle governing the near horizon limit of BPS black holes in matter-coupled $N = 2$ supergravity and observe that the black hole entropy is the Legendre transform of the Hesse potential encoding the geometry of the scalar fields.

1 Introduction

Special geometry was discovered more than 20 years ago [1]. While the term special geometry originally referred to the geometry of vector multiplet scalars in four-dimensional $N = 2$ supergravity, today it is used more generally for the geometries encoding the scalar couplings of vector and hypermultiplets in theories with 8 real supercharges. It applies to rigidly and locally supersymmetric theories in ≤ 6 space-time dimensions, both in Lorentzian and in Euclidean signature. The scalar geometries occuring in these cases are indeed closely related. In particular, they are all much more restricted than the Kähler geometry of scalars in theories with 4 supercharges, while still depending on arbitrary functions. In contrast, the scalar geometries of theories with 16 or more supercharges are completely fixed by their matter content. Theories with 8 supercharges have a rich dynamics, which is still constrained enough to allow one to answer many questions exactly. Special geometry lies at the heart of the Seiberg-Witten solution of $N = 2$ gauge theories [2] and of the non-perturbative dualities between $N = 2$ string compactifications [3, 4].

While the subject has now been studied for more than twenty years, there are still new aspects to be discovered. One, which will be the topic of this paper, is the role of real coordinates. Many special geometries, in particular the special Kähler manifolds of four-dimensional vector multiplets and the hyper-Kähler geometries of rigid hypermultiplets are complex geometries. Nevertheless, they also possess distinguished real parametrizations, which are natural to use for certain physical problems. Our first example illustrates this in the context of special geometries in theories with Euclidean supersymmetry. This part reviews the results of [5, 6], and gives us the opportunity to explore another less studied aspect of special geometry, namely the scalar geometries of $N = 2$ supersymmetric theories in Euclidean space-time. It turns out that the relation between the scalar geometries of theories with Lorentzian and Euclidean space-time geometry is (roughly) given by replacing complex structures by para-complex structures. One technique for deriving the scalar geometry of a Euclidean theory in D dimensions is to start with a Lorentzian theory in $D+1$ dimensions and to perform a dimensional reduction along the time-like direction. The specific example we will review is to start with vector multiplets in four Lorentzian dimensions, which gives, by reduction over time, hypermultiplets in three Euclidean dimensions. This provides us with a Euclidean version of the so-called c-map. The original c-map [7, 8] maps any scalar manifold of four-dimensional vector multiplet scalars to a scalar manifold of hypermultiplets. For rigid supersymmetry, this relates affine special Kähler manifolds to hyper-Kähler manifolds, while for local supersymmetry this relates projective special Kähler manifolds to quaternion-Kähler manifolds. By using dimensional reduction with respect to time rather than space, we will derive the scalar geometry of Euclidean hypermultiplets. As we will see, the underlying geometry is particularly transparent when using real scalar fields rather than complex ones. The geometries of Euclidean supermultiplets are relevant for the study of instantons, and, by 'dimensional oxidation over time' also for solitons, as outlined in [5]. In this paper we will restrict ourselves to the geometrical aspects.

Our second example is taken from a different context, namely BPS black hole solutions of matter-coupled $N = 2$ supergravity. The laws of black hole mechanics suggest to assign an entropy to black holes, which is, at least to leading order, proportional to the area of the event horizon. Since (super-)gravity presumably is the low-energy effective theory of an underlying quantum theory of gravity, the black hole entropy is analogous to the macroscopic or thermodynamic entropy in thermodynamics. A quantum theory of gravity should provide the fundamental or microscopic level of description of a black hole and, in particular, should allow one to identify the microstates of a black hole and to compute the corresponding microscopic or statistical entropy. The microscopic entropy is the missing information if one only knows the macrostate but not the microstate of the black hole. In other words, if a black hole with given mass, charge(s) and angular momentum (which characterise the macrostate) can be in d different microstates, then the microscopic entropy is $S_{\text{micro}} = \log d$. If the area of the event horizon really is the corresponding macroscopic entropy, then these two quantities must be equal, at least to leading order in the semi-classical limit. In string theory it has been shown that the two entropies are indeed equal in this limit [9], at least for BPS states (also called supersymmetric states). These are states which sit in special representations of the supersymmetry algebra, where part of the generators act trivially. These BPS (also called short) representations saturate the lower bound set for the mass by the supersymmetry algebra, and, as a consequence, the mass is exactly equal to a central charge of the algebra.[1] In this paper we will be interested in the macroscopic part of the story, which is the construction of BPS black hole solutions and the computation of their entropy. The near horizon limit of such solutions, which is all one needs to know in order to compute the entropy, is determined by the so-called black hole attractor equations [11], whose derivation is based on the special geometry of vector multiplets. The attractor equa-

[1] See [10] Chapter 2.

tions are another example where real coordinates on the scalar manifold appear in a natural way. In the second part of the paper we review how the attractor equations and the entropy can be obtained from a variational principle. When expressed in terms of real coordinates, the variational principle states that the black hole entropy is the Legendre transform of the Hesse potential of the scalar manifold. We also discuss how the black hole free energy introduced by Ooguri, Strominger and Vafa [12] fits into the picture, and indicate how higher curvature and non-holomorphic corrections to the effective action can be incorporated naturally. This part of the paper is based on [13] and on older work including [14, 15, 16].

Finally we would like to point out how our two subjects are connected to pseudo-Riemannian geometry. In both parts of the paper we have two relevant geometries, the geometry of space-time and the geometry of the target manifold of the scalar fields. In the first case, space-time is Euclidean, but, as we will see, the scalar manifold is pseudo-Riemannian with split signature. In the second case the scalar geometry is positive definite, but space-time is pseudo-Riemannian with Lorentz signature.

2 Euclidean special geometry

We start by reviewing the geometry of vector multiplets in rigid four-dimensional $N = 2$ supersymmetry.[2] A vector multiplet consists of a gauge field A_m, ($m = 0, \dots, 3$ is the Lorentz index), two Majorana spinors λ^i ($i = 1, 2$) and one complex scalar X. We consider n such multiplets, labeled by an index $I = 1, \dots, n$. The field equations for the gauge fields are invariant under $Sp(2n, \mathbb{R})$ rotations which act linearly on the field strength F^I_{mn} and the dual field strength $G_{I|mn} = \frac{\delta L}{\delta F^I_{mn}}$, where L denotes the Lagrangian. These symplectic rotations generalize the electric-magnetic duality rotations of Maxwell theory and are in fact invariances of the full field equations. A thorough analysis shows that this has the important consequence that all vector multiplet couplings are encoded in a single holomorphic function of the scalars, $F(X^I)$, which is called the prepotential [1]. Using superspace methods the general action for vector multiplets can be derived to be a chiral superspace integral of the prepotential F, considered as a superspace function of n so-called restricted chiral multiplets $(X^I, \lambda^{I+}, F^{I-}_{mn})$, which encode the gauge invariant quantities of the n vector multiplets. Here λ^{I+} are the positive chirality projections of the spinors and F^{I-}_{mn} are the antiselfdual

[2] Some more background material and references on vector multiplets can be found in [17].

projections of the field strength. To be precise, the Lagrangian is the sum of a chiral and an antichiral superspace integral, the latter depending on the complex conjugated multiplets $(\overline{X}^I, \lambda^{I-}, F^{I+}_{mn})$. When working out the Lagrangian in components, all couplings can be expressed in terms of $F(X^I)$, its derivatives, which we denote $F_I, F_{IJ}, \ldots$ and their complex conjugates $\overline{F}_I, \overline{F}_{IJ}, \ldots$. For later use we specify the bosonic part of the Lagrangian:

$$L^{\text{4d VM}}_{\text{bos}} = -\tfrac{1}{2} N_{IJ} \partial_m X^I \partial^m \overline{X}^J - \tfrac{i}{2} \left(F_{IJ} F^{I-}_{mn} F^{J-mn} - \text{c.c.} \right) , \tag{2.1}$$

where

$$N_{IJ} = \partial_I \partial_{\overline{J}} \left(-i(X^I \overline{F}_I - F_I \overline{X}^I) \right) \tag{2.2}$$

can be interpreted as a Riemannian metric on the target space M_{VM} of the scalars X^I.[3] $N = 1$ supersymmetry requires this metric to be a Kähler metric, which is obviously the case, the Kähler potential being $K = -i(X^I \overline{F}_I - F_I \overline{X}^I)$. As a consequence of $N = 2$ supersymmetry the metric is not a generic Kähler metric, since the Kähler potential can be expressed in terms of the holomorphic prepotential $F(X^I)$. The resulting geometry is known as affine (also: rigid) special Kähler geometry. The intrinsic characterization of this geometry is the existence of a flat, torsionfree, symplectic connection ∇, called the special connection, such that

$$(\nabla_U I) V = (\nabla_V I) U , \tag{2.3}$$

where I is the complex structure and U, V are arbitrary vector fields [18]. It has been shown that all such manifolds can be constructed locally as holomorphic Langrangian immersions into the complex symplectic vector space $T^*\mathbb{C}^n \simeq \mathbb{C}^{2n}$ [20]. In this context X^I, F_I are flat complex symplectic coordinates on $T^*\mathbb{C}^n$ and the prepotential is the generating function of the immersion $\Phi : M_{VM} \to T^*\mathbb{C}^n$, i.e., $\Phi = dF$. For generic choice of Φ, the X^I provide coordinates on the immersed M_{VM}, while $F_I = \partial_I F = F_I(X)$ along M_{VM} . The X^I are non-generic coordinates, physically, because they are the lowest components of vector multiplets, mathematically, because they are adapted to the immersion. They are called special coordinates.

So far we have considered vector multiplets in a four-dimensional Minkowski space-time. In four-dimensional Euclidean space the theory has the same form, except that the complex structure I, $I^2 = -\mathbb{1}$ is replaced by a para-complex structure J. This is defined to be an endomorphism of TM_{VM}

[3] In general, the scalar fields X^I will only provide local coordinates. We will work in a single coordinate patch throughout.

such that $J^2 = \mathbb{1}$, with the eigendistributions corresponding to the eigenvalues ± 1 having equal rank. Many notions of complex geometry, including Kähler and special Kähler geometry can be adapted to the para-complex realm. We refer to [5, 6] for the details. In particular, it can be shown that the target space geometry of rigid Euclidean vector multiplets is affine special para-Kähler. Such manifolds are the para-complex analogues of affine special Kähler manifolds. When using an appropriate notation, the expressions for the Lagrangian, the equations of motion and the supersymmetry transformation rules take the same form as for Lorentzian supersymmetry, except that complex quantities have to be re-interpreted as para-complex ones. For example, the analogue of complex coordinates $X^I = x^I + iu^I$, where x^I, u^I are real and i is the imaginary unit, are para-complex coordinates $X^I = x^I + eu^I$, where e is the para-complex unit characterized by $e^2 = 1$ and $\overline{e} = -e$, where the 'bar' denotes para-complex conjugation.[4] While in Lorentzian signature the selfdual and antiselfdual projections of the field strength are related by complex conjugation, in the Euclidean theory one can re-define the selfdual and antiselfdual projections by appropriate factors of e such that they are related by para-complex conjugation. One can also define para-complex spinor fields such that the fermionic terms of the Euclidean theory take the same form as in the Lorentzian one. The Euclidean bosonic Lagrangian takes the same form (2.1) as the Lorentzian one, with (2.2) replaced by

$$N_{IJ} = \partial_I \partial_{\overline{J}} \left(-e(X^I \overline{F}_I - F_I \overline{X}^I) \right) \ . \tag{2.4}$$

Note that the Euclidean Lagrangian is real-valued, although the fields X^I and F^{I-}_{mn} are para-complex. We also remark that a para-Kähler metric always has split signature. The full Lagrangian, including fermionic terms, and the supersymmetry transformation rules can be found in [5]. There we also verified that it is related to the rigid limit of the general Lorentzian signature vector multiplet Lagrangian [22, 23] by replacing $i \to e$ (together with additional field redefinitions, which account for different normalizations and conventions).

Our next step is to construct the geometry of Euclidean hypermultiplets. This can be done by either reducing the Lorentzian vector multiplet Lagrangian with respect to time or the Euclidean vector multiplet Lagrangian with respect to space [6]. Here we start from the Lorentzian Lagrangian and perform the reduction over space and over time in parallel. This is

[4] It has been known for quite a while that the Euclidean version of a supersymmetric theory can sometimes be obtained by replacing $i \to e$ [21].

instructive, because the reduction over space corresponds to the standard c-map and gives us hypermultiplets in three-dimensional Minkowski space-time, while the reduction over time is the new para-c-map and gives us hypermultiplets in three-dimensional Euclidean space.

Before performing the reduction, we rewrite the Lorentzian vector multiplet Lagrangian in terms of real fields. Above we noted that the intrinsic characteristic of an affine special Kähler manifold is the existence of the special connection ∇, which is, in particular, flat, torsionfree and symplectic [18]. The corresponding flat symplectic coordinates are

$$x^I = \mathrm{Re}X^I \quad y_I = \mathrm{Re}F_I \ . \tag{2.5}$$

Note that since F is an arbitrary holomorphic function, these real coordinates are related in a complicated way to the special coordinates X^I. The real coordinates x^I, y_I are flat (or affine) coordinates with respect to ∇, i.e., $\nabla dx^I = 0 = \nabla dy_I$, and they are symplectic (or Darboux coordinates), because the symplectic form on M_{VM} is $\omega = 2dx^I \wedge dy_I$. While in special coordinates the metric of M_{VM} can be expressed in terms of the prepotential by (2.2), the metric has a Hesse potential when using the real coordinates $q^a = (x^I, y_I)$, where $a = 1, \ldots, 2n$ [18, 19]:

$$g_{ab} = \frac{\partial^2 H}{\partial q^a \partial q^b} \ . \tag{2.6}$$

The Hesse potential is related to the imaginary part of the prepotential by a Legendre transform [24]:

$$H(x, y) = 2\mathrm{Im}F(x + iu) - 2u^I y_I \ . \tag{2.7}$$

The two parametrizations of the metric on M_{VM} are related by

$$ds^2 = -\tfrac{1}{2} N_{IJ} dX^I d\overline{X}^J = -g_{ab} dq^a dq^b \ . \tag{2.8}$$

In order to rewrite the Lagrangian (2.1) completely in terms of real fields, we express the (anti)selfdual field strength $F^{I\pm}_{mn}$ in terms of the field strength $F^I_{mn} = F^{I+}_{mn} + F^{I-}_{mn}$ and their Hodge-duals $\tilde{F}^I_{mn} = i(F^{I+}_{mn} - F^{I-}_{mn})$. The result is

$$L^{\text{4d VM}}_{\text{bos}} = -g_{ab} \partial_m q^a \partial^m q^b - \tfrac{1}{4} N_{IJ} F^I_{mn} F^{Jmn} + \tfrac{1}{4} R_{IJ} F^I_{mn} \tilde{F}^{Jmn} \ , \tag{2.9}$$

where

$$\begin{aligned} R_{IJ} &= F_{IJ} + \overline{F}_{IJ} \ , \\ N_{IJ} &= i(F_{IJ} - \overline{F}_{IJ}) = \partial_I \partial_{\overline{J}} \left(-i(X^I \overline{F}_I - F_I \overline{X}^I) \right) \ . \end{aligned} \tag{2.10}$$

We now perform the reduction of the Lagrangian (2.9) from four to three dimensions. We treat the reduction over space and over time in parallel. In the following formulae, $\epsilon = 1$ refers to reduction over time, which gives a Euclidean three-dimensional theory, while $\epsilon = -1$ refers to reduction over space. By reduction, one component of each gauge field becomes a scalar. We define:

$$p^I = A^{I|0} \text{ for } \epsilon = 1 \;, \quad p^I = A^{I|3} \text{ for } \epsilon = -1 \;. \tag{2.11}$$

Moreover, the n three-dimensional gauge fields $A^{I|\hat{m}}$ obtained from dimensional reduction[5] can be dualized into n further real scalars s_I. Denoting the new scalars by

$$(\hat{q}_a) = (s_I, 2p^I) \;, \tag{2.12}$$

the reduced bosonic Lagrangian takes the following, remarkably simple form:

$$L_{HM} = -g_{ab}(q)\partial_i q^a \partial^i q_b + \epsilon g^{ab}(q)\partial_i \hat{q}_a \partial^i \hat{q}_b \;, \tag{2.13}$$

where $g^{ab}(q)$ is the inverse of $g_{ab}(q)$. In this parametrization it is manifest that the hypermultiplet target space with metric $(g_{ab}(q)) \oplus (-\epsilon g^{ab}(q))$ is $N = M_{HM} = T^*M_{VM}$. The geometry underlying this Lagrangian was presented in detail in [6] for $\epsilon = 1$, and works analogously for $\epsilon = -1$. Here we give a brief summary. The special connection ∇ on $M = M_{VM}$, can be used to define a decomposition

$$T_\xi N = \mathcal{H}^\nabla_\xi \oplus T^v_\xi N \simeq T_q M \oplus T^*_q M \;, \tag{2.14}$$

where $\xi \in N$ is a point on N (with local coordinates $(q^a, \hat{q}_a)$), $q = \pi(\xi) \in M$ is its projection onto M, $\mathcal{H}^\nabla_\xi$ is the horizontal subspace with respect to the connection ∇ and $T^v_\xi N$ is the vertical subspace. The identification with $T_qM \oplus T^*_qM$ is canonical, and the scalar fields $q^a, \hat{q}_a$ obtained by dimensional reduction are adapted to the decomposition. One can then define a complex structure J_1 on N, which acts on $T_\xi N \simeq T_qM \oplus T^*_qM$ by multiplication with

$$J_1 := J_1^\nabla = \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix} \;, \tag{2.15}$$

where J, J^* denote the action of the complex structure J of M on TM and T^*M, respectively. Let us now consider the Euclidean case $\epsilon = 1$ for definiteness. Using the Kähler form ω on M, one can further define

$$J_2 = \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix} \;, \tag{2.16}$$

[5]The three-dimensional vector index takes values $\hat{m} = 0, 1, 2$ for $\epsilon = -1$ and $\hat{m} = 1, 2, 3$ for $\epsilon = 1$.

where ω is interpreted as a map $T_q M \to T_q^* M$. This is a para-complex structure, $J_2^2 = \mathbb{1}$. Moreover, $J_3 = J_1 J_2$ is a second para-complex structure, and J_1, J_2, J_3 satisfy a modified version of the quaternionic algebra known as the para-quaternionic algebra. Thus, (J_1, J_2, J_3) is a para-hyper-complex structure on N. When defining, as in (2.13), the metric on N by

$$g_N = \begin{pmatrix} g & 0 \\ 0 & -g^{-1} \end{pmatrix} , \tag{2.17}$$

where g is the metric on M, then J_1 is an isometry, while J_2, J_3 are anti-isometries. This means that (J_1, J_2, J_3, g_N) is a para-hyper Hermitian structure.[6] Moreover, the structures J_α, $\alpha = 1, 2, 3$ are parallel with respect to the Levi-Civita connection on N. Thus the metric g_N is para-hyper Kähler, meaning that it is Kähler with respect to J_1 and para-Kähler with respect to J_2, J_3. The case $\epsilon = -1$ works analogously. Here one finds three complex structures satisfying the quaternionic algebra, and the metric defined by (2.13) is hyper-Kähler.

One can introduce (para-)complex fields such that one of the complex or (para-)complex structures becomes manifest in the three-dimensional Lagrangian [7, 6]. In these coordinates the Lagrangian is more complicated, and the geometrical structure reviewed above is less clear. Moreover one has singled out one of the three (para-)complex structures. Thus working in real coordinates has advantages, which should be exploited further in the future. Note in particular that for the c-map in local supersymmetry, the target space of hypermultiplets is quaternion-Kähler for Lorentzian space-time, while it is expected to be para-quaternion-Kähler for Euclidean space-time. In general, the structures J_α occuring in this case will not be integrable. Hence, combining real scalar fields into (para-)complex fields is not natural, as these fields do not define local (para-)complex coordinates.

3 The black hole variational principle

We now turn to our second topic, which is BPS black hole solutions in $N = 2$ supergravity coupled to n vector multiplets. The underlying Lagrangian was constructed using the superconformal calculus [22].[7] The idea of this method is to start with a theory of $n + 1$ rigidly supersymmetric vector multiplets and to impose that the theory is invariant under superconformal

[6] Also note that J_1, J_2, J_3 are integrable, which follows from the integrability of J.

[7] Further references on $N = 2$ vector multiplet Lagrangians and the superconformal calculus include [25, 26, 1, 27].

transformations. This implies that the prepotential has to be homogenous of degree 2 in addition to being holomorphic:

$$F(\lambda X^I) = \lambda^2 F(X^I)\ , \quad \lambda \in \mathbb{C}^*\ , \tag{3.18}$$

where now $I = 0, 1, \ldots, n$. Next one 'gauges' the superconformal transformation, that is one makes the Lagrangian locally superconformally invariant by introducing suitable connections. The new fields entering through this process are encoded in the so-called Weyl multiplet.[8] Finally, one imposes gauge conditions which reduce the local superconformal invariance to a local invariance under standard (Poincaré) supersymmetry. Through the gauge conditions some of the fields become functions of the others. In particular, only n out of the $n+1$ complex scalars are independent. A convenient choice for the independent scalars is

$$z^A = \frac{X^A}{X^0}\ , \tag{3.19}$$

where $A = 1, \ldots, n$. This provides a set of special coordinates for the scalar manifold M_{VM}. In contrast, all $n+1$ gauge fields remain independent. While one particular linear combination, the so-called graviphoton, belongs to the Poincaré supergravity multiplet, the other n gauge fields sit in vector multiplets, together with the scalars z^A. The Weyl multiplet also provides physical degrees of freedom, namely the graviton and two gravitini.

From the underlying rigidly superconformal theory the supergravity theory inherits the invariance under symplectic rotations. For the gauge fields this is manifest, as $(F^I_{mn}, G_{I|mn})$ transforms as a vector under $Sp(2(n+1), \mathbb{R})$.[9] In the scalar sector (X^I, F_I), where $F_I = \partial_I F$, also transforms as a vector, while the gravitational degrees of freedom are invariant. To maintain manifest symplectic invariance, it is advantagous to work with (X^I, F_I) instead of z^A.

The underlying geometry can be described as follows [18, 19, 20]: the fields X^I provide coordinates on the scalar manifold of the associated rigidly superconformal theory. This manifold has complex dimension $n+1$, and can be immersed into $T^*\mathbb{C}^{n+1} \simeq \mathbb{C}^{2(n+1)}$ just as described in the previous section. The additional feature imposed by insisting on superconformal invariance is that the prepotential is homogenous of degree 2. Geometrically

[8] One also needs to add a further 'compensating multiplet', which can be taken to be a hypermultiplet. We won't need to discuss this technical detail here. See for example [17] for more background material and references.

[9] The dual gauge fields $G_{I|mn}$ were introduced at the beginning of section 2.

this implies that the resulting affine special Kähler manifold is a complex cone. The scalar manifold of the supergravity theory is parametrized by the scalars z^A and has complex dimension n. It is obtained from the manifold of the rigidly superconformal theory by gauge-fixing the dilatation and $U(1)$ symmetry contained in the superconformal algebra. This amounts to taking the quotient of the complex cone with respect to the $\mathbb{C}^*$-action $X^I \to \lambda X^I$. Thus the scalar manifold M_{VM} is the base of the conical affine special Kähler manifold $C(M_{VM})$ of the rigid theory. For many purposes, including the study of black hole solutions, it is advantagous to work on $C(M_{VM})$ instead of M_{VM}. In particular, this allows to maintain manifest symplectic covariance, as we already noted. In physical terms this means that one can postpone the gauge-fixing of the dilatation and $U(1)$ transformations. The manifolds which can be obtained from conical affine special Kähler manifolds by a $\mathbb{C}^*$-quotient are called projective special Kähler manifolds. These are the target spaces of vector multiplets coupled to supergravity. All couplings in the Lagrangian and all relevant geometrical data of M_{VM} are encoded in the prepotential. In particular, the affine special Kähler metric on $C(M_{VM})$ has Kähler potential

$$K_C(X^I, \overline{X}^I) = -i(X^I \overline{F}_I - F_I \overline{X}^I) \ , \tag{3.20}$$

while the projective special Kähler metric on M_{VM} has Kähler potential

$$K(z^A, \overline{z}^{\overline{B}}) = -\log \left(-i(X^I \overline{F}_I - F_I \overline{X}^I) \right) \ , \tag{3.21}$$

with corresponding metric

$$g_{a\overline{b}} = \frac{\partial^2 K(z^A, \overline{z}^{\overline{B}})}{\partial z^a \partial \overline{z}^b} \ . \tag{3.22}$$

In string theory the four-dimensional supergravity Lagrangians considered here are obtained by dimensional reduction of the ten-dimensional string theory on a compact six-dimensional manifold X and restriction to the massless modes. Then the scalar manifold M_{VM} is the moduli space of X. It turns out that the moduli spaces of Calabi-Yau threefolds provide natural realizations of special Kähler geometry [33]. Consider for instance the Calabi-Yau compactification of type-IIB string theory. In this case M_{VM} is the moduli space of complex structures of X, the cone M_{VM} is the moduli space of complex structures together with a choice of the holomorphic top-form, and $T^*\mathbb{C}^{n+1} \simeq \mathbb{C}^{2(n+1)}$ is $H^3(X, \mathbb{C})$, see [34].

Let us then discuss BPS black hole solutions of $N = 2$ supergravity with n vector multiplets. These are static, spherically symmetric solutions of the field equations, which are asymptotically flat, have regular event horizons, and possess 4 Killing spinors. The concept of a Killing spinor is analogous to that of a Killing vector. Let us denote the dynamical fields collectively by Φ, and denote a supersymmetry transformation with parameter $\varepsilon(x)$ by $\delta_{\varepsilon(x)}\Phi$. The supersymmetry transformation parameter is a spinor, and in supergravity it depends on space-time. If Φ_0 is a solution to the field equations such that

$$\delta_{\varepsilon(x)}\Phi_0 = 0 \,, \tag{3.23}$$

for some non-vanishing spinor field $\varepsilon(x)$, then Φ_0 is called a BPS solution (or supersymmetric solution). The corresponding spinor field $\varepsilon(x)$ is called a Killing spinor field. We restrict our attention to purely bosonic solutions, that is all fermionic fields are identically zero in the background.

Let us first have a look at pure four-dimensional $N = 2$ supergravity, i.e., we drop the vector multiplets, $n = 0$. The bosonic part of this theory is precisely the Einstein-Maxwell theory. In pure $N = 2$ supergravity, BPS solutions have been classified [35, 36, 37]. The number of linearly independent Killing spinor fields can be 8,4 or 0. This can be seen, for example, by investigating the integrability conditions of the Killing spinor equation. Solutions with 8 Killing spinors are maximally supersymmetric and therefore considered as supersymmetric ground states. Examples are Minkowski space and $AdS^2 \times S^2$. Solutions with 4 Killing spinors are called $\frac{1}{2}$-BPS, because they are invariant under half as many supersymmetries as the ground state. They are solitonic realisations of states sitting in BPS representations. For static $\frac{1}{2}$-BPS solutions the space-time metric takes the form [35, 36]

$$ds^2 = -e^{-2f(\vec{x})}dt^2 + e^{2f(\vec{x})}d\vec{x}^2 \,, \tag{3.24}$$

where $\vec{x} = (x_1, x_2, x_3)$ are space-like coordinates and the function $f(\vec{x})$ must be such that $e^{f(\vec{x})}$ is a harmonic function with respect to $\vec{x}$. The solutions also have a non-trivial gauge field, which likewise can be expressed in terms of $e^{f(\vec{x})}$. This class of solutions of Einstein-Maxwell theory is known as the Majumdar-Papapetrou solutions. The only Majumdar-Papapetrou solutions without naked singularities are the multi-centered extremal Reissner-Nordstrom solutions, which describe static configurations of extremal black holes, see for example [38]. If one imposes in addition spherical symmetry, one arrives at the extremal Reissner-Nordstrom solution describing a single charged black hole. In this case the metric takes the form

$$ds^2 = -e^{-2f(r)}dt^2 + e^{2f(r)}(dr^2 + r^2 d\Omega^2) \,, \tag{3.25}$$

where r is a radial coordinate and $d\Omega^2$ is the line element on the unit two-sphere. The harmonic function takes the form

$$e^{f(r)} = 1 + \frac{q^2 + p^2}{r} \ , \tag{3.26}$$

where q, p are the electric and magnetic charge with respect to the graviphoton. The solution has two asymptotic regimes. In one limit, $r \to \infty$, it becomes asymptotically flat: $e^f \to 1$. In the other limit, $r \to 0$, which is the near-horizon limit, it takes the form

$$ds^2 = -\frac{r^2}{q^2 + p^2} dt^2 + \frac{q^2 + p^2}{r^2} dr^2 + (q^2 + p^2) d\Omega^2 \ . \tag{3.27}$$

This is a standard form for the metric of $AdS^2 \times S^2$. The area of the two-sphere, which is the area of the event horizon of the black hole, is given by $A = 4\pi(q^2 + p^2)$. The two limiting solutions, flat Minkowski space-time and $AdS^2 \times S^2$ are among the fully supersymmetric solutions with 8 Killing spinors that we mentioned before. Thus, the extremal Reissner-Nordstrom black hole interpolates between two supersymmetric vacua [28]. This is a property familiar from two-dimensional kink solutions, and motivates the interpretation of supersymmetric black hole solutions as solitons, i.e., as particle-like collective excitations.

Let us now return to $N = 2$ supergravity with an arbitrary number n of vector fields. We are interested in solutions which generalize the extremal Reissner-Nordstrom solution. Therefore we impose that the solution should be $\frac{1}{2}$-BPS, static, spherically symmetric, asymptotically flat, and that it should have a regular event horizon.[10] Besides a non-flat metric, the solution can now contain $n+1$ non-vanishing gauge fields and n non-constant scalar fields. For any $\frac{1}{2}$-BPS solution, which is static and spherically symmetric, the metric can be brought to the form (3.26) [16]. The condition that the solution is static and spherically symmetric is understood in the strong sense, i.e., it also applies to the gauge fields and scalars. Thus gauge fields and scalars are functions of the radial coordinate r, only. Moreover the electric and magnetic fields are spherically symmetric, which implies that each field strength $F^I_{mn}(r)$ has only two independent components (see for example Appendix A of [17] for more details).

The electric and magnetic charges carried by the solution are defined

[10] This excludes both naked singularties and null singularities, where the horizon coincides with the singularity and has vanishing area.

through flux integrals of the field strength over asymptotic two-spheres:

$$(p^I, q_I) = \frac{1}{4\pi} \left(\oint F^I , \oint G_I \right) , \tag{3.28}$$

where F^I, G_I are the two-forms associated with the field strength F^I_{mn} and their duals G_{Imn}. As a consequence, the charges transform as a vector under symplectic transformations. By contracting the charges with the scalars one obtains the symplectic function

$$Z = p^I F_I - q_I X^I . \tag{3.29}$$

This field is often called the central charge, which is a bit misleading because Z is a function of the fields X^I and F_I and therefore a function of the scalar fields z^A, which are space-time dependent.[11] Hence, in the backgrounds we consider, Z is a function of the radial coordinate r. However, when evaluating this field in the asymptotically flat limit $r \to \infty$, it computes the electric and magnetic charge carried by the graviphoton, which combine into the complex central charge of the $N = 2$ algebra [39].

In particular, the mass of the black hole is given by

$$M = |Z|_\infty = M(p^I, q_I, z^A(\infty)) . \tag{3.30}$$

Thus BPS black holes saturate the mass bound implied by the supersymmetry algebra. Note that the mass does not only depend on the charges, but also on the values of the scalars at infinity, which can be changed continuously.

The other asymptotic regime is the event horizon. If the horizon is regular, then the solution must be fully supersymmetric in this limit [11]. Thus, while the bulk solution has 4 Killing spinors, both asymptotic limits have 8. In the near horizon limit, the metric (3.26) takes the form

$$ds^2 = -\frac{r^2}{|Z|^2_{\text{hor}}} dt^2 + \frac{|Z|^2_{\text{hor}}}{r^2} dr^2 + |Z|^2_{\text{hor}} d\Omega^2 , \tag{3.31}$$

where $|Z|^2_{\text{hor}}$ is the value of $|Z|^2$ at the horizon. As in the extremal Reissner-Nordstrom solution, this is $AdS^2 \times S^2$. The area of the two-sphere, which is

[11] One can analyse BPS solutions without imposing the gauge conditions which fix the superconformal symmetry, and in fact it is advantagous to do so [15, 16]. Then the scalars are encoded in the fields $X^I(r)$, which are subject to gauge transformations. Once gauge conditions are imposed, one can express $Z(r)$ in terms of the physical scalar fields $z^A(r)$. See [17] for more details.

the area of the event horizon, is given by $A = 4\pi|Z|^2_{\text{hor}}$. Hence the Bekenstein Hawking entropy is

$$\mathcal{S}_{\text{macro}} = \frac{A}{4} = \pi|Z|^2_{\text{hor}} \; . \tag{3.32}$$

A priori, $\mathcal{S}_{\text{macro}}$ depends on both the charges and the values of the scalars at the horizon, and one might expect that one can change the latter continuously. This would be incompatible with relating $\mathcal{S}_{\text{macro}}$ to a statistical entropy $\mathcal{S}_{\text{micro}}$ which counts states. But it turns out that the values of the scalar fields at the horizon are themselves determined in terms of the charges. Here, it is convenient to define $Y^I = \overline{Z} X^I$ and $F_I = F_I(Y) = \overline{Z} F_I(X)$.[12] In terms of these variables, the black hole attractor equations [11], which express the horizon values of the scalar fields in terms of the charges, take the following form:

$$\left(\begin{array}{c} Y^I - \overline{Y}^I \\ F_I - \overline{F}_I \end{array} \right)_{\text{hor}} = i \left(\begin{array}{c} p^I \\ q_I \end{array} \right) \; . \tag{3.33}$$

The name attractor equations refers to the behaviour of the scalar fields as functions of the space-time radial coordinate r. While the scalars can take arbitrary values at $r \to \infty$, they flow to fixed points, which are determined by the charges, for $r \to 0$. This fixed point behaviour follows when imposing that the event horizon is regular. Alternatively, one can show that to obtain a fully supersymmetric solution with geometry $AdS^2 \times S^2$ the scalars need to take the specific values dictated by the attractor equations [16]. This is due to the presence of non-vanishing gauge fields. The gauge fields in $AdS^2 \times S^2$ are covariantly constant, so that this can be viewed as an example of a flux compactification. In contrast, Minkowski space is also maximally supersymmetric, but the scalars can take arbitrary constant values, because the gauge fields vanish. In type-II Calabi-Yau compactifications, the radial dependence of the scalar fields defines a flow on the moduli space, which starts at an arbitrary point and terminates at a fixed point corresponding to an 'attractor Calabi Yau.' Since the electric and magnetic charges (p^I, q_I), which determine the fixed point, take discrete values, such attractor threefolds sit at very special points in the moduli space. This has been studied in detail in [29].

Using the fields Y^I instead of X^I to parametrize the scalars simplifies formulae and has the advantage that the Y^I are invariant under the $U(1)$ transformations of the superconformal algebra. Note that

$$|Z|^2 = p^I F_I - q_I Y^I \; , \tag{3.34}$$

[12]Note that F_I is homogenous of degree 1.

which is easily seen using the homogeneity properties of the prepotential. Geometrically, going from X^I to Y^I corresponds to a non-holomorphic diffeomorphism of $C(M_{VM})$, which, however, acts trivially on M_{VM}. Note in particular that

$$z^A = \frac{X^A}{X^0} = \frac{Y^A}{Y^0} \ . \tag{3.35}$$

We now turn to the black hole variational principle, which was found in [14] and generalized recently in [13], motivated by the observations of [12]. First, define the entropy function

$$\Sigma(Y^I, \overline{Y}^I, p^I, q_I) = \mathcal{F}(Y^I, \overline{Y}^I) - q_I(Y^I + \overline{Y}^I) + p^I(F_I + \overline{F}_I) \tag{3.36}$$

and the black hole free energy

$$\mathcal{F}(Y^I, \overline{Y}^I) = -i\left(\overline{Y}^I F_I - Y^I \overline{F}_I\right) \ . \tag{3.37}$$

The reason for our choice of terminology will become clear later. Now we impose that the entropy function is stationary, $\delta\Sigma = 0$, under variations of the scalar fields $Y^I \to Y^I + \delta Y^I$. Using that the prepotential is homogenous of degree two, it is easy to see that the conditions for Σ being stationary are precisely the black hole attractor equations (3.33). Furthermore, at the attractor point we find that[13]

$$\begin{aligned} \mathcal{F}_{\rm attr} &= -i\left(\overline{Y}^I F_I - Y^I \overline{F}_I\right)_{\rm attr} = \left(q_I Y^I - p^I F_I\right)_{\rm attr} \\ &= \left(q_I \overline{Y}^I - p^I \overline{F}_I\right)_{\rm attr} = -|Z|^2_{\rm attr} \end{aligned} \tag{3.38}$$

and therefore

$$\Sigma_{\rm attr} = |Z|^2_{\rm attr} = \tfrac{1}{\pi}\mathcal{S}_{\rm macro}(p^I, q_I) \ . \tag{3.39}$$

Thus, up to a constant factor, the entropy is obtained by evaluating the entropy function at its critical point. Moreover, a closer look at the variational principle shows us that, again up to a factor, the black hole entropy $\mathcal{S}_{\rm macro}(p^I, q_I)$ is the Legendre transform of the free energy $\mathcal{F}(Y^I, \overline{Y}^I)$, where the latter is considered as a function of $x^I = \mathrm{Re}(Y^I)$ and $y_I = \mathrm{Re}(F_I)$. At this point the real variables discussed in the previous section become important again. Note that the change of variables $(Y^I, \overline{Y}^I) \to (x^I, y_I)$ is well defined provided that $\mathrm{Im}(F_{IJ})$ is non-degenerate. This assumption will be

[13] From now on we use the label 'attr' instead of 'hor' to indicate that quantities are evaluated at the black hole horizon.

satisfied in general, but breaks down in certain string theory applications, where one reaches the boundary of the moduli space.[14]

We are therefore led to rewrite the variational principle in terms of real variables. First, recall that the Hesse potential $H(x^I, y_I)$ is the Legendre transform of (two times) the imaginary part of the prepotential, see (2.7).[15] This Legendre transform replaces the independent variables (x^I, u^I)= (Re(Y^I), Im(Y^I)) by the independent variables (x^I, y_I)=(Re(Y^I), Re(F_I)) and therefore implements the change of variables $(Y^I, \overline{Y}^I) \to (x^I, y_I)$. Using (2.7) we find

$$H(x^I, y_I) = -\tfrac{i}{2}(\overline{Y}^I F_I - \overline{F}_I Y^I) = \tfrac{1}{2}\mathcal{F}(Y^I, \overline{Y}^I) \ . \tag{3.40}$$

Thus, up to a factor, the Hesse potential is the black hole free energy. We can now express the entropy function in terms of the real variables:

$$\Sigma(x^I, y_I, p^I, q_I) = 2H(x^I, y_I) - 2q_I x^I + 2p^I y_I \ . \tag{3.41}$$

If we impose that Σ is stationary with respect to variations of x^I and y_I, we get the black hole attractor equations in real variables:

$$\frac{\partial H}{\partial x^I} = q_I \ , \quad \frac{\partial H}{\partial y_I} = -p^I \ . \tag{3.42}$$

Plugging this back into the entropy function we obtain

$$\mathcal{S}_{\text{macro}} = 2\pi \left(H - x^I \frac{\partial H}{\partial x^I} - y_I \frac{\partial H}{\partial y_I} \right)_{\text{attr}} \ . \tag{3.43}$$

Thus, up to a factor, the black hole entropy is the Legendre transform of the Hesse potential. This is an intriguing observation, because it relates the black hole entropy, which is a space-time quantity, in a very direct way to the special geometry encoding the scalar dynamics. In string theory compactifications this relates the geometry of four-dimensional space-time to the geometry of the compact internal space X.

We can also relate the black hole free energy to another quantity of special geometry. In terms of complex variables we observe that

$$\mathcal{F}(Y^I, \overline{Y}^I) = K_C(Y^I, \overline{Y}^I) := i(\overline{Y}^I F_I - \overline{F}_I Y^I) \ . \tag{3.44}$$

[14]See for example [13] for a discussion of some of the implications.

[15]Note that this is the Hesse potential of the affine special Kähler metric on $C(M_{VM})$. The projective special Kähler metric on M_{VM} is obtained by the $\mathbb{C}^*$-quotient.

Comparing to (3.20) it appears that we should interpet $K_C(Y^I, \overline{Y}^I)$ as the Kähler potential of an affine special Kähler metric on $C(M_{VM})$. Since the diffeomorphism $X^I \rightarrow Y^I$ is non-holomorphic, this is not the same special Kähler structure as with (3.20). However, we already noted that the diffeomorphism acts trivially on M_{VM}, see (3.35). Moreover it is easy to see that when taking the quotient with respect to the $\mathbb{C}^*$-action $Y^I \rightarrow \lambda Y^I$, then the resulting projective special Kähler metric with Kähler potential $K(Y^I, \overline{Y}^I) = -\log K_C(Y^I, \overline{Y}^I)$ is the same as the one derived from (3.21), because the two Kähler potentials differ only by a Kähler transformation. It appears that in the context of black hole solutions the affine special Kähler metric associated with the rescaled scalars Y^I is of more direct importance than the one based on the X^I. The same remark applies to the Hesse potential, which depends on the real coordinates associated to Y^I.

Note that it is more natural to identify the free energy with the Hesse potential than the Kähler potential. The first reason is that the various Legendre transforms involve the real and not the complex coordinates. The second reason is that, as we will discuss below, we need to generalize the supergravity Lagrangian in order to take into account certain corrections appearing in string theory. We will see that this works naturally by introducing a generalized Hesse potential.

Before turning to this subject, we also remark that the terms in the entropy function (3.36) which are linear in the charges, and which induce the Legendre transform, have a further interpretation in terms of supersymmetric field theory. Namely,

$$W = q_I Y^I - p^I F_I \tag{3.45}$$

has the form of an $N = 2$ superpotential. The four-dimensional supergravity Lagrangian we are studying does not have a superpotential. However, the near-horizon solution has the form $AdS^2 \times S^2$ and carries non-vanishing, covariantly constant gauge fields. The dimensional reduction on S^2 is a flux compactification, with fluxes parametrized by (p^I, q_I), and the resulting two-dimensional theory will possess a superpotential. This also provides an alternative interpretation of the attractor mechanism, as the resulting scalar potential will lift the degeneracy of the moduli.

So far we only considered supergravity Lagrangians which contain terms with at most two derivatives. The effective Lagrangians derived from string theory also contain higher derivative terms, which modify the dynamics at short distances. These terms describe interactions between the massless states which are mediated by massive string states. While the effective

Lagrangian does not contain the massive string states explicitly, it is still possible to describe their impact on the dynamics of the massless states.

In $N = 2$ supergravity a particular class of higher derivative terms can be taken into account by giving the prepotential an explicit dependence on an additional complex variable Υ, which is proportional to the lowest component of the Weyl multiplet [32, 40]. The resulting function $F(Y^I, \Upsilon)$ is required to be holomorphic in all its variables, and to be (graded) homogenous of degree two:[16]

$$F(\lambda Y^I, \lambda^2 \Upsilon) = \lambda^2 F(Y^I, \Upsilon) \ . \tag{3.46}$$

Assuming that it is analytic at $\Upsilon = 0$ one can expand it as

$$F(Y^I, \Upsilon) = \sum_{g=0}^{\infty} F^{(g)}(Y^I) \Upsilon^g \ . \tag{3.47}$$

Then $F^{(0)}(Y^I)$ is the prepotential, while the functions $F^{(g)}(Y^I)$ with $g > 0$ appear in the Lagrangian as the coefficients of various higher-derivative terms. These include in particular terms quadratic in the space-time curvature, and therefore one often loosely refers to the higher derivative tems as R^2-terms.

In type-II Calabi Yau compactifications the functions $F^{(g)}(Y^I)$ can be computed using (one of) the topologically twisted version(s) of the theory [31]. They are related to the partition functions $Z^{(g)}_{\text{top}}$ of the topologically twisted string on a world sheet with genus g by $F^{(g)} = \log Z^{(g)}_{\text{top}}$. Therefore they are called the (genus-g) topological free energies.

It was shown in [15, 16] that the black hole attractor mechanism can be generalized to the case of Lagrangians based on a general function $F(Y^I, \Upsilon)$. The attractor equations still take the form (3.33), but the prepotential is replaced by the full function $F(Y^I, \Upsilon)$. The additional variable Υ takes the value $\Upsilon = -64$ at the horizon. In gravitational theories with higher derivative terms the black hole entropy is no longer given by the Bekenstein-Hawking area law $\mathcal{S}_{\text{macro}} = \frac{A}{4}$. A generalized formula was derived in [41] by insisting on the validity of the first law of black hole mechanics. The evaluation of the resulting formula for $N = 2$ supergravity gives [15]

$$\mathcal{S}_{\text{macro}}(q^I, p_I) = \pi \left(|Z|^2 + 4 \text{Im}(\Upsilon F_\Upsilon) \right)_{\text{attr}} \ , \tag{3.48}$$

[16] Since we are interested in black hole solutions, we take this function to depend on the rescaled fields Y^I, Υ.

where $F_\Upsilon = \partial_\Upsilon F$.[17]

While the first term corresponds to the area law, the second term is an explicit modification which depends on the coefficients $F^{(g)}$, $g > 0$ of the higher derivative terms.

It was shown in [13] that the variational principle generalizes to the case with R^2-terms. The black hole free energy $\mathcal{F}$ is now proportional to a generalized Hesse potential $H(x^I, y_I, \Upsilon, \overline{\Upsilon})$, which in turn is proportional to the Legendre transform of the imaginary part of the function $F(Y^I, \Upsilon)$:

$$H(x^I, y_I, \Upsilon, \overline{\Upsilon}) = 2\text{Im}F(x^I + iu^I, \Upsilon) - 2y_I u^I \ .$$

In terms of complex fields Y^I this becomes

$$\begin{aligned} H(x^I, y_I, \Upsilon, \overline{\Upsilon}) &= -\tfrac{i}{2}(\overline{Y}^I F_I - \overline{F}_I Y^I) - i(\Upsilon F_\Upsilon - \overline{\Upsilon F}_{\overline{\Upsilon}}) \\ &= \tfrac{1}{2}\mathcal{F}(Y^I, \overline{Y}^I, \Upsilon, \overline{\Upsilon}) \ . \end{aligned} \tag{3.49}$$

The entropy function (3.41), the attractor equations (3.42) and the formula for the entropy (3.43), which now includes correction terms to the area law, remain the same, except that one uses the generalized Hesse potential. From (3.49) it is obvious that the black hole free energy naturally corresponds to a generalized Hesse potential (defined by the Legendre transform of the prepotential) and not to a 'generalized Kähler potential', which would only give rise to the first term on the right hand side of (3.49).

There is a second class of correction terms in string-effective supergravity Lagrangians. Quantum corrections involving the massless fields lead to modifications which correspond to adding non-holomorphic terms to the function $F(Y^I, \Upsilon)$. The necessity of such non-holomorphic terms can be seen by observing that otherwise the invariance of the full string theory under T-duality and S-duality is not captured by the effective field theory. In particular, one can show that the black hole entropy can only be T- and S-duality invariant if non-holomorphic corrections are taken into account [30].[18] From the point of view of string theory the presence of these terms is related to a holomorphic anomaly [31, 32].

As the holomorphic R^2-corrections, the non-holomorphic corrections can be incorporated into the black hole attractor equations and the black hole variational principle [30, 13]. The non-holomorphic terms are encoded in

[17] At the attractor point, Υ takes the value $\Upsilon = -64$.

[18] We are referring to compactifications with exact T- and S-duality symmetry. These are mostly compactifications with $N = 4$ supersymmetry, which, however, can be studied in the $N = 2$ framework. We refer to [30, 42, 13] for details.

a function $\Omega(Y^I, \overline{Y}^I, \Upsilon, \overline{\Upsilon})$, which is real valued and homogenous of degree two. To incorporate non-holomorphic terms into the variational principle one has to define the generalized Hesse potential as the Legendre transform of $2\mathrm{Im}F + 2\Omega$:

$$H(x^I, \hat{y}_I, \Upsilon, \overline{\Upsilon}) = 2\mathrm{Im}F(x^I + iu^I, \Upsilon, \overline{\Upsilon}) + 2\Omega(x^I, u^I, \Upsilon, \overline{\Upsilon}) - 2\hat{y}_I u^I \ , \quad (3.50)$$

where $\hat{y}_I = y_I + i(\Omega_I - \Omega_{\overline{I}})$ and $\Omega_I = \frac{\partial \Omega}{\partial Y^I}$ and $\Omega_{\overline{I}} = \frac{\partial \Omega}{\partial \overline{Y}^I}$. Up to these modifications, the attractor equations, the entropy function, and the entropy remain as in (3.42), (3.41) and (3.43). Also note from (3.50) that if Ω is harmonic, it can be absorbed into $\mathrm{Im}F$, because it then is the imaginary part of holomorphic function. Thus, the non-holomorphic modifications of the prepotentail correspond to non-harmonic functions Ω.

In terms of the complex variables the attractor equation are

$$\begin{pmatrix} Y^I - \overline{Y}^I \\ F_I + 2i\Omega_I - F_I + 2i\Omega_{\overline{I}} \end{pmatrix} = i \begin{pmatrix} p^I \\ q_I \end{pmatrix} . \quad (3.51)$$

The modified expressions for the free energy and the entropy function can be found in [13].

At this point it is not quite clear what the R^2-corrections and the non-holomorphic corrections mean in terms of special geometry. Since they correspond to higher derivative terms in the Lagrangian, they do not give rise to modifications of the metric on the scalar manifold, which, by definition, is the coefficient of the scalar two-derivative term.[19] It would be very interesting to extend the framework of special geometry such that the functions $F^{(g)}$ get an intrinsic geometrical meaning.

Let us now discuss how the black hole variational principle is related to the results of [12]. It is possible to start from the generalized Hesse potential and to perform partial Legendre transforms by imposing only part of the attractor equations. If this subset of fields is chosen such that the variational principle remains valid, then further extremisation yields the black hole entropy. Specifically, one can solve the magnetic attractor equations $Y^I - \overline{Y}^I = ip^I$ by setting[20]

$$Y^I = \tfrac{1}{2}(\phi^I + ip^I) \ . \quad (3.52)$$

Plugging this back, the entropy function becomes

$$\Sigma(p^I, \phi^I, q_I) = \mathcal{F}_E(p^I, \phi^I, \Upsilon, \overline{\Upsilon}) - q_I \phi^I \ , \quad (3.53)$$

[19] See however [43], where such an interpretation was proposed.

[20] Obviously, $\phi^I = 2x^I$. We use ϕ^I to be consistent with the notation used in [13]. The conventions of [12] are slightly different.

where[21]

$$\mathcal{F}_E(p^I,\phi^I,\Upsilon,\overline{\Upsilon}) = 4\left(\mathrm{Im}F(Y^I,\Upsilon)+\Omega(Y^I,\overline{Y}^I,\Upsilon,\overline{\Upsilon})\right)_{\mathrm{mgn}} \tag{3.54}$$

Here the label 'mgn' indicates that the magnetic attractor equations have been imposed, i.e., $Y^I=\frac{1}{2}(\phi^I+ip^I)$. Both $\mathcal{F}(Y^I,\overline{Y}^I,\Upsilon,\overline{\Upsilon})=2H(x^I,y_I,\Upsilon,\overline{\Upsilon})$ and $\mathcal{F}_E(p^I,\phi^I,\Upsilon,\overline{\Upsilon})$ are interpreted as free energies, which, however, refer to different statistical ensembles. While the microscopic entropy, i.e., the state degeneracy, is defined within a microcanonical ensemble, where the electric and magnetic charges are fixed, the free energy $\mathcal{F}$ belongs to a canonical ensemble, where both electric and magnetic charges fluctuate. The free energy $\mathcal{F}_E$ belongs to a mixed ensemble, where the magnetic charges are fixed while the electric charges fluctuate.

If one imposes that $\Sigma(p^I,\phi^I,q_I)$ is stationary with respect to variations of ϕ^I, then one obtains the electric attractor equations $(F_I-2i\Omega_I)-(\overline{F}_I+2i\Omega_{\overline{I}})=iq_I$ (3.51). Plugging these back one sees that at the stationary point $\Sigma_{\mathrm{attr}}=\frac{1}{\pi}\mathcal{S}_{\mathrm{macro}}(p^I,q_I)$ and that the macroscopic entropy is the partial Legendre transform of the free energy $\mathcal{F}_E(p^I,\phi^I,\Upsilon,\overline{\Upsilon})$.

The observation that the black hole entropy is the Legendre transform of the free energy $\mathcal{F}_E(p^I,\phi^I,\Upsilon,\overline{\Upsilon})$ was made in [12] and restarted the interest in black hole entropy in string theory. A particularly intriguing observation made in [12] is that there appears to be a direct link between the free energy $\mathcal{F}_E(p^I,\phi^I,\Upsilon,\overline{\Upsilon})$ and the topological string. In [12] the holomorphic higher derivative corrections are taken into account, but not the non-holomorphic ones. In this case the free energy is related to a holomorphic prepotential $F(Y^I,\Upsilon)$. Then the black hole free energy is related to the partition function Z_{top} of the topological string by [12]

$$e^{\pi\mathcal{F}_E(p,\phi,\Upsilon,\overline{\Upsilon})}=|Z_{\mathrm{top}}|^2\ . \tag{3.55}$$

The topological partition function is given by $Z_{\mathrm{top}}=e^{F_{\mathrm{top}}}$, where the topological free energy F_{top} equals the generalized prepotential (3.47) up to a conventional (imaginary) prefactor. Therefore (3.55) follows, for holomorphic prepotentials, because the black hole free energy $\mathcal{F}_E$ is the imaginary part of the prepotential. If the thermodynamical interpretation of $\mathcal{F}_E$ is correct, then the number $d(p,q)$ of black hole microstates with charges (p^I,q_I) should be given, at least in the semiclassical limit corresponding to large

[21]We suppressed the dependence of Σ on Υ, but indicated it for $\mathcal{F}_E$ in order to make explicit that we included the higher derivative corrections.

charges, by

$$d(p,q) \approx \int d\phi e^{\pi[\mathcal{F}_E - q\phi]} \ . \tag{3.56}$$

Here $d\phi = \prod_I \phi^I$, and the ϕ^I are taken to be complex and integrated along a contour encircling the origin. The relation (3.56) is intriguing, as it relates the black hole microstates in a very direct way to the topological string partition function. Note that a saddle point evalation of the integral gives

$$d(p,q) \approx e^{\mathcal{S}_{\text{macro}}(p,q)} \ , \tag{3.57}$$

because at the critical point of the integrand we have $\pi[\mathcal{F}_E - q_I \phi^I]_{\text{attr}} = \mathcal{S}_{\text{macro}}(p,q)$. Thus the microscopic entropy $\mathcal{S}_{\text{micro}}(p,q) = \log d(p,q)$ and the macroscopic entropy $\mathcal{S}_{\text{macro}}(p,q)$ argree to leading order in the semiclassical limit.[22]

There are several points concerning the proposal (3.56) which deserve further study. The number of states $d(p,q)$ should certainly be invariant under stringy symmetries such as S-duality and T-duality. In the context of compactifications with $N \geq 2$ supersymmetry, where duality symmetries are realized as symplectic transformation, this also means that $d(p,q)$ should be a symplectic function. However, in the approach of [12] the electric and magnetic charges are treated differently, so that there is no manifest symplectic covariance. A related issue is how to take into account non-holomorphic corrections. While [12] is based on the holomorphic function $F(Y^I, \Upsilon)$, it is clear that non-holomorphic terms have to enter one way or another, because they are needed in order that $d(p,q)$ is duality invariant. A concrete proposal for modifying (3.56) was made in [13]. It is based on the free energy $\mathcal{F}$, i.e., on the generalized Hesse potential, instead of $\mathcal{F}_E$. This allows one to treat electric and magnetic charges on equal footing and to keep manifest symplectic covariance. Then (3.56) is replaced by

$$d(p,q) \approx \int dx d\hat{y} e^{\pi \Sigma(x,\hat{y},p,q)} \ . \tag{3.58}$$

Note that, in absence of R^2- and non-holomorphic corrections, the measure $dxdy = \prod_{I,J} dx^I dy_J$ is proportional to the top power of the symplectic form $dx^I \wedge dy_I$ on $C(M_{VM})$ and therefore symplectically invariant. In the

[22] In general $\mathcal{S}_{\text{macro}}$ and $\mathcal{S}_{\text{micro}}$ are expected to be different, once subleading terms are taken into account. The reason is that $\mathcal{S}_{\text{micro}}$ refers to the microcanonical ensemble, while, according to [12], $\mathcal{S}_{\text{macro}}$ corresponds to the mixed ensemble. It is possible to determine the exact relation between both quantities, at least in principle.

presence of R^2 and non-holomorphic corrections, $dx d\hat{y}$ is the appropriate generalization. Also note that Σ is a symplectic function.

As above, the variational principle ensures that in saddle point approximation we have $d(p,q) \approx \exp(\mathcal{S}_{\rm macro})$, as $\mathcal{S}_{\rm macro}$ is the Legendre transform of the Hesse potential and hence the saddle point value of $\pi\Sigma$. In order to compare (3.58) to (3.56), we can rewrite (3.58) in terms of the complex variables and perform the integral over ImY^I in saddle point approximation, i.e., we perform a Gaussian integration with respect to the subspace where the magnetic attractor equations are satisfied. The result is [13]

$$d(p,q) \approx \int d\phi \sqrt{\Delta^-(p,\phi)} e^{\pi[\mathcal{F}_E - q\phi]} \tag{3.59}$$

and modifies (3.56) in two ways: first, in contrast to [12] we have included non-holomorphic terms into the free energy $\mathcal{F}_E$; second, the integral contains a measure factor $\Delta^-(p,\phi)$, whose explicit form can be found in [13]. The measure factor is needed in order to be consistent with symplectic covariance.

The proposals (3.56) and (3.58) can be tested by comparing the black hole entropy to the microscopic state degeneracy. There are some cases where these are either known exactly, or where at least subleading contributions are accessible. While this chapter is far from being closed, there seems to be agreement by now that (3.56) needs to be modified by a measure factor [44, 43, 13]. In particular, the measure factors extracted from the evaluation of exact dyonic state degeneracies in $N = 4$ compactifications [45] are consistent, at the semiclassical level, with the proposal (3.58) [13].

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