Fuzzy Randomness Simulation of Long Term Infrastructure Projects

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Abstract

The conventional simulation model used in the prediction of long term infrastructure development systems such as Public Private Partnership (PPP)-Build Operate Transfer (BOT) projects assume single probabilistic values for all of the input variables. Traditionally, all the input risks and uncertainties in Monte Carlo Simulation (MCS) are modelled based on probability theory. Its result is shown by a probability distribution function (PDF) and a cumulative distribution function (CDF) which are utilized for analyzing and decision making. In reality, however, some of the variables are estimated based on the expert judgment and others are derived from historical data. Further, the parameters' data of the probability distribution for the simulation model input are subject to change and difficult to predict. Therefore, a simulation model which is capable of handling both types of fuzzy and probabilistic input variables is needed and vital. Recently fuzzy randomness, which is an extension of classical probability theory, provides additional features and improvements for combining fuzzy and probabilistic data to overcome aforementioned shortcomings.

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Fuzzy Randomness Monte Carlo Simulation (FR-MCS) technique is a hybrid simulation method used for risk and uncertainty evaluation. The proposed approach permits any type of risk and uncertainty in the input values to be explicitly defined prior to the analysis and decision making. It extends the practical use of the conventional MCS by providing the capability of choosing between fuzzy sets and probability distributions. This is done to quantify the input risks and uncertainties in a simulation. A new algorithm for generating fuzzy random variables is developed as part of the proposed FR-MCS technique based on the α-cut. FR-MCS output results are represented by fuzzy probability and the decision variables are modelled by fuzzy CDF. The FR-MCS technique is demonstrated in a PPP-BOT case study. The FR-MCS results are compared with those obtained from conventional MCS. It is shown that FR-MCS technique facilitates decision making for both the public and private sectors' decision makers involved in PPP-BOT projects. This is done by determining a negotiation bound for negotiable concession items (NCIs) instead of precise values as are used in conventional MCS's results. This approach prevents prolonged and costly negotiations in development phase of PPP-BOT projects by providing more flexibility for decision makers. Both parties could take advantage of this technique at the negotiation table.

Introduction

A majority of decision making in real projects takes place in an environment in which the objective functions, the constraints and the consequences of possible actions are not precisely known. Moreover, the historical data for long term infrastructure development systems are not normally available and therefore are not directly determinable. Even the available data from previous projects cannot be used directly since in general each project is unique. Difficulties arise if the available information is limited and is of a *fuzzy* rather than of a *stochastic* nature. To use historical data (pervious projects), expert knowledge must be applied. Expert knowledge is especially useful in the development phase when insufficient data are available for negotiations (Attarzadeh, 2007 and 2014).

In order to achieve an appropriate simulation modelling in accordance with the nature of the underlying input data, it is common to use non-deterministic methods. Typically, there are two types of uncertainties: *randomness* due to inherent variability and *fuzziness* due to imprecision and lack of knowledge and information. The former type of uncertainty is often referred to as objective, aleatory and stochastic whereas the latter is often referred to as subjective, imprecise and being a major source of imprecision in many decision processes. The argument in this paper is that there is a need for a differentiation between these two types of imprecision modelling. The distinction between aleatory and imprecise uncertainty plays a particularly important role in the quantitative risk assessment framework (e.g., MCS) that is applied to complex and long term infrastructure development systems.

Risk (randomness characteristic) that refers to probabilistic features is expressed by stochastic models (probability theory and statistical methods) and uncertainty (fuzziness characteristic) that refers to non-probabilistic, also called possibilistic, features is represented by fuzzy sets (theory of possibility). In this research for simplicity, the former is called *stochastic* and the latter is called *fuzzy*.

A fuzzy set (Zadeh, 1965) is a non-probabilistic method used in subjective modelling which overcomes the short comings of the probabilistic methods. Briefly, fuzzy approach is used due to unique aspects of a project, lack of data and subjectivity. In these circumstances subjective judgment and linguistic information obtained from the practitioners of a PPP-BOT project, is often necessary and leads to non-probabilistic uncertainty modelling, or fuzziness.

The distinction between risk (*stochastic*) and uncertainty (*fuzzy*) helps to avoid inappropriate modelling of the non-deterministic input data, especially when both probabilistic and non-probabilistic components appear simultaneously. Because practical situations of risk computation often involve both types of vagueness, methods are needed to combine these two modes of ambiguity representation in the propagation step of simulation. Also, a more vigorous investment decision method that incorporates both risk and uncertainty in simulation and financial modelling and evaluation is needed.

In the current risk assessment practice, both types of uncertainties are represented by means of probability distributions. In other words, to deal quantitatively with imprecision, traditionally the concepts and techniques of probability theory have been employed. This approach has some shortcomings to overcome uncertainties in decision makings (Ferrero and Salicone (2002, 2004, 2005, 2006, 2007); Klir and Yuan (1995); Klir et al. (1997)). The conventional simulation approach presented in the literature review is incapable of fuzzy modelling. Hence, the estimation and simulation of the project data and decision variables can be unreliable. Therefore, other theories and computational methods that propagate uncertainty and variability in exposure and risk assessment are needed.

Having a simulation approach that can deal with stochastic and fuzzy process is fundamental and crucial in risk analysis process of PPP-BOT projects. This paper proposes FR-MCS technique as an adequate hybrid simulation method for uncertainty and risk modelling and their propagation in the simulation model. It presents the procedure regarding risk analysis process and uncertainty propagation in PPP-BOT projects using non-deterministic approaches. The proposed technique generalizes conventional MCS and it can be utilized as an alternative in risk assessment. A comparison of the two approaches relative to their computational requirements, data requirements and availability is provided. Determining negotiation bound and maximizing gains within the bound are the main benefit and advantage of this approach.

The focus of this paper is non-probabilistic features of the simulation input data and the representation of the uncertainty by fuzzy numbers. This approach leads to better informed decision making in negotiations for main parties involved in long term infrastructure projects. In the proposed fuzzy randomness simulation model, random variables and random processes are utilized to cater for the objective input variables and their assessment. Furthermore, fuzzy variables and fuzzy inference system (FIS) are utilized to cater for the subjective input variables and their assessment. Fuzzy probability approach is used to combine these two variables in the simulation process. Then hybrid probabilistic and possibilistic risk and uncertainty assessment technique is carried out instead of the conventional probabilistic risk assessment (PRA). This approach introduces a new concept for the uncertain characterization method that is called uncertainty modelling.

The negotiation simulation problem, including parameters with undeclared and vague probabilities, is solved by a combination of stochastic simulation and fuzzy analysis. The simulation output is then captured in terms of fuzzy probability which denotes success/failure in the project objectives based on the predetermined criteria. In this context, fuzzy probability approach provides a powerful tool to combine the observed data and judgmental information. Fuzzy randomness simultaneously describes objective and subjective information as a fuzzy set of possible probabilistic models over some range of imprecision. This generalized uncertainty and risk model contains fuzziness and randomness as special cases.

The output of a risk analysis based on the conventional MCS is therefore a probability distribution (PDF, CDF) of all probable expected returns. This provides the prospective investors with an incomplete return profile, or risk profile of the project giving all probable outcomes from the investment decision. Conversely, the output of a risk and uncertainty analysis based on the hybrid simulation, FR-MCS, is a set or range of probability distribution (PDF, CDF) of all probable and possible expected returns. This provides the prospective investors with a complete return profile or risk and uncertainty profile of the project showing all probable and possible outcomes from the investment decision.

If sufficient information to generate PDFs and CDFs of the parameters as random variables is not available, but only expert knowledge or scarce data is available to represent the PDF and CDF of the parameters as fuzzy numbers with appropriate membership function, then fuzzy set theory can be utilized to treat the uncertainties in these parameters. In the subjective probabilities approach, there are two cases for possibility risk assessment. In the first case, instead of describing the parameters of PDFs and CDFs as crisp value, e.g. mean (μ) and standard deviation (σ) for normal distribution, they can be described as fuzzy numbers. This case is called Alternative 1, fuzzy randomness. Alternatively, in the context of PPP-BOT projects, fuzzy numbers and parameters are directly used to address lack of data or subjective issues. This case is called Alternative 2, pure fuzzy.

The remaining of this paper is organized as follows: firstly, after a discussion on decision making under uncertainty and risk, the related works in the literature are reviewed. Secondly, conventional MCS and value at risk are considered. Thirdly, FR-MCS technique is proposed and studied in detail. A new algorithm is proposed to generate fuzzy random variables. Finally, FR-MCS is applied for decision making under uncertainty and risk in a real case of PPP-BOT project.

Literature Review

In the previous researches, the risks and uncertainties affecting PPP-BOT projects are not properly considered. In the literature, probabilistic approach of risk modelling is well established for risk analysis (Weiler, 1965; Kalos and Whitlock, 1986; Pawlak, 1991; Ahuja et al., 1994; Maio, 1998; Mun, 2006; Vose, 1996 and 2008; Attarzadeh, 2007). However, the recent criticisms of the probabilistic characterization of uncertainty claim that traditional probability theory is not capable of capturing subjective uncertainty. Thus, the use of probability theory is not a reasonable approach to model the uncertainty. In this case, the possibility theory should be considered (Dubois and Prade, 1988; Pedrycz and Gomide, 1998; Ferrero and Salicone 2002, 2004, 2005, 2006, 2007; Klir and Yuan, 1995; Klir et al., 1997; Moore et al., 2009; Attarzadeh, 2014).

Most researchers attempt to eliminate or transform one type of uncertainty to another before performing a simulation. Wonneberger et al. (1995) Dubois and et al. (2004) presented a possibility to probability transformation. Since fuzzy logic and probability theory reflect different types of uncertainty, conceptually this transformation is not acceptable (Pedrycz and Gomide, 1998; Ferrero and Salicone (2002, 2004, 2005, 2006, 2007); Klir and Yuan (1995); Klir et al. (1997)).

Guyonnet et al. (2003) and Baudrit et al. (2005) proposed a hybrid approach for addressing uncertainty in risk assessment without transforming one type to another which is critiqued by Sadeghi et al. (2010). There are three main shortcomings on Guyonnet et al. (2003) and Baudrit et al. (2005)'s approaches. Firstly, the α-cuts of a fuzzy set cannot always be represented by infimum and supremum values. Secondly, they do not mention why a 5% probability of getting lower and higher values of the histograms of the α-cuts will generate the Inf and Sup of the output α-cut. Thirdly, if only random inputs are considered as the extreme case for this model, the result will not be similar to the traditional MCS approach (Sadeghi et al., 2010).

Alternatively Sadeghi et al. (2010) proposed a method for dealing with both fuzzy and probabilistic uncertainty in the input of a simulation model. However, it is not also free from limitations and shortcomings. A cautious study exposes some features of the approach that need further modification and improvement. Firstly, they did not provide any method for fuzzy random generation to produce appropriate sample sets. Secondly, they have used the probability-possibility transformation method to transform some of the probability distributions in the simulation input into fuzzy sets. Thirdly, they perform fuzzy arithmetic to calculate the output in the form of fuzzy set. Fuzzy arithmetic implementation is not easy and straightforward for a complex simulation such as a PPP-BOT project.

Since, our goal is not to convert probability density functions into membership functions or vice versa or to use one in place of the other, no proper direct numerical comparisons for the calculated risk estimates are provided. Further, no attempt to provide such a comparison due to inherent differences in the definition, meaning and treatment of the uncertainty as utilized in each method should be made.

As can be seen, varieties of mathematical models have been developed to address risk and uncertainty modelling. In this paper, fuzzy randomness (Moller and Beer, 2004) is used as an appropriate approach. The proposed fuzzy randomness simulation of long term infrastructure projects is a modification of Moller and Beer (2004). Uncertainty of the simulation input data can be modelled appropriately with the aid of non-probabilistic methods under possibility theory. Fuzzy set is common non-probabilistic approach for uncertainty modelling. Furthermore, fuzzy probability which is the focus of this paper is applied properly when risk and uncertainty appear simultaneously.

The possibility theory is utilized directly to reflect uncertainties based on the experts judgments. Fuzzy set theory is used in combination with probabilistic method to generate hybrid approach for risk and uncertainty assessment studies. Vague probabilistic models for the uncertain variables are determined with the aid of fuzzy numbers. However, the proposed algorithm for generating fuzzy random variable and FR-MCS is simpler to implement because it is an interval analysis based on the α-level sets (α-cuts) of the input fuzzy sets. FR-MCS is carried out for finding the Inf and Sup values of the output α-cuts intervals.

Monte Carlo Simulation (MCS) Technique

MCS is a method for analyzing risk propagation, where the goal is to study the outcome variability of a system (Wittwer, 2004). MCS is currently regarded as a powerful technique for cash flow analysis and its associated problems, especially for long term infrastructure projects. To do this the conventional PRA technique is carried out. (Reilly, 2005; Dey and Ogunlana, 2004; Stock and Watson, 2005). Full statistical analysis of outcomes using MCS, incorporating sensitivity analysis and scenario analysis (worst/best cases), gives a more realistic risk analysis and representation in terms of range (confidence intervals) of probable outcomes, and provides the most detailed comparisons. Sensitivity analysis measures the impact on project outcomes of changing one or more key input values about which there is uncertainty. (Akintoye et al., 2001a, b, 2003; Grimsey and Lewis, 2005; Stock and Watson, 2005).

Since MCS can only treat its parameters as random variables by using stochastic models, its main problem is when its parameters are a mixture of stochastic and fuzzy. MCS is unable to address this situation. Mathematically, random variable X is represented by: $X_{R.V.} = \mu + z * SD$ where μ is mean, SD is standard deviation; z is the number of SD. A key task in the application of

MCS is the generation of the appropriate values of the random variables in accordance with the respective prescribed probability distributions. This can be accomplished systematically for each variable by first generating a uniformly distributed random number between 0 and 1, and through an appropriate transformation the corresponding random number with the specified probability distribution is then obtained (Ang and Tang, 1984).

Value-at-risk

Value-at-risk (VaR) is related to the percentiles of probability distributions and measures the predicted maximum portfolio loss at a specified probability level over a certain period. Mathematically, VaR at a probability level $100(1-\theta)\%$ is defined as the value γ such that the probability that the negative of the investment return will exceed γ is not more than θ:

$$VaR_{1-\theta}(\tilde{r}) = min\{\gamma | P(-\tilde{r} > \gamma) \leq \theta\}$$

where $\tilde{r}$ denotes the random variable representing the investment return, and $-\tilde{r}$ is associated with the portfolio loss. (e.g., $\theta = 0.05$, then $100(1-\theta)\% = 95\%$ means that decision maker is interested in the 95% VaR which is the level of the investment losses that will not be exceeded with probability of more than 5%).

VaR is the difference between the mean value and a multiple of standard deviations. It can be expressed as deviations from the mean VaR in units of the standard deviation. Every percentile can be expressed as a sum of the mean of the distribution and the standard deviation scaled by a multiplier as confidence coefficient indicating the degree of confidence for an individual risk level (number of standard deviations) with general form: $VaR_{(1-\theta)} = -\mu + \beta\sigma$ and in the case of the normal distribution: $VaR_{(1-\theta)} = -\mu + Z_{1-\theta}\sigma$, where μ is the mean and σ is the standard deviation of the underlying data distribution, respectively. The number $Z_{1-\theta}$ is the $100(1-\theta)^{\text{th}}$

percentile of a standard normal distribution (e.g.: $Z_{0.95}$ corresponding to the 95th percentile is 1.64).

VaR could be generated for a PPP-BOT project from different perspective at a specific confidence level. VaR in the PPP-BOT projects context, is defined as the minimum expected value at a given confidence level. Figure 1 presents the cumulative probability for the VaR of a PPP-BOT project with low risks. In the context of PPP-BOT projects, a project manager as a decision maker is typically interested in two important statistics issues aimed to decision-making: (1) an arbitrary and subjective quantile, and (2) the probability of exceeding (or not exceeding) a specific threshold. In most cases, project managers are concerned in finding the probability that a project will exceed a certain value (a specific threshold) of interest (meet the target on cost or time). At the given confidence level, (1-θ)%, the value-at-risk (VaR_θ) is shown in Figure 2. VaR^* is defined as acceptable threshold value from party's perspective based on its objective. It represents the worth of Value-at-Risk at confidence level of $1-\theta^*$. θ^* represents the confidence level at the point of VaR^* (See Figure 1). In this case VaR_θ is greater than VaR^*. Value-at-risk at a given confidence level, $1-\theta$, is computed by integrating between $-\infty$ and VaR_θ equal to θ, and the confidence level at the point of VaR^* is obtained by integrating between $-\infty$ and VaR^* (See Figure 2).

A literature review of the current simulation and financial risk evaluation methods shows that VaR system provides decision criteria with a confidence level. Ye and Tiong (2000) defined the NPV-at-risk based on the VaR system as a particular NPV generated for a project at some specific confidence level. Their definition of NPV-at-risk can be used to derive the decision rule: the project is acceptable with a confidence level of 1-θ if the NPV-at-risk at given confidence is greater than zero. According to the requirements of decision rules, there are two approaches to investment decision making: calculation of NPV at a given confidence level and calculation of a confidence level at the point of zero NPV. NPV-at-risk at a given confidence θ and the confidence level at the point of zero NPV can be obtained using percentile analysis on the cumulative distribution function (CDF). The NPV-at-risk method takes into account all probable returns resulting from various risks associated with PPP-BOT projects.

The decision rule emerging from the use of this criterion indicates that a PPP-BOT project investment should be selected for implementation if its indicator at risk (IND-at-risk) as VaR expected shortfall exceeds an investor defined limit. As can be seen, although VaR analysis has been successfully performed in previous research projects, it could only take randomness into account and cannot deal with fuzziness involved. The following sections will address this essential gap.

Fuzzy Variables/Numbers

Fuzzy set theory introduced by Zadeh (1965) permits the gradual assessment of the membership of the elements in relation to a set. It provides a suitable basis for relaxing the need for precise values or bounds. It allows the specification of a smooth transition for elements from belonging to a set to not belonging to a set. This is described with the aid of a membership function. Membership values are assigned to the estimation results by subjective assessment. A fuzzy set $\tilde{A}$ is defined as follows; $\tilde{A} = \{(x, \mu_A(x)), x \in X, 0 \leq \mu_A(x) \leq 1\}$. Membership function, $\mu_A(x)$, associates each $x \in \tilde{A}$ to a real number in the interval $[0,1]$. $\mu_A(x)$ represents the membership degree of x in set $\tilde{A}$. The fuzzy set $\tilde{A}$ is referred to fuzzy variable $\tilde{x}$ (Moller and Beer, 2004). A fuzzy number is said to be normal if there is an $x \in A$ such that $\mu_A(x) = 1$ and it is a convex fuzzy subset of the real line if $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$, for $\lambda \in [0,1]$. The definition of fuzzy random variables (FRVs) is due to Kwakernaak (1978, 1979); "*fuzzy random variables (FRVs) are random variables whose values are not real, but fuzzy numbers*". Fuzzy numbers are a generalization and refinement of intervals for representing imprecise parameters. This modelling corresponds to the theory of fuzzy random variables and to fuzzy probability theory (Kratschmer, 2001; Beer, 2009).

α-level set (α-cut)

α-level set or α-cut is one of the important features of fuzzy set $\widetilde{A}$ and is useful in processing fuzzy variables through engineering computation. For fuzzy set $\widetilde{A}$, the crisp sets $A_{\alpha_k} = \{ x \in X, \mu_A(x) \geq \alpha_k \}$ can be extracted for real numbers $\alpha_k \in (0,1]$. These crisp sets are called α-level sets. All α-level sets A_{α_k} are crisp subsets of the support $S(\widetilde{A})$. The support $S(\widetilde{A})$ is defined as: $S(\widetilde{A}) = \{x \in \mathbb{R}, \mu_A(x) > 0 \}$. For a convex fuzzy set, its α-level sets are intervals $A_{\alpha_k} = [x^L_{\alpha_k}, x^R_{\alpha_k}]$, see Figure 3. This aids the illustration of the fuzzy set $\widetilde{A}$ using its α-level sets as follow:

$$\widetilde{A} = \left\{ \left(A_{\alpha_k}, \mu\left(A_{\alpha_k}\right)\right), \mu\left(A_{\alpha_k}\right) = \alpha_k \ \forall \alpha_k \in (0,1] \right\}, A_{\alpha_k} \subseteq A_{\alpha_i} \ \forall \alpha_i, \alpha_k \in (0,1], \alpha_i \leq \alpha_k$$

If the Fuzzy set $\widetilde{A}$ is convex, each α-level set A_{α_k} is a connected interval $[x^L_{\alpha_k}, x^R_{\alpha_k}]$ in which:

$$x^L_{\alpha_k} = min[x \in X, \mu_A(x) > \alpha_k] \ , \ x^R_{\alpha_k} = max[x \in X, \mu_A(x) > \alpha_k].$$

In other words, the α-cut of a continuous convex possibility distribution, $\widetilde{A}$, may be understood as the inequality $\widetilde{A}_{\alpha_k} = \{x | p(x \in [x^L_{\alpha_k}, x^R_{\alpha_k}]) \geq 1 - \alpha_k\}$.

α-level set of each fuzzy input parameter represents a set of values within an interval with max-min values which is called Supremum-Infimum values corresponding to specific α-level set. Fuzzy alpha-cut (FAC) technique uses fuzzy set theory to represent uncertainty or imprecision in the concerned parameters at different level of uncertainties (α-levels). Uncertain parameters are considered to be fuzzy numbers with some assumed membership functions. There are many types of functional formulations for the membership functions. Two common used membership functions are triangular and trapezoidal functional formulations and corresponding fuzzy numbers/variables can be represented by the following notations; Triangular fuzzy number "T.F.N" $\tilde{x}_{Tri}: \langle a_1, a_2, a_3 \rangle$, Trapezoidal fuzzy number "Tr.F.N" $\tilde{x}_{Trap}: \langle a_1, a_2, a_3, a_4 \rangle$. Figure 4 and Figure 5 show parameter x represented as a triangular and trapezoid fuzzy number with support of A_0. The wider the support of the membership function, the higher the uncertainty. The fuzzy set that contains all elements with a membership of $\alpha \in [0,1]$ and above is called the α-cut of the membership function. At a resolution level of α, it will have support of A_α and the higher the value of α, the higher the confidence in the parameter.

Defining the α-cut, the interval of confidence at level α, T.F.N is characterized as follows: $\forall \alpha \in (0,1], a_1 \leq a_2 \leq a_3$.

$$f(x; a_1, a_2, a_3) = max\left(min\left(\frac{x - a_1}{a_2 - a_1}, \frac{a_3 - x}{a_3 - a_2}\right), 0\right)$$

$$A_\alpha = [x_\alpha^L, x_\alpha^R] = [(a_2 - a_1)\alpha + a_1, -(a_3 - a_2)\alpha + a_3],$$

Defining the α-cut, the interval of confidence at level α, Tr.F.N is characterized as follows: $\forall \alpha \in (0,1], a_1 \leq a_2 \leq a_3 \leq a_4$.

$$f(x; a_1, a_2, a_3, a_4) = max\left(min\left(\frac{x - a_1}{a_2 - a_1}, 1, \frac{a_4 - x}{a_4 - a_3}\right), 0\right)$$

$$A_\alpha = [x_\alpha^L, x_\alpha^R] = [(a_2 - a_1)\alpha + a_1, -(a_4 - a_3)\alpha + a_4],$$

The proposed FAC method is based on the fuzzy extension principle (Zadeh, 1975a, b, c, d), which implies that functional relationships can be extended to involve fuzzy arguments and can be used to map the dependent variable as a fuzzy set. In simple arithmetic operations, this principle can be used analytically. However, in most practical modelling applications where relationships involve partial differential equations and other complex structures, the analytical application of this principle is difficult. Therefore, interval arithmetic is used to carry out the analysis. Interval arithmetic is a method of finding lower and upper bounds for the possible values of a result by performing a computation in a manner which preserves these bounds.

Fuzzy Randomness-Monte Carlo Simulation (FR-MCS) Technique

To address aforementioned shortcomings, this paper proposes a new simulation method, ***Fuzzy Randomness-Monte Carlo simulation (FR-MCS) technique.*** The structure of FR-MCS technique is demonstrated in Figure 6. Numerical processing of fuzzy probabilities can be realized with a combination of stochastic and fuzzy analysis. Whilst a probabilistic model is analyzed using a traditional stochastic approach, the imprecision of the probabilistic model is transferred to the simulation results via fuzzy analysis. The purpose of proposing FR-MCS is to provide an alternative approach to the conventional MCS for dealing with uncertainties in the simulation input including the parameters of the PDFs using fuzzy set theory. This technique can model uncertainties involved in simulation input effectively, accompanied with random variables and deterministic input parameters. For instance $y = f(x_1, \ldots, x_m, \tilde{x}_1, \ldots, \tilde{x}_{n-m})$ is a function of n variables includes of both types of non-deterministic variables: risky and uncertain variables, Risky (randomness) variables group: $x_1, \ldots, x_m$, uncertain (fuzziness) variables group: $\tilde{x}_1, \ldots, \tilde{x}_{n-m}$.

FR-MCS, which is used to combine multiple PDFs and CDFs in risk and uncertainty calculations, is a means of quantifying uncertainty or variability in a hybrid fuzzy-probabilistic framework using simulation. The simulation output, based on the conventional MCS, will be exactly a CDF. On the other hand, FR-MCS is proposed as a general form of MCS technique. The output of a FR-MCS analysis is a collection of CDFs for each simulation and it results in a bound of CDFs (CDF series).

FR-MCS combines MCS (Random Sampling) with the extension principle of fuzzy set theory (Zadeh, 1975a, b, c, d; Gerla and Scarpati, 1998; Moller and Beer 2004, 2008). FR-MCS utilizes a combination of probability and possibility theories to include probabilistic and possibilistic information in the risk analysis model. Fuzzy approach provides the likelihood of occurrence of each risk value for all the possible risks. The risk value corresponding to a membership value of 1.0 is the most possible/likely risk. Higher uncertainty and variability involved can be seen from the supports of the membership functions of fuzzy risks generated for various percentiles. The resulting fuzzy risk has a larger range of possibilities (i.e., the support of the membership function is larger). Fuzzy calculations take into consideration all possible combinations of parameter values rather than random sampling. Similar to conventional MCS, the variability in the random variables of the risk equation (i.e., exposure frequency/probability and consequence) is treated using normal PDFs and the uncertainty associated with them is treated by using fuzzy numbers for the parameters of these random variables. That is, the means and the standard deviations of these PDFs are modelled as fuzzy numbers. Similar to MCS, the independence of the input parameters has been assumed in generating fuzzy random variables and producing fuzzy randomness; the output may be overestimated when using fuzzy randomness for a function with dependent input parameters. Algorithms are required to generate random variables and fuzzy random variables to implement FR-MCS. In the following section an algorithm is proposed for generating fuzzy random variables.

FR-MCS technique results in multiple CDF of function y which is called F(y) series. It considers the spread of CDFs membership functions. Based on the resulted F(y) series, lower bound, $\underline{F}(y)$, as inferior value and upper bound, $\overline{F}(y)$, as superior value of CDFs are determined.

The appropriate membership degree, μ, corresponding to each CDF is then determined. This procedure is demonstrated in Figure 7.

The FR-MCS produces two CDFs (i.e., one for upper and one for lower bound) for each alpha-cut level except for alpha-cut 1.0 since the lower and the upper bound at alpha-cut 1.0 is the same. For each specific value of y e.g.: y', based on the lower and upper bounds, fuzzy probability of y' can be calculated and drawn. Also, for each membership degree, lower and upper bound of CDFs are determined. Consequently, a corresponding fuzzy probability is established which is represented as a confidence level interval $[CL^{\alpha}_{L}, CL^{\alpha}_{R}]$ as demonstrated in Figure 8. For each specific value of F(y) as a confidence level e.g.: θ, based on the lower and upper bounds, fuzzy probability of y' can be calculated and drawn. Further, for each membership degree, lower and upper bound of CDFs are determined. Consequently, a corresponding fuzzy probability is established which is represented as negotiation interval $\left(y'^{\alpha,\theta}_{L}, y'^{\alpha,\theta}_{R}\right)$ as demonstrated in Figure 9.

Compatible decisions that are made using conventional MCS can be made based on FR-MCS technique only for the case of pure random variables of simulation input. In the case of pure probabilistic in the input of FR-MCS technique the result of simulation will be a CDF. As the number of fuzzy variables in the simulation input is increased, the CDF function in the simulation output increases in fuzziness Consequently CDF bound is wider. In the case of pure fuzzy random variables of simulation input, the results are similar with the fuzzy set theory analysis. In this case the CDF bound is widest. The fuzzy inference mechanism is an applicable technique for this case. Mamdani and Sugeno are two types of fuzzy inference mechanism (Sivanandam et al., 2007). The Mamdani style is the most famous type of fuzzy controllers. α-cut levels signify uncertain level and represent the amount of uncertainty involved. On the contrary, α-confidence levels signify risky level and represent the amount of risk involved. Thus if the decision maker is optimistic and assumes high precision ($\mu = 1$), he works with the cores of the fuzzy intervals, but, if is cautious, he may choose $\mu = 0$ and use corresponding supports. In the case of in between, a corresponding value within $\mu = [0, 1]$ is chosen by decision maker.

The method of decision making using fuzzy sets is based on the confidence level between 0 and 1 to obtain a range of values for the simulation final output. This range is calculated by finding the α-cut at the value of 1 minus the confidence level. In this manner, the decision maker can choose from a range of values (interval) instead of a crisp output which is the result of conventional MCS. An arbitrary quantile can also be determined using the inverse of the fuzzy CDF. Fuzzy CDF has the unique feature of representing both fuzzy and probabilistic (uncertainty and risk) in a single diagram.

Algorithm for generating Fuzzy random variable

The procedure of generating fuzzy random variable is not the same as that for generating random variable described earlier, in the section *Monte Carlo Simulation technique*. Current literature provides some knowledge on specific procedure for generating fuzzy random variable. Moller and Beer (2004, 2008) proposed a procedure which could be summarized as follows. They argue that fuzzy variables need to be treated separately. The fuzzy variables (assume n fuzzy variables), for each alpha-level (alpha cut), form an n-dimensional hypercube. For each point and vector out of this hypercube Monte Carlo can be performed with the random variables and a CDF obtained for the result, e.g. a failure probability or some other probability of interest. It is now needed to select another point out of the hypercube and repeat the Monte Carlo simulation to get another result. The aim of repeating this analysis is to find those points in the hypercube, which give max and min final results (e.g. the failure probability). This is called global optimization (Moller and Beer, 2004, 2008). When some knowledge about simulation function is available, this analysis may be significantly simplified. For example, when the simulation function is *monotonic* in every direction, then the extreme points are the corners of a hypercube. Only these points need to be checked for optimization.

In this paper, a modified and simplified procedure is developed for generating fuzzy random variable. Its procedure is explained in detail for two main types of fuzzy numbers and variables: triangular fuzzy number T.F.N, $\tilde{x}_{Tri}$: $\langle a_1, a_2, a_3 \rangle$, trapezoidal fuzzy number Tr.F.N, $\tilde{x}_{Trap}$: $\langle a_1, a_2, a_3, a_4 \rangle$ in 4 operative steps for a hybrid function of both randomness and fuzziness type of variables: $y = f(x_1, \ldots, x_m, \tilde{x}_1, \ldots, \tilde{x}_{n-m})$. Randomness variables group: $x_1, \ldots, x_m$, which is characterized by probability distributions, and fuzziness variables group: $\tilde{x}_1, \ldots, \tilde{x}_{n-m}$ which is represented in terms of possibility distributions (membership function) measuring the degree of possibility that the linguistic variables are.

Step 1: The membership function is cut horizontally at a finite number of α-levels between 0 and 1, $\alpha = \{\alpha^1, \alpha^2, \ldots, \alpha^i, \alpha^j, \ldots, \alpha^N\}$. Consequently, for each α-level, an interval (a boundary) of concerned fuzzy values is achieved. For each α-level of the parameter, the model is run to determine the minimum and maximum possible values of the concerned output. This information is then directly used to construct the corresponding membership function (fuzziness) of the output which is used as a measure of uncertainty. If the output function is *monotonic* with respect to the dependent fuzzy variables, the process is rather simple since only two simulations will be enough for each α-level (one for each boundary in left and right). Otherwise, optimization routines have to be carried out to determine the minimum and maximum values of the output for each α-level. This approach is used to model the interested output subject to imprecise boundary conditions and properties. The α-cut can be repeated for a number of iteration, N. Apply α-level set (α-cut) for a set of α to a fuzzy number, T.F.N or Tr.F.N (Figure 10). The resulted intervals are varied, when the membership function is characterized by convex and concave shape instead of common linear shape.

Step 2: The boundary and resulted interval corresponding to α-level is demonstrated as follows: $A_\alpha = [x_\alpha^L, x_\alpha^R]$, The resulted intervals for T.F.N are characterized as follows:

$A_\alpha = [x_\alpha^L, x_\alpha^R] = [(a_2 - a_1)\alpha + a_1, -(a_3 - a_2)\alpha + a_3]\, , \forall\alpha \in (0,1]$.

The resulted intervals for Tr.F.N are characterized as follows:

$A_\alpha = [x_\alpha^L, x_\alpha^R] = [(a_2 - a_1)\alpha + a_1, -(a_4 - a_3)\alpha + a_4]\, , \forall\alpha \in (0,1]$.

Step 3: Generate random variables from resulted intervals: $A_\alpha = [x_\alpha^L, x_\alpha^R]$, corresponding to each set of α- level (α-cut) e.g.: $x_\alpha^r = x_\alpha^L + RAND() * (x_\alpha^R - x_\alpha^L)$, (This procedure is demonstrated in Figure 10). $RAND()$ is a function to generate random numbers in the interval (0,1), by assuming a uniform distribution function. These random numbers multiply by the range of resulted intervals. Having more information, other type of distribution function may apply.

Step 4: Take the resulted values in steps 1, 2 and 3, including the boundary values in left and right and random variables generated for each α-level, as a set of Fuzzy random variables: $FRV = \{x_\alpha^L, x_\alpha^r, x_\alpha^R\}$.

Fuzzy probability distribution

Fuzzy probability provides a suitable framework for a realistic modelling of risk and uncertainty to ensure that both risky and uncertain input data type is appropriately reflected in computation results. In the framework of fuzzy probability, both the probabilistic and the possiblistic data can be considered simultaneously and transferred and reflected combinedly and jointly to the results (Moller and Beer, 2004; Baudrit et al., 2006).

The processing of fuzzy randomness simulation can be realized with a combination of stochastic simulation and fuzzy analysis in a nested form. Fuzzy numbers with appropriate membership function as uncertain variables are input parameters for a fuzzy analysis. With each set of crisp values and random variables for the simulation input parameters, a traditional stochastic analysis is performed. The extreme results from the various conventional stochastic computations and also incorporating the uncertain variables subsequently define the bounds on probability or fuzzy probabilities respectively. This issue is important for the loss caused by catastrophic risks, project bankruptcy and negotiation failure. Negotiation failure and bankruptcy probability are obtained as fuzzy variables. Their range of possible values reflects the non-probabilistic feature of uncertain variables from the specification of the probabilistic model for the input variables. This topic is discussed in full by Moller and Beer (2004). For the propagation of probabilistic and possibilistic uncertainty information, the conventional MCS technique (Kalos & Whitlock, 1986) can be combined with the extension principle of fuzzy set theory (Zadeh, 1965, 1975a, b, c, d) by means of the following 3 main steps:

I. Repeat Monte Carlo sampling of the probabilistic variables to process their risk (generating random variable).
II. Apply fuzzy interval analysis to process the uncertainty connected with the possibilistic variables.
III. Employ fuzzy probability procedure for joint propagation of probabilistic and possibilistic uncertainty.

A possibility value α as uncertain level is selected. The generic kth random values for ith iteration, x_k^i, $k = 1, \ldots, m$, are sampled by Monte Carlo from the probabilistic distributions. A fuzzy set π_l^f, estimate of possibility distribution for l^{th} possibilistic variables $\tilde{x}_l^i$ of $y = f(X)$, $l =$

$1, \ldots, n-m$, is constructed by fuzzy interval analysis according to the assumed α-level. After m repeated samplings of the probabilistic variables, x_k^i, the fuzzy set estimates $\pi_l^f, l = 1,2, \ldots, \text{n} - \text{m}$, are combined with those of random values to give an estimate of $y = f(X)$ as a fuzzy random variable (or random possibility distribution) according to the rules of evidence theory (Shafer, 1976). This is repeated for a number of iteration (i=1,..,N).

The steps of the fuzzy probability distribution procedure are as follows: (Baraldi and Zio, 2008; Guyonnet et al., 2003)

Step 1: Select a possibility value α and the corresponding cut of the possibility distributions $(\pi_1^f, \ldots, \pi_{n-m}^f)$ as intervals of possible values $\mathrm{A}_\alpha = [x_\alpha^L, x_\alpha^R]$ of the possibilistic variables $\tilde{x}_l^i$ $(\tilde{x}_1^i, \ldots, \tilde{x}_{n-m}^i)$.

Step 2: Sample the i[th] realization of the probabilistic variables $x_k^i(x_1^i, \ldots, x_m^i)$. (Generating random variable for i[th] iteration)

Step 3: Interval calculation, compute the supremum and infimum (largest and smallest) values of $y^i = f\left(x_1^i, \ldots, x_m^i, \tilde{x}_1^i, \ldots, \tilde{x}_{n-m}^i\right)$, denoted by $\underline{f}_\alpha^i$ and $\overline{f}_\alpha^i$, respectively.

Step 4: Return to Step 2 to generate a new realization of the random variables. The above procedure is repeated for $\text{i} = 1,2, \ldots, \text{N}$; at the end of the procedure an ensemble of realizations of fuzzy intervals is obtained, that is, $(\pi_1^F, \ldots, \pi_N^F)$.

Step 5: Return to step 1, choose another α-cut and repeat the process for new α-cut; after having repeated steps 2 to 4 for all the α-cuts of interest, the fuzzy random realization (fuzzy interval) π_i^F of $y = f(X)$ is obtained as the collection of the values $\underline{f}_\alpha^i$ and $\overline{f}_\alpha^i$. Then, take the extreme values of $\underline{f}_\alpha^i$ and $\overline{f}_\alpha^i$, found in this step, as lower and upper limits of α-cuts of $y =$

$f(x_1, \ldots, x_m, \tilde{x}_1, \ldots, \tilde{x}_{n-m})$ and denote them by $\underline{F}_\alpha^i$ and $\overline{F}_\alpha^i$. In other words, π_i^F is defined by all its α-cut intervals $\left[\underline{F}_\alpha^i, \overline{F}_\alpha^i\right]$ (Refer to Figure 12).

Hence, a fuzzy probability distribution function $\tilde{\mathrm{F}}(x)$ can be formulated as a fuzzy set of traditional probability distribution function $\mathrm{F}(x)$ of random variable X, which is given by:

$$\tilde{\mathrm{F}}(\mathrm{x}) = \left\{\left(\mathrm{F}(\mathrm{x}), \mu\left(\mathrm{F}(\mathrm{x})\right)\right) \mid \mathrm{X} \in \tilde{\mathrm{X}}, \mu\left(\mathrm{F}(\mathrm{x})\right) = \mu(\mathrm{X}) \right\}$$

The functional values of $\tilde{\mathrm{F}}(\mathrm{x})$ are fuzzy variables and possess membership functions. Interval probabilities $\mathrm{F}_\alpha(\mathrm{x}) = \left[\underline{\mathrm{F}}_{\alpha\mathrm{l}}(\mathrm{x}), \overline{\mathrm{F}}_{\alpha\mathrm{r}}(\mathrm{x})\right]$ weighted by the membership degree $\mu\left(\mathrm{F}_\alpha(\mathrm{x})\right)$ that can be obtained for each α-level. This interval probability contains the probability of all possible states describing the occurrence of $\mathrm{X} \in \tilde{\mathrm{X}}$. Thus, a fuzzy probability function can be described as a fuzzy set of interval probabilities. For introducing the $\tilde{\mathrm{F}}(\mathrm{x})$ in numerical procedures the α-discretization is applied. This leads to fuzzy functional value for each specified x.

$$\tilde{\mathrm{F}}(\mathrm{x}) = \left\{\begin{matrix} \left(\mathrm{F}_\alpha(\mathrm{x}), \mu\left(\mathrm{F}_\alpha(\mathrm{x})\right)\right) \mid \mathrm{F}_\alpha(\mathrm{x}) = \left[\underline{\mathrm{F}}_{\alpha\,\mathrm{l}}(\mathrm{x}), \overline{\mathrm{F}}_{\alpha\,\mathrm{r}}(\mathrm{x})\right], \\ \mu\left(\mathrm{F}_\alpha(\mathrm{x})\right) = \alpha \;\forall \alpha \in (0,1] \end{matrix}\right\}$$

$$\overline{\mathrm{F}}_{\alpha\,\mathrm{r}}(\mathrm{x}) = \max \tilde{\mathrm{F}}(\mathrm{x}), \underline{\mathrm{F}}_{\alpha\,\mathrm{l}}(\mathrm{x}) = \min \tilde{\mathrm{F}}(\mathrm{x})$$

The fuzzy probability distribution function $\tilde{\mathrm{F}}(\mathrm{x})$ of $\tilde{\mathrm{X}}$ may thus be interpreted as being the set of the probability distribution functions F(x) of all originals X of $\tilde{\mathrm{X}}$ with the membership values $\mu\left(\mathrm{F}(\mathrm{x})\right)$. This representation is suitable for numerical processing of fuzzy probabilistic variables. The description of fuzzy probability distribution functions can be realized with the aid of fuzzy variables for parameters in the probability functions. For instance, if the underlying uncertain random variable X is assumed to be normal distribution $\mathrm{N}(\tilde{m}, \tilde{\sigma})$ with fuzzy expected value $\tilde{m}_x = \langle 5.5, 6.0, 6.8 \rangle$ and fuzzy standard deviation $\tilde{\sigma}_x = \langle 0.8, 1.0, 1.1 \rangle$, then fuzzy PDF and fuzzy CDF can be specified as,

$$\tilde{f}(x) = \frac{1}{\tilde{\sigma}\sqrt{2\pi}} e^{-0.5\left[\frac{(x-\tilde{m}_x)}{\sigma_x}\right]^2}, \tilde{F}(x) = \frac{1}{\tilde{\sigma}\sqrt{2\pi}} \int_{-\infty}^{x} e^{-0.5\left[\frac{(t-\tilde{m}_x)}{\sigma_x}\right]^2} dt$$

and are shown in Figure 11. The functional value of $\tilde{F}(x)$ at a specified value x is a fuzzy variable. For instance, $\tilde{F}(6) = \langle 0.15, 0.5, 0.75 \rangle$. All PDFs and CDFs used to describe the variability in a fuzzy probability model have a certain degree of uncertainty (μ: membership function).

Reliability modelling and evaluation with Fuzzy data

Fuzzy probability can be generalized as is represented in Figure 12. Two ways to fuzzify the series curves $\tilde{F}(x)$ are shown. $\overline{F}(x)$ and $\underline{F}(x)$ are upper and lower CDF bounds. $F_1(x)$ is the expected CDF. As a rule, minimization and maximization algorithm can be used for finding *Inf* and *Sup* values of a general model. However, when the simulation model is a simple *monotonic* function, as in our study, the *Inf* and *Sup* values are identified directly without using minimization or maximization algorithms.

When it is known which combination of parameters from the alpha-level sets of fuzzy variables in simulation input leads to the boundary/extremes curves in simulation output, any software can be utilized to plot the output, fuzzy probability curves, and gray out the area in between. When it is unknown which combination of parameters leads to the extremes, the best way to get a figure is to perform FR-MCS over the parameter space and plot curve by curve for the result. Now we consider the membership function of the series curves $\tilde{F}(x)$ as follows.

$$\mu(F(x)) = 0, if\ F(x) \le \underline{F}(x), \mu(F(x)) = 0, if\ F(x) \ge \overline{F}(x)$$

$$\mu(F(x)) = \frac{F(x) - \underline{F}(x)}{F_1(x) - \underline{F}(x)}, if\ \underline{F}(x) \le F(x) \le F_1(x),$$

$$\mu(F(x)) = \frac{\overline{F}(x) - F(x)}{\overline{F}(x) - F_1(x)}, if\ F_1(x) \le F(x) \le \overline{F}(x)$$

and using the α-cuts:

$$\tilde{F}_\alpha(x) = \left[\underline{F}(x) + \left(F_1(x) - \underline{F}(x)\right)\alpha, \overline{F}(x) - \left(\overline{F}(x) - F_1(x)\right)\alpha\right]$$

In this section it was shown that when an uncertainty is associated with the estimates, the simulation output function and other related concepts can be modelled using the intervals of confidence, and fuzzy numbers instead of the probabilistic characterization. The extension principle, which is one of the most important concepts of fuzzy set theory, is used to conduct arithmetic operations on interval of confidence and fuzzy numbers. As can be seen the simulation and financial evaluation method based on the Value-at-risk and uncertainty (VaRaU) approach, which incorporates both risk and uncertainty analysis using confidence and uncertain levels and discount rate concept give more equitable results for all parties involved in the PPP-BOT project. Therefore by these simulation results, negotiations objectives will be promptly obtained.

Illustrative Case Study- MCS vs. FR-MCS

Typically case studies assume deterministic assumption. FR-MCS has been employed to estimate volatility of parties' objectives like volatility of investment project value and the impact of uncertainties on the project cost estimation, contract decision variables/indicators and the optimal outcomes. This is to simulate cash flows of a PPP-BOT investment project with appropriate risk and uncertainty models and further to describe fuzzy probability distribution of cost estimation and returns by iterating large number of simulations. The application of the FR-MCS model for the evaluation of uncertainties including demand uncertainty for a BOT toll road and bridge project is demonstrated with a realistic case study. To achieve this, a special program has been developed using MATLAB (The MathWorks, Inc., Natick, Massachusetts). In this study the focus is on the representation of the uncertainties by fuzzy numbers. Basic input data of the project comprises deterministic, uncertain and risky parameters. Uncertain and risky parameters consist of three components i.e. *macroeconomic indicators and indexes*, *fuzzy-stochastic variables (FSV)* and *negotiable concession items (NCIs).* Main project data including expected value of parameters and their distribution or membership function is tabulated in Table 1.

The expected value of parameters is taken from *The Toolkit for Public-Private Partnerships in Roads and Highways* provided by the World Bank-PPIAF V1.1 (World Bank, 2009). The distribution or membership function of parameters is taken based on the expert knowledge through interview. The fuzzy approach has been used as a measurement of uncertainty, e.g., demand uncertainty (See Figure 13). The level of uncertainty is represented and considered by membership value between 0 and 1. The membership function of operating revenue by considering demand uncertainty is represented in Figure 14.

Figure 15 and Figure 16 represent PDF and CDF of total project costs for the same case resulted from conventional MCS by considering no uncertainties. The result is just a PDF/CDF that does not take into account any uncertainty. Consequently by taking a value for probability (θ) in CDF, it will result in only a deterministic value. Based on this result, as engineering implication, a decision maker will come to the negotiation table with a deterministic value of the decision variables.

In this case a total of 1000 iterations are performed to carry out a FR-MCS and generate a fuzzy CDFs. Figure 17 illustrates three dimensional view of fuzzy CDF for total project costs (TPC) resulting from the FR-MCS that are generated by MATLAB. Figure 18 and Figure 19 represent the x-y and x-z views of fuzzy CDF resulted in Figure 17 respectively.

The procedure is the same for the decision variables. Figure 20 shows the three dimensional view of fuzzy CDF for the debt service cover ratio (DSCR) resulting from the FR-MCS. Figure 21 and Figure 22 represent the x-y and x-z views of fuzzy CDF resulted in Figure 20 respectively.

As can be seen, the result of conventional MCS is a CDF which has no uncertainty taken into account while the result of FR-MCS is fuzzy CDFs and has taken uncertainties into account i.e. means to take into account the possibility that uncertainty may increase or reduce. As a result, by taking a specific value of the confidence level in fuzzy CDF, an interval for the decision variable will be obtained. On the contrary, by the same approach for CDF resulted from MCS, just a deterministic value will be obtained. Decision makers are more comfortable with an effective interval (negotiation bound) for NCIs on the negotiation table (Ferrero and Salicone (2002, 2004, 2005, 2006, 2007); Klir and Yuan (1995); Klir et al. (1997)).

Sensitivity analysis of FR-MCS technique

The results of FR-MCS are sensitive to fuzziness of the input variables. In the absence of fuzziness (pure probability in inputs) the result of FR-MCS is exactly equal to a CDF which is the same with the results of conventional MCS. In the absence of randomness (pure fuzziness in inputs) the result of FR-MCS is represented by CDF bound. It can be shown that the fuzziness of the output expands when the number of fuzzy random variables increases. Reasonably, for smaller number of fuzzy random variables, the CDF function has less fuzziness, and the CDF bound is narrower. More detailed discussion was illustrated in Figure 8 and Figure 9.

Decision making based on the generated Fuzzy probability distributions

Similar to the CDF function concluding from conventional MCS (refer to value-at-risk section), a decision maker can use the fuzzy CDF of the decision variable/indicator in the simulation output to do decision making on not just probability but also possibility of acquired desirable output (i.e. probability and possibility that the decision variable/indicator will be more/less than a specific amount/value) and probability and possibility of success. Furthermore, it can be used to find an appropriate contingency value (arbitrary quantile) of project decision variable/indicator. Figure 23 represents intersecting of x-y view of fuzzy CDF of return on equity (EIRR) resulted from FR-MCS with hurdle rate. The hurdle rate or minimum acceptable rate of return (MARR) is defined as the minimum rate of return required on a project to cover costs and profit. It indicates the probability that the rate of return on equity will not be less than hurdle rate, 14%. This probability is in the form of a fuzzy set, as shown in Figure 23. The Level Rank defuzzification method (Moller and Beer, 2004) is used to convert the output fuzzy variable into a crisp value. By defuzzifying the output in Figure 23, it can be stated the probability that the rate of return on equity will not be less than hurdle rate, 14%, is around 79.5% (=1-20.5%).

The arbitrary quantile in a Fuzzy CDF is represented as a fuzzy set. Figure 24 illustrates intersecting of x-y view of fuzzy CDF of return on equity (EIRR) resulted from FR-MCS with specific confidence levels, 0.10 and 0.50, to find the appropriate contingency values (arbitrary quantile). It represents the 10th and 50th quantile of return on equity (EIRR). By defuzzifying the outputs in Figure 24, it can be stated that with 10% and 50% probability the rate of return on equity are around 17.10% and 15.20% respectively which are much greater than hurdle rate, 14%.

As can be seen, the FR-MCS technique and obtained fuzzy CDF have improved decision making based on the conventional MCS by incorporating the uncertainties involved in the project. FR-MCS helps and facilitates decision makers to come up with negotiation interval for negotiable concession items (NCIs) that takes players' characteristics into account.

Conclusion Remarks

Probability theory has been successfully used in modelling random variables; however, this is insufficient for modelling imprecise information. Currently, the most popular method to carry out the PRA is MCS and its analysis. Typically the data required to conduct the conventional MCS is not readily available or it is too costly to collect the required data. However, available data can be utilized through other mathematical tools such as fuzzy set theory. Thus, it is risk analysts responsibility to investigate, gather and efficiently include all the existing information using the most appropriate methods and mathematical tools.

This paper introduced a new approach to simulation techniques under risk and uncertainty, which is termed FR-MCS technique. The aim of this development is for a generalization of the conventional MCS to make decisions based on the hybrid simulation approach of randomness and fuzziness of input parameters. The basic requirement of FR-MCS is to be able to randomly produce random/fuzzy/crisp number in simulation procedure. Consequently, determine inferior and superior of output values of simulation function by using fuzzy probability (fuzzy CDF). The proposed methodology has been introduced to integrate fuzzy set theory into PRA studies. α-cut method is used to perform algorithm for generating fuzzy random variable and to implement FR-MCS. Practically, given enough iterations of FR-MCS technique, it will produce a sufficiently small error.

The main idea proposed here is to utilize subjective probabilities, i.e. available information to represent the uncertain variable as a fuzzy number, and produce outputs which reflect all variable and uncertain information (i.e., uncertainty due to randomness, imprecision or due to both). In this approach, random variables parameters are treated as fuzzy numbers (Alternative 1). Alternatively, by using subjective approach, random variables are treated as pure fuzzy numbers (Alternative 2).

For cases where the necessity of conventional MCS and its analysis is justified but necessary information to conduct this analysis does not exit, the new approach proposed in this paper can be conducted as an alternative to conventional MCS. The proposed FR-MCS technique allows fuzzy and probabilistic uncertainty to be considered simultaneously for the risk analysis of PPP-BOT projects. Depending on the project's host country, the decision maker can adjust the conservative nature of FR-MCS using lower percentiles of risks and uncertainties. As for FR-MCS, the decision making will be based on the intervals while in MCS the decision making is based on the deterministic values. This advantage facilitates decision making of long term infrastructure projects.

The proposed technique is applied to a BOT toll road and bridge case, whose data requirements are comparatively less difficult or easier to obtain. The membership functions of the parameters of the fuzzy random variables can be formed using imprecise, vague information or expert judgment. Thus, application of the FR-MCS approach to risk assessment problems instead of conventional MCS approaches may be more realistic for many PPP-BOT cases and may provide decision makers with sufficient information for decision making. The results of conventional MCS and its analysis cannot easily be compared with fuzzy probability results of FR-MCS. It is not straightforward. Extensions of possibilistic concepts to various situations of reliability evaluation expand these results in the PPP-BOT context.

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Figure Captions List

Figure 19 x-z view of fuzzy CDF resulted in Figure 17

Figure 20 three dimensional view of fuzzy CDF for debt service cover ratio (DSCR) resulting from the output of FR-MCS, x-axis: TPC, y-axis: probability, z-axis: α-value

Figure 21 x-y view of fuzzy CDF resulted in Figure 20

Figure 22 x-z view of fuzzy CDF resulted in Figure 20

Figure 23 Intersecting of x-y view of fuzzy CDF of return on equity ($EIRR$) resulted from FR-MCS with specific amount to find the probability of interest (a), fuzzy set of probability of acquired desirable output (b)

Figure 24 Intersecting of x-y view of fuzzy CDF of return on equity ($EIRR$) resulted from FR-MCS with 10% and 50% confidence levels to find the appropriate contingency values (arbitrary quantile) (a), fuzzy set of the appropriate contingency values (arbitrary quantile) with 10% confidence level (b) and with 50% confidence level (c)

Tables

Table 1 Basic input data of the case study

Input data	**Expected Value**	**Distribution/Membership function**
Macroeconomic indicators and indexes		
Project Economic life, project life cycle (yrs)	40	Deterministic
Costs regime during construction	-	<0.1,0.3,0.5,0.1>
Escalation rate during construction/inflation rate during operation period (%)	4	Log Normal distribution, LnN(4,1)
Amortization period (yrs)	20	Deterministic
Tax rate (%)	30	Deterministic
Gov. discount rate (%)	8.16	Deterministic
Cost of debt (%)	5.25	Deterministic
Cost of equity (hurdle rate) (%)	14	Deterministic
Loan Interest rate (%)	7.5	Deterministic
Loan repayment period/debt maturity (yrs)	10	Deterministic
Annual growth rate of unit price (%)	5	Normal distribution, N(5,1)
Annual growth rate of quantity of demand (%)	5	Normal distribution, N(5,1)
Cost of finance coefficient for Pre concession period costs calculation	0.05	Deterministic
Cost of tender coefficient for Pre concession period costs calculation	0.05	Deterministic
Annual revenue coefficient for O&M calculation	0.07	Deterministic
Increasing rate of annual growth rate of unit price (%)	10	Normal distribution, N(10,1)
Expected Base Cost coefficient for Asset value calculation at transfer date	0.1	Normal distribution, N(0.1,0.01)
Fuzzy-Stochastic Variables (FSV)		
Total project costs (M$)	170	Normal distribution, N(170,25)
Operation and maintenance costs (M$/year)	1.8907	Normal distribution, N(1.8907,0.25)
Annual growth rate of O&M costs (%)	5	Normal distribution, N(5,1)
Initial daily traffic (vehicles/day)	20,000	***Fuzzy variable***: Tr.F.N, ⟨19,178, 20,000, 20,000, 20,822⟩
Quantity of demand (vehicle per year)	7,300,000	***Fuzzy variable***: Tr.F.N, ⟨7,000,000, 7,300,000, 7,300,000, 7,600,000⟩
Operating revenue (M$/year)	27.01	***Fuzzy variable***: Tr.F.N, ⟨25.9, 27.01, 27.01, 28.12⟩
Pre concession period (yrs)	2	Log Normal distribution, LnN(2,0.5)
Negotiable concession items (NCIs)		
Construction period (yrs)	4	
Operation period (yrs)	21	
Concession period (yrs)	25	
Unit price of services (service in first year of operation) ($)	3.7	
Debt, Equity (%)	40%,30%	
Government subsidy/contribution, grant fraction, Royalty	30%	