The Topology of Postsingularly Finite Exponential Maps

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Stephen John Paton Worsley

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Abstract

We study the family of exponential maps, $E_a(z) := ae^z$ for $a \in \mathbb{C} \setminus \{0\}$. It was shown by Schleicher and Zimmer [18] that the escaping set of E_a can be described using a collection of disjoint injective curves called *dynamic rays*. Though each such dynamic ray is an injective curve, its closure in the *Riemann sphere* $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ may exhibit interesting topological behaviour. For example, it is known that this closure may be an *indecomposable continuum*[17, 6]. In fact, this is the case when a is a *Misiurewicz parameter*, a parameter a such that the singular value 0 is preperiodic to a repelling periodic orbit of E_a. In this thesis we focus on the specific map $E_{2\pi i}$, perhaps the simplest Misiurewicz parameter, though the techniques used should extend to a more general class of maps.

According to Schleicher and Zimmer's description of dynamic rays, for each dynamic ray γ there is a unique *address* $\underline{s} \in \mathbb{Z}^{\mathbb{N}}$ which encodes its dynamical behaviour. A natural question arises, for which addresses $\underline{s}$ does there exist a ray γ with address $\underline{s}$ whose closure in $\hat{\mathbb{C}}$ is indecomposable? In this thesis we give a complete answer to this question. We find a necessary and sufficient condition on $\underline{s}$ which determines if the closure of γ in $\hat{\mathbb{C}}$ is indecomposable. We also find a similar condition for whether the closure of some subset of γ is indecomposable.

We also prove that the closures of every dynamic ray in $\mathbb{C}$, when suitably compactified, is an *arclike continuum*.

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Chapter 1

Introduction

In dynamics, we study the evolution of systems which change over time according to some fixed rule. We may model many such systems as a space of possible states X with a function $f\colon X \to X$ acting on that space. The intuition here is that we start in a certain state at time 0 which we may describe as a point $x_0 \in X$. After an applying f once, we are at time 1 and our state is $x_1 \coloneqq f(x_0)$. After each discrete interval of time we advance our state by another iterate of f. Writing the nth iterate $f \circ ... \circ f$ as f^n, we have that the image of x_0 at time n is $x_n \coloneqq f^n(x_0)$. We may then ask questions about what sort of behaviour these sequences of x_n may have. For what values of x_0 does the sequence x_n tend to some stable orbit? Is the behaviour of x_n predictable when we restrict x_0 to a small neighbourhood? It often turns out that these systems turn out to be unpredictable even when the rules of the system described by f are quite simple. This accounts for what is popularly known as *chaos theory*. Broadly speaking, *chaos* describes when a small change in the initial conditions, here described by x_0, can lead to large differences in eventual behaviour, described by the sequence x_n.

One particularly rich area of study in dynamics is that of complex dynamics. This considers the case where X is the complex plane $\mathbb{C}$. When f is a holomorphic function, then we may divide $\mathbb{C}$ into two sets which categorises the eventual behaviour in the neighbourhood of points within those sets. Roughly speaking, the *Julia set* $J(f)$ is the set of all points $z \in \mathbb{C}$ such that f acts *chaotically* in a neighbourhood of z. The *Fatou set* $F(f) := \mathbb{C} \setminus J(f)$ contains all the points z such that there is a neighbourhood of z in which the behaviour of f is in some way stable. For even the simplest holomorphic functions, for example the quadratic family $f_c\colon z \mapsto z^2 + c$ for $c \in \mathbb{C}$, we find that the Julia set $J(f)$ appears to have a remarkably intricate topological structure.

One of the motivations for studying dynamics in general is the fact that topological objects which could otherwise be considered pathological often occur quite naturally as dynamical objects of relatively simple systems. One example of such strange topological behaviour is that of *indecomposable continua.*

1.0.1. Definition (Indecomposable Continua)**.**
A continuum X is said to be *decomposable* if there are two proper subcontinua $C_a, C_b \subset X$ such that $X = C_a \cup C_b$. If no two such subcontinua exist, X is said to be *indecomposable.*

The *buckethandle* is one such example of such an *indecomposable continuum.* One way of constructing the buckethandle is as the closure of a ray which accumulates upon itself. We may give a rough definition of such a ray $\gamma\colon [0,\infty) \to \mathbb{C}$ here. Let $\gamma([0,1])$ be a straight line. For $n \in \mathbb{N}$ with $n > 0$, we may then define $\gamma([n, n+1])$ in such a way that it runs back along $\gamma([0,n])$ from $\gamma(n)$ to $\gamma(0)$, all the while staying close to $\gamma([0,n])$. As n increases, so $\gamma([n, n+1])$ is defined to be closer to $\gamma([0,n])$. Such a ray is shown in the figure below, defined up to $\gamma([0,4])$. One of the interesting topological properties of such a construction can be seen by considering a vertical cross section, which in this case will be a Cantor set. It turns out that there are uncountably many path components in the buckethandle, each of which is a dense ray of the form $\gamma\colon [0,\infty) \to \mathbb{C}$ for the initial ray and a two sided dense ray of the form $\gamma\colon (-\infty,\infty) \to \mathbb{C}$ for each other component. Such topological

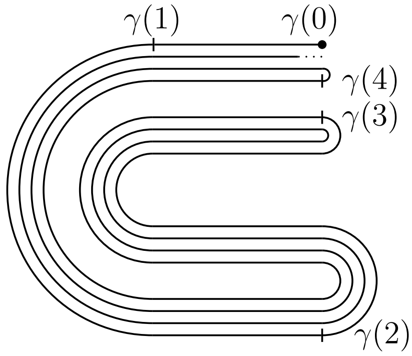

Figure 1.1: A ray accumulating on itself, its closure is the *buckethandle*.

objects have been shown to occur quite naturally in a dynamical context. For example, it was shown by Smale in 1965 [19] that the buckethandle can be constructed as the attractor of the horseshoe map, a map with simple folding behaviour.

In this thesis, we study the topology of certain dynamical objects associated with the map $E_{2\pi i}\colon \mathbb{C} \to \mathbb{C}; z \mapsto 2\pi i e^z$. In this case, the Julia set $J(E_{2\pi i})$ is in fact the whole plane. We are instead interested in the *escaping set*, the set of points which tend to infinity under iteration. The escaping set in some sense gives structure to the Julia set, inasmuch as for entire functions the Julia set is the boundary of the escaping set. For a holomorphic function f we write the escaping set as $I(f)$ and define it to be $I(f) := \{z \in \mathbb{C}\colon \lim_{n\to\infty} |f^n(z)| = \infty\}$. The escaping set $I(E_{2\pi i})$ is a natural candidate for study since it has been shown that there exist components of $I(E_{2\pi i})$ which exhibit interesting topological behaviour. There exist, for example, rays whose closure is an *indecomposable continuum*.

In the case of transcendental dynamics, the escaping set has been the subject of much interest. It is an open question, for example, if for any transcendental entire function f every connected component of $I(f)$ is unbounded. This was conjectured by Eremenko in 1987 [8]. A stronger form of Eremenko's conjecture, where all points in $I(f)$ are connected to infinity by a curve in

$I(f)$, is known not to hold. In fact, there exist transcendental functions f for which $I(f)$ contains no curves and where the closures in $\hat{\mathbb{C}}$ of the components of $I(f)$ are homeomorphic to the pseudo-arc [18].

The exponential family, $E_a\colon \mathbb{C} \to \mathbb{C}; z \mapsto ae^z$ for $a \in \mathbb{C} \setminus \{0\}$, is in some ways the simplest family of transcendental entire functions and perhaps the most well studied. Exponential maps have precisely one singular value. By a singular value we mean a point that is in the closure of the set of critical and asymptotic values. The singular values of a function in some way determine the dynamics of a function in as much as limits on the number of and distribution of singular values on a function generally limit the kind of dynamic behaviour that function can exhibit. In this sense the exponential family is analogous to the quadratic family for polynomial dynamics. A quadratic map has just one single singular value, a critical value. Similarly, the only singular value of E_a is an asymptotic value at 0. The exponential family in fact classifies this simple behaviour in the following sense. For all transcendental functions f with only one singular value, there exists some $a \in \mathbb{C} \setminus \{0\}$ such that f is conjugate to E_a [16][Theorem 2.3.5].

A description of the escaping set of E_a has been given by Schleicher and Zimmer which introduces the *dynamic ray* [18]. According to this description, every component of $I(E_a)$ is either the image of some injective curve $\gamma\colon (0, \infty) \to \mathbb{C}$ with $\lim_{x\to\infty} |\gamma(x)| = +\infty$ or the image of such a curve and its *landing point* $\lim_{x\to 0} \gamma(x)$ (we may note here that the landing point does not necessarily exist). Such curves, for which $E_a^n(\gamma((0, \infty)))$ is also unbounded for all $n \in \mathbb{N}$, are called the *dynamic rays* of E_a. The orbit of every point $z \in I(E_a)$ is eventually contained in the union of the dynamic rays and their landing points. These dynamic rays may be indexed as $\gamma_{\underline{s}}$ by their address $\underline{s} := (s_0, s_1, ...)$ where $s_j \in \mathbb{Z}$. These addresses encode dynamical information about $\gamma_{\underline{s}}$. Specifically, the address is defined in such a way that for all $n \in \mathbb{N}$ we have

$$\lim_{x\to\infty} \operatorname{Im}(E_a^n \gamma(x)) = 2\pi s_n - \arg(a)$$

while

$$\lim_{x\to\infty} \mathrm{Re}(E_a^n \gamma(x)) = +\infty.$$

In fact, for all addresses $\underline{s}$ for which $|s_n|$ does not grow too fast, it is known that there exists a unique dynamic ray with that address. A fuller description of Schleicher and Zimmer's results is given in the preliminaries.

For many parameters a, the topology of these rays is well understood. For example, when E_a has an attracting fixed point then the collection of dynamic rays is a Cantor bouquet and all rays land at a unique point [7] (we say a ray γ lands when there exists in $\hat{\mathbb{C}}$ a limit of $\gamma(x)$ as $x \to 0$). For other parameters, the behaviour of these dynamic rays is more wild. When the singular value 0 is pre-periodic to a repelling periodic point of E_a, we say a is a Misiurewicz parameter (it is in fact the case that whenever 0 is pre-periodic in E_a, the cycle it is pre-periodic to must be repelling). In this case, it is known that the dynamic rays of E_a are dense in $\mathbb{C}$. Furthermore not all rays land in $\hat{\mathbb{C}}$. Instead, some rays have a non-trivial *accumulation set*.

1.0.2. Definition.
The *Accumulation set* of a dynamic ray γ is defined as

$$\bigcap_{x>0} \overline{\gamma((0,x))}$$

There exist some rays who accumulate on themselves, by which we mean they are contained within their own accumulation set. The closure of such rays in the Riemann sphere $\hat{\mathbb{C}}$ has been shown to be an indecomposable continuum [17]. A well studied example of a parameter where E_a exhibits such behaviour is that of $a = 2\pi i$. This is perhaps the simplest example of a Misiurewicz parameter, we have $E_{2\pi i}(0) = 2\pi i$ and $2\pi i$ is a fixed point.It has been shown by Devaney, Rocha and Jarque [6] that $E_{2\pi i}$ also contains examples of pairs of rays such that one ray accumulates on the other. It has been shown by Jianxun Fu and Gaofei Zhang that there exist rays whose accumulation sets are bounded indecomposable continua and rays whose accumulation sets are arcs [10].One of the aims of this thesis is to give a deeper topological description for dynamic rays of $E_{2\pi i}$.

One such description we find is in terms of *arclike continua*. We use the following definition in terms of *ϵ-maps*. Note that some references use the term *chainable continuum* which has a different, but equivalent, definition.

1.0.3. Definition (Arclike Continua and ϵ-maps)**.**
A continuum X is said to be *arclike* if for every ϵ there exists a continuous surjective map $g_\epsilon \colon X \to [0,1]$ such that for all $x \in [0,1]$ we have $\operatorname{diam}(g_\epsilon^{-1}(x)) < \epsilon$. Such maps we call *$\epsilon$-maps* of X.

There are a rich variety of arclike continua. Famous examples include the $\sin \frac{1}{x}$ continuum and the aforementioned buckethandle. There are in fact uncountably many non-homeomorphic arclike continua which can be constructed as the closure of a ray. For example, an uncountable collection of such non-homeomorphic arclike continua are constructed by the now proved Ingram Conjecture [3]. Each of these continua are constructed as the inverse limit of some tent map $F_s \colon [0,1] \to [0,1]; x \mapsto \min(sx, s(1-x))$ for $s \in [1,2]$. For values of $s > \sqrt{2}$, the resulting continuum contains an indecomposable subcontinuum. The inverse limit of F_2 is in fact the buckethandle. For values of $s < \sqrt{2}$, the resulting continuum is decomposable. In each of these cases, there are uncountably many such homeomorphically unique arclike continua. Each of these can be constructed as the closure of a ray $\gamma \colon [0,\infty) \to \mathbb{C}$ and hints at the potential richness of such a class of topological objects.

One of our main results in this thesis, Theorem 3.3.2, states that for any dynamic ray γ of $E_{2\pi i}$, there is a certain compactification of the closure of γ in $\mathbb{C}$ which is an arclike continuum. We note that this compactification is not necessarily the closure of γ in $\hat{\mathbb{C}}$. There exist, for example, dynamic rays whose closure in $\hat{\mathbb{C}}$ is a topological circle, a non-arclike continuum. We instead take the closure of γ with respect to a different compactification of $\mathbb{C}$.

We add to $\mathbb{C}$ the *circle of addresses* at infinity to form the compactification $\tilde{\mathbb{C}}$. The circle of addresses are so called because they contain the address $\underline{s}$ of each ray $\gamma_{\underline{s}}$ in such a way that $\gamma_{\underline{s}}(x)$ tends to $\underline{s}$ in $\tilde{\mathbb{C}}$ as x increases. This compactification has been used before in the study of exponential maps [17]. In this thesis, we give a full description of its construction which generalises to a compactification on $\mathbb{C}$ with respect to other similar classes of families of rays in $\mathbb{C}$. The power of this compactification is that it in some way allows us to distinguish between accumulation points at infinity of different rays. We recall that there exist pairs of rays γ_a, γ_b whose closures in $\mathbb{C}$ intersect [6]. Theorem 3.3.2 also gives us that the closure of $\gamma_a \cup \gamma_b$ in $\tilde{\mathbb{C}}$ is arclike.

Ultimately, it is the same kind of "folding back" type behaviour in the tent map and the Smale horseshoe which gives rise to this rich topological behaviour. Similarly, we find that much of the rich topological behaviour of rays of $E_{2\pi i}$ happens due to the fact that for a ray γ and large $n \in \mathbb{N}$, certain branches of $E_{2\pi i}^{-n}(\gamma)$ fold it back in some sense (by a branch of $E_{2\pi i}^{-n}(\gamma)$, we mean the image of a branch of $E_{2\pi i}^{-n}$ which is defined and continuous on γ). We can here give a sketch of how this folding behaviour arises.

For any dynamic ray $\gamma_{\underline{s}}$ with address $\underline{s} := (s_0, s_1, ...)$, there exists a branch of $E_{2\pi i}^{-1}(\gamma)$ for which $2\pi i$ acts as an attracting fixed point. We find that this branch of $E_{2\pi i}^{-1}(\gamma)$ is a dynamic ray with imaginary part asymptotic to $2\pi s_n - \arg(a)$ i.e. it has an address of the form $(1, s_0, s_1, ...)$. After many such preimages, we find $E_{2\pi i}^{-n}(\gamma_{\underline{s}})$ becomes arbitrarily close to $2\pi i$ and has address $(1, 1, ..., 1, s_0, s_1, ...)$. Similarly, there is then a branch of $E_{2\pi i}^{-(n+1)}(\gamma_{\underline{s}})$ which is arbitrarily close to 0. This branch is such that $E_{2\pi i}^{-(n+1)}(\gamma_{\underline{s}})$ has address $(0, 1, 1, ..., 1, s_0, s_1, ...)$. Any branch of the preimage of this ray must have points with large negative real part. Any further branch of these points must map them to points with large positive real value. There is then a branch of $E_{2\pi i}^{-(n+3)}(\gamma_{\underline{s}})$ which has address $(a, b, 0, 1, 1, ..., 1, s_0, s_1, ...)$ for some $a, b \in \mathbb{N}$ which maps certain points in $\gamma_{\underline{s}}$ with small real part to points with large real part. When a, b are chosen to be small, we also have that the minimum real part of $E_{2\pi i}^{-(n+3)}(\gamma_{\underline{s}})$ is bounded and small. Let S_k be the finite sequence of the form $0, 1, 1, ..., 1$ of length k. The examples of non-landing rays in $E_{2\pi i}$ we have mentioned so far [17, 6, 10] are all constructed by taking dynamic rays $\gamma_{\underline{s}}$ whose addresses contain a sequence of such S_{k_n} where k_n is chosen to increase "sufficiently quickly". This thesis gives a precise description as to what is meant by "sufficiently quickly". We give necessary and sufficient conditions in terms of the address $\underline{s}$ for when the closure of $\gamma_{\underline{s}}$ is arclike and when we have the following topological property for $\gamma_{\underline{s}}$:

1.0.4. Definition (Indecomposability Beyond a Point)**.**
We say a dynamic ray γ is *indecomposable beyond* $x \in (0, \infty)$ if $\overline{\gamma(0, x)}$ is indecomposable and $\gamma((x, \infty))$ does not intersect $\overline{\gamma(0, x)}$.

We recall that there is a class of dynamic rays $\gamma_{\underline{s}}$ of $E_{2\pi i}$ whose accumulation set intersects the accumulation set of some other ray; such rays we call *binary type.* All other dynamic rays of $E_{2\pi i}$ we call *non-binary type.* Within each class of rays we may determine the topology of $\gamma_{\underline{s}}$ entirely in terms of its address $\underline{s}$. We note that we give, in Section 9.4, a complete description of which addresses $\underline{s}$ have binary or non-binary type rays associated with them. In this way, it is possible to determine these topological properties of $\gamma_{\underline{s}}$ entirely from $\underline{s}$.

1.0.5. Theorem.
Let $\gamma_{\underline{s}}$ be a binary type ray with address $\underline{s}$. The set $\overline{\gamma_{\underline{s}}} \subset \hat{\mathbb{C}}$ is indecomposable if and only if, for all $k \in \mathbb{Z}$, there exist infinitely many $m' \in \mathbb{N}$ with $m' \geq k$ such that the following conditions hold:

- $s_{m'} \in \{0, 2\}$;
- $s_{m'+1} = 0$;
- $s_j = 1$ *for all* $j > m' + 1$ *with* $j < \exp^{m'-k}(1)$.

If and only if the above conditions are satisfied for some, but not all $k \in \mathbb{Z}$, *then $\gamma_{\underline{s}}$ is indecomposable beyond some $x \in (0, \infty)$.*

1.0.6. Theorem.
Let $\gamma_{\underline{s}}$ be a non-binary type ray with address $\underline{s}$. The set $\overline{\gamma_{\underline{s}}} \subset \hat{\mathbb{C}}$ is indecomposable if and only if, for all $k \in \mathbb{Z}$, there exist infinitely many $m' \in \mathbb{N}$ with $m' \geq k$ such that the following conditions hold:

- *$s_{m'+1} = 0$;*
- *$s_j = 1$ for all $j > m' + 1$ with $j < \exp^{m'-k}(1)$;*
- *Let $M := \lfloor \exp^{m'-k}(1) \rfloor$. Then for all $j > M$ we have $|s_j| \leq \exp^{j-M}(1)$.*

If and only if the above conditions are satisfied for some, but not all $k \in \mathbb{Z}$, then $\gamma_{\underline{s}}$ is indecomposable beyond some $x \in (0, \infty)$.

These follow directly from, and are essentially restatements of, Corollaries 9.4.2, 9.4.3 and 9.5.3.

Additionally, Theorem 9.3.6 gives conditions, which can be expressed in terms of addresses, for precisely when there exist pairs of rays γ_a, γ_b whose closures in $\mathbb{C}$ intersect and any one of the following is true of γ_a and γ_b when taking closures in $\tilde{C}$ or $\hat{\mathbb{C}}$:

- $\overline{\gamma_a \cup \gamma_b}$ is indecomposable;
- $\overline{\gamma_a}$ is indecomposable and $\overline{\gamma_b} \setminus \gamma_b = \overline{\gamma_a}$;
- $\overline{\gamma_a}$ is indecomposable and γ_b is indecomposable beyond some $x \in (0, \infty)$ and $\overline{\gamma_b}((0, x)) = \overline{\gamma_a}$;
- γ_a is indecomposable beyond some $x_a \in (0, \infty)$, γ_b is indecomposable beyond some $x_b \in (0, \infty)$, and $\overline{\gamma_a}((0, x_a)) = \overline{\gamma_b}((0, x_b))$;
- γ_a is indecomposable beyond some $x \in (0, \infty)$, and $\overline{\gamma_b} \setminus \gamma_b = \overline{\gamma_a((0, x))}$.

The first two cases have already been shown to exist [6]. The last three show an entirely new kind of behaviour. We can illustrate examples of this kind of behaviour as follows:

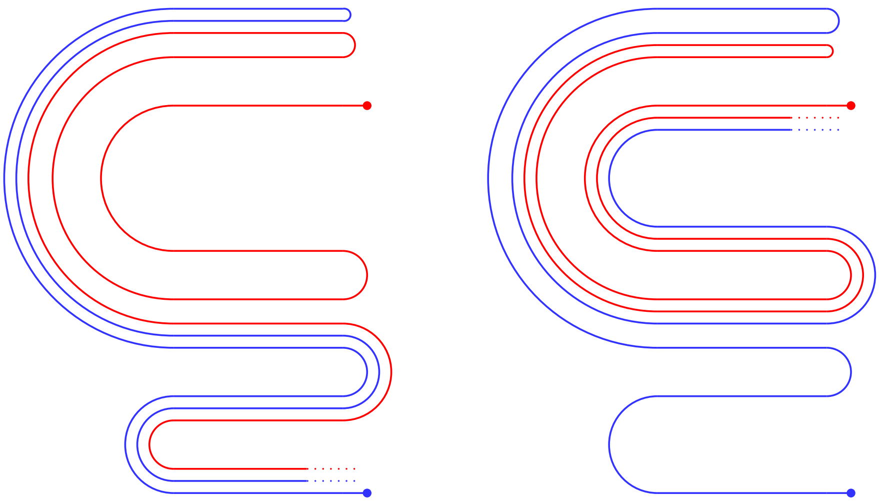

Figure 1.2: An example of a pair of rays exhibiting the first kind two kinds of behaviour. In the first case, we see an example of two rays accumulating on one another. The closure of the red and blue rays here is the same indecomposable continuum. In the second case, we see the red ray accumulates on itself and the blue ray accumulates on the red ray. The closure of the red ray here is an indecomposable continuum, and the accumulation set of the blue ray is the closure of the red ray.

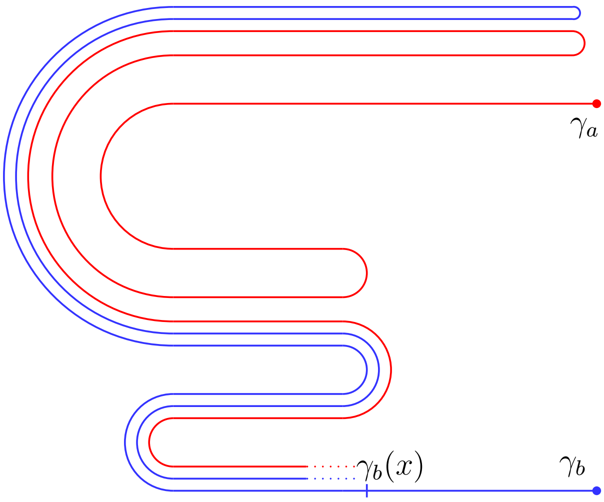

Figure 1.3: This is an example of the third kind of behaviour mentioned. The red ray γ_a accumulates on itself, and on $\gamma_b((0, x])$, but nowhere on $\gamma_b((x, \infty))$. Its closure is indecomposable. The blue ray γ_b is indecomposable beyond x. The closure of $\gamma_b((0, x])$ is precisely the closure of γ_a.

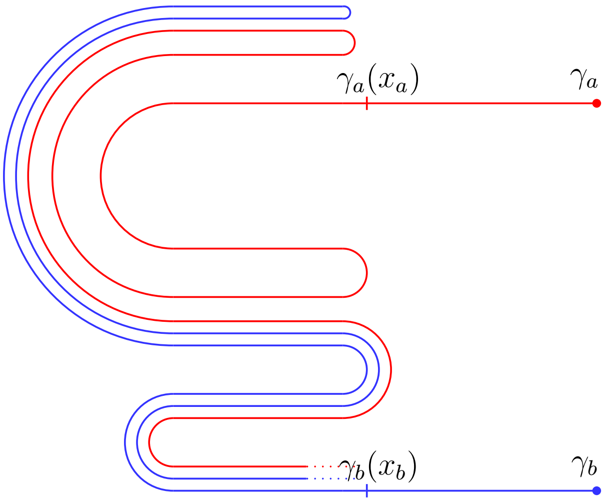

Figure 1.4: This is an example of the fourth kind of behaviour mentioned. The red ray γ_a is indecomposable beyond x_a, the blue ray γ_b is indecomposable beyond x_b. The accumulation sets of both rays are equal.

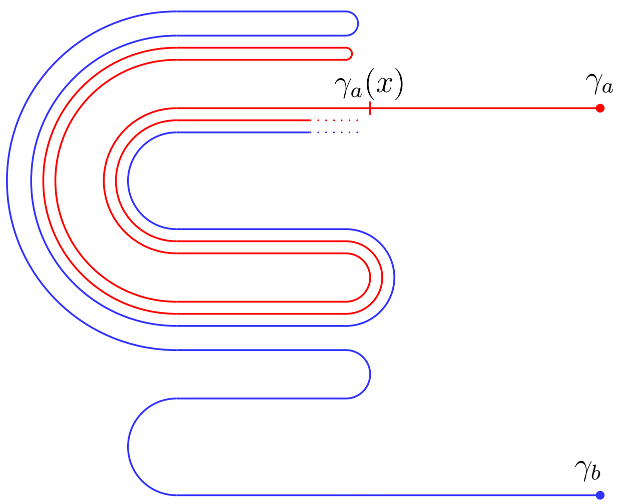

Figure 1.5: This is an example of the fifth kind of behaviour mentioned. The red ray γ_a is indecomposable beyond x. The blue ray γ_b is not indecomposable beyond any $x \in (0, \infty)$. The accumulation sets of both sets are precisely the closure of $\gamma_a((0, x])$.

Chapter 2

Preliminaries

2.1 General Notation

For a function $F\colon X \to X$, we write the nth iterate as F^n. In other words, F^n is the composition $F \circ ... \circ F$ of n copies of F.

We write as $\mathbb{N}$ the set of natural numbers $\{0, 1, 2, ...\}$. Let $\hat{\mathbb{C}}$ denote the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Let $\mathbb{D}$ denote the open unit disc $\{z \in \mathbb{C}\colon |z| < 1\}$. We will write the strictly positive real numbers as the interval $(0, \infty)$. For $R \in (0, \infty)$, let B_R denote the the open ball centred at 0 of radius R, that is, $B_R \coloneqq \{z \in \mathbb{C}\colon |z| < R\}$.

For $z \in \mathbb{C}$, we write as $\mathrm{Re}(z)$ the real value of z and write as $\mathrm{Im}(z)$ the imaginary value.

We write as $\underline{x}$ the sequence of objects x_j indexed by $j \in \mathbb{N}$. We also write this sequence as $(x_0, x_1, ...)$. We write as σ the one sided shift map $(x_0, x_1, x_2, ...) \mapsto (x_1, x_2, ...)$ defined on whichever space is appropriate, depending on what space x_j is chosen from.

For $x \in \mathbb{R}$, we write as $\lfloor x \rfloor$ the largest integer $n \in \mathbb{Z}$ such that $n \leq x$.

2.2 Ideas from Complex Dynamics

We assume the reader is familiar with basic complex analysis. We will here introduce the necessary facts and ideas from complex dynamics used in this thesis.

2.2.1. Theorem (Kœnigs Linearisation)**.**
Let F be holomorphic in some neighbourhood V of 0. Let $F(0) = 0$ and let the derivative $F'(0) = \lambda$ such that $|\lambda|$ is neither 0 nor 1. Then there exists a neighbourhood $U \subset V$ which contains 0, and some injective holomorphic function $\varphi\colon U \to \mathbb{C}$ such that $\varphi \circ F|_U = \lambda\varphi$. Furthermore, we may assume that $\varphi'(0) = 1$.

Proofs of this can be found in, for example, [11][Theorem 8.2].

We establish here some definitions and facts about *dynamic rays*. We give here a description of dynamic rays which we argue is equivalent to the description given by [9][Proposition 2.1] (although we use a slightly different parameterisation). We refer the reader to [18] for a more detailed construction and proof of that description.

2.2.2. Definition (Dynamic Rays, Landing Points, Accumulation sets and Forward Tails)**.**
Let $\gamma\colon (0,\infty) \to \mathbb{C}$ be an injective curve. Abusing notation slightly, we may also write γ as the image $\gamma((0,\infty))$. We say γ is a *dynamic ray* of E_a if γ is a path connected component of $I(E_a)$ and for all $n \in \mathbb{N}$,

$$\lim_{x\to\infty} |E_a^n(\gamma(x))| = \infty.$$

If $\lim_{x\to 0}\gamma(x)$ exists in $\hat{\mathbb{C}}$ we say that the ray γ lands. When γ lands, we call $\lim_{x\to 0}\gamma(x)$ the *landing point* of γ. We recall again the *accumulation set* of a dynamic ray γ, as defined in the introduction, is defined to be

$$\bigcap_{x>0} \overline{\gamma((0,x))}.$$

We note that the landing point of a ray is precisely the accumulation set in the case where the accumulation set contains only one point.

For $x \in (0,\infty)$, we call $\gamma((x,\infty))$ a *forward tail* of γ.

In the literature quoted, dynamic rays have a different definition. It is necessary then to justify that the two definitions are indeed equivalent, at least with respect to the images of these rays. Following the notation of [9], we write as $g_{\underline{s}}$ the image of a ray which agrees with the description of a dynamic ray in [9][Proposition 2.1]. According to [9][Corollary 4.3], we can derive that all path connected components X of $I(E_a)$ (ignoring their endpoints) are either dynamic rays of the form $g_{\underline{s}}$, or else are path connected components of the iterated preimage of a ray $g_{\underline{s}}$ which contains 0. We may parameterise this $g_{\underline{s}}$ containing 0 as $\gamma\colon (0,\infty) \to g_{\underline{s}}$ and divide $g_{\underline{s}}$ into two parts, $\gamma((0,c))$ and $\gamma((c,\infty))$, where c is chosen so that $\gamma(c) = 0$. According to the description of dynamic rays in [9][Proposition 2.1], it is straightforward to see that $g_{\underline{s}}$ may be parameterised in such a way that all iterated preimage components of $E_a^n(\gamma((c,\infty)))$ are in fact dynamic rays and all iterated preimage components of $E_a^n(\gamma((0,c)))$ are not dynamic rays (this parameterisation will have the same orientation as the parameterisation of $g_{\underline{s}}$ in [9]).

Taking the definition of dynamic rays in [9][Proposition 2.1], we note that $|g_{\underline{s}}(t)|$ must be large when t is large. Similarly, for all $n \in \mathbb{N}$, we have that $|E_a^n g_{\underline{s}}(t)|$ will also be large when t is large. It follows that all such dynamic rays satisfy Definition 2.2.2. In order to show that all of our dynamic rays also satisfy the description in [9][Proposition 2.1], we must show that the path connected components $X \subset I(E_a)$ such that $E_a^n(X) = \gamma((0,c))$ holds, which are not dynamic rays according to [9], are not dynamic rays according to Definition 2.2.2 either. We do this by establishing that for all dynamic rays

γ parameterising some set $g_{\underline{s}}$, (with $g_{\underline{s}}$ as defined in [9]) there is some $m \in \mathbb{N}$ such that $E_a^m(\gamma)$ does not land at infinity. This fact can be derived, as we will show in Lemma 2.2.3 below, from [17][Lemma 3.3 and Lemma 6.5]. Given that $E_a^n(X) = \gamma((0, c))$ and $E_a^m(\gamma)$ does not land, we can show that $E_a^{n+m}(X)$ cannot be parameterised in a way that allows X to satisfy Definition 2.2.2. This is due to the fact that for $x \in (0, c)$, we have that $\lim_{x\to 0} |E_a^m(\gamma(x))| \neq \infty$ and $\lim_{x\to c} |E_a^m(\gamma(x))|$ is finite. Therefore, the two definitions of dynamic rays are equivalent (ignoring parameterisation).

2.2.3. Lemma.
Let γ be a dynamic ray of E_a. There exists some $m \in \mathbb{N}$ such that $E_a^m(\gamma)$ does not land at infinity.

Proof. Let $g_{\underline{s}}$ be a dynamic ray as described by [17][Proposition 2.1]. Suppose the first case of [17][Lemma 6.5] holds for $\underline{s}$. Then by [17][Lemma 3.3], it follows that the accumulation set of $g_{\underline{s}}$ contains another ray and $g_{\underline{s}}$ does not land. Suppose the second case of [17][Lemma 6.5] holds for $\underline{s}$. Then for some $m \in \mathbb{N}$ we have that $E_a^m(g_{\underline{s}})$ does not accumulate at infinity. In either case, there is some $m \in \mathbb{N}$ such that $E_a^m(g_{\underline{s}})$ does not land at infinity.

This proves the lemma for dynamic rays, as described in [17][Proposition 2.1] and [9][Proposition 2.1]. As argued above, if this lemma holds for such dynamic rays then it follows that the two definitions are equivalent with respect to their images. If $E_a^m(g_{\underline{s}})$ does not land at infinity with respect to the parameterisation in [17], there can be no parameterisation γ of the set $g_{\underline{s}}$ such that γ satisfies Definition 2.2.2 while also having $E_a^m(\gamma)$ land at infinity. This lemma therefore also holds for our definition of dynamic rays. ∎

We note that these definitions are equivalent in the sense that for every dynamic ray defined for one definition, there is a dynamic ray defined for the other definition with the same image. Furthermore, these dynamic rays will both have the same unique orientation. By this, we mean the following. Let $g_{\underline{s}}$ and γ be rays as defined by [17] and Definition 2.2.2 respectively, both sharing the same image. Then if x, x', y, y' are such that $\gamma(x) = g_{\underline{s}}(x')$ and $\gamma(y) = g_{\underline{s}}(y')$, then $x < y$ if and only if $x' < y'$. From the properties of $g_{\underline{s}}$, it is always possible to induce a dynamic ray γ with the same orientation. It remains to show that the orientation on γ is unique. To be more precise, we say the orientation is unique in the sense that for any two parameterisations $\gamma', \gamma'' \colon (0, \infty) \to \mathbb{C}$ of the same image of a ray γ, there is an order preserving homeomorphism $h \colon (0, \infty) \to (0, \infty)$ such that $\gamma' = \gamma'' \circ h$. This follows again from the above lemma. If γ' is a parameterisation of a dynamic ray satisfying Definition 2.2.2, then for some $m \in \mathbb{N}$ we have that $E_a^m(\gamma')$ does not land at infinity. Let γ'' be a parameterisation of the image of γ' but with opposite orientation. Then $\lim_{x\to\infty} |E_a^m(\gamma'')| = \lim_{x\to 0} |E_a^m(\gamma')| \neq \infty$ and so γ'' does not satisfy Definition 2.2.2.

Furthermore, dynamic rays of exponential functions are known to have the following properties.

- A dynamic ray does not land on itself. From the description, we also know a pair of distinct rays cannot land on or intersect each other. This follows from [15][Lemma 5.1]. There are, however, examples of pairs of rays sharing a landing point.

- If a dynamic ray γ does not contain 0, then every continuous branch of $E_a^{-1}(\gamma)$ is also a dynamic ray. This follows from Definition 2.2.2 as each branch of $E_a^{-1}(\gamma)$ is a path connected component of $I(E_a)$ (ignoring endpoints) for which $E_a^{-1}(\gamma(x))$ tends to infinity as $x \to \infty$.

- The image of a dynamic ray is either a dynamic ray or the forward tail of some dynamic ray. That is, for all rays γ, there is some ray γ' and some $x_0 \in [0, \infty)$ such that $E_a(\gamma) = \gamma'((x_0, \infty))$. The image $E_a(\gamma)$ is a forward tail (i.e. $x_0 \neq 0$) precisely when γ' contains 0. In this case, we have that $\gamma'(x_0) = 0$.

  Proof. From the definition, γ is (ignoring endpoints) a path connected component of $I(E_a)$. Therefore $E_a(\gamma)$ is contained in a path connected component of $I(E_a)$. It is easy to use the parameterisation of $E_a(\gamma)$ to induce a parameterisation of this path connected component (ignoring endpoints where necessary) which satisfies Definition 2.2.2. It follows that $E_a(\gamma)$ is contained in some dynamic ray γ'. Furthermore, the orientation of $E_a(\gamma)$ and γ' must agree. In order for $E_a(\gamma(x))$ to tend to infinity as $x \to \infty$, then $E_a(\gamma)$ it must either be γ' or else be some forward tail of γ'. Suppose 0 is not contained in γ'. Then all connected components of the preimage of γ' are continuous branches of E_a^{-1} on γ'. In this case, $E_a(\gamma) = \gamma'$. The only time $E_a(\gamma) \neq \gamma'$ is when a continuous branch of E_a^{-1} on γ' cannot be taken. This happens precisely when there is some $x_0 \in (0, \infty)$ such that $\gamma'(x_0) = 0$. In this case, all connected components of the preimage of γ' are the image of a continuous branch of E_a^{-1} on either $\gamma'((x_0, \infty))$ or $\gamma'((0, x_0))$. As we have implied before by Lemma 2.2.3, only the components of the preimage of $\gamma'((x_0, \infty))$ can be dynamic rays. ■

- Every dynamic ray γ is asymptotic to some horizontal line such that

$$\lim_{x\to\infty} \operatorname{Re}(\gamma(x)) = +\infty$$

and for some $s \in \mathbb{Z}$, we have

$$\lim_{x\to\infty} \operatorname{Im}(\gamma(x)) = 2\pi s - \arg(a).$$

Proof. Since $\lim_{x\to\infty} |E_a(\gamma(x))| = \infty$, it follows that $\lim_{x\to\infty} \operatorname{Re}(\gamma(x)) = +\infty$. For sufficiently large x, it follows that $E_a^2(\gamma((x, \infty)))$ is contained in the right half plane so that $E_a(\gamma((x, \infty)))$ is contained in some horizontal strip of the form $\{z \in \mathbb{C}\colon s - \arg(a) - \frac{\pi}{2} \operatorname{Im}(z) < s - \arg(a) + \frac{\pi}{2}\}$ for some $s \in \mathbb{Z}$. In particular, $\lim_{x\to\infty} \arg(E_a(\gamma(x))) = 0$ so that $\lim_{x\to\infty} \operatorname{Im}(\gamma(x)) = 2\pi s - \arg(a)$. ■

Since this last point also holds for all iterates of γ, this fact allows us to define an *address* for each dynamic ray.

2.2.4. Definition (Addresses and Exponential Bounded Sequences)**.**
Let $\underline{s} := (s_0, s_1, ...)$ be a sequence of integers. If we have that

$$\lim_{x\to\infty} \operatorname{Im}(E_a^n \gamma(x)) = 2\pi s_n - \arg(a)$$

for all $n \in \mathbb{N}$, then we say γ has *address* $\underline{s}$.

Let $\underline{s}$ again be a sequence of integers. If there exists some $r \in (0, \infty)$ such that for all $n \in \mathbb{N}$ we have

$$|s_n| < \exp^n(r),$$

then we say the sequence $\underline{s}$ is *exponentially bounded.* We write as $\mathcal{S}_e$ the set of exponentially bounded integer sequences.

2.2.5. Theorem (Schleicher-Zimmer)**.**
Let $a \in \mathbb{C} \setminus \{0\}$. The set of dynamic rays can be indexed as $\{\gamma_{\underline{s}}\}_{\underline{s}\in\mathcal{S}_e}$ so that $\gamma_{\underline{s}}$ has address $\underline{s}$ for each $\underline{s} \in \mathcal{S}_e$. For each $z \in I(E_a)$, one of the following holds:

- *There is some unique $\underline{s} \in \mathcal{S}_e$ such that $z \in \gamma_{\underline{s}}$;*
- *There is some unique $\underline{s} \in \mathcal{S}_e$ such that z is the landing point of $\gamma_{\underline{s}}$;*
- *There is some $\underline{s} \in \mathcal{S}_e$ such that $0 \in \gamma_{\underline{s}}$ and there is some $n \in \mathbb{N}$ such that $E_a^n(z)$ is contained in, or is a landing point of $\gamma_{\underline{s}}$.*

From [5], we know that for each Misiurewicz parameter $a \in \mathbb{C} \setminus \{0\}$ there exists a dynamic ray of E_a which lands at 0. In particular, in [6] it is shown that for $E_{2\pi i}$ we have the following.

2.2.6. Lemma.
There exists some γ_1, a dynamic ray of $E_{2\pi i}$ which lands at $2\pi i$ and for which $E_{2\pi i}(\gamma_1) = \gamma_1$.

We may here sketch a proof for the existence of such a ray.

Sketch of Proof. Let $T := \{z \in \mathbb{C} : \frac{\pi}{2} < \operatorname{Im}(z) < \frac{5\pi}{2}\}$. Then there exists a branch of $E_{2\pi i}^{-1}$ which maps $\mathbb{C} \setminus (-\infty, 0]$ into T. We call this branch L_1. We note that L_1 is defined over all of T and $2\pi i$ is an attracting fixed point of L_1. Furthermore, for the derivative L_1' of L_1 we have that $|L_1'(z)| \leq \frac{2}{\pi} < 1$ for all $z \in T$. In this way we see that the basin of attraction of $2\pi i$ is in fact all of $\mathbb{C}\setminus(-\infty, 0]$. That is, for every point $z \in \mathbb{C}\setminus(-\infty, 0]$ we have that $L_1^n(z) \to 2\pi i$ as $n \to \infty$. Furthermore, we have that all points of $\bigcap_{n\in\mathbb{N}} L_1^n(T) \setminus \{2\pi i\}$ are escaping. It turns out that $\bigcap_{n\in\mathbb{N}} L_1^n(T)$ is precisely $\gamma_1 \cup \{2\pi i\}$.

It remains to parameterise γ_1 as a ray. Let

$$F : [0, \infty) \to [0, \infty), x \mapsto \exp(x) - 1$$

be a model for exponential growth. We may define a sequence of rays

$$g_n : [0, \infty) \to \mathbb{C}, x \mapsto L_1^n(2\pi i + F^n(x)).$$

We then use the fact that $|(2\pi i + F^n(x)) - L_1(2\pi i + F^{n+1}(x))|$ is bounded and the derivative of L_1^n is small (and is in fact bounded above by a decreasing geometric series) to show that g_n converges uniformly on $[0, \infty)$ as $n \to \infty$. We then take γ_1 to be the limit of $g_n|_{(0,\infty)}$. We note that when defined this way γ_1 lands at 0 and is contained in $\bigcap_{n\in\mathbb{N}} L_1^n(T) \setminus \{2\pi i\}$. Its image is mapped by L_1 onto itself and is therefore also mapped by $E_{2\pi i}$ onto itself. Furthermore, for any point in $z \in \bigcap_{n\in\mathbb{N}} L_1^n(T) \setminus \{2\pi i\}$ we have that $E_{2\pi i}^n(z)$ is within a bounded distance to γ_1. Pulling back by L_1^n we find that z is within $(\frac{2}{\pi})^n$ of that distance to γ_1, this distance will be arbitrarily small for arbitrarily large n. It follows that $\gamma_1 = \bigcap_{n\in\mathbb{N}} L_1^n(T) \setminus \{2\pi i\}$.

Let C be the path connected component of $I(E_{2\pi i})$ containing γ_1. If C is not equal to γ_1, then there is a closed path $P \subset C$ landing on γ_1 such that P is disjoint from γ_1. It is difficult to prove here directly from first principles, but as a consequence of [9][Corollary 4.3], we know that such a path cannot exist. So we have that γ_1 is indeed a path connected component of $I(E_a)$ and all properties of a dynamic ray are satisfied. ■

We note that this construction gives us that the address of γ_1 is $(1, 1, ...)$. The construction in this proof is similar to the method used to construct rays of arbitrary address. It must be noted though, the particular properties of the strip T make the construction of γ_1 simpler. In particular, the fact that $|L_1'(z)| \leq \frac{2}{\pi} < 1$ for all $z \in T$.

Given the existence of γ_1, we may now label each dynamic ray which is a branch of the preimage of γ_1. We note that since $E_{2\pi i}(\gamma_1) = \gamma_1$, one of these branches is itself. Each branch is then some $2k\pi i$ translate of γ_1.

2.2.7. Definition.
For $k \in \mathbb{Z}$ let $\gamma_k := 2(k-1)\pi i + \gamma_1$.

We note that γ_0 is a dynamic ray landing at 0. The existence of such a ray allows for much of the behaviour we find in $E_{2\pi i}$. In particular, its preimage divides the plane into strips and allows us to define an *itinerary.*

2.2.8. Definition (Itineraries and Itinerary Sets)**.**
Let t_1 be the dynamic ray which is the component of $E_{2\pi i}^{-1}(\gamma_0)$ which lies between γ_0 and γ_1 (in the sense that it lies above γ_0 and below γ_1 according to the vertical order defined in 3.2). Let $t_k := 2(k-1)\pi i + t_1$ denote the translates of t_1 in $E_{2\pi i}^{-1}(\gamma_0)$. Let T_k be the connected component of $E_{2\pi i}^{-1}(\mathbb{C} \setminus (\gamma_0 \cup \{0\}))$ whose boundary is $t_k \cup t_{k+1}$. We note here that since γ_k lies between t_k and t_{k+1}, then γ_k and $2\pi i k$ are contained in T_k.

Let $\underline{u} := (u_0, u_1, ...)$ be some sequence of integers. We say a point $z \in \mathbb{C}$ has *itinerary* $\underline{u}$ if for all $n \in N$

$$E_{2\pi i}^n(z) \in T_{u_n} \cup t_{u_n}.$$

We write as $C_{\underline{u}}$ the set of all points in $\mathbb{C}$ with itinerary $\underline{u}$. We call this the *itinerary set* of $\underline{u}$.

2.2.9. Remark.
This definition assigns a unique itinerary for every point in $\mathbb{C}$. For any dynamic ray γ and any $n \in \mathbb{N}$, we have that either $E^n(\gamma)$ is disjoint from $\bigcup_{k\in\mathbb{Z}} t_k$, or $E^n(\gamma) = t_k$ for some $k \in \mathbb{Z}$. In either case, we can see that all points of γ must have the same itinerary. In this sense, we may talk about the itinerary of a given ray.

In [6], itineraries are divided into two classes, A sequences and B sequences. For itineraries $\underline{u}$ with B sequences it is possible that $C_{\underline{u}}$ contains two dynamic rays. In this thesis, we use instead the term *binary type itinerary* to refer to B sequences and *non-binary type itinerary* to refer to A sequences. This replaces the definition used in the introduction of binary type rays and non-binary type rays. Binary type rays will have binary type itineraries and non-binary type rays will have non-binary type itineraries.

2.2.10. Definition (Binary Type Sequences)**.**
Let $\underline{u}$ be an itinerary. If $u_j \in \{0, 1\}$ for all but finitely many j, we say $\underline{u}$ is of *binary type.* If $u_j = 1$ for all but finitely many j, we say $\underline{u}$ is of *singular type.*

To be precise, we may classify how many rays are contained within a given type of itinerary as follows.

2.2.11. Theorem (Classification of Sequences)**.**
We may classify $C_{\underline{u}}$ into three cases:

- *If $\underline{u}$ is an exponentially bounded and non-binary itinerary, then $C_{\underline{u}}$ contains precisely one ray;*
- *If $\underline{u}$ is binary and non-singular, then $C_{\underline{u}}$ contains precisely two rays;*
- *If $\underline{u}$ is singular, then $C_{\underline{u}}$ contains precisely one ray. This ray upon iteration is eventually mapped onto γ_1. Furthermore, $C_{\underline{u}}$ only contains such a ray when $\underline{u}$ is singular.*

We give our own proof of this in Chapter 5.

2.3 Topology

We give here some assorted basic topological facts and definitions that we will use. Proof of the following facts can be found in [13].

2.3.1. Definition (Jordan Curves, Interiors and Orientation)**.**
Let $J\colon [0,1] \to \mathbb{C}$ be a continuous curve such that J is injective on $[0,1)$ and $J(0) = J(1)$. Then we say J is a *Jordan curve*. If there exists a bounded open subset of $\mathbb{C}$ whose boundary is J, then we call that set the *interior* of J. We note that by the Jordan curve theorem this set exists and is unique. We write as $|J|$ this interior of J.

We say J is *positively oriented* if for all points $z \in |J|$, the winding number of J around z is 1. Similarly, if for all $z \in |J|$ the winding number of J around z is -1, we say J is *negatively oriented.* It is a fact that every Jordan curve in $\mathbb{C}$ must be either positively or negatively oriented. We note here that being positively oriented is an equivalent definition to being anti-clockwise.

2.3.2. Theorem (Jordan–Schoenflies Theorem)**.**
Let J be a Jordan curve in $\mathbb{C}$. There exists a homeomorphism of $\mathbb{C}$ which maps J onto the unit circle and the interior of J onto the open unit disc.

A proof of this is given in [4]. We will make use of the following corollary which allows us to extend any homeomorphism between two Jordan curves to a homeomorphism between their interiors as well.

2.3.3. Corollary.
Let J_a, J_b be two Jordan curves in $\mathbb{C}$ and let $f\colon J_a \to J_b$ be a homeomorphism. Then there exists a homeomorphism F on $\mathbb{C}$ which maps $|J_a|$ to $|J_b|$ and is equal to f on J_a.

Proof. Let H_a be a homeomorphism on $\mathbb{C}$ which maps $|J_a|$ to the open unit disc and J_a to the unit circle $\partial\mathbb{D}$. Let H_b be similarly defined for J_b. Now the map $g \coloneqq H_b|_{J_b} \circ f \circ H_a^{-1}|_{J_a}$ is a homeomorphism of the unit circle which can be extended to a homeomorphism on $\mathbb{C}$ as follows. Let $G(0) = 0$ and for $z \in \mathbb{C} \setminus \{0\}$ let $G(z) = |z|g(\frac{z}{|z|})$. We note that this maps the open unit circle to the open unit circle. Now let $F \coloneqq H_b^{-1} \circ G \circ H_a$. This map is the composition of three homeomorphisms and so is a homeomorphism. It maps $|J_a|$ to $|J_b|$ and is equal to f on J_a. ■

Let $p\colon [0,1] \to X$ be a continuous path onto some space X. We may define $-p$ to be the path defined as $-p\colon x \mapsto p(1-x)$. If p, q are two paths in $\hat{\mathbb{C}}$ with $p(1) = q(0)$, then we may define their addition as follows:

$$p + q\colon x \mapsto \begin{cases} p(2x) & x \in [0, \frac{1}{2}] \\ q(2x-1) & x \in [\frac{1}{2}, 1] \end{cases}.$$

We write $p + (-q)$ as $p - q$ where such a path exists. Suppose we have three paths p_1, p_2, p_3 in $\mathbb{C}$ which intersect only at their common endpoints. To be precise, we mean $p_1(0) = p_2(0) = p_3(0)$ and $p_1(1) = p_2(1) = p_3(1)$. Then we find that $p_j - p_k$ is a Jordan curve for $j, k \in \{1, 2, 3\}$ with $j \neq k$. We use the following theorems to compare their orientations.

2.3.4. Theorem ([13] Chapter VII, Theorem 9.5).
Let p_1, p_2, p_3 be paths in $\mathbb{C}$ which intersect only at their common endpoints.If $p_3((0,1)) \subset |p_1 - p_2|$, then $|p_1 - p_3| \subset |p_1 - p_2|$ and both $p_1 - p_3$ and $p_1 - p_2$ have the same orientation.

2.3.5. Theorem ([13] Chapter VII, Theorem 9.6).
Let p_1, p_2, p_3 again be paths in $\mathbb{C}$ which intersect only at their common endpoints. Then $p_3((0,1)) \subset |p_1 - p_2|$ if and only if $p_1 - p_3$ and $p_2 - p_3$ have opposite orientations.

2.3.1 Continuum Theory

We recall that a continuum is a compact connected metric space. We will use the following facts about continua.

2.3.6. Theorem (Boundary Bumping Theorem)**.**
Let C be a continuum and and let $U \subset C$ be a proper subset. Let K be a connected component of $C \setminus U$. Then $\partial U \cap \partial K$ is non-empty.

A proof for this is given in [12].

We recall the definition of *indecomposable continua* from the introduction in Definition 1.0.1. We clarify here that by *decomposing* a continuum X into some number of subcontinua $\{C_k\}_{k\in\mathcal{K}}$ we mean that each C_k is a proper subcontinuum of X and $\bigcup_{k\in\mathcal{K}} C_k = X$. Having said that, we may prove the following.

2.3.7. Theorem.
If a continuum X can be decomposed into finitely many proper subcontinua then it can be decomposed into two proper subcontinua.

Proof. Suppose X can be decomposed into N proper subcontinua where $N > 2$. Let $\{C_j\}_{j\in\mathbb{N}, j<N}$ denote these subcontinua. Choose one of these subcontinua $C_0 \subset X$. Suppose there exist no other subcontinua C_j such that C_0 intersects C_j. Then the closed sets $\bigcup_{0<j<N} C_j$ and C_0 are disjoint from one another, contradicting the connectedness of $X = \bigcup_{0<j<N} C_j \cup C_0$. There therefore exists one of these subcontinua which intersects C_0, without loss of generality let this be C_1. Then either $C_0 \cup C_1 = X$, in which case we are done, or $C_0 \cup C_1$ is a proper subcontinuum of X. Then the continua $C_0 \cup C_1$ and C_j for $1 < j < N$ are a decomposition of X into $N - 1$ proper subcontinua. We may continue this process until we get a decomposition into two proper subcontinua. ■

In this sense, a continuum can be said to be indecomposable if it cannot be decomposed into any finite number of proper subcontinua.

2.3.8. Definition (Terminal Points)**.**
A point p is said to be a *terminal point* on a continuum X if for every two subcontinua $A, B \subset X$ with $p \in A$ and $p \in B$, we have that either $A \subset B$ or

$B \subset A$.

We recall again the definition of *arclike continua* from the introduction in Definition 1.0.3. It is known that terminal points on an arclike continuuum can be determined by the following theorem from [12][Theorem 12.8].

2.3.9. Theorem.
Let X be a continuum and let $p \in X$. X is arclike and p is a terminal point on X if and only if for every $\epsilon > 0$ there exists an ϵ-map $g_\epsilon \colon X \to [0, 1]$ such that $g_\epsilon(p) = 1$.

Chapter 3

The Circle of Addresses

In this chapter, we define $\tilde{\mathbb{C}}$, the compactification of $\mathbb{C}$ by adding the circle of addresses. This compactification has been used before in the study of exponential maps, for example in [17]. This construction uses the family of rays $\{\gamma_{\underline{s}}\}_{\underline{s}\in\mathcal{S}_e}$. We give here a full description of the construction of $\tilde{\mathbb{C}}$ such that it can be defined on a more general class of families of rays which tend to infinity. We define our compactification $\tilde{\mathbb{C}}$ using a family of rays $\mathcal{F}$ in such a way that each ray $\gamma \in \mathcal{F}$ tends to a point on $\tilde{\mathbb{C}} \setminus \mathbb{C}$ and no two rays tend to the same point. We find then in Theorem 3.1.14 that for all such families of rays, the associated compactification $\tilde{\mathbb{C}}$ is actually homeomorphic to the closed disc $\overline{\mathbb{D}}$. Under such a homeomorphism $h\colon \tilde{\mathbb{C}} \to \overline{\mathbb{D}}$, we find that the rays in $h\mathcal{F}$ tend to a set of points dense in $\partial\mathbb{D}$.

3.1 Compactification of General Indexed Ray Families

Let $\{\gamma_j\}_{j\in\mathcal{K}}$ be a family of disjoint rays $\gamma_j\colon (0,\infty) \to \mathbb{C}$ such that $|\gamma_j(x)| \to \infty$ as $x \to \infty$ for all $j \in \mathcal{K}$. For $j \in \mathcal{K}$, let $R \in \mathbb{R}$ be large enough that the ball B_R intersects the ray γ_j. Then let $\delta_{j,R} \in (0,\infty)$ be the largest value such that

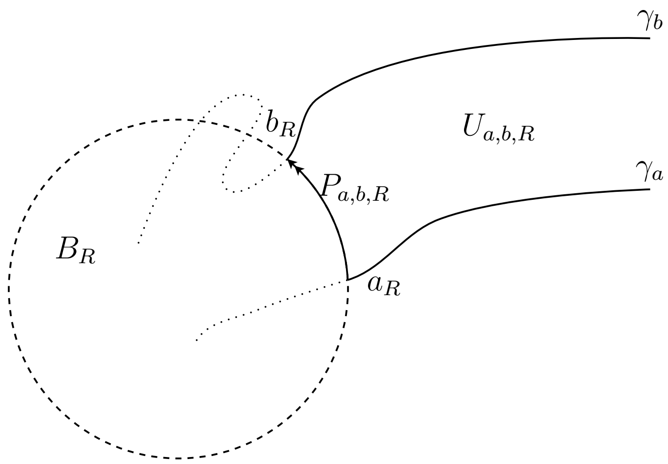

Figure 3.1: A picture of $U_{a,b,R}$.

$|\gamma_j(\delta_{j,R})| = R$ and let $j_R := \gamma_j(\delta_{j,R})$.

Let $a, b \in \mathcal{K}, a \neq b$ and let R be large enough that a_R, b_R exist. Then the set

$$\mathbb{C} \setminus (\overline{B_R} \cup \gamma_a((\delta_{a,R}, \infty)) \cup \gamma_b((\delta_{b,R}, \infty)))$$

consists of two connected components. Let $P_{a,b,R}$ be the path in ∂B_R from a_R to b_R which is positively oriented. This $P_{a,b,R}$ is contained in the boundary of precisely one of the components of $\mathbb{C} \setminus (B_R \cup \gamma_a \cup \gamma_b)$. We shall call this component $U_{a,b,R}$. In the figure above we give a picture of the construction of $U_{a,b,R}$.

We may now induce the following ternary relation.

3.1.1. Definition (Cyclic Order of Rays)**.**
For $a, b, c \in \mathcal{K}$, we say b *lies between* a and c when for some $R \in \mathbb{R}$ we have $\gamma_b((\delta_{b,R}, \infty)) \subset U_{a,c,R}$. When b lies between a and c we write this as $[a, b, c]$.

A ternary relation written in the form $[a, b, c]$ is a *cyclic order* if the following axioms are satisfied:

- Cyclicity: if $[a, b, c]$, then $[b, c, a]$;
- Asymmetry: if $[a, b, c]$, then not $[c, b, a]$;
- Transitivity: if $[a, b, c]$ and $[a, c, d]$, then $[b, c, d]$;
- Totality: if a, b, c are distinct, then either $[a, b, c]$ or $[c, b, a]$.

Cyclicity and totality are straightforward to show. Asymmetry and transitivity can be proved as a consequence of the following claim on the independence of the ordering with respect to R. If the conditions for $[a, b, c]$ are satisfied some $R \in \mathbb{R}$ such that γ_a, γ_b and γ_c all intersect B_R, then they are satisfied for all $R \in \mathbb{R}$ such that γ_a, γ_b and γ_c all intersect B_R.

In order to show the independence of our ordering on the choice of R, we aim to show the following: for $R \neq R'$, when $U_{a,c,R}$ contains some forward tail of some ray γ_b, then it follows that if $U_{a,c,R'}$ exists, it also contains some forward tail of γ_b. In this sense, we are aiming to show that the two sets are similar near infinity. More precisely we are interested in showing the following property:

3.1.2. Definition (Agreeing at Infinity)**.**
Let $A, B \subset \mathbb{C}$ be two sets. We say they *agree at infinity* if there is some neighbourhood U of infinity (of the form $U = \{z \in \mathbb{C} \colon |z| > R\}$ for example) for which $U \cap A = U \cap B$.

We then wish to prove the following:

3.1.3. Lemma.
Let $a, b \in \mathcal{K}$ and let $R, R' \in \mathbb{R}$ be sufficiently large that $U_{a,b,R}$ and $U_{a,b,R'}$ exist. Then $U_{a,b,R}$ and $U_{a,b,R'}$ agree at infinity.

This is a direct corollary of Lemma 3.1.5 and will be proved later. We state this lemma here in order to show how the asymmetry and transitivity axioms are satisfied.

Proof of Asymmetry and Transitivity. Suppose $a, b, c \in \mathcal{K}$ and $R, R' \in \mathbb{R}$ are such that $U_{a,c,R}$, $U_{a,c,R'}$, $\delta_{b,R}$ and $\delta_{b,R'}$ exist. We also note that $\gamma_b((\delta_{b,R}, \infty))$ and $\gamma_b((\delta_{b,R'}, \infty))$ agree at infinity. Similarly, by Lemma 3.1.3, $U_{a,c,R}$ and $U_{a,c,R'}$ agree at infinity. It follows that $\gamma_b((\delta_{b,R}, \infty)) \cup U_{a,c,R}$ and $\gamma_b((\delta_{b,R'}, \infty)) \cup U_{a,c,R'}$ agree at infinity. We note that $\gamma_b((\delta_{b,R}, \infty))$ is either disjoint from or entirely contained in $U_{a,c,R}$. Because of this, $\gamma_b((\delta_{b,R}, \infty)) \subset U_{a,c,R}$ if and only if $\gamma_b((\delta_{b,R'}, \infty)) \subset U_{a,c,R'}$. The ordering is therefore independent of R. If $\gamma_b((\delta_{b,R}, \infty)) \subset U_{a,c,R}$ for some R, then $\gamma_b((\delta_{b,R'}, \infty))$ is disjoint from $U_{c,a,R'}$ for all appropriate R', and so the asymmetry axiom holds.

Suppose that $[a, b, c]$ and $[a, c, d]$ both hold and let R be large enough that γ_a, γ_b and γ_c all intersect B_R. Then by the same independence from R, then $\gamma_c((\delta_{c,R}, \infty)) \subset U_{a,d,R}$ and $\gamma_b((\delta_{b,R}, \infty)) \subset U_{a,c,R}$. Equivalently, we have that $P_{a,d,R}$ contains c_R and $P_{a,c,R}$ contains b_R. We note that $P_{a,c,R}$ is a subpath of $P_{a,d,R}$. Travelling along $P_{a,d,R}$ we pass though, in order: a_R, b_R, c_R, d_R. It is clear then that $P_{b,d,R}$ contains c_R and, equivalently, $\gamma_c((\delta_{c,R}, \infty)) \subset U_{b,d,R}$ so that $[b, c, d]$ holds. Therefore the transitivity axiom also holds so that this ordering is cyclic. ∎

Before we prove Lemma 3.1.3 we first generalise the sets $U_{a,b,R}$ in the following way:

3.1.4. Definition.
Let $a, b \in \mathcal{K}$ and let J be a positively oriented Jordan curve with $0 \in |J|$ and such that J intersects both γ_a and γ_b. Let $\delta_{a,J}$ be the largest value that $\gamma_a(\delta_{a,J}) \in J$ and let $a_J := \gamma_a(\delta_{a,J})$. We may define $\delta_{b,J}$ and b_J similarly.

Let $P_{a,b,J}$ be the path in J from a_J to b_J which is oriented in the same direction as J. Then we may define $U_{a,b,J}$ to be the connected component of $\mathbb{C} \setminus (|J| \cup \gamma_a([\delta_{a,J}, \infty)) \cup \gamma_a([\delta_{b,J}, \infty)))$ which contains $P_{a,b,J}$ on its boundary.

When J is a positively oriented circle of radius R centered at 0, then $U_{a,b,J} = U_{a,b,R}$. The following lemma is therefore a generalization of Lemma 3.1.3.

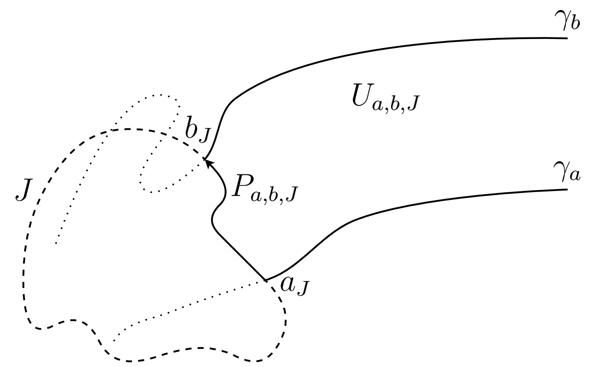

Figure 3.2: A picture of $U_{a,b,J}$.

3.1.5. Lemma.
Let $a, b \in \mathcal{K}$ and let J and J' be Jordan curves such that $U_{a,b,J}$ exists and $U_{a,b,J'}$ exist. Then $U_{a,b,J}$ and $U_{a,b,J'}$ agree at infinity.

Proof. For $a, b \in \mathcal{K}$ and a Jordan curve J such that $U_{a,b,J}$ exists, we will define three paths from a_J to b_J. Let q_1 be the path $P_{a,b,J}$, let q_2 be the path $-P_{b,a,J}$ and let q_3 be the path defined as

$$q_3\colon x \mapsto \begin{cases} \gamma_a(\delta_{a,J} - 1 + \frac{1}{1-2x}) & x \in [0, \frac{1}{2}) \\ \infty & x = \frac{1}{2} \\ \gamma_b(\delta_{b,J} - 1 + \frac{1}{2x-1}) & x \in (\frac{1}{2}, 1] \end{cases}$$

which follows a component of $(\gamma_a((0,\infty)) \cup \gamma_b((0,\infty)) \cup \{\infty\}) \setminus |J|$ from a_J to b_J.

We may transform these paths by the Möbius transformation $M\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, which we define as $M(z) := \frac{1}{z}$ for $z \in \mathbb{C}\backslash\{0\}$, with $M(0) := \infty$ and $M(\infty) := 0$. Let $p_1 := M \circ q_1$, let $p_2 := M \circ q_2$, and let $p_3 := M \circ q_3$.

Given that $M \circ (p_1 - p_3)$ is the boundary of $U_{a,b,J}$, and given that $M(|p_1 - p_3|)$ must be the unique connected component of $\mathbb{C} \setminus (q_1 - q_3)$ which does not contain 0, it follows that $M(|p_1 - p_3|) = U_{a,b,R}$. We may note that $p_1 - p_2$ is positively oriented. By Theorem 2.3.4, we know that $p_1 - p_3$ is always positively oriented also. Taking some other J' such that $U_{a,b,J'}$ exists, let R be large enough that

$$R > |\gamma_a([\delta_{a,J}, \delta_{a,J'}]) \cup \gamma_b([\delta_{b,J}, \delta_{b,J'}]) \cup P_{a,b,J} \cup P_{a,b,J'}|.$$

There then exists a path $\chi \colon [0,1] \to \mathbb{C}$ which is a cross cut of $U_{a,b,J}$. By a cross cut, we mean that $\chi((0,1)) \subset U_{a,b,J}$ and $\chi(\{0,1\}) \subset \partial U_{a,b,J}$. We choose this cross cut such that $|\chi(x)| \geq R$ for all $x \in [0,1]$ and for such that $\chi(0) \in \gamma_a((\delta_{a,J'}, \infty) \cap \gamma_a((\delta_{a,J}, \infty))$ and $\chi(1) \in \gamma_b((\delta_{b,J'}, \infty) \cap \gamma_b((\delta_{b,J}, \infty))$.

As in Theorem 2.3.5, we may divide the boundary of $M(U_{a,b,J})$ into two paths from $M(\chi(0))$ to $M(\chi(1))$. One path contains 0, we shall label this path r_1. The other path contains a positively oriented part of $\partial B_{\frac{1}{R}}$, we label this path r_2. Let $r_3 = M \circ \chi$. We may use Theorem 2.3.4 in a similar way as before to show that $r_2 - r_3$ is positively oriented. We find that $r_2 - r_3$ has the same orientation as some equivalent Jordan curve which can be written as $p_1 - P$ where P is a cross cut of the circle which follows the same path as $r_2 - r_3$. As before, we find the orientation of $p_1 - P$ is the same as the orientation of $p_1 - p_2$, that is, positive. By Theorem 2.3.5, $r_1 - r_3$ is then negatively oriented.

We then claim that χ is also a cross cut of $U_{a,b,J'}$. By construction, χ does not intersect the boundary of $U_{a,b,J'}$ except at the endpoints. It remains to show that χ is contained within $U_{a,b,R'}$. We may define r_1', r_2', r_3' in a similar way as with $U_{a,b,J}$, noting that we can define $r_1' = r_1$ and $r_3' = r_3$. Thus $r_1' - r_3'$ is negatively oriented. By the same argument as before, $r_2' - r_3'$ is positively oriented. Theorem 2.3.5 shows that $r_3'((0,1)) \subset |r_1' - r_2'|$ and therefore χ is indeed a cross cut of $U_{a,b,J'}$.

We have that $M(|r_2 - r_3|) = M(|r_2' - r_3'|)$ is a neighbourhood of ∞ for both $U_{a,b,J}$ and $U_{a,b,J'}$. It follows that $U_{a,b,J}$ and $U_{a,b,J'}$ agree at infinity. ■

We note that Lemma 3.1.3 follows immediately, completing our proof of the asymmetry and transitivity of the cyclic order $[a, b, c]$.

In order to use $\mathcal{K}$ to construct a compactification of $\mathbb{C}$ with desirable topological properties, we will require that $\mathcal{K}$ is *dense* in the following sense. We say a cyclically ordered set $\mathcal{K}$ is *dense* if for all $a, c \in \mathcal{K}$ there exists some $b \in \mathcal{K}$ such that $[a, b, c]$.

The set we wish to add to $\mathbb{C}$ to form our compactification is the *completion of $\mathcal{K}$ by cuts* (or some set homeomorphic to this completion). The construction of such a completion of a cyclically ordered set is described in [14].

3.1.6. Definition (Cuts, Dedekind Cuts and Complete Sets)**.**
A *cut* of a cyclically ordered set $\mathcal{K}$ is a linear order $<$ on $\mathcal{K}$ such that for all $a, b, c \in \mathcal{K}$ we have $a < b < c \Rightarrow [a, b, c]$.

A cut is said to be *Dedekind* if either there is a maximum element but no minimum element or there is a minimum element but no maximum element.

If all cuts of $\mathcal{K}$ are Dedekind cuts then we say $\mathcal{K}$ is *complete*.

It is described in [14] how the natural *completion* of any dense cyclically ordered set $\mathcal{K}$ is constructed. We define this completion here as follows:

Let $\mathcal{K}_r$ be the set of cuts of $\mathcal{K}$ with no minimum element. We may write these cuts as linear orders $<_k$ indexed by some $k \in X$. We may also suppose that $\mathcal{K}$ is contained in this index set X, and when $k \in \mathcal{K}$, then $<_k$ refers to the cut in which k is the maximum element in $\mathcal{K}$ under the relation $<_k$. There is a natural cyclic order on $\mathcal{K}_r$ where for $<_a, <_b, <_c \in \mathcal{K}_r$, we say $[<_a, <_b, <_c]$ if and only if there exist $x, y, z \in \mathcal{K}$ such that

$$x <_a y <_a z, \qquad y <_b z <_b x, \qquad z <_c x <_c y.$$

We call $\mathcal{K}_r$ the *completion of $\mathcal{K}$ by cuts.*

If S is a subset of a cyclically ordered set $\mathcal{K}$, then we say S is *densely contained* in $\mathcal{K}$ if for all distinct pairs $a, c \in \mathcal{K}$, there exists some $b \in S$ such that $[a, b, c]$. This definition is equivalent to S being dense in $\mathcal{K}$ with respect to the order topology. We use this terminology here to avoid confusion with our definition of dense cyclically ordered sets.

We may define the order topology on a cyclically ordered set $\mathcal{K}$ as follows. For $a, b \in \mathcal{K}$, we may define the interval (a, b) to be the set of points $x \in \mathcal{K}$ such that $[a, x, b]$. Taking the set of such intervals as a sub-base defines a topology on $\mathcal{K}$. By this definition, (a, b) will be an open set. We call (a, b) an *open interval.* It is worth noting that since $\mathcal{K}$ is densely contained in $\mathcal{K}_r$, the order topology on $\mathcal{K}_r$ is induced by the sub-base of open intervals in $\mathcal{K}_r$ whose endpoints are in $\mathcal{K}$.

3.1.7. Proposition.
Let $\mathcal{K}$ be a dense cyclically ordered index set of some family of non-intersecting rays in $\mathbb{C}$, each ray tending to infinity. Let $\mathcal{K}$ also be non-trivial in the sense that it contains more than one point. Then $\mathcal{K}_r$ is homeomorphic to a circle.

Sketch of Proof. We prove this proposition in steps. First, showing that there exists in each such $\mathcal{K}$ a countable subset S densely contained in $\mathcal{K}$. Then, we show that all such subsets S of all such $\mathcal{K}$ are *isomorphic* to one another. We show that the completion of each S is *isomorphic* to the completion of each $\mathcal{K}$. Finally, we show that there exists at least one such S whose completion is the circle. ■

3.1.8. Lemma.
Let $\mathcal{K}$ be as described in Proposition 3.1.7. If $\mathcal{K}$ is dense then there exists a countable subset S densely contained in $\mathcal{K}$.

Proof. We define a sequence $(S_n)_{n\in\mathbb{N}}$ of finite subsets of $\mathcal{K}$ in the following way:

For $n \in \mathbb{N}$ we take the n circles with radius $R \in \{1, 2, ..., n\}$ and centred at 0. We divide each of them into n sections $S_n^{m,q}$ of arc length $\frac{2\pi}{n}$ of the form $S_n^{m,q} := \{qe^{i\vartheta} : \vartheta \in [2\pi\frac{m}{n}, 2\pi\frac{m+1}{n})\}$ for $q, m \in \mathbb{N}$ with $q, m \leq n$. For each of these sections $S_n^{m,q}$, if there exists some $x \in \mathcal{K}$ such that $x_R \in S_n^{m,q}$, we chose precisely one such of those such x to be in S_n. We then have that $|S_n| \leq (n+1)^2$.

Then S is defined to be the union of each of these S_n. Since S_n is finite, S is countable. To see it is dense, consider $a, b, c \in \mathcal{K}$ such that $[a, b, c]$. Then for some $R \in \mathbb{N}$ with $R > 0$, we have that a_R, b_R, c_R exist. Furthermore, there exists some $n, m, q \in \mathbb{N}$ such that there is some section $S_n^{m,q}$ which contains b_R and is contained within the positively oriented arc from a_R to c_R. There is therefore some $x \in S$ such that x_R is also contained within this section. It follows that $[a, x, c]$. Given that $\mathcal{K}$ is dense, it follows that such an x exists for all distinct points $a, c \in \mathcal{K}$. We therefore have that S is densely contained in $\mathcal{K}$. ∎

3.1.9. Theorem.
All countable dense cyclically ordered sets are isomorphic *in the following sense: for any two such sets X, Y there exists a bijective map $f: X \to Y$ such that for all $a, b, c \in X$ we have $[a, b, c] \Leftrightarrow [f(a), f(b), f(c)]$.*

Proof. Let X, Y be two countable dense cyclically ordered sets. Let $\underline{x}, \underline{y}$ be enumerations of those sets. We may define a homeomorphism f as follows: Let $f(x_0) = y_0$ and $f(x_1) = y_1$. For $n > 1$ let $j_0^n, j_1^n, ..., j_{n-1}^n$ be a re-ordering of $0, 1, ..., n-1$. We say $x_{j_0^n}, x_{j_1^n}, ..., x_{j_{n-1}^n}$ are *consecutive* if each of the intervals $I_0^x := (x_{j_0^n}, x_{j_1^n}), I_1^x := (x_{j_1^n}, x_{j_2^n}), ..., I_n^x := (x_{j_{n-1}^n}, x_{j_0^n})$ do not intersect $\{x_j\}_{j<n}$. Suppose that all consecutive sequences $x_{j_0^n}, x_{j_1^n}, ..., x_{j_{n-1}^n}$ of length n are mapped by f to consecutive sequences $f(x_{j_0^n}), f(x_{j_1^n}), ..., f(x_{j_{n-1}^n})$. Such is the case for $n = 2$. We may define the intervals $I_0^y, ..., I_n^y$ similarly as $I_0^y := (f(x_{j_0^n}), f(x_{j_1^n}))$ etc. Let $k \leq n$ be such that $x_n \in I_k^x$. We define m to be the first value such that $y_m \in I_k^y$. Since Y is dense, this is guaranteed to exist. Then we define $f(x_n) := y_m$. We note that by this definition, consecutive se-

quences are still mapped to consecutive sequences for $n+1$. This, therefore, also holds for all n. This is equivalent to saying that f is order preserving over $\{x_n\}_{n \leq n}$ for all n, which in turn is equivalent to saying that f is order preserving over X.

It remains to show that f is surjective. Let $n > 1$ and let m be the first such value that $y_m \neq f(x_j)$ for any $j < n$. Let k and I_k^y be defined in such a way that $y_m \in I_k^y$. Then by the density of X, there will exist some least $n' \geq n$ such that $x_{n'} \in I_k^x$. It will follow that $f(x_{n'}) = y_m$. So for all y_m there will eventually be some $x_{n'}$ which maps to it and f is therefore an order preserving bijection. Thus X and Y are isomorphic. ■

For a dense cyclically ordered set $\mathcal{K}$, we can show that the completion by cuts yields a dense cyclically ordered set $\mathcal{K}_r$ in which $\mathcal{K}$ is densely contained. More generally, we call $\overline{\mathcal{K}}$ a *completion* of $\mathcal{K}$ when $\overline{\mathcal{K}}$ is a dense cyclically ordered set in which $\mathcal{K}$ is densely contained. We find that for such $\mathcal{K}$, all completions, including the completion by cuts, are unique up to isomorphism.

3.1.10. Theorem.
Let $\overline{\mathcal{K}}$ be a dense complete cyclically ordered set. Let $\mathcal{K}$ be densely contained in $\overline{\mathcal{K}}$. Then $\mathcal{K}_r$, the completion by cuts of $\mathcal{K}$, is order isomorphic to $\overline{\mathcal{K}}$.

Proof. For every element $x \in \overline{\mathcal{K}}$, there exists a cut of $\overline{\mathcal{K}}$ for which x is a maximum element (taking $a < b$ precisely when $[a, b, x]$). This induces a cut $<_x$ of $\mathcal{K}$. This cut will be an element of $\mathcal{K}_r$ since it has no minimum.

We wish to show then that the map $f\colon \overline{\mathcal{K}} \to \mathcal{K}_r$, $x \mapsto <_x$ is an order preserving isomorphism.

To show that f is injective, take $a, b \in \overline{\mathcal{K}}$ and suppose $a \neq b$. By the density of $\overline{\mathcal{K}}$, there exists $y, z \in \mathcal{K}$ such that $[a, y, b]$ and $[b, z, a]$. Then $y <_a z$ and $z <_b y$, so these cuts are not equal.

To prove f is surjective, we show that for every cut in $\mathcal{K}_r$ there is some $x \in \overline{\mathcal{K}}$ inducing it. Let $<_k \in \mathcal{K}_r$ be a cut on $\mathcal{K}$. Then we may induce the following ordering on $\overline{\mathcal{K}}$: For $a, b \in \overline{\mathcal{K}}$ with $a \neq b$, if $(a, b) \cap \mathcal{K}$ has a lower bound in $\mathcal{K}$ with respect to the ordering $<_k$, then we write $a <_{\overline{k}} b$. We will then show that this ordering is a cut of $\overline{\mathcal{K}}$ with a maximum element but no minimum element.

To see that this is a proper linear ordering, we must prove trichotomy (for all $a, b \in \overline{\mathcal{K}}$ precisely one of the following holds, $a <_{\overline{k}} b, b_{\overline{k}} < a$ or $a = b$) and transitivity (for all $a, b, c \in \overline{\mathcal{K}}$, then if $a <_{\overline{k}} b$ and $b <_{\overline{k}} c$, then $a <_{\overline{k}} c$). To prove trichotomy, we show that when $a \neq b$, then of the two intervals (a, b) and (b, a), precisely one of them has a lower bound in $\mathcal{K}$.

If one of these intervals, say (a, b), has a lower bound c in $\mathcal{K}$, then all values $c' \in \mathcal{K}$ with $c' <_k c$ are contained in the interval (b, a). It follows that (b, a) has no lower bound.

Suppose neither (a, b) nor (b, a) have a lower bound. Then for all $c_1 \in (a, b) \cap \mathcal{K}$ there exists some $c_2 \in (b, a) \cap \mathcal{K}$ such that $c_2 <_k c_1$. Similarly, there would exist some $c_3 \in (a, b) \cap \mathcal{K}$ and $c_4 \in (b, a) \cap \mathcal{K}$ such that $c_4 <_k c_3 <_k c_2 <_k c_1$. This implies that $[c_3, c_2, c_1]$ and $[c_1, c_4, c_3]$. Since it is the case that either $(c_1, c_3) \subset (a, b)$ or $(c_3, c_1) \subset (a, b)$, this further implies that either $c_2 \in (a, b)$ or $c_4 \in (a, b)$. However, this is also a point in (b, a), which is a contradiction of the cyclic ordering of $\overline{\mathcal{K}}$.

To prove transitivity, we note that if $a <_{\overline{k}} b$ and $b <_{\overline{k}} c$, then (a, b) and (b, c) both have a lower bound in $\mathcal{K}$. In fact they share a lower bound (by taking the minimum lower bound of the two of them) and this lower bound must also be a lower bound of (a, c) so that $a <_{\overline{k}} c$.

Given $a <_{\overline{k}} b <_{\overline{k}} c$, it naturally follows that $[a, b, c]$, so that this ordering is in fact a cut of $\overline{\mathcal{K}}$.

Furthermore, we can show there is no minimum element of $\overline{\mathcal{K}}$ under the order $<_{\overline{k}}$. For any $a \in \overline{\mathcal{K}}$, we wish to show it is not a minimum. Taking some $b \in \overline{\mathcal{K}}$ with $a \neq b$, if $b <_{\overline{k}} a$, then we are done, otherwise $a <_{\overline{k}} b$ and $(a, b) \cap \mathcal{K}$ has a lower bound in $\mathcal{K}$. We may take some $c \in \mathcal{K}$ less than this lower bound, then we have that $c <_{\overline{k}} a$ and a is not a minimum.

Since $\overline{\mathcal{K}}$ is complete, there must be some maximum element $\overline{k} \in \overline{\mathcal{K}}$ which induces the order $<_{\overline{k}}$ and therefore the cut $<_k \in \mathcal{K}_r$.

It is possible to show that f is order preserving as follows. Let $x, y, z \in \overline{\mathcal{K}}$ such that $[x, y, z]$ and let $a, b, c \in \mathcal{K}$ such that $[x, a, y], [y, b, z]$ and $[z, c, x]$. Then we have that

$$a <_x b <_x c, \qquad b <_y c <_y a, \qquad c <_z a <_z b$$

so that $[<_x, <_y, <_z]$. ■

3.1.11. Theorem.
Let $\overline{\mathcal{K}}$ be a cyclically ordered complete dense set with a subset $\mathcal{K}$ be which is countable and densely contained in $\overline{\mathcal{K}}$. Then $\overline{\mathcal{K}}$ is homeomorphic to a circle and isomorphic to the cyclically ordered circle.

Proof. $\overline{\mathcal{K}}$ is isomorphic to the set $\mathcal{K}_r$ of cuts on $\mathcal{K}$ which do not have a maximum element.

Similarly, there is a natural cyclic order on $\mathbb{Q}$ for which the normal linear ordering on $\mathbb{Q}$ is a cut. That is, for $a, b, c \in \mathbb{Q}$ we have $[a, b, c]$ if and only if $a < b < c, b < c < a$ or $c < a < b$. The completion by cuts of this cyclically ordered set is precisely the cyclically ordered circle S.

There exists an order preserving isomorphism f from $\mathcal{K}$ to the cyclically ordered $\mathbb{Q}$. This isomorphism induces an isomorphism of cuts in such a way that $\mathcal{K}_r$ is isomorphic to S. Given $<_k \in \mathcal{K}_r$, a cut on $\mathcal{K}$, then we write as $\overline{f}$ the map from $<_k$ to the cut $<_{f(k)}$ on $\mathbb{Q}$. We define $<_{f(k)}$ to be the unique cut on $\mathbb{Q}$ such that for $a, b \in \mathcal{K}$ we have $a <_k b \Leftrightarrow f(a) <_{f(k)} f(b)$. Therefore $\overline{\mathcal{K}}$ is isomorphic and consequently homeomorphic to S. ■

Proposition 3.1.7 thus follows as described in the sketch of the proof.

It is now possible to define $\tilde{\mathbb{C}}$, the completion of $\mathbb{C}$ with respect to $\mathcal{K}$.

3.1.12. Definition.
We define $\tilde{\mathbb{C}}$ to be the set $\mathbb{C} \cup \mathcal{K}_r$ equipped with the topology defined from

the sub-base consisting of open sets in $\mathbb{C}$ and sets of the form $U_{a,b,R} \cup (a,b)$ for $a, b \in \mathcal{K}$ (where (a,b) is the interval in $\mathcal{K}_r$).

3.1.13. Remark.
We may note that for a set U to be a neighbourhood of $x \in \mathcal{K}_r$, it is sufficient that there is some other neighbourhood U' of x such that $U \cap \mathbb{C}$ and $U' \cap \mathbb{C}$ agree at infinity and $U \cap \mathcal{K}_r = U' \cap \mathcal{K}_r$.

Proof. Suppose $U \cap \mathbb{C}$ and $U' \cap \mathbb{C}$ agree past some R. By this, we mean that $(U \cap \mathbb{C}) \setminus B_R = (U' \cap \mathbb{C}) \setminus B_R$. Then there exists some neighbourhood of x of the form $U_{a,b,R}$. We then find that $U' \cap U_{a,b,R}$ is a neighbourhood of x and it follows that $U \cap U_{a,b,R} = U' \cap U_{a,b,R} \subset U$ so that U' is indeed a neighbourhood of x. ■

3.1.14. Theorem.
$\tilde{\mathbb{C}}$ is homeomorphic to a closed disk.

Proof. Let K be a countable dense subset of $\mathcal{K}$. We may enumerate K as $(k_n)_{n\in\mathbb{N}}$. It is possible to construct a sequence $(J_n)_{n\in\mathbb{N}}$ of mutually disjoint Jordan curves such that J_n intersects $\gamma_{k_j}((1,\infty))$ at precisely one point for each $j \leq n$ and such that $\min |J_n| \to \infty$ as $n \to \infty$. We now define a structure on $\mathbb{C}$. Let δ_{k_n,J_n} be as in Definition 3.1.4 and let

$$G \coloneqq \bigcup_{n\in\mathbb{N}} (J_n \cup \gamma((\delta_{k_n,J_n}, \infty)) \cup \{k_n\}) \setminus |J_2|.$$

Let $\mathbb{D}$ be the closed unit disc and let f_0 be a homeomorphism from $\mathcal{K}_r$ to $\partial\mathbb{D}$. Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of circles in $\mathbb{D}$ centred at the origin and tending to $\partial\mathbb{D}$. For a straight line connecting $f_0(k_n)$ to the origin of $\mathbb{D}$, let p_n be the subpath which has endpoints at S_n and $f_0(k_n)$. We may now define an equivalent structure on $\mathbb{D}$. Let

$$G' \coloneqq \bigcup_{n\in\mathbb{N}} (S_n \cup p_n \cup f_0(k_n)) \setminus |S_2|.$$

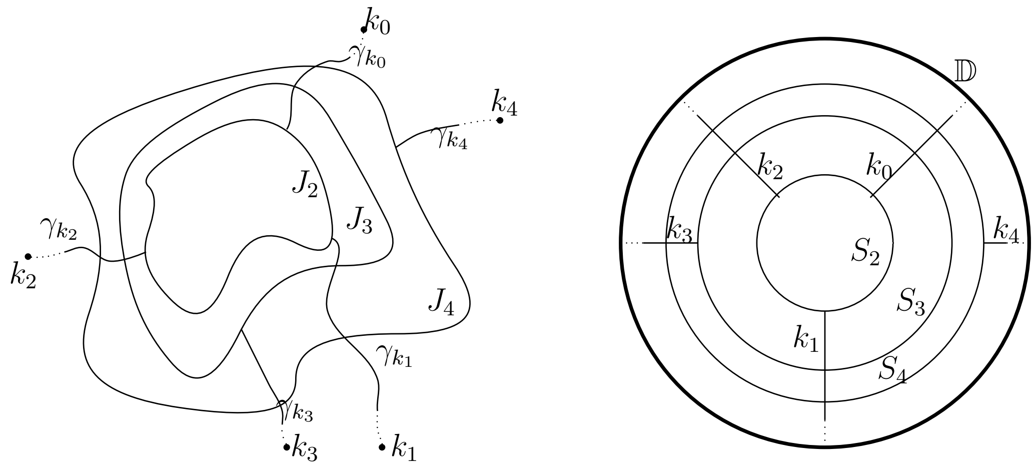

Figure 3.3: The structure of G and G'.

Then there is a homeomorphism f_1 from G to G' which maps J_n onto S_n, maps $\gamma((\delta_{k_n,J_n},\infty))$ onto p_n and is equal to f_0 on S. This maps the intersection points $\gamma((\delta_{k_n,J_n},\infty)) \cap J_q$ onto $p_n \cap S_q$. Since the linear order of these intersections is the same on $\gamma((\delta_{k_n,J_n},\infty))$ as it is on p_n, and since the cyclic order of them is the same on J_q as it is on S_q, then appropriate homeomorphisms may be found on $\gamma((\delta_{k_n,J_n},\infty))$ and J_q which construct the whole of f_1. We note that this homeomorphism preserves orientation.

It is possible to extend this further to a homeomorphism from $\tilde{\mathbb{C}}$ to $\overline{\mathbb{D}}$ in the following way. Every connected component $C \subset \mathbb{C} \setminus G$ has as its boundary either J_2 or some sections of γ_n, γ_m, J_q and J_{q+1} for some $n, m, q \in \mathbb{N}$ which uniquely determine the connected component. In a similar way, there is a corresponding connected component $C' \subset \mathbb{D} \setminus G'$ bound by either S_2 or p_n, p_m, S_q and S_{q+1}. By construction, f' is a homeomorphism from ∂C to $\partial C'$. As a consequence of Corollary 2.3.3, the closure $\overline{C}$ is homeomorphic to the corresponding closure $\overline{C}'$ in such a way that this homeomorphism agrees with f_1 on the boundary. We may define such a map f_C for each component. We can define f_2 to map every point $z \in C$ to $f_C(z)$, to map $z \in G$ to $f_1(z)$, and to map $z \in \mathcal{K}_r$ to $f(z)$. Since every point of $\mathbb{C}$ is either on G or one of these components C, and since by our construction every point on the interior of $\mathbb{D}$ either lies on G' or on some C', then this map is a bijection.

It remains to show that f_2 is continuous. It is already continuous by definition on $\mathbb{C}$ so it remains to show it is continuous on $\mathcal{K}_r$. Let $k \in \mathcal{K}_r$ and let V be some neighbourhood of $f_0(k)$. Since $f_2(K)$ is densely contained in the circle, there exist $n, m, q \in \mathbb{N}$ such that the connected component V' of $\overline{\mathbb{D}} \setminus (p_n \cup p_m \cup S_q)$ which contains $f_0(k)$ is a subset of V; without loss of generality suppose $[k_n, k, k_m]$. Then the preimage of V' is precisely $U_{k_n, k_m, J_q} \cup (k_n, k_m)$, a neighbourhood of k. This proves that f_2 is continuous and therefore a homeomorphism from $\tilde{\mathbb{C}}$ to the disk. ■

As a consequence, it follows naturally that $\tilde{\mathbb{C}}$ is a continuum.

3.2 Compactification for Dynamic Rays of the Exponential Map

In particular, we will take our index set to be $\mathcal{S}_e$, the set of exponentially bounded addresses of rays.

In the case of dynamic rays of $E_{2\pi i}$, the more natural order is perhaps the vertical linear order which, we will show, agrees with the lexicographical order on addresses. More generally, we can define a linear order for generic rays whose real part tends to infinity.

Let $H_R := \{z \in \mathbb{C} \colon \operatorname{Re}(z) > R\}$. Let γ_a be a ray in $\mathbb{C}$ for which $\operatorname{Re}(\gamma(x))$ tends to $+\infty$ as x increases. Let $R \in \mathbb{R}$ be sufficiently large that H_R intersects $\gamma_a((0, \infty))$ and let $\delta_a > 0$ be the maximum value such that $\operatorname{Re}(\gamma(\delta)) = R$. Let $\gamma_a^R := \gamma_a((\delta_a, \infty))$. Then $H_R \setminus \gamma_a^R$ contains precisely two unbounded components, $H_R^+(\gamma_a)$ and $H_R^-(\gamma_a)$, the upper and lower parts respectively, in the sense that $\partial H_R^+(\gamma_a) \cap \partial H_R$ has imaginary part unbounded above and $\partial H_R^-(\gamma_a) \cap \partial H_R$ has imaginary part unbounded below.

Given two non-intersecting rays, γ_a and γ_b, we say γ_a lies above γ_b if there

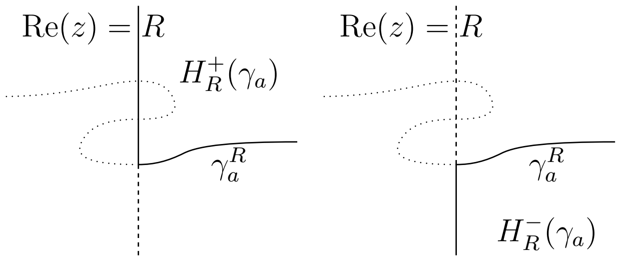

Figure 3.4: The upper and lower components $H_R^+(\gamma_a)$ and $H_R^-(\gamma_a)$.

is some $R > 0$ such that $H_R^+(\gamma_a) \subset H_R^+(\gamma_b)$. It is clear that this ordering is independent of the choice of R.

3.2.1. Theorem (Vertical Ordering of Rays)**.**
Let $\underline{s}^-$ and $\underline{s}^+$ be exponentially bounded addresses with $\underline{s}^- < \underline{s}^+$ according to the lexicographical order. Then $\gamma_{\underline{s}^+}$ lies above $\gamma_{\underline{s}^-}$.

Proof. When $s_0^- < s_0^+$, it is clear that $\gamma_{\underline{s}^+}$ lies above $\gamma_{\underline{s}^-}$. Recall that for the first index n for which s_n^- and s_n^+ differ we have $s_n^- < s_n^+$. It is, therefore, sufficient to show that when two rays share an asymptote, $E_{2\pi i}$ preserves their order. When this happens, the map $E_{2\pi i}^n$ then preserves the order of $\gamma_{\underline{s}^+}$ and $\gamma_{\underline{s}^-}$ and $E_{2\pi i}^n(\gamma_{\underline{s}^+})$ lies above $E_{2\pi i}^n(\gamma_{\underline{s}^-})$. It must then be the case that $\gamma_{\underline{s}^+}$ lies above $\gamma_{\underline{s}^-}$.

Suppose $E_{2\pi i}(\gamma_{\underline{s}^+})$ lies above $E_{2\pi i}(\gamma_{\underline{s}^-})$ and $s_0^+ = s_0^-$. Let R be large enough that $H_R^+(E_{2\pi i}(\gamma_{\underline{s}^-}))$ and $H_R^+(E_{2\pi i}(\gamma_{\underline{s}^+}))$ exist and let U^-, U^+ be the connected components of their respective preimages whose boundaries intersect $\gamma_{\underline{s}^-}$ and $\gamma_{\underline{s}^+}$ respectively. Since $H_R^+(E_{2\pi i}(\gamma_{\underline{s}^+})) \subset H_R^+(E_{2\pi i}(\gamma_{\underline{s}^-}))$, then it is either the case that U^- is disjoint from U^+, or that $U^+ \subset U^-$. In the case that U^- is disjoint from U^+, then U^+ must be contained in some $2\pi ik$ translate of U^- for some $k \in \mathbb{Z} \setminus \{0\}$. Each of these translates are again contained within separate components of $E_{2\pi i}^{-1}(H_R)$, each of which is contained within a strip of height π and is separated from every other component by a vertical distance of at least π. In this case it is impossible for two rays on the boundary of U^- and U^+ respectively to have the same asymptote. It must therefore be the case that $U^- \subset U^+$. In particular, we have that U^- contains some forward tail of $\gamma_{\underline{s}^+}$ which is part of the boundary of U^+.

For any $R > 0$ so that $H_R^+(\gamma_{\underline{s}^-})$ and $H_R^+(\gamma_{\underline{s}^+})$ exist, we can find some similarly defined $U^- \subset E_{2\pi i}^{-1}(H_{R'}^+(E_{2\pi i}(\gamma_{\underline{s}^-})))$. So long as R' is defined so that $\min \operatorname{Re}(E_{2\pi i}^{-1}(H_{R'})) > R$, then we find that U^- is in fact a component of $E_{2\pi i}^{-1}(H_{R'}) \cap H_R^+(\gamma_{\underline{s}^-})$. Now as before, we have that some forward tail of $\gamma_{\underline{s}^+}$ is contained in U^-, it is therefore also contained in $H_R^+(\gamma_{\underline{s}^-})$. This now implies that $H_R^+(\gamma_{\underline{s}^+}) \subset H_R^+(\gamma_{\underline{s}^-})$, so that $\gamma_{\underline{s}^+}$ lies above $\gamma_{\underline{s}^-}$ as required. ■

This linear order induces a cyclic order on $\mathcal{S}_e$. By this, we mean there exists a unique cyclic order on $\mathcal{S}_e$ such that for $a, b, c \in \mathcal{S}_e$, whenever $a < b < c$, then in the cyclic order we have $[a, b, c]$ (to determine if $[a, b, c]$, it is sufficient to check if one of $a < b < c$, $b < c < a$ or $c < b < a$ holds. This uniquely determines the cyclic ordering of every triple in $\mathcal{S}_e$).

When a family of rays $\{\gamma_k\}_{k\in\mathcal{K}}$ can be ordered vertically, they may also be ordered cyclically by following Definition 3.1.1. We show that the vertical ordering is a cut of the cyclical ordering by showing the construction of two orderings are related in the following way.

3.2.2. Lemma.
Let $\{\gamma_k\}_{k\in\mathcal{K}}$ be a family of non-intersecting rays tending to infinity to the right. Suppose $a, b \in \mathcal{K}$ and $a < b$ in the vertical order. Then for any $R \in \mathbb{R}$ and $R' > 0$ sufficiently large that both $H_R^+(\gamma_a), H_R^-(\gamma_b)$ and $U_{a,b,R'}$ exist, we have that $H_R^+(\gamma_a) \cap H_R^-(\gamma_b)$ and $U_{a,b,R'} \cap \mathbb{C}$ agree at infinity.

Proof. For any $R, R' \in \mathbb{R}$ large enough that $H_R^+(\gamma_a) \cap H_R^-(\gamma_b)$ and $H_{R'}^+(\gamma_a) \cap H_{R'}^-(\gamma_b)$ exist, we can show that they agree at infinity. Without loss of generality, let $R < R'$. We have that $H_R^+(\gamma_a)$ and $H_{R'}^+(\gamma_a)$ agree for values of

$z \in \mathbb{C}$ such that $|z| > \max |\gamma_a^R \setminus \gamma_a^{R'}|$. Therefore $H_R^+(\gamma_a)$ and $H_{R'}^+(\gamma_a)$ agree at infinity. In a similar way, we can show that $H_R^-(\gamma_a)$ and $H_{R'}^-(\gamma_a)$ also agree at infinity, as do $H_R^+(\gamma_a) \cap H_R^-(\gamma_b)$ and $H_{R'}^+(\gamma_a) \cap H_{R'}^-(\gamma_b)$. We may note that agreeing at infinity is transitive in the sense that, if for sets $A, B, C \subset \mathbb{C}$ we have that A and B agree at infinity and B and C agree at infinity, then we also have that A and C agree at infinity. Since $H_R^+(\gamma_a) \cap H_R^-(\gamma_b)$ agrees at infinity with all $H_{R'}^+(\gamma_a) \cap H_{R'}^-(\gamma_b)$, and similarly, $U_{a,b,R} \cap \mathbb{C}$ agrees at infinity with $U_{a,b,R'} \cap \mathbb{C}$ for all sufficiently large R and R', then it is sufficient to prove that $H_R^+(\gamma_a) \cap H_R^-(\gamma_b)$ agrees at infinity with $U_{a,b,R'} \cap \mathbb{C}$ for some R and R'. We therefore fix $R > 0$ to be sufficiently large that $H_R^+(\gamma_a), H_R^+(\gamma_b), H_R^+(\gamma_c)$ exist.

Let P be the path in $\partial H_R^+(\gamma_a) \cap H_R^-(\gamma_b)$ which connects δ_a to δ_b. There then exists some Jordan curve around 0 and contained in $\{z \in \mathbb{C} \colon \operatorname{Re}(z) \leq R\}$ for which P is a sub-arc, since P is vertically increasing it follows that J is positively oriented around 0. The points δ_a, δ_b are respectively the last points for which γ_a, γ_b intersect J.

It follows that $U_{a,c,J} = H_R^+(\gamma_a) \cap H_R^-(\gamma_c)$ and by Lemma 3.1.5 then for sufficiently large R' we have that $U_{a,b,J}$ agrees with $U_{a,c,R'}$ at infinity. ■

3.2.3. Theorem.
Let $\{\gamma_k\}_{k \in \mathcal{K}}$ be a family of non-intersecting rays tending to infinity to the right. The linear order induced on $\mathcal{K}$ is a cut of the cyclic order induced on $\mathcal{K}$.

Proof. It is sufficient to show that for $a, b, c \in \mathcal{K}$ when $a < b < c$ in the vertical order then $\gamma_a, \gamma_b, \gamma_c$ lie succesively above each other in the cyclic order.

Let $R > 0$ be sufficiently large that $H_R^+(\gamma_a), H_R^+(\gamma_c), U_{a,c,R}$ exist. When $a < b < c$ in the vertical order then some forward tail of γ_b is contained in $H_R^+(\gamma_a) \cap H_R^+(\gamma_c)$. Since $H_R^+(\gamma_a) \cap H_R^+(\gamma_c)$ agrees at infinity with $U_{a,c,R}$, then some forward tail of γ_b is contained in $U_{a,c,R}$ and $[a, b, c]$ holds. Therefore $a < b < c$ implies $[a, b, c]$ as required. ■

3.2.4. Theorem.
$\mathcal{S}_e$ is dense with respect to this cyclic order.

Proof. Let $\underline{s}^a, \underline{s}^c \in \mathcal{S}_e$. If $\underline{s}^c < \underline{s}^a$, then there exists some $\underline{s}^b := (s_0^a + 1, 0, 0, ...) \in \mathcal{S}_e$ such that $\underline{s}^c < \underline{s}^a < \underline{s}^b$. It follows that $[\underline{s}^b, \underline{s}^c, \underline{s}^a]$, and therefore $[\underline{s}^a, \underline{s}^b, \underline{s}^c]$ follows by cyclicity. If $\underline{s}^a < \underline{s}^c$, then let n be the first point such that $s_n^a \neq s_n^c$. Let $\underline{s}_b := (s_0^a, ..., s_n^a, s_{n+1}^a + 1, 0, 0, ...)$. Then $\underline{s}^a < \underline{s}^b < \underline{s}^c$, and therefore $[\underline{s}^a, \underline{s}^b, \underline{s}^c]$ as required. ■

We will here define $\overline{\mathcal{S}}$, the completion of $\mathcal{S}_e$. Each of the following sets is equipped with a cyclic order induced from the lexicographical linear order.

3.2.5. Definition (Completion of $\mathcal{S}_e$ and Intermediate Addresses)**.**
Let $\mathcal{S}_0$ be the set $\mathbb{Z}^{\mathbb{N}}$ of all integer sequences. $\mathcal{S}_0$ can be completed by adding finite length *intermediate addresses* with the following form:

$\underline{s} = (s_0, s_1, ..., s_n)$ for $n \in \mathbb{N}$, where $s_n = \infty$, $s_{n-1} \in \mathbb{Z} + \frac{1}{2}$, and $s_j \in \mathbb{Z}$ for $0 \leq j \leq n-2$. We define $\overline{\mathcal{S}}$ to be the union of $\mathcal{S}_0$ with the set of all such intermediate addresses, and we write

$$\mathcal{S} := \overline{\mathcal{S}} \setminus \{(\infty)\},$$

where (∞) is the intermediate address of length 1.

3.2.6. Lemma.
$\overline{\mathcal{S}}$ is complete with respect to the cyclic order.

Proof. We consider the cut $<$ on $\overline{\mathcal{S}}$ for which (∞) is the maximum. There is an equivalence between cuts on the cyclic order of $\overline{\mathcal{S}}$ and *linear cuts*. By *linear cuts*, we mean subsets $S \in \overline{\mathcal{S}}$ such that when $a \in S$ and $b \in \overline{\mathcal{S}} \setminus S$, then $a < b$. For every cyclic cut $<_k$, there is an associated linear cut S_k, which is the set of all points $a \in \overline{\mathcal{S}}$ such that for all $b \in \overline{\mathcal{S}}$ with $a < b$, we have $a <_k b$. For every linear cut S_k there is an associated cyclic cut $<_k$ which is defined as follows: When $a \in S$, then $a <_k b$ precisely when both $a < b$ and $b \in S$.

When $a \in \overline{\mathcal{S}} \setminus S$, then $a <_k b$ precisely when either $a < b$ or $b \in S$.

We find that $\overline{\mathcal{S}}$ is complete precisely when all linear cuts S have a supremum in $\overline{\mathcal{S}}$. For a cut S, let s_0^* be the supremum of s_0^k where $\underline{s}^k \in S$. For $n \in \mathbb{N}$, if s_n^* is not ∞, then let s_n^* be the supremum of s_n^k where $\underline{s}^k \in S$ and $s_j^k = s_j^*$ for all $j < n$. When $s_n^* = \infty$, then let $s_{n+1}^* := \infty$. When s_n^* is finite for all $n \in \mathbb{N}$, let $\underline{s} := \underline{s}^*$. When n is the first value for $\underline{s}^*$ such that $s_n^* = \infty$, we may define $\underline{s}$ to be the intermediate address of length $n+1$ such that $s_{n-1} := s_{n-1}^* + \frac{1}{2}$ (where $n-1 \geq 0$) and such that $s_j := s_j^*$ for $j \leq n, j \neq n-1$. In either case, we find that $\underline{s} \in \overline{\mathcal{S}}$ is the supremum of S. ■

To show that $\overline{\mathcal{S}}$ is a completion of $\mathcal{S}_e$, it is remains to prove $\mathcal{S}_e$ is densely contained in $\overline{\mathcal{S}}$.

3.2.7. Theorem.
$\mathcal{S}_e$ is densely contained in $\overline{\mathcal{S}}$.

Proof. Let $\underline{s}^a, \underline{s}^c \in \overline{\mathcal{S}}$ be two different addresses. Let n be the first point at which $s_n^a \neq s_n^c$. If $s_n^a < s_n^c$, there exists some integer s_n^b such that $s_n^a \leq s_n^b \leq s_n^c$. If $s_n^b = s_n^a$, let $s_{n+1}^b = \lfloor s_{n+1}^a \rfloor + 1$, if $s_n^b = s_n^c$, let $s_{n+1}^b = \lfloor s_{n+1}^c \rfloor - 1$, otherwise let $s_{n+1}^b = 0$. Then for $\underline{s}_b := (s_0^a, ... s_{n-1}^a, s_n^b, s_{n+1}^b, 0, 0, ...) \in \mathcal{S}_e$ we have $[\underline{s}_a, \underline{s}_b, \underline{s}_c]$.

A similar argument follows when $s_n^c < s_n^a$ by choosing s_n^b outside the closed interval $[s_n^c, s_n^a]$. ■

3.3 Statement of Arclikeness properties

Recall the definition of the itinerary set $C_{\underline{u}}$ from Definition 2.2.8. We may now extend this definition to $\tilde{\mathbb{C}}$.

3.3.1. Definition.
Let $\tilde{C}_{\underline{u}}$ be the closure of $C_{\underline{u}}$ in $\tilde{\mathbb{C}}$.

We may now properly state our result on the arclikeness of $C_{\underline{u}}$.

3.3.2. Theorem (General Arclike Itinerary Sets)**.**
Let $\underline{u} \in \mathcal{S}_0$ *be exponentially bounded. Then* $\tilde{C}_{\underline{u}}$ *is an arclike continuum and all points in* $\tilde{C}_{\underline{u}} \cap \overline{\mathcal{S}}$ *are terminal points of* $\tilde{C}_{\underline{u}}$*.*

A proof of Theorem 3.3.2 is given at the end of Chapter 8. We note that from here on, when we refer to the closure $\overline{X}$ of a set $X \subset \tilde{\mathbb{C}}$ we refer to its closure in $\tilde{\mathbb{C}}$.

Chapter 4

Continua in $\tilde{\mathbb{C}}$

4.1 Continuity Properties

In order to prove that $\tilde{C}_{\underline{u}}$ is an arclike continuum, we must first show that $\tilde{C}_{\underline{u}}$ is in fact a continuum. We plan to do this by showing that it is the nested intersection of a sequence of sets, each of which is known to be a continuum because it is the image of a continuum under some continuous function. This function is, in some sense, an extension of a branch of the inverse of $E_{2\pi i}$. To this end, we extend $E_{2\pi i}$ to maps on the spaces $\tilde{\mathbb{C}}$ and $\tilde{\mathbb{C}} \setminus \{(\infty)\}$ and show that these maps have the necessary properties that such a continuous branch of the inverse does indeed exist.

4.1.1. Definition.
Let $E \colon \tilde{\mathbb{C}} \setminus (\infty) \to \tilde{\mathbb{C}}$ be defined so that

$$E|_{\mathbb{C}} = E_{2\pi i},$$

and

$$E|_{\mathcal{S}} = \sigma$$

where σ is the shift map. We may further extend this map to $\tilde{E} \colon \tilde{\mathbb{C}} \to \tilde{\mathbb{C}}$ by setting $\tilde{E}((\infty)) \coloneqq 0$.

We note that $\tilde{E}$ is not continuous on $\tilde{\mathbb{C}}$. However, we will show it is continuous when restricted to certain subsets of $\tilde{C}$. Specifically, when it is restricted to a connected component of $\tilde{E}^{-1}(\tilde{\mathbb{C}} \setminus \overline{\gamma_0})$, where γ_0 is defined as in Definition 2.2.7. The map E on the other hand has very strong continuity properties.

4.1.2. Theorem.
The map E is a covering map of $\tilde{\mathbb{C}} \setminus \{0\}$.

Proof. We know that $E_{2\pi i}$ is a cover of $\mathbb{C} \setminus \{0\}$. It remains to show that for every point $\underline{s} \in \overline{\mathcal{S}}$, there is an open neighbourhood V of $\underline{s}$ such that $E^{-1}(V)$ is the union of disjoint open sets U_j, where for each U_j, we have that E is a homeomorphism from U_j onto V.

Let $\underline{s}^-, \underline{s}^+ \in \mathcal{S}_e$ be such that $\underline{s} \in (\underline{s}^-, \underline{s}^+)$. Let $R > 0$ be large enough that $U_{\underline{s}^-,\underline{s}^+,R}$ exists. Let $V \coloneqq U_{\underline{s}^-,\underline{s}^+,R}$. Now let $\underline{t}^{j,-} \coloneqq (j, s_0^-, s_1^-, ...)$. When $\underline{s}^- < \underline{s}^+$, we define $\underline{t}^{j,+} = (j, s_0^+, s_1^+, ...)$ and when $\underline{s}^- > \underline{s}^+$ we define $\underline{t}^{j,+} = (j+1, s_0^+, s_1^+, ...)$. We note that E maps $(\underline{t}^{j,-}, \underline{t}^{j,+})$ bijectively onto $(\underline{s}^-, \underline{s}^+)$.

Now let $R' \coloneqq \frac{1}{2\pi} \ln R$ and let $U_j \coloneqq H^+_{R'}(\gamma_{\underline{t}^{j,-}}) \cap H^-_{R'}(\gamma_{\underline{t}^{j,+}}) \cup (\underline{t}^{j,-}, \underline{t}^{t+})$ for $j \in \mathbb{Z}$. We find that $\underline{t}^{j,-} < \underline{t}^{j,+}$, there is an upward vertical path with real value R' which goes from $\gamma_{\underline{t}^{j,-}}$ to $\gamma_{\underline{t}^{j,+}}$, and this path is mapped by E to a positively oriented path (that is, a subpath of the positively oriented circle, centred at 0 with radius R) which goes from $\gamma_{\underline{s}^-}$ to $\gamma_{\underline{s}^+}$. In this way, we can show that $\partial V = E(\partial U_j)$.

We note that, from our definition, we have that $\underline{t}^{j,+} < \underline{t}^{j+1,-}$. In this way, U_{j+1} lies strictly above U_j, in as much as $H^+_{R'}(\gamma_{\underline{t}^{j+1,-}})$ does not intersect $U_j \cap \mathbb{C}$, and so U_{j+1} and U_j are disjoint. We note that $U_{j+1} \cap \mathbb{C}$ is in fact the $2\pi i$ translate of $U_j \cap \mathbb{C}$. Since $U_j \cap \mathbb{C}$ does not intersect any of its $2\pi ik$ translates for any $k \in \mathbb{Z}$, it must be mapped injectively by E. Since U_j is open and is mapped injectively and continuously by E, we must have that

$$\partial E(U_j \cap \mathbb{C}) \cap \mathbb{C} = E(\partial U_j) \cap \mathbb{C} = (\partial V) \cap \mathbb{C}.$$

Therefore either $E(U_j) \cap \mathbb{C} = V \cap \mathbb{C}$ or $E(U_j) \cap \mathbb{C} = \mathbb{C} \setminus \overline{V}$. There exists some $\underline{t}^* \in (\underline{t}^{j,-}, \underline{t}^{t,+}) \cap \mathcal{S}_e$. We note that $E(\underline{t}^*) \in (\underline{s}^-, \underline{s}^+)$. By the vertical ordering of $\mathcal{S}_e$, we will have that some forward tail of $\gamma_{\underline{t}^*}$ will be contained in U_j. By the cyclic ordering of $\mathcal{S}_e$, we will have that some forward tail of $E(\gamma_{\underline{t}^*})$ will be contained in V. Therefore it must be the case that $E(U_j) \cap \mathbb{C}$ intersects $V \cap \mathbb{C}$, and so $E(U_j) \cap \mathbb{C} = V \cap \mathbb{C}$, with E here acting as a bijection.

We have now that E is a bijection from each U_j to V. Since $\bigcup_{j\in\mathbb{Z}} U_j$ contains all $2\pi ik$ translates of $U_j \cap \mathbb{C}$ and equivalent translates of $U_j \cap \mathcal{S}$, we see that $\bigcup_{j\in\mathbb{Z}} U_j$ is the entire preimage of V. It remains to show that E is continuous on U_j, and the branch of E^{-1} from V onto U_j is also continuous. We know that both of these maps are continuous for all $z \in \mathbb{C}$, so it remains to show they are also continuous on $\overline{\mathcal{S}}$. We note that for any $\underline{s} \in \mathcal{S}$, when $\underline{s}^-, \underline{s}^+$ are chosen arbitrarily close to $E(\underline{s})$, then $\underline{t}^{j,+}, \underline{t}^{j,-}$ are arbitrarily close to $\underline{s}$.

Now let X be an arbitrarily small neighbourhood of $E(\underline{s})$. Then there is some V, as defined before, such that V is contained in X near infinity, by which we mean $(V \setminus \{z \in \mathbb{C} \colon |z| > R\}) \subset X$ for some $R > 0$. Let $V' = V \cap X$. Then there is some neighbourhood U_j of $\underline{s}$, as defined before, which maps onto V. Since $V \setminus V'$ is bounded in $\mathbb{C}$, we find that $U' \coloneqq E^{-1}(V')$ agrees with U_j at infinity where E^{-1} is the branch defined onto U_j. Since U' agrees with a neighbourhood of $\underline{s}$ at infinity, U' is also a neighbourhood of $\underline{s}$ itself. So we have that $E(V') \subset X$ and E is continuous on $\mathcal{S}$.

Let L be defined as a branch of E^{-1} from some open set V onto some neighbourhood U_j of $\underline{s}$. Let $X \subset U_j$ be an arbitrarily small neighbourhood of $\underline{s}$. Similarly, let

$$U \coloneqq H^+_{R'}(\gamma_{\underline{t}^{j,-}}) \cap H^-_{R'}(\gamma_{\underline{t}^{j,+}}) \cup (\underline{t}^{j,-}, \underline{t}^{j,+})$$

be chosen with $\underline{t}^{j,-}, \underline{t}^{j,+}$ near enough $\underline{s}$ that U is contained in X near infinity (by which we mean U agrees at infinity with some subset of X). Let $U' := U \cap X$ and let $V' := E(U')$. We have, similarly, that V' is a neighbourhood of $E(\underline{s})$ and $L(V') \subset X$, so that L is continuous on all $E(\underline{s}) \in V$. ■

As mentioned before, $\tilde{E}$ is not continuous. The following property, however, is sufficient for our purposes.

4.1.3. Lemma.
Let $X \subset \tilde{\mathbb{C}}$ be such that there exists a simple path $P\colon [0,1] \to \tilde{\mathbb{C}}$ with $P(0) = 0$ and $P(1) \in \overline{\mathcal{S}}$ such that $P((0,1)) \subset \mathbb{C} \setminus X$. Then there exists a branch of $\tilde{E}^{-1}$ on X which is a homeomorphism. This branch may be chosen to map X into a connected component of $\tilde{E}^{-1}(\tilde{\mathbb{C}} \setminus P([0,1]))$

Proof. We note that the set $X_P := \tilde{\mathbb{C}} \setminus P([0,1])$ is a simply connected subset of $\tilde{\mathbb{C}} \setminus \{0\}$ and so there is a branch of E^{-1} which is a homeomorphism from X_P to some $X_P^{-1} \subset \tilde{\mathbb{C}}$. This branch of E^{-1} will map small neighbourhoods of 0 to neighbourhoods of (∞) with real part bounded by large negative R. Extending this branch of E^{-1} to a branch of $\tilde{E}^{-1}$ we find that this branch is continuous at 0.

Since X_P is open in $\tilde{\mathbb{C}}$, so is X_P^{-1}. The boundary of X_P^{-1} in $\tilde{\mathbb{C}} \setminus \{0\}$ maps under E to the the boundary of X_P in $\tilde{\mathbb{C}} \setminus \{(\infty)\}$. The boundary components of X_P^{-1} are therefore images of paths of the form $P^-\colon (0,1] \to \tilde{C} \setminus \{(\infty)\}$ and $P^+ = P^- + 2\pi i$. We will also have that $P^-(1) \in \mathcal{S}$ and $P^+(1) \in \mathcal{S}$ and we will have $P^-((0,1)) \subset \mathbb{C}$ and $P^-((0,1)) \subset \mathbb{C}$. As x tends to 0 the real values of $P^-(x)$ and $P^+(x)$ tend to $-\infty$.

For every $R < 0$, we have that the set $\{z \in X_P^{-1} \cap \mathbb{C}\colon |z| > R\}$ accumulates only on an interval of $[P^-(1), P^+(1)]$ and not on (∞). For a sufficiently small neighbourhood U of (∞), we have that the maximum real part of $z \in X_P^{-1} \cap U$ is less than R. It follows that $E(X_P^{-1} \cap U)$ is contained within an arbitrarily small ball of size $2\pi e^{-R}$ around 0. It follows that $\tilde{E}$ is continuous on $X_P^{-1} \cap \{(\infty)\}$.

This branch of $\tilde{E}^{-1}$ is then a homeomorphism on X_P and therefore also on X. ■

4.1.4. Remark.
In particular, when X is disjoint from $\overline{\gamma_0}$, then any branch of E^{-1} which maps into the interior of $\overline{T}_j$ is a homeomorphism

4.2 Itineraries on $\tilde{\mathbb{C}}$

Since we have extended $E_{2\pi i}$ to a map on $\tilde{\mathbb{C}}$, we may now also extend our definition of itineraries to $\tilde{\mathbb{C}}$. We may do this in such a way that $\tilde{C}_{\underline{u}}$ is precisely the set of points in $\tilde{\mathbb{C}}$ with itinerary $\underline{u}$. This fact is shown here in Proposition 4.2.4.

4.2.1. Definition (Itineraries of $\tilde{\mathbb{C}}$)**.**
Recall that t_j is defined in Definition 2.2.8 as a dynamic ray which maps onto γ_0. Let $\underline{t}_j = (j, 0, 1, 1, ...)$. This is defined such that $\underline{t}_j$ is the address of t_j. Let $\tilde{T}_j := T_j \cup [\underline{t}_j, \underline{t}_{j+1}) \cup (\infty)$ where T_j is similarly taken from Definition 2.2.8. We say $\underline{u}$ is an *itinerary* of a point $z \in \tilde{\mathbb{C}}$ if for all $j \in \mathbb{N}$

$$\tilde{E}^j(z) \in \tilde{T}_{u_j}.$$

We eventually plan on invoking Lemma 4.1.3 to give us a homeomorphic branch of the preimage on $\tilde{C}_{\underline{u}}$. We use the fact that for most itineraries $\underline{u}$, we have that $\tilde{C}_{\underline{u}}$ is disjoint from $\overline{\gamma_0}$. However, it is first necessary to consider separately the special case of the itinerary $\underline{u} = (0, 1, 1, ...)$. This is the only case where $\tilde{C}_{\underline{u}}$ intersects $\overline{\gamma_0}$ and, in fact, is equal to it.

4.2.2. Lemma.
$\tilde{C}_{(0,1,1,...)}$ is precisely the closure of γ_0. For all other itineraries $\underline{u}$, we have that $\tilde{C}_{\underline{u}}$ is disjoint from $\overline{\gamma_0}$.

Proof. We will later give a proof in Theorem 8.0.1 that the set of escaping points in $C_{(0,1,1,...)}$ is dense in $C_{(0,1,1,...)}$. It follows that $\tilde{C}_{(0,1,1,...)}$ is the closure of some collection of dynamic rays. Suppose there were another ray in $C_{(0,1,1,...)}$ that wasn't γ_1. There would also exist a ray in $C_{(1,1,...)}$ which was not γ_1. Let this ray have address $\underline{s} \neq (1,1,...)$ and let n be the first entry such that $s_n \neq 1$. If $s_n < 1$, then $E^n(\underline{s})$ is not in $\overline{T_1}$. If $s_n > 2$, then $E^n(\underline{s})$ is not in $\overline{T_1}$. If $s_n = 2$, then either $s_{n+1} > 0$ and $E^n(\underline{s})$ is not in $\overline{T_1}$, or $s_{n+1} < 0$ and $E^{n+1}(\underline{s})$ is not in $\overline{T_1}$. Either way, there is some image of $\gamma_{\underline{s}}$ whose forward tail lands outside $\overline{T_1}$ and therefore this image of $\gamma_{\underline{s}}$ cannot be contained in T_1 and $\gamma_{\underline{s}}$ cannot be in $C_{(1,1,...)}$. Therefore $C_{(0,1,1,...)}$ can only contain γ_0 and 0.

Supposing there were some $\underline{u}$ such that $\tilde{C}_{\underline{u}}$ contained some point of $\overline{\gamma_0}$. We must then have that $u_0 = 0$, since $\overline{\gamma_0}$ is contained in the interior of $\tilde{T}_0$. By the continuity of E, for all $n \geq 1$ we would have $E^n(C_{\underline{u}})$ would intersect $\overline{\gamma_1}$ and so $u_n = 1$. Therefore $\underline{u}$ must be $(0,1,1,...)$ and $\tilde{C}_{\underline{u}}$ is disjoint from $\overline{\gamma_0}$ for all other values of $\underline{u}$ ■

The following theorem illustrates the advantages of using $\tilde{\mathbb{C}}$ over $\hat{\mathbb{C}}$. For each $n, m \in \mathbb{N}$, we have that $\tilde{C}_{\sigma^n(\underline{u})}$ and $\tilde{C}_{\sigma^m(\underline{u})}$ are homeomorphic. We note that this would not be the case if we took our closure in $\hat{\mathbb{C}}$, since the closure of $C_{(1,0,1,1...)}$ in $\hat{\mathbb{C}}$ is a topological circle and the closure of $C_{(0,1,1...)}$ in $\hat{\mathbb{C}}$ is a topological arc. Contrary to this, we see that it is only the eventual behaviour of $\underline{u}$ which determines the topology of $\tilde{C}_{\underline{u}}$.

4.2.3. Theorem.
For all $\underline{u}$ and $n \in \mathbb{N}$, we have that $\tilde{C}_{\underline{u}}$ is homeomorphic to $\tilde{E}^n(\tilde{C}_{\underline{u}}) = \tilde{C}_{\sigma^n(\underline{u})}$

Proof. If $\sigma(\underline{u}) \neq (0,1,1,...)$, then we may apply Lemma 4.1.3, taking our path P along γ_0. There then exists a branch of $\tilde{E}^{-1}$ which is a homeomorphism on $\underline{C}_{\sigma(\underline{u})}$ and which maps into the interior of some $\tilde{T}_{u_0}$. We will have that all points of $C_{\sigma(\underline{u})}$ will be mapped by this branch to points with itinerary $\underline{u}$. Since for each point in $\mathbb{C}$ there is only one preimage in T_{u_0}, then we must have that $\tilde{E}^{-1}(C_{\sigma(\underline{u})}) = C_{\underline{u}}$ for this branch. Since $C_{\sigma(\underline{u})}$ is dense in $\tilde{C}_{\sigma(\underline{u})}$ and our branch of $\tilde{E}^{-1}$ is a homeomorphism, it follows that $C_{\underline{u}}$ is dense in the compact set

$\tilde{E}^{-1}(\tilde{C}_{\sigma(\underline{u})})$. It follows that $\tilde{E}^{-1}(\tilde{C}_{\sigma(\underline{u})})$ is precisely $\tilde{C}_{\underline{u}}$, the closure of $C_{\underline{u}}$.

When $\underline{u} = (u_0, 0, 1, 1, ...)$ and $\sigma(\underline{u}) = (0, 1, 1, ...)$, then we have that $\tilde{C}_{\sigma(\underline{u})} = \overline{\gamma_0}$. This also satisfies the conditions for Lemma 4.1.3 with some other path P. The branches of $\tilde{E}^{-1}$ which are homeomorphisms must be the branches which take 0 to (∞) and take $\gamma_0 \cup \{(0, 1, 1, ...)\}$ to some path connected component, which must be $t_j \cup \{\underline{t}_j\}$ for some j. There is therefore a homeomorphism under this branch from $\tilde{C}_{\sigma(\underline{u})} = \overline{\gamma_0}$ to $\tilde{C}_{\underline{u}} = \overline{t_{u_0}}$. We note also that $\tilde{E}(\tilde{C}_{\sigma(\underline{u})}) = \tilde{C}_{\underline{u}}$ in this case also. It follows that for any n, the map $\tilde{E}^n$ is a homeomorphism from $\tilde{C}_{\underline{u}}$ to $\tilde{C}_{\sigma^n(\underline{u})}$. ■

4.2.4. Proposition.
Let $\underline{u}$ be exponentially bounded. Then $\tilde{C}_{\underline{u}}$ is precisely the set of points in $\tilde{\mathbb{C}}$ with itinerary $\underline{u}$ with respect to $\{\tilde{T}_j\}_{j\in\mathbb{Z}}$ and $\tilde{E}$.

Proof. If a point $\underline{s} \in \overline{\mathcal{S}}$ has exponentially bounded itinerary $\underline{u}$, then $\underline{s}$ is either an intermediate address or is also exponentially unbounded. If $\underline{s}$ is an intermediate address, then $\tilde{E}^n(\underline{s}) = 0$ for some $n \in \mathbb{N}$. Then $\sigma^n(\underline{u}) = (0, 1, 1, ...)$ and $\tilde{E}^n(\tilde{C}_{\underline{u}}) = \overline{\gamma_0}$. Then we see that $\tilde{C}_{\underline{u}}$ contains some branch of $\tilde{E}^{-n}(\gamma_0)$ which lands on $\underline{s}$. If $\underline{s}$ is not intermediate and is exponentially bounded, then there exists a ray $\gamma_{\underline{s}}$ with that address whose forward tail tends to $\underline{s}$. Furthermore, by the continuity of E, we have that $E^n(\gamma_{\underline{s}})$ tends to $E^n(\underline{s})$ for all $n \in \mathbb{N}$. When $E^n(\underline{s}) \in (\underline{t}_j, \underline{t}_{j+1})$, then by the ordering of rays we have that $\gamma_{\underline{s}}$ lies between t_j and t_{j+1}. When $E^n(\underline{s}) = \underline{t}_j$, then $\gamma_{\underline{s}} = t_j$. So $\gamma_{\underline{s}}$ has the same itinerary as $\underline{s}$ and its closure contains $\underline{s}$. When $\underline{s}$ has itinerary $\underline{u}$, then $\underline{s} \in \tilde{C}_{\underline{u}}$. So every point in $\tilde{\mathbb{C}}$ with itinerary $\underline{u}$ is in $\tilde{C}_{\underline{u}}$.

It remains to show that every point of $\tilde{C}_{\underline{u}}$ has itinerary $\underline{u}$. If this were not the case for some $\underline{u}$, then there would be some n such that $\tilde{E}^n(\tilde{C}_{\underline{u}})$ is not contained in $\tilde{T}_{u_n}$. This will only occur when $C_{\underline{u}}$ accumulates somewhere on $t_{u_0} \cup \{\underline{t}_{u_0}\}$ in the sense that the closure of $C_{\underline{u}}$ contains some point of $t_{u_0} \cup \{\underline{t}_{u_0}\}$.

Supposing there were some $\underline{u}$ such that $C_{\underline{u}}$ accumulated somewhere on $\partial\tilde{T}_{u_0}$. Then $E(C_{\underline{u}})$ must accumulate somewhere on the closure of γ_0 by the continuity of E. By Lemma 4.2.2, this only occurs when $\tilde{E}(\tilde{C}_{\underline{u}}) = \overline{\gamma_0}$. This, however, implies that $\tilde{C}_{\underline{u}} = \overline{t}_{u_0} \subset \tilde{T}_{u_0}$. Therefore the closure of $\tilde{E}^n(C_{\underline{u}})$ is always contained in $\tilde{T}_{u_n}$ and all points of $\tilde{C}_{\underline{u}}$ have itinerary $\underline{u}$. ■

In particular, we note that a ray $\gamma_{\underline{s}}$ has the same itinerary as its address $\underline{s}$. We note here that when $\underline{s} \in \mathcal{S}_e$, then the following are equivalent:

- $\underline{s} \in \tilde{C}_{\underline{u}} \cap \mathcal{S}_0$;
- $\forall n \in \mathbb{N}, \underline{t}_{u_n} \leq \sigma^n(\underline{s}) < \underline{t}_{u_n+1}$;
- $\gamma_{\underline{s}} \subset C_{\underline{u}}$.

4.2.5. Lemma.
Let $\underline{u}$ be exponentially bounded. Then $\tilde{C}_{\underline{u}}$ is a continuum.

Proof. Let $\hat{C}_{\underline{u}}$ be the set of points in $z \in \tilde{\mathbb{C}}$ for which $E^n(z)$ is in $\overline{T}_{u_n}$, the closure of $\tilde{T}_{u_n}$, for all $n \in \mathbb{N}$. Let $\hat{C}_{\underline{u},n}$ be the set of points in $z \in \tilde{\mathbb{C}}$ for which for which $E^m(z)$ is in $\overline{T}_{u_n}$ for all $m \in \mathbb{N}, m \leq n$.

To show that $\hat{C}_{\underline{u},n}$ is a continuum consider sets X of $\tilde{\mathbb{C}}$ with the following properties:

- X is a continuum (is closed and connected);
- X either contains $\overline{\gamma_0}$ in its interior or is disjoint from it;
- X either contains $\overline{\gamma_1}$ in its interior or is disjoint from it.

If these properties hold for X, then they hold for $X_j := \tilde{E}^{-1}(X) \cap \overline{T}_j$. When j is neither 0 nor 1, then the second and third properties always hold for X_j. When $j = 1$, then the second property always holds for X_j, and the third property holds for X_j whenever X satisfies the third property. When $j = 0$, the third property always holds for X_j, and the second property holds for X_j whenever X satisfies the third property. We can now show that the first property holds for X_j whenever the first and second property hold for X.

Suppose X were a continuum disjoint from $\overline{\gamma_0}$. Then there is a homeomorphic branch of $\tilde{E}^{-1}$ from X into the interior of $\tilde{T}_j$. We have that $\tilde{E}^{-1}(X)$ is the union of the vertical translates of this branch of the preimage, none of which intersect $\partial\overline{T}_j$ and therefore none of which are contained in $\overline{T}_j$. Therefore X_j is precisely this homeomorphic image of X and is also a continuum.

Suppose X is a continuum and contains $\overline{\gamma_0}$ in its interior. Then $\tilde{\mathbb{C}} \setminus X$ is open and disjoint from $\overline{\gamma_0}$. It follows from E being a covering map that $\tilde{E}^{-1}(\tilde{\mathbb{C}} \setminus X) = E^{-1}(\tilde{\mathbb{C}} \setminus X)$ is open in $\tilde{\mathbb{C}} \setminus \{(\infty)\}$ and therefore open in $\tilde{\mathbb{C}}$. Similarly, neither $\tilde{\mathbb{C}} \setminus X$ nor $\tilde{E}^{-1}(\tilde{\mathbb{C}} \setminus X)$ have any connected components which are not simply connected. It follows that $\tilde{E}^{-1}(X)$ is closed and connected in $\tilde{\mathbb{C}}$. We find that X_j, as the intersection of two closed sets, is also closed. We both that $\tilde{E}^{-1}(X)$ and X_j contain $\partial\overline{T}_j$. In fact $\partial\overline{T}_j$ is the boundary of X_j in $\tilde{E}^{-1}(X)$. By the boundary bumping theorem, every connected component of X_j shares part of its boundary with $\partial\overline{T}_j$. Since X_j contains $\partial\overline{T}_j$, that means that all these components are connected, or rather, there is only one connected component. X_j is therefore a closed connected subset of the continuum $\tilde{\mathbb{C}}$ and is therefore a continuum.

For all $\underline{u}$, the above properties hold for $\hat{C}_{\underline{u},0}$. By induction, since all $\hat{C}_{\underline{u},n+1}$ are the preimage of $\hat{C}_{\sigma(\underline{u}),n}$ in $\overline{T}_{u_0}$, these properties must also hold for $\hat{C}_{\underline{u},n}$ for all $\underline{u}$ and $n \in \mathbb{N}$. $\hat{C}_{\underline{u}}$ is the intersection of the nested sequence of continua $\{\hat{C}_{\underline{u},n}\}_{n\in\mathbb{N}}$ and is therefore a continuum itself.

It remains to show that for exponentially bounded non-singular $\underline{u}$ we have that $\hat{C}_{\underline{u}} = \tilde{C}_{\underline{u}}$. We note that we immediately have that $\tilde{C}_{\underline{u}} \subset \hat{C}_{\underline{u}}$, and a point in $\hat{C}_{\underline{u}}$ only fails to be a point in $\tilde{C}_{\underline{u}}$ if its orbit intersects some $\overline{t}_{u_n+1}$. In this case the orbit of such a point is eventually contained in $\overline{\gamma}_1$ and $\underline{u}$ is singular. If $\underline{u}$ is non-singular then $\tilde{C}_{\underline{u}}$ is some homeomorphic branch of some preimage of $\overline{\gamma_1}$ and is therefore also a continuum. ■

We note here that when $\underline{s} \in \mathcal{S}_e$, then the following are equivalent:

- $\underline{s} \in \tilde{C}_{\underline{u}} \cap \mathcal{S}_0$;

- $\forall n \in \mathbb{N}, \underline{t}_{u_n} \leq \sigma^n(\underline{s}) < \underline{t}_{u_n+1}$;
- $\gamma_{\underline{s}} \subset C_{\underline{u}}$.

We can therefore determine the itinerary of a ray $\gamma_{\underline{s}}$ entirely from a combinatorial analysis of its address $\underline{s}$. We do this in the following chapter.

Chapter 5

Augmented Itineraries

In order to give a proof of Theorem 2.2.11 and determine how many rays belong to each type of itinerary, we will introduce the *augmented itinerary* for points on $\mathcal{S}_0$.

5.0.1. Definition (Augmented Itineraries)**.**
Let $\underline{s} \in \mathcal{S}_0$ be non-singular and let $\underline{u}$ be the itinerary of $\underline{s}$. We define $\underline{\chi}$ to be the sequence $(\chi_0, \chi_1, ...)$ given by

$$\chi_j := \begin{cases} - & \sigma^j(\underline{s}) < (0,1,1,...) \\ 0 & (0,1,1,...) < \sigma^j(\underline{s}) < (1,1,...) \\ + & \sigma^j(\underline{s}) > (1,1,...) \end{cases}$$

and we call the sequence $\underline{u}^* := ((u_0, \chi_0), (u_1, \chi_1), ...)$ the *augmented itinerary* of $\underline{s}$.

We use augmented itineraries because, unlike regular itineraries, they have the following uniqueness property.

5.0.2. Lemma.
Let $\underline{u}^*$ be an augmented itinerary. Then there exists at most one address $\underline{s} \in \mathcal{S}_0$ with augmented itinerary $\underline{u}^*$. Specifically, when it exists, $\underline{s}$ is determined

by

$$s_j = \begin{cases} u_j & \chi_{j+1} \in \{0, +\} \\ u_j + 1 & \chi_{j+1} = - \end{cases}$$

for all $j \in \mathbb{N}$.

Proof. This follows directly from the definition of augmented addresses. Assuming $\underline{s}$ has augmented itinerary $\underline{u}^*$, then whenever $\chi_{j+1} \in \{0, +\}$, then $\sigma^{j+1}(\underline{s}) > (0, 1, 1, ...)$ and given that $\underline{t}_{u_j} \le \sigma^j(\underline{s}) < \underline{t}_{u_j+1}$, it must be the case that $u_j = s_j$. Similarly, whenever $\chi_{j+1} = -$, then $\sigma^{j+1}(\underline{s}) < (0, 1, 1, ...)$ and $u_j + 1 = s_j$. ■

5.0.3. Lemma.
Let $\underline{u}$ *be a non-singular itinerary with an associated augmented itinerary* $\underline{u}^*$. *There exists some address* $\underline{s} \in \mathcal{S}_0$ *with augmented itinerary* $\underline{u}^*$ *precisely when for all* $j \in \mathbb{N}$, *we have*

$$\chi_j = \begin{cases} - & u_j < 0 \vee (u_j = 0 \wedge \chi_{j+1} = 0) \\ 0 & (u_j = 0 \wedge \chi_{j+1} \in \{-, +\}) \vee (u_j = 1 \wedge \chi_{j+1} = 0) \\ + & u_j > 1 \vee (u_j = 1 \wedge \chi_{j+1} \in \{-, +\}) \end{cases} . \tag{5.0.1}$$

To unpack Equation 5.0.1, whenever $u_j < 0$, then $\chi_j = -$; whenever $u_j > 1$, then $\chi_j = +$; whenever $u_j = 0$, then χ_j is determined by χ_{j+1} so that $\chi_j = -$ when $\chi_{j+1} = 0$ and $\chi_j = 0$ otherwise; whenever $u_j = 0$, then $\chi_j = 0$ when $\chi_{j+1} = 0$ and $\chi_j = +$ otherwise.

Proof. Let $I_+ \subset \mathcal{S}_0$ be the open interval $((1, 1, ...), (\infty))$. Similarly, we define $I_0 := ((0, 1, 1, ...), (1, 1, ...))$ and $I_- := ((\infty), (0, 1, 1, ...))$.

We note that when $u > 1$, then E maps $\tilde{T}_u \cap I_+$ bijectively onto $\mathcal{S}_0$, while $\tilde{T}_u \cap I_0$ and $\tilde{T}_u \cap I_-$ are empty. Similarly when $u < 0$, then E maps $\tilde{T}_u \cap I_-$ bijectively onto $\mathcal{S}_0$, while $\tilde{T}_u \cap I_0$ and $\tilde{T}_u \cap I_+$ are empty.

When $u = 1$, then E maps $\tilde{T}_u \cap I_+$ bijectively onto $I_+ \cup I_-$ and maps

$\tilde{T}_u \cap I_0$ onto I_0, while $\tilde{T}_u \cap I_-$ is empty. When $u = 0$, then E maps $\tilde{T}_u \cap I_0$ bijectively onto $I_+ \cup I_-$ and maps $\tilde{T}_u \cap I_-$ onto I_0, while $\tilde{T}_u \cap I_+$ is empty.

In this way, we can see that for any $j \in \mathbb{N}$, some fixed $\chi_{j+1} \in \{-, 0, +\}$ and some $\underline{s}^{j+1} \in I_{\chi_{j+1}}$, then for u_j, χ_j, there exists some $\underline{s}^j \in \tilde{T}_{u_j} \cap I_{\chi_j}$ such that $E(\underline{s}^j) = \underline{s}^{j+1}$ if and only if u_j, χ_j and χ_{j+1} satisfy Equation 5.0.1. Fixing $u_0, ..., u_n$ and $\chi_0, ..., \chi_{n+1}$, there then exists an address $\underline{s}$ with augmented itinerary $\underline{u}^*$ which agrees with these u_j, χ_j if and only if u_j, χ_j, χ_{j+1} satisfy Equation 5.0.1 for all $0 \leq j \leq n$.

Now we fix some non-singular augmented itinerary $\underline{u}^*$. If Equation 5.0.1 is not satisfied, then there will be no address with augmented itinerary $\underline{u}^*$. Suppose Equation 5.0.1 is satisfied. Then there exists a sequence of addresses $(\underline{s}^n)$ such that the augmented itinerary of $\underline{s}^n$ agrees with the first n entries of $\underline{u}^*$. By Lemma 5.0.2, these addresses converge to some address $\underline{s}$. Since $\underline{u}^*$ is non-singular then $\underline{s}$ is also non-singular. Therefore $E^j(\underline{s})$ will not lie on the boundary of any $\tilde{T}_{u_j} \cap I_{\chi_j}$ and so, since $E^j(\underline{s}) = \lim_{n\to\infty} E^j(\underline{s}^n)$ and for all $n > j$ we have $E^j(\underline{s}^n) \in \tilde{T}_{u_j} \cap I_{\chi_j}$, then we have $E^j(\underline{s}) \in \tilde{T}_{u_j} \cap I_{\chi_j}$ for all j. Therefore $\underline{u}^*$ has some address for which it is an augmented address. ■

5.0.4. Lemma.
Let $\underline{u}$ be a non-singular binary itinerary such that for all $n \in \mathbb{N}$ we have $u_n \in \{0, 1\}$. Then there are precisely two addresses $\underline{a}, \underline{b} \in \overline{\mathcal{S}}$ with itinerary $\underline{u}$.

In particular, a and b have augmented itineraries $\underline{u}^{a*}$ and $\underline{u}^{b*}$ respectively, and for all $n \in \mathbb{N}$ we have that one of χ_n^a or χ_n^b is 0 and the other is in $\{-, +\}$.

Proof. We first note that if $\underline{s} \in \overline{\mathcal{S}} \setminus \mathcal{S}_0$, then $\underline{s}$ has a singular itinerary.

For $n \in \mathbb{N}$, let k_n be the $n+1$th index such that $u_{k_n} = 0$. We note that if $\underline{u}^*$ is the augmented address of some $\underline{s} \in \mathcal{S}_0$, then by the above lemma, χ_{k_n} is either 0 or $-$. When $\chi_{k_{n+1}} = 0$, then if $u_{k_{n+1}-1} = 1$, then $\chi_{k_{n+1}-1} = 0$. In this way, $\chi_{k_j} = 0$ for all $k_n < j \leq k_{n+1}$ so we have that $\chi_{k_n} = -$.

Similarly, when $\chi_{k_{n+1}} = -$, then if $u_{k_{n+1}-1} = 1$, then $\chi_{k_{n+1}-1} = +$ and similarly $\chi_{k_j} \in \{-, +\}$ for all $k_n < j \leq k_{n+1}$ so we have that $\chi_{k_n} = 0$.

If we let $\chi_{k_{2n}} = 0$ for all $n \in \mathbb{N}$, then we can define a unique augmented itinerary $\underline{u}^{a*}$ which satisfies Equation 5.0.1 and which agrees with u_j for all $j \in \mathbb{N}$ and agrees with $\chi_{k_{2n}}$ for all $n \in \mathbb{N}$. Similarly, we can define a unique valid augmented itinerary $\underline{u}^{b*}$ for which $\chi_{k_{2n}} = -$. For any other determination of $(\chi_{k_{2n}})_{n\in\mathbb{N}}$, there will be some n such that $\chi_{k_{2n}} \neq \chi_{k_{2(n+1)}}$ and there will be no valid augmented itinerary with this determination of $(\chi_{k_{2n}})_{n\in\mathbb{N}}$.

There are then only two valid augmented itineraries, $\underline{u}^{a*}$ and $\underline{u}^{b*}$. We note that for all $n \in \mathbb{N}$, whenever $\chi^a_n = 0$, then $\chi^b_n \in \{-, +\}$ and whenever $\chi^b_n = 0$, then $\chi^a_n \in \{-, +\}$. ■

Proof of Theorem 2.2.11. Let $\underline{u} := (u_0, ..., u_{n-1}, 1, 1, ...)$ be a singular itinerary. Then $\tilde{E}^n(\tilde{C}_{\underline{u}})$ has itinerary $(1, 1, ...)$ and must contain only the ray γ_1. We therefore have that $\tilde{C}_{\underline{u}}$ must also only contain one ray.

Suppose $\underline{u}$ is a non-singular binary address. Then there is some $n \in \mathbb{N}$ such that $\sigma^n(\underline{u})$ contains only 0 and 1. Then by Lemma 5.0.4, there exist precisely two addresses $\underline{a}^n$ and $\underline{b}^n$ with itinerary $\sigma^n(\underline{u})$. There therefore exist precisely two addresses $\underline{a}, \underline{b}$ with $E^n(\underline{a}) = \underline{a}^n$ and $E^n(\underline{b}) = \underline{b}^n$ such that $\underline{a}$ and $\underline{b}$ have itinerary $\underline{u}$. These addresses will have associated rays $\gamma_{\underline{a}}, \gamma_{\underline{b}}$ with itinerary $\underline{u}$.

Suppose $\underline{u}$ is a non-binary address and $\underline{s} \in \mathcal{S}_0$ has itinerary $\underline{u}$. Let k_n be the $n + 1$th index such that u_{k_n} is neither 0 nor 1. Then in the augmented itinerary $\underline{u}^*$ of $\underline{s}$ there must be a unique determination of χ_{k_n} which satisfies Equation 5.0.1. Each χ_{k_n}, along with $\underline{u}$, uniquely determines χ_j for $j \leq k_n$. In this way, for all $j \in \mathbb{N}$, there is some sufficiently large k_n which uniquely determines χ_j, and so there is a unique augmented address which satisfies Equation 5.0.1. By Lemma 5.0.2, this implies that there is a unique address $\underline{s}$ with this itinerary. If $\underline{u}$ is exponentially bounded, then $\underline{s}$ will also be exponentially bounded so that there will exist a unique ray $\gamma_{\underline{s}}$ which has itinerary $\underline{u}$. ■

Chapter 6

Defining ϵ-maps

We can now describe the ϵ-maps which will prove the arclikeness of $\tilde{C}_{\underline{u}}$ following Theorem 2.3.9. In particular, we find ϵ-maps to prove arclikeness for non-singular itineraries (where *non-singular* is defined in Defenition 2.2.10). For singular itineraries $\underline{u}$, we intend to show in the Chapter 8 that $\tilde{C}_{\underline{u}}$ is homeomorphic to an arc. In particular, $\tilde{C}_{\underline{u}}$ is homeomorphic to $\overline{\gamma_1}$ by some branch of E^{-n} for some $n \in \mathbb{N}$.

We first note that while $\tilde{\mathbb{C}}$ is a metric space, an explicit metric is difficult to find. We therefore use the following lemma to allow us to use a metric on a subset of $\mathbb{C}$.

6.0.1. Lemma.
Let $X \subset \tilde{\mathbb{C}}$ be a continuum and let $\operatorname{diam}_{X\cap\mathbb{C}}$ be the diameter function of some metric on $X \cap \mathbb{C}$ compatible with the topology on $X \cap \mathbb{C}$. Suppose for all $\epsilon > 0$ there exist surjective continuous maps $g_\epsilon \colon X \to [0,1]$ such that the following hold:

- *If either $g_\epsilon^{-1}(0)$ or $g_\epsilon^{-1}(1)$ intersect $\overline{\mathcal{S}}$, then they are single points;*
- *$g_\epsilon^{-1}((0,1)) \subset \mathbb{C}$;*
- *$\operatorname{diam}_{X\cap\mathbb{C}}(g_\epsilon^{-1}(x)) < \epsilon$ for all $x \in [0,1]$ with $g_\epsilon^{-1}(x) \in \mathbb{C}$.*

Then X is arclike and all points on $X \cap \overline{\mathcal{S}}$ are terminal points in X.

Proof. There are at most two points p_0, p_1 on X which are not in $\mathbb{C}$. Fixing some metric $\mathrm{dist}_{\tilde{\mathbb{C}}}$ on $\tilde{\mathbb{C}}$, let $\delta > 0$ and let B_0, B_1 be open balls of diameter δ in $\tilde{\mathbb{C}}$ around p_0, p_1 where p_0, p_1 exist. Let B be the union of these balls when they exist (otherwise B is the empty set). Then $X \setminus B$ is bounded in $\mathbb{C}$ and therefore compact.

For any metric $\mathrm{dist}_{X\cap\mathbb{C}}$ on $\mathbb{C}$, the metrics $\mathrm{dist}_{\tilde{\mathbb{C}}}$ and $\mathrm{dist}_{X\cap\mathbb{C}}$ are comparable on $X \setminus B$ in the following way. There exists some $\epsilon > 0$ such that for $a, b \in X \setminus B$ when $\mathrm{dist}_{X\cap\mathbb{C}}(a, b) < \epsilon$, then $\mathrm{dist}_{\tilde{C}}(a, b) < \delta$ (if it were not the case, there would exist a sequence of pairs $(a_n, b_n)_{n\in\mathbb{N}} \subset X \setminus B$ such that $\mathrm{dist}(\mathbb{C})(a_n, b_n)$ tends to 0 and $\mathrm{dist}_{\tilde{\mathbb{C}}}(a_n, b_n) > \delta$. Since $X \setminus B$ is bounded, there exists a subsequence of (a_n, b_n) which tends to a point, this contradicts $\mathrm{dist}_{\tilde{\mathbb{C}}}(a_n, b_n) > \delta$).

If it is not the case that B_0 and B_1 exist and are disjoint, then we have that $\mathrm{diam}_{\tilde{\mathbb{C}}}(B) \leq 2\delta$. Let g_ϵ be a map satisfying the hypothesis. Then for any $x \in [0, 1]$ we have that $\mathrm{diam}_{\tilde{\mathbb{C}}}(g_\epsilon^{-1}(x)) < 3\delta$.

Now suppose B_0 and B_1 are disjoint. For $j \in \{0, 1\}$ and $\epsilon > 0$ we define $B_{j,\epsilon} := \{z \in B_j \colon \mathrm{dist}_{X\cap\mathbb{C}}(z, X \setminus B) < \epsilon\}$. We note that these sets are bounded for sufficiently small ϵ. Then we may define ϵ to be sufficiently small that additionally

$$\epsilon < \min(\mathrm{dist}_{X\cap\mathbb{C}}(X \cap B_{0,\epsilon}, X \cap B_1), \mathrm{dist}_{X\cap\mathbb{C}}(X \cap B_{1,\epsilon}, X \cap B_0)).$$

Then I claim that for $x \in (0, 1)$, we have that $g_\epsilon^{-1}(x)$ intersects at most one component of B. Supposing there exists some x such that $g_\epsilon^{-1}(x)$ intersects both B_0, B_1, then by the above inequality $g_\epsilon^{-1}(x)$ does not intersect $X \setminus B$. In fact $\mathrm{dist}_{X\cap\mathbb{C}}(g_\epsilon^{-1}(x), X \setminus B) \geq \epsilon$.

Let I_x be the union of all closed intervals I'_x in $[0, 1]$ containing x such that $g_\epsilon^{-1}(I'_x)$ intersects both B_0, B_1. Let $C_{j,x} := g_\epsilon^{-1}(I_x) \cap B_j$ for $j \in \{0, 1\}$. Then there exists a component of $X \setminus g_\epsilon^{-1}(I_x)$ which connects $C_{0,x}$ to $C_{1,x}$

which we shall call C. Now since $g_\epsilon^{-1}(I_x)$ is disconnected from $X \setminus B$, then C must intersect both B_0 and B_1. By continuity, we know that $g_\epsilon(C \cap B_0)$ and $g_\epsilon(C \cap B_1)$ must be connected to either the upper or lower bound of I_x (by which we mean there exists a closed interval in $g_\epsilon(C \cap B_j) \cup I_x$ which contains I_x as a proper subset). If both $g_\epsilon(C \cap B_0)$ and $g_\epsilon(C \cap B_1)$ are connected to the upper or lower bound, then I_x will not be maximum. If $g_\epsilon(C \cap B_0)$ and $g_\epsilon(C \cap B_1)$ are connected one to the upper and one to the lower bound then since $g_\epsilon(C)$ does not contain I_x, then $g_\epsilon(C)$ will be disconnected. Either case is a contradiction, therefore such an x cannot exist.

Therefore we have that $g_\epsilon^{-1}(x)$ intersects at most one of B_0, B_1 and

$$\operatorname{diam}_{\tilde{\mathbb{C}}}(g_\epsilon^{-1}(x)) < 2\delta$$

for any $x \in [0, 1]$. In either case, we may find a δ-map for arbitrarily small $\delta > 0$ so that X is therefore arclike.

Let $p \in X \cap \overline{\mathcal{S}}$. Then under each of these δ-maps $g_\delta \colon X \to [0, 1]$, then either $g_\delta(p) = 1$ or $g_\delta(p) = 0$. If $g_\delta(p) = 0$, then $1 - g_\delta$ is also a δ-map which maps p to 1. In either case, there exists a δ-map which satisfies Theorem 2.3.9 such that p is a terminal point. ■

Let $\tilde{A} = \tilde{\mathbb{C}} \setminus (\gamma_0 \cup \gamma_1 \cup (0, 1, 1, ...) \cup (1, 1, 1, ...))$. Let h be the infimum of the real part of γ_1, let

$$H_0 := \{z \in \mathbb{C}\colon \ \operatorname{Re}(z) < h\} \cup (\infty),$$

let T^+ be the union of the connected components of $\tilde{A} \setminus H_0$ with unbounded imaginary part and let $T^-\colon = \tilde{A} \setminus \{H_0 \cup T^+\}$.

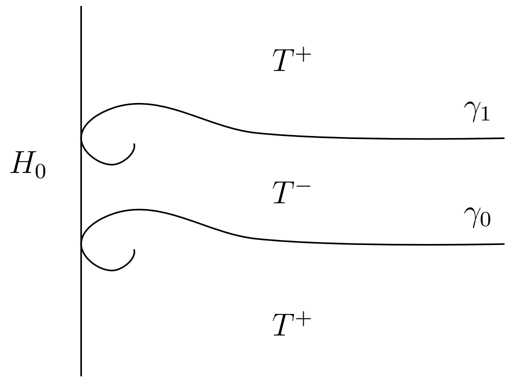

Figure 6.1: A picture of H_0, T^- and T^+ as they lie in $\mathbb{C}$.

We define the map

$$\Gamma\colon \tilde{A} \to [0,1] \qquad z \mapsto \begin{cases} 0 & z \in T^- \setminus \mathbb{C} \\ \frac{1}{2(1+\operatorname{Re}(z)-h)} & z \in T^- \cap \mathbb{C} \\ \frac{1}{2} & z \in H_0 \\ 1 - \frac{1}{2(1+\operatorname{Re}(z)-h)} & z \in T^+ \cap \mathbb{C} \\ 1 & z \in T^+ \setminus \mathbb{C} \end{cases} \tag{6.0.1}$$

Given $\tilde{C}_{\underline{u}}$ for non-singular exponentially bounded $\underline{u}$ and $n \in \mathbb{N}$, we may define the map

$$g_{\underline{u},n}\colon \tilde{C}_{\underline{u}} \to [0,1], \qquad z \mapsto \Gamma \circ \tilde{E}^n(z).$$

It is easy to check that these maps are continuous and non-trivial in the sense that their image in $[0,1]$ contains more than one point. There then must exist linear maps $\tau_{\underline{u},n}\colon \mathbb{R} \to \mathbb{R}$ which map the image of $g_{\underline{u},n}$ to $[0,1]$. We may also choose $\tau_{\underline{u},n}$ in such a way that there exists a point $\underline{s} \in \tilde{C}_{\underline{u}}$ on the circle of addresses for which $\tau_{\underline{u},n} \circ g_{\underline{u},n}(\underline{s}) = 1$ for all $n \in \mathbb{N}$.

To prove Theorem 3.3.2 for non-singular itineraries, it is sufficient then to show the following.

6.0.2. Proposition.
Let $\underline{u}$ be non-singular and exponentially bounded and let j_k be a strictly increasing sequence such that $u_{j_k+2} \neq 1$ for all $k \in \mathbb{N}$. Then there exists some metric on $\tilde{C}_{\underline{u}} \cap \mathbb{C}$ such that the maps

$$g_k := \tau_{\underline{u},j_k} \circ g_{\underline{u},j_k}$$

satisfy the hypothesis of Lemma 6.0.1 for some sequence of $\epsilon_k > 0$ tending to 0. That is, for sufficiently large k, we have that:

- $g_k^{-1}((0,1)) \subset \mathbb{C}$;
- *If $g_k^{-1}(0)$ or $g_k^{-1}(1)$ intersect $\overline{\mathcal{S}}$, they consist of a single point;*
- $\operatorname{diam}(g_k^{-1}(x)) < \epsilon_k$ *for all $x \in [0,1]$ with $g_k^{-1}(x) \subset \mathbb{C}$.*

The first of the conditions in the hypothesis, i.e. $g_k^{-1}((0,1)) \subset \mathbb{C}$, follows immediately from the definition of g_k^{-1}.

The second holds because of the following:

6.0.3. Lemma.
Let $\underline{u}$ be non-singular and exponentially bounded. Then for sufficiently large n, if either $g_{\underline{u},n}^{-1}(0)$ or $g_{\underline{u},n}^{-1}(1)$ intersect $\overline{\mathcal{S}}$, they consist of a single point.

Proof. From our definition of $g_{\underline{u},n}$ and our choice of $\tau_{\underline{u},n}$ and from Theorem 2.2.11, it follows that for non-binary $\underline{u}$, we have that $g_{\underline{u},n}^{-1}(1)$ contains the unique point on $\tilde{C}_{\underline{u}}$ which lies on $\overline{\mathcal{S}}$.

When $\underline{u}$ is binary type and non-singular type, then we may take n to be large enough that $\sigma^n(\underline{u})$ contains only 0s and 1s. Then by Lemma 5.0.4, $g_{\underline{u},n}^{-1}(0)$ and $g_{\underline{u},n}^{-1}(1)$ both contain precisely one point. ∎

It remains to find an appropriate metric on $\tilde{C}_{\underline{u}} \cap \mathbb{C} = C_{\underline{u}}$ such that the final condition holds.

Chapter 7

Expansion on E

Let $A \coloneqq \mathbb{C} \setminus (\overline{\gamma_0} \cup \overline{\gamma_1})$ and $A' \colon = \mathbb{C} \setminus (\{0\} \cup \{2\pi i\})$. In this chapter we define a metric on A whose restriction to $C_{\underline{u}}$ gives us an appropriate metric to prove Proposition 6.0.2. We will define here a metric on A whose distance function we write as $\mathrm{dist}_{A;A'}$. We write it in this way because it is defined on A using the *hyperbolic metric* on A'. This particular metric is chosen because it has certain nice expansion properties with respect to E, which we will prove in this chapter. For any two points $a, b \in C_{\underline{u}}$, we show that

$$\mathrm{dist}_{A;A'}(E(a), E(b)) > \mathrm{dist}_{A;A'}(a, b).$$

Furthermore, for any bounds $r > 0, R \in \mathbb{R}$ such that a, b are at least a distance r from $\{0, 2\pi i\}$ and have real part larger than R, then there exists some $\lambda > 1$ such that $\mathrm{dist}_{A;A'}(E(a), E(b)) > \lambda \, \mathrm{dist}_{A;A'}(a, b)$.

7.1 Hyperbolic Metrics

The hyperbolic metric can be defined as follows.

7.1.1. Definition (Hyperbolic Domains and Metrics).
Let $U \subset \mathbb{C}$ be a connected open subset of $\mathbb{C}$ which omits at least two points

of $\mathbb{C}$. We call such a set a *hyperbolic domain*. Let $\rho_U \colon U \to (0,\infty)$ be an integrable function. For a piecewise smooth path P, we may define the length $l_U(P)$ to be the line integral of ρ_U along P. We may define the distance $\mathrm{dist}_U(a,b)$ between two points $a,b \in U$ to be the infimum of $l_U(P)$ over paths $P \subset U$ with endpoints a and b. This defines a metric on U.

In particular, for the unit disc $\mathbb{D}$, we define the *hyperbolic metric density* to be $\rho_{\mathbb{D}} := \frac{2}{1-|z|^2}$. Let $f \colon \mathbb{D} \to U$ be a holomorphic covering map of U. Then by the following lemma, there is a unique choice of ρ_U such that for all $z \in \mathbb{D}$, we have $|\frac{df}{dz}|\rho_U(f(z)) = \rho_{\mathbb{D}}(z)$. It follows from the Uniformisation Theorem [20][Theorem 1.2.6] that so long as we omit at least two points, the universal covering space of U is conformally isomorphic to $\mathbb{D}$. In other words, there always exists such a covering map from $\mathbb{D}$ to U when U is a hyperbolic domain. There is therefore a choice of ρ_U, induced by some f, which defines a metric for all such U. We will show that this choice of ρ_U is in fact invariant with respect to different covering maps, and ρ_U is therefore unique. We may then say that ρ_U defines the *hyperbolic metric* of U.

7.1.2. Lemma (Uniqueness of the Hyperbolic Metric Density)**.**
Let U be a hyperbolic domain. Let $f_a \colon \mathbb{D} \to U$ and $f_b \colon \mathbb{D} \to U$ be two a holomorphic covering maps. Suppose $x_a, x_b \in \mathbb{D}$ and $z \in U$ are such that $f_a(x_a) = f_b(x_b) = z$. Let $\rho_U^a(z) := \frac{\rho_{\mathbb{D}}(x_a)}{|f_a'(x_a)|}$ and $\rho_U^b(z) := \frac{\rho_{\mathbb{D}}(x_b)}{|f_b'(x_b)|}$. Then $\rho_U^a(z) = \rho_U^a(z)$.

Proof. It is possible to continuously extend a branch of $f_b^{-1} \circ f_a$ to a holomorphic map $\varphi \colon \mathbb{D} \to \mathbb{D}$ in such a way that $\varphi(x_a) = x_b$. This map will be a bijection. We may then define a metric density

$$\rho_{\mathbb{D}}^{\varphi}(x) := \frac{\rho(\varphi^{-1}(x))}{|\varphi'(\varphi^{-1}(x))|}.$$

Note that

$$|\varphi'(\varphi^{-1}(x))| = \frac{|f_a'(\varphi^{-1}(x))|}{|f_b'(x)|}.$$

Now letting $z \in U$ be the point $f_a(\varphi^{-1}(x)) = f_b(x)$, we have that

$$|\varphi'(\varphi^{-1}(x))| = \frac{\rho_U^b(z)}{\rho_U^a(z)} \frac{\rho_{\mathbb{D}}(\varphi^{-1}(x))}{\rho_{\mathbb{D}}(x)}.$$

It follows that

$$\frac{\rho_{\mathbb{D}}^{\varphi}(x)}{\rho_{\mathbb{D}}(x)} = \frac{\rho_U^a(z)}{\rho_U^b(z)}.$$

To prove the lemma, it is therefore sufficient to prove that for every bijective holomorphic map $\varphi\colon \mathbb{D} \to \mathbb{D}$ we have that $\rho_{\mathbb{D}}^{\varphi}(x) = \rho_{\mathbb{D}}(x)$.

It is shown in [11][Theorem 1.8] that the hyperbolic metric is invariant in the sense that $\rho_{\mathbb{D}}^{\varphi}(x) = \rho_{\mathbb{D}}(x)$ for all φ. We can also show this due to the fact, shown in [2][Theorem 13.15], that the only bijective holomorphic self maps of $\mathbb{D}$ are Möbius transformations of the form

$$\varphi(x) = e^{i\vartheta} \frac{x + c}{\bar{c}x + 1}$$

where $\vartheta \in \mathbb{R}, c \in \mathbb{C}$ with $|c| < 1$. Note that $\varphi'(x) = e^{i\vartheta} \frac{1-|c|^2}{(\bar{c}x+1)^2}$ so that

$$\rho_{\mathbb{D}}^{\varphi}(\varphi(x)) = \frac{\frac{2}{1-|x|^2}}{\frac{1-|c|^2}{|\bar{c}x+1|^2}} = \frac{2|\bar{c}x + 1|^2}{(1 - |x|^2)(1 - |c|^2)}.$$

Similarly,

$$\rho_{\mathbb{D}}(\varphi(x)) = \frac{2}{1 - |\frac{x+c}{\bar{c}x+1}|^2} = \frac{2|\bar{c}x + 1|^2}{|\bar{c}x + 1|^2 - |x + c|^2}.$$

We note that

$$|\bar{c}x + 1|^2 - |x + c|^2 = (\bar{c}x + 1)(c\bar{x} + 1) - (x + c)(\bar{x} + \bar{c}) =$$

$$= (\bar{c}c\bar{x}x + 1 + \bar{c}x + c\bar{x}) - (\bar{x}x + \bar{c}c + \bar{c}x + c\bar{x}) =$$

$$= \bar{c}c\bar{x}x + 1 - \bar{x}x - \bar{c}c = |cx|^2 + 1 - |x|^2 - |c|^2 = (1 - |x|^2)(1 - |c|^2).$$

In this way we see that $\rho_{\mathbb{D}}^{\varphi}(\varphi(x)) = \rho_{\mathbb{D}}(\varphi(x))$ and the lemma is proved. ■

7.1.3. Definition (Hyperbolic Expansion)**.**
Let U and V be hyperbolic domains equipped with metric density functions ρ_U and ρ_V respectively. Let F be a function defined on a neighbouhood of some x in U, whose image is in V, and which is differentiable at x. Then we define

$$\|\mathrm{D}F(x)\|_V^U := \frac{F'(x)\rho_V(F(x))}{\rho_U(x)}$$

to be the *hyperbolic expansion* of F at x.

We find that from Pick's Theorem as stated in [11][Theorem 2.11], we may derive the following:

7.1.4. Theorem (Pick Theorem)**.**
Let F be a holomorphic map from hyperbolic domain U to hyperbolic domain V. Let ρ_U, ρ_V be their respective hyperbolic metrics. Then

$$\|DF(x)\|_V^U \geq 1$$

for all $x \in U$. If F is a covering map, then

$$\|DF(x)\|_V^U = 1.$$

If F is not a covering map, then

$$\|DF(x)\|_V^U > 1$$

for all $x \in U$.

In the case where U is contained in V, F is a covering map from U to V, and $x \in U$, we note that

$$\|\mathrm{D}F(x)\|_V^V = \frac{\rho_U(x)}{\rho_V(x)}\|\mathrm{D}F(x)\|_V^U = \frac{\rho_U(x)}{\rho_V(x)}.$$

We will use the following lemma to estimate the hyperbolic metric density of a given hyperbolic domain.

7.1.5. Lemma (Hyperbolic Metric Estimate)**.**
Let $U \in \mathbb{C}$ be a hyperbolic domain and let $z \in U$. Then $\rho_U(z) \leq \frac{2}{\mathrm{dist}(z,\partial U)}$, and if $(z_0, z_1, ...)$ is a sequence of points tending to a point on ∂U, then $\rho_U(z_n)$ tends to infinity.

Proof. Let $z \in U$. Let B be an open ball centred at z with radius $\mathrm{dist}(z, \partial U)$. We note that there is a linear map φ from $\mathbb{D}$ to B with $\varphi(0) = z$ and $\varphi'(0) = \mathrm{dist}(z, \partial U)$. Since we know that $\rho_{\mathbb{D}}(0) = 2$, it follows that $\rho_B(z) = \frac{2}{\mathrm{dist}(z,\partial U)}$. Noting that B is contained in U and letting F be the inclusion of B in U, then by Theorem 7.1.4, we know that $\frac{\rho_B(x)}{\rho_U(x)} \geq 1$. It follows that $\rho_U(z) \leq \frac{2}{\mathrm{dist}(z,\partial U)}$.

Let $(z_0, z_1, ...)$ be a sequence of points tending to a point $a \in \partial U$, let b be a point in $\mathbb{C} \setminus (U \cup \{a\})$, and let $V := \mathbb{C} \setminus \{a \cup b\}$. By [1][Theorem 1.12], we know that $\rho_V(z_n) \to \infty$ as $n \to \infty$. Since $U \subset V$, then by Theorem 7.1.4 again, we know that $\rho_U(z_n)$ also tends to infinity as $n \to \infty$. ■

7.2 Constructing a Metric

Let P be a path in A'. We write $l_{A'}(P)$ as the length of P with respect to $\rho_{A'}$.

We may define a metric distance $\mathrm{dist}_{A;A'} \colon A \times A \to [0, \infty)$ on A in the following way.

7.2.1. Definition.
Let P be a path in A between points $a, b \in A$. We define $\mathrm{dist}_{A;A'}(a, b)$ to be the infimum of $l_{A'}(P')$ for all paths $P' \subset A'$ which are homotopic in A' to P. We note that all paths $P \subset A$ between a and b are in the same homotopy class with respect to A'. Because of this, $\mathrm{dist}_{A;A'}(a, b)$ does not depend on the choice of P.

We define $\mathrm{diam}_{A;A'} \colon \mathcal{P}(A) \to [0, \infty)$ to be the diameter operator with respect to $\mathrm{dist}_{A;A'}$.

We note that we have the following expansion properties with respect to $\rho_{A'}$:

7.2.2. Lemma.
For all $z \in A'' := E^{-1}(A')$,

$$\|DE(z)\|_{A'}^{A'} > 1.$$

Let U be a neighbourhood of $\partial A'$. Then there exists some $\lambda > 1$ such that for all $z \in A'' \setminus U$

$$\|D(z)\|_{A'}^{A'} > \lambda$$

Proof. The first part follows Theorem 7.1.4 and the fact that $E_{2\pi i}$ is of cover from A'' to A'.

Supposing the second part does not hold, then there exists a sequence (z_n) of points in $A'' \setminus U$ for which $\|\mathrm{D}E(z)\|_{A'}^{A'}$ tends to 1. By taking subsequences where necessary we may assume (z_n) converges in $\hat{\mathbb{C}}$.

If (z_n) converges to a point z in A'', then by continuity $\|\mathrm{D}E(z)\|_{A'}^{A'} = 1$ which contradicts the first part.

If (z_n) converges to a point in $\partial A'' \setminus \partial A'$, then $\rho_{A''}(z)$ tends to infinity while $\rho_{A'}(z)$ is bounded above. Therefore $\|\mathrm{D}E(z)\|_{A'}^{A'} = \frac{\rho_{A''}(z)}{\rho_{A'}(z)}$ must tend to infinity.

If (z_n) tends to infinity then for sufficiently large z_n consider some z'_n such that $E(z'_n) = z_n$. Let $A''' := E^{-1}(A'')$. Then by the equivalence $E'(z'_n)\rho_{A'''}(z'_n) = \rho_{A''}(z_n)$ and $E'(z'_n)\rho_{A''}(z'_n) = \rho_{A'}(z_n)$ we have

$$\|\mathrm{D}E(z_n)\|_{A'}^{A'} = \frac{\rho_{A''}(z_n)}{\rho_{A'}(z_n)} = \frac{E'(z'_n)\rho_{A'''}(z'_n)}{E'(z'_n)\rho_{A''}(z'_n)} = \frac{\rho_{A'''}(z'_n)}{\rho_{A''}(z'_n)}.$$

We know that z'_n must have large real part. Therefore by Lemma 7.1.5, $\rho_{A''}(z'_n)$ is small. The set $\partial A'''$ contains all points with imaginary part πik

and real part $\ln(j)$ for $k \in \mathbb{Z}$ and $j \in \mathbb{N}$ with $j > 0$. There must therefore exist, within a bounded distance of z'_n, a pair of points in $\partial A'''$, one of which is a $2\pi i$ translate of the other. There is then a translate of A' which contains A''' and for which z'_n is within a bounded distance of the boundary. We consider the embedding of a translate of A''' into A' that maps z'_n to a point with bounded modulus. That is, the map $f_n \colon A''' \to A', z \to z - (\ln(j) + \pi i k)$ where $\ln(j) + \pi i k$ is within a bounded distance of z'_n. It follows from Theorem 7.1.4 that if $\rho_{A'''}(z'_n)$ is not bounded below away from 0, then neither would $\rho_{A'}(f_n(z'_n))$ be. Either $f_n(z'_n)$ would not tend to $\partial A'$, and there would be an accumulation point $z \in A'$ with $\rho_{A'}(z) = 0$, or else $f_n(z'_n)$ would tend to $\partial A'$ and Lemma 7.1.5 would fail. In either case we have a contradiction. Therefore, $\|\mathrm{D}E(z_n)\|_{A'}^{A'}$ tends to infinity when $|z_n|$ tends to infinity. ■

The metric $\mathrm{dist}_{A;A'}$ is the metric on $C_{\underline{u}} \cap \mathbb{C}$ we will use to complete the proof of Proposition 6.0.2. Specifically, it suffices to prove the following two propositions:

7.2.3. Proposition.
There exists $K > 0$ such that for all non-singular exponentially bounded itineraries $\underline{u}$, there are infinitely many $n \in \mathbb{N}$ such that for all $x \in [0,1]$ with $g^{-1}_{\sigma^n(\underline{u}),0}(x) \subset \mathbb{C}$, we have

$$\mathrm{diam}_{A;A'}(g^{-1}_{\sigma^n(\underline{u}),0}(x)) < K.$$

7.2.4. Proposition.
For all non-singular itineraries $\underline{u}$ and $\Lambda > 1$, there exists some $m \in \mathbb{N}$ such that for all $a, b \in C_{\underline{u}}$ and $n > m$,

$$\mathrm{dist}_{A;A'}(E^n(a), E^n(b)) > \Lambda\, \mathrm{dist}_{A;A'}(a,b).$$

To see that these two propositions imply Proposition 6.0.2, let $\epsilon > 0$. We may choose Λ sufficiently large that $\frac{K}{\Lambda} < \epsilon$. Since $E^n(g^{-1}_{\underline{u},n}(x)) = g^{-1}_{\sigma^n(\underline{u}),0}(x)$, there exists some n which satisfies satisfies both Proposition 7.2.3 and 7.2.4. Then

$$\operatorname{diam}_{A;A'}(g_{\underline{u},n}^{-1}(x)) < \Lambda \operatorname{diam}_{A;A'}(g_{\sigma^n(\underline{u}),0}^{-1}(x)) < \Lambda K < \epsilon$$

holds whenever $g_{\sigma^n(\underline{u}),0}^{-1}(x) \subset \mathbb{C}$. Proving Proposition 7.2.3 and 7.2.4 will therefore complete the proof of Proposition 6.0.2 as required.

7.3 Geometric Bounds

7.3.1. Definition.

For $R > 1$, let $A_R := \{z \in \mathbb{C}\colon \operatorname{Re}(z) > -R, |z| > \frac{1}{R}, |z - 2\pi i| > \frac{1}{R}\}$. We note that if U is a neighbourhood of $2\pi i$, then there is some $R > 1$ such that $E^2(\mathbb{C} \setminus A_R) \subset U$. From here on, when we use A_R, we will set R to be large enough that $E^2(\mathbb{C} \setminus A_R) \subset T_1$, where we recall from Definition 2.2.8 that T_1 is the strip containing $2\pi i$.

7.3.2. Lemma.

There is some $R > 1$ such that, for the set A_R, the following holds. For all non-singular exponentially bounded itineraries $\underline{u}$, there are infinitely many $n \in \mathbb{N}$ such that

$$E^n(C_{\underline{u}}) \subset A_R.$$

Proof. We can choose n such that $u_{n+2} \neq 1$. Since we choose R such that $E^2(\mathbb{C} \setminus A_R) \subset T_1$, we have that $C_{\sigma^n(\underline{u})}$ must be contained entirely in A_R. ■

7.3.3. Lemma.

There exists a linear function of the form $M\colon x \mapsto C + Dx$ with $C, D \in \mathbb{R}, a > 0$ such that the following holds.

Let $u \in \mathbb{N}$. Let S be one of the sets $T_u \cap T^- \cap A_R, T_u \cap T^+ \cap A_R$ or $T_u \cap H_0 \cap A_R$. If $a, b \in S$, then $\operatorname{dist}_{A;A'}(a, b) < M(|a - b|)$.

Proof. We note that each choice of S has precisely one unbounded component. Since each choice of S is at least some positive distance from $\partial A'$, it follows that $\rho_{A'}$ is bounded above on S by some upper bound independent of the choice of S.

For sufficiently large K, let $S^+ := \{z \in S\colon |z| \geq K\}$. We may choose K large enough that the convex hull of S^+ does not intersect $\partial A'$. We may also choose K such that S^+ is contained in a connected component S^0 of $\{z \in S\colon \operatorname{Re}(z) \neq 0\}$. The union of S^0 and the convex hull of S^+ is a simply connected subset of A'. Therefore the straight line between any two points in $x, y \in S^+$ is homotopic to a path between x and y in S^0. This path is also a path in A. We therefore find that $\operatorname{dist}_{A;A'}(x, y)$ is bounded above by $\sup_{z \in A_R}(\rho_{A'}(z))|x - y|$ for $x, y \in S^+$.

It remains to show that the diameter of $S^- := \{z \in S\colon |z| < K\}$ is bounded. For this, it is sufficient to show that $S^* := \{z \in A_R \cap A\colon |z| < K\}$ has finite diameter with respect to $\operatorname{dist}_{A;A'}$. To do this, we show first that there is some neighbourhood of $(\partial A) \cap S^*$ with finite diameter.

For $j \in \{0, 1\}$, let δ_j^- be the least value such that $\gamma_j(\delta_j^-)$ intersects ∂S^* and let δ_j^+ be the greatest value such that $\gamma_j(\delta_j^-)$ intersects ∂S^*. Then let $\gamma_j^* := \gamma_j((\delta_j^-, \delta_j^+))$.

We now claim that γ_j^* has the following property. There exists a simply connected neighbourhood $U_j \subset A'$ of γ_j^* and some bound M such that between any two points in U, there exists a path $P \subset U$ between them with $l_{A'}(P) < M$.

This can be shown by the fact that for some sufficiently large n, the path $E^n(\gamma_j^*)$ will be close to some section of the asymptote of γ_1. We find that the above property then holds for $E^n(\gamma_j^*)$. There exists some bounded simply connected neighbourhood U_n around $E^n(\gamma_j^*)$ such that between every pair of points in U_n, there is a path in U_n of bounded length (we may take a rectangle around the appropriate section of the asymptote for example). This neighbourhood satisfies the above property.

Taking the appropriate branch of E^{-n} which maps U_n to a neighbourhood U_j of γ_j^*, we have that since $\|\mathrm{D}E^{-n}(z)\|_{A'}^{A'}$ is bounded for $z \in U_n$, then the above property holds for U_j and some M.

Let $\epsilon > 0$ be sufficiently small that $U_{j,\epsilon} \coloneqq \{z \in A \colon \operatorname{dist}_{A;A'}(z, \gamma_S) < \epsilon\}$ contains two components and is contained within U. Then each component of $U_{j,\epsilon}$ has bounded diameter with respect to $\operatorname{dist}_{A;A'}$. This follows from the fact that all points in a component can be connected by a path $P \subset U_{j,\epsilon} \subset A$ and a path $P' \subset U_j$ of finite length. These paths will be homotopic to one another since they are both contained in a simply connected subset of A'.

To show that the diameter of $U_{j,\epsilon}$ is finite, we note that for every $z \in \gamma_j^*$, there exists a path J in A starting and ending at z which is a Jordan curve, the interior of which contains one of 0 or $2\pi i$. The path J must intersect both components of $U_{j,\epsilon}$, and the homotopy class of J in A' will contain a path of finite length. Similarly, by taking some vertical line of euclidean length at most 2π, there exists a path of finite length connecting $U_{0,\epsilon}$ to $U_{1,\epsilon}$. The set $U_{0,\epsilon} \cap U_{1,\epsilon}$ therefore has finite diameter with respect to $\operatorname{dist}_{A;A'}$.

Since every point in S^* can by connected to $\gamma_0^* \cap \gamma_1^*$ by a path in S^* of bounded length, it follows that the diameter of S^* is also finite.

Therefore for $x, y \in S$, we have that

$$\operatorname{dist}_{A;A'}(x, y) \leq \operatorname{diam}_{A,A'}(S^*) + \sup_{z \in A_R} (\rho_{A'}(z)) \operatorname{dist}(x, y)$$

as required. ■

Proof of Proposition 7.2.3. Let $\underline{u}$ be an exponentially bounded non-singular itinerary. There exist infinitely many $n \in \mathbb{N}$ such that $u_{n+2} \neq 1$. For such an n, we have that $E^n(C_{\underline{u}}) \in T_{u_n} \cap A_R \cap A$. We can see from the geometry of the strip that the euclidean diameter of $\Gamma^{-1}(x) \cap T_u \cap A_R \cap A$ is bounded for all $x \in [0, 1]$ and $u \in \mathbb{Z}$. By the above lemma, this gives us an upper bound $K > 0$ such that $\operatorname{diam}_{A;A'}(\Gamma^{-1}(x) \cap T_u \cap A_R \cap A) < K$ for all $x \in [0, 1]$.

We know that $g^{-1}_{\sigma^n(\underline{u}),0}(x) \subset \Gamma^{-1}(\tau^{-1}_{\sigma^n(\underline{u}),0}(x)) \cap T_u \cap A_R \cap A$. Therefore K is an upper bound such that for all $x \in [0,1]$, we have

$$\operatorname{diam}_{A;A'}(g^{-1}_{\sigma^n(\underline{u}),0}(x)) < K.$$

■

7.4 General Expansion Properties

7.4.1. Lemma (Expansion on $\operatorname{dist}_{A;A'}$)**.**
Let a and b be on the interior of the same strip T_j, then

$$\operatorname{dist}_{A;A'}(E(a), E(b)) > \operatorname{dist}_{A;A'}(a, b).$$

Additionally, for all $R > 0$, there exists some $\lambda = \lambda(R) > 1$ such that if we also have that $a, b \in A_R$, then

$$\operatorname{dist}_{A;A'}(E(a), E(b)) > \lambda \operatorname{dist}_{A;A'}(a, b).$$

Proof. We begin by fixing points a_0, b_0 on the interior of some T_j, letting $a_1 := E(a_0), b_1 := E(b_0)$ and fixing P to be a path contained in A between a_1 and b_1.

Let P' be any path in A' homotopic in A' to P. Then there exist a paths Q, Q' in A' such that the following hold:

- Q and Q' have endpoints at a_0, b_0;
- Q and Q' are homotopic in A';
- $E(Q) = P, E(Q') = P'$.

Q is guaranteed to exist since there is a branch of E^{-1} which maps A continuously into $T_j \cap A$, mapping P onto Q and a_0, b_0 onto a_1, b_1 respectively. Q' can be seen to exist since the homotopy between P and P' can be pulled back continuously along some branch of E^{-1} to a homotopy between Q and Q'.

If there is some λ such that for all $z \in Q'$, we have

$$\|\mathrm{D}E(z)\|_{A'}^{A'} > \lambda;$$

it follows then that

$$l_{A'}(P') > \lambda l_{A'}(Q').$$

This would immediately imply that $\mathrm{dist}_{A;A'}(a_1, b_1) > \lambda \,\mathrm{dist}_{A;A'}(a_0, b_0)$. By the first part of Lemma 7.2.2, we know that there exists some such $\lambda \geq 1$ which proves the first part of our lemma.

To prove the second part of the lemma, we use the following lemma, which we will prove later.

7.4.2. Lemma.
Let U_0, U_1 be disjoint closed neighbourhoods of $0, 2\pi i$ respectively. Then there exist neighbourhoods V_0, V_1 of $0, 2\pi i$ such that if P is a path in A with endpoints in $A \setminus (U_0 \cup U_1)$ and P' a path in A' homotopic to P, then there exists some $P'' \subset \mathbb{C} \setminus (V_0 \cup V_1)$ homotopic to P in A' with $l_{A'}(P'') \leq l_{A'}(P')$.

Let $U_0 \cup U_1 \subset E(A_R)$ and let V_0, V_1 satisfy the above lemma. For each pair P', Q' as before, there exists an associated pair P'', Q'' subject to the same properties as P', Q', but with $l_{A'}(P'') \leq l_{A'}(P')$ and $P'' \subset \mathbb{C} \setminus (V_0 \cup V_1)$ so that $Q'' \subset E^{-1}(\mathbb{C} \setminus (V_0 \cup V_1))$. There exists some fixed $\lambda > 1$ so that for $z \in E^{-1}(\mathbb{C} \setminus (V_0 \cup V_1))$,

$$\|\mathrm{D}E(z)\|_{A'}^{A'} > \lambda.$$

Therefore $l_{A'}(P') \geq l_{A'}(P'') > \lambda l_{A'}(Q'')$ and $\mathrm{dist}_{A;A'}(a_1, b_1) > \lambda \,\mathrm{dist}_{A;A'}(a_0, b_0)$ when $a_0, b_0 \in A_R$. ■

We will fix R so that it satisfies Definition 7.3.1, and therefore also satisfies Lemma 7.3.2. This also fixes $\lambda = \lambda(R)$.

Proof of Lemma 7.4.2. I wish to show that there exists some bound $M \in \mathbb{R}$ such that for any two points $a, b \in A \setminus (U_0 \cup U_1)$ with a path $P \subset A$ between them, there exists a path $P^* \subset A'$ homotopic to P with respect to A' such that

$$l_{A'}(P') < \text{dist}_{A'}(a, (U_0 \cup U_1)) + \text{dist}_{A'}(b, (U_0 \cup U_1)) + M.$$

That is to say

$$\text{dist}_{A;A'} < \text{dist}_{A'}(a, (U_0 \cup U_1)) + \text{dist}_{A'}(b, (U_0 \cup U_1)) + M. \qquad (7.4.1)$$

From such an inequality, it follows that if we define V_0, V_1 such that

$$\text{dist}_{A'}((\partial U_0 \cup \partial U_1), (V_0 \cup V_1)) > M,$$

then any P' which intersects $V_0 \cup V_1$ must have length greater than that of P^*, so we set $P'' := P^*$. Otherwise, when P' does not intersect $V_0 \cup V_1$ we set $P'' := P'$. Either way our lemma is satisfied.

It remains to prove the inequality 7.4.1 holds. Let R be sufficiently large that the ball B_R contains $U_0 \cup U_1$, and such that $\gamma_0 \setminus B_R$ and $\gamma_1 \setminus B_R$ are both contained in the half plane $H^+ := \{z \in \mathbb{C} \colon \text{Re}(z) \geq 0\}$. It is sufficient to show the following conditions hold:

- Let $Q := (A \cup B_R) \setminus (U_0 \cup U_1)$. Then $\text{diam}_{A;A'}(Q)$ is finite;
- If $z \in A \setminus B_R$, then $dist_{A;A'}(z, B_R) - \text{dist}_{A'}(z, B_R)$ is bounded.

We note that for any simply connected set $U \subset A'$, given $a, b \in U$, then all paths contained in U from a to b are homotopic with respect to A'. For all $z \in A'$, there is some $\delta > 0$ such that all balls (measured by $\text{dist}_{A'}$) centered at z with radius less than or equal to δ are simply connected. Let U_z be a path connected neighbourhood of z contained in a ball of radius $\frac{\delta}{2}$. The shortest path between any two points in U_z is contained within a ball of radius δ, and is therefore be homotopic to any other path contained in U_z with the same endpoints.

It follows that if we have that $U_z \subset A$, then $\operatorname{dist}_{A'}$ and $\operatorname{dist} A; A'$ are identical when restricted to U_z and $\operatorname{diam}_{A;A'}$ is finite. If $z \in \partial A \setminus \{0, 2\pi i\}$, then since the rays in ∂A do not intersect, $U_z \cap A$ has precisely two connected components containing z on their boundary. We may further restrict U_z to be open and have the property that these are the only components of $U_z \cap A$. When we restrict the metrics $\operatorname{dist}_{A'}$ and $\operatorname{dist}_{A;A'}$ to either of these components, we find they are identical. Since $\operatorname{diam}_{A'}$ is finite for either of these components, it follows that $\operatorname{diam}_{A;A'}$ is also finite.

To show the first condition holds, let γ_0^* be the smallest subpath of γ_0 containing $\gamma_0 \cap Q$, let γ_1^* be defined similarly. Taking a cover of $(\gamma_0^* \cup \gamma_1^*)$ by such U_z, then by compactness, there exists a finite subcover of such neighbourhoods. Let W be the union of this subcover. $W \cap A$ is therefore composed of at most four connected components and has a finite cover of open sets with finite $\operatorname{diam}_{A;A'}$. There is similarly a finite cover of $Q \cup (W \cap A)$ by sets with finite $\operatorname{diam}_{A;A'}$. It can be shown that $Q \cup (W \cap A)$ is connected, therefore it and Q have finite $\operatorname{diam}_{A;A'}$.

To show the second condition holds, recall from earlier in the proof that $H^+ := \{z \in \mathbb{C} \colon \operatorname{Re}(z) \geq 0\}$ and let $H^- := \{z \in \mathbb{C} \colon \operatorname{Re}(z) \leq 0\}$. For $j \in \{-, +\}$ and $z \in A \cap H^j \setminus B_R$, since the metric $\rho_{A'}$ and the boundary ∂B_R are symmetric between H^- and H^+, then the shortest path Γ_a from z to ∂B_R may be chosen such that it lies entirely within H^j. Similarly, a path Γ_b in A from z to ∂B_R may also be chosen to lie entirely within H^j. Let p_a and p_b be the endpoints of Γ_a and Γ_b on B_R respectively. We may extend Γ_a by adding the section of ∂B_R between p_a and p_b to form a new path Γ. It is easy to see that Γ is homotopic to Γ_b, and since the length of ∂B_R is bounded, then $l_{A'}(\Gamma) - \operatorname{dist}_{A'}(z, B_R) \geq \operatorname{dist}_{A;A'}(z, B_R)$ is also bounded. ∎

Proof of Proposition 7.2.4. Let $\Lambda > 1$ be as in Proposition 7.2.4 and let λ satisfy the expansion properties in Lemma 7.4.1 for A_R. Let $\lambda_m >$ be the lower bound of expansion as described in Lemma 7.4.1 for E on $E^m(C_{\underline{u}})$. That is, for all $a, b \in E^m(C_{\underline{u}})$, then $\operatorname{dist}_{A;A'}(E(a), E(b)) > \lambda_m \operatorname{dist}_{A;A'}(a, b)$. When $E^m(C_{\underline{u}})$ is contained in A_R, we may set $\lambda_m = \lambda$, otherwise we may set $\lambda_m = 1$.

The expansion of E^n on $C_{\underline{u}}$ is then $\prod_{m=0}^{n-1} \lambda_m$. By Lemma 7.3.2, there are infinitely many m such that $E^m(C_{\underline{u}}) \subset A_R$. Choosing n large enough that $E^m(C_{\underline{u}}) \subset A_R$ for k different values of $m < n$, then we have that $\operatorname{dist}_{A;A'}(E^n(a), E^n(b)) > \Lambda_m \operatorname{dist}_{A;A'}(a, b)$ for all $a, b \in C_{\underline{u}}$ as required. ∎

This completes the proof of Proposition 6.0.2.

Chapter 8

Density of Escape

In this section we show that the itinerary sets $C_{\underline{u}}$ for any exponentially bounded $\underline{u}$ are precisely the closure of the rays contained within them. We may use this fact to show that when $\underline{u}$ is singular, then $\tilde{C}_{\underline{u}}$ is precisely an arc, completing our proof of Theorem 3.3.2.

8.0.1. Theorem.
For any exponentially bounded $\underline{u}$, then $I(E) \cap C_{\underline{u}}$ is dense in $C_{\underline{u}}$.

Proof. We aim to prove this by showing that for infinitely many n, there exists some escaping subset of $E^n(C_{\underline{u}})$ which is a bounded distance away from every other point in $E^n(C_{\underline{u}})$, and there is sufficient expansion in E^n that escaping points must be arbitrarily close to all points of $C_{\underline{u}}$ when we consider n to be arbitrarily large. The existence of such escaping subsets is guaranteed by the following lemma, which we will prove at the end of this chapter.

8.0.2. Lemma.
There exists some $R_0 > 0$ such that the following holds for any exponentially bounded $\underline{u}$. Let X be a connected subset of $C_{\underline{u}}$ such that the closure $\overline{X}$ contains a point of $\mathcal{S}_0$, and $\min(\mathrm{Re}(X)) > \max(0, \min(\mathrm{Re}(C_{\underline{u}}))) + R_0$. Then $X \subset I(E)$.

If $\underline{u}$ is singular, then it suffices to show that the escaping set is dense in

$C_{(1,1,\ldots)}$. Within a euclidean ball B of radius $r < -\ln(\frac{1}{2\pi})$ around the point 2π, the map E is strictly expanding with respect to the euclidean metric. Now since $C_{(1,1,\ldots)} \subset E^{-1}(\mathbb{C} \setminus T_0)$, then it follows that

$$C_{(1,1,\ldots)} \subset T' := \{z \in T_1 \colon \operatorname{Re}(z) > -K\}$$

for some $K > 0$. Let L_1 be the branch of E^{-1} which maps $\mathbb{C} \setminus \overline{\gamma_0}$ into T_1. We let K be sufficiently large that $L_1(T') \subset L_1(T_1) \subset T'$.

We may then define a metric density $\rho_{T'}$ on $\mathbb{C} \setminus \overline{\gamma_0}$ which is equal to $\rho_{A'}$ outside of the ball B, and is a constant $h < \inf(\rho_{A'}(B \setminus L_1(B)))$ inside the ball B. By Lemma 7.2.2, there exists some $\lambda > 1$ such that if p is a path in $\mathbb{C} \setminus (\overline{\gamma_1} \cup B)$ whose length with respect to $\rho_{A'}$ is l, then the length of $p' := L_1(p)$ with respect to $\rho_{A'}$ is less than $\frac{l}{\lambda}$. The length of p' with respect to $\rho_{T'}$ is therefore also less than $\frac{l}{\lambda}$. Similarly, the preimage of any path in B also strictly shrinks the length with respect to $\rho_{T'}$ by at least a facto for some constant $\lambda > 1$. There is then some $\lambda > 1$ such that for any path p in $\mathbb{C} \setminus \overline{\gamma_1}$ with length l with respect to $\rho_{T'}$, then $L_1(p)$ has length less than $\frac{l}{\lambda}$.

By the above lemma, there exists some X, a component of the escaping set in $C_{(1,1,\ldots)}$, which is connected to infinity. For any $z \in C_{(1,1,\ldots)}$ and $n \in \mathbb{N}$, there is a path P_n in $\mathbb{C} \setminus \overline{\gamma_0}$ of bounded length connecting $E^n(z)$ to X. We can see this is the case because both $E^n(z)$ and X lie in T', and all but a bounded part of T' lies above γ_0 in some sense. To be more precise, we mean that for sufficiently large r, we have that when $x \in T'$ and $y \in \gamma_0$ with $\operatorname{Re}(x) = \operatorname{Re}(y)$, then $\operatorname{Im}(x) > \operatorname{Im}(y)$. Because of this, when $\operatorname{Re}(E^n(z))$ is sufficiently large, we may connect it to X by some straight vertical path which is contained in $\mathbb{C} \setminus \gamma_0$ and whose length is bounded both in the euclidean metric and with respect to $\rho_{T'}$. If $\operatorname{Re}(E^n(z))$ is not sufficiently large, then it lies in a bounded part of T' and the distance to X is also bounded. We find that $L_n^n(P_n)$ connects z to $L_1^n(X)$. Then as n increases, $L_1^n(P_n)$ becomes arbitrarily small. We note that the points in $L_1^n(X)$ are escaping points of $C_{(1,1,\ldots)}$ and so the escaping set is dense in $C_{(1,1,\ldots)}$.

For all other singular itineraries $\underline{u}$, we find that there is some branch of $\tilde{E}^{-n}$ which maps $\tilde{C}_{(1,1,\ldots)}$ onto $\tilde{C}_{\underline{u}}$ homeomorphically. Escaping points in $\tilde{C}_{(1,1,\ldots)}$ are mapped by this branch onto escaping points of $\tilde{C}_{\underline{u}}$. Since escaping points are dense in $\tilde{C}_{(1,1,\ldots)}$, they are also dense in $\tilde{C}_{\underline{u}}$ and $C_{\underline{u}}$.

For $E^n(C_{\underline{u}})$, let X_n be the union of the connected components of the set $\{z \in E^n(C_{\underline{u}})\colon z \geq \max(0, \min(\mathrm{Re}(E^n(C_{\underline{u}})))) + R_0\}$ which contain a forward tail of some dynamic ray. By the above lemma, X_n is escaping for each n. By the boundary bumping theorem, it is possible to show that the minimum real part of each component of X_n is $\max(0, \min(\mathrm{Re}(C_{\underline{u}}))) + R_0$.

If $\underline{u}$ is non-singular and binary, then whenever $u_{n+2} \neq 1$, then X_n is a bounded distance (with respect to $\mathrm{dist}_{A;A'}$) from all points in $E^n(C_{\underline{u}})$. For points with real value less than R_0, this follows from the fact that the set $\{z \in E^n(C_{\underline{u}})\colon \mathrm{Re}(z) \leq R_0\}$ has a bounded diameter and shares a boundary with X_n. For points z with real value greater than or equal to R_0, we know that $z \in g^{-1}_{\sigma^n(\underline{u}),0}(x)$ for some $x \in (0,1)$. Furthermore, $g^{-1}_{\sigma^n(\underline{u}),0}(x)$ contains a point of X_n (since the connected components of $g_{\sigma^n(\underline{u}),0}(X_n)$ accumulate on both 0 and 1) and as we have shown, the $\mathrm{diam}_{A;A'}(g^{-1}_{\sigma^n(\underline{u}),0}(x))$ is bounded.

If $\underline{u}$ is non-binary, let n be some point such that $u_n \neq 0, 1$ and let $m \geq n$ be some point such that $u_{m+2} \neq 1$. Let $X_{n,m}$ be the branch of $E^{n-m}(X_m)$ contained in $E^n(C_{\underline{u}})$. Then the distance from a point $E^n(C_{\underline{u}})$ to $X_{n,m}$ is bounded. As before, $\{z \in E^m(C_{\underline{u}})\colon \mathrm{Re}(z) \leq \min(\mathrm{Re}(X_m))\}$ has a bounded diameter and shares a boundary with X_m. For points $z_m \in E^m(C_{\underline{u}})$ with $\mathrm{Re}(z_m) \geq \min(\mathrm{Re}(X_m))$, it may be the case that z_m and X_m lie in different components of T^+, T^- so that $\mathrm{dist}_{A;A'}(z_m, X_m)$ is arbitrarily large. However, $\mathrm{dist}_{A'}(z_m, X_m)$ is bounded. This is because $E^m(C_{\underline{u}}) \subset A_R$. We can use the following lemma to show that for the corresponding point $z_n \in E^n(C_{\underline{u}})$ such that $E^{m-n}(z_n) = z_m$, we have that $\mathrm{dist}_{A;A'}(z_n, X_n)$ is bounded. We give a proof of this lemma after concluding the proof of this theorem.

8.0.3. Lemma.
There exists some bound M such that when $a_1, b_1 \in A_R$, then the following

holds: If $E(a_0) = a_1, E(b_0) = b_1$ and a_0, b_0 are in the same strip T_u, then

$$\operatorname{dist}_{A'}(a_0, b_0) < \max(\operatorname{dist}_{A'}(a_1, b_1), M).$$

We may set M so that it is also an upper bound of $\operatorname{dist}_{A'}(z_m, X_m)$. Let $x_m \in X_m$ be such that $\operatorname{dist}_{A'}(z_m, x_m) < M$. For $j < m$, let $x_j \in E^j(C_{\underline{u}})$ be such that $E^{m-j}(x_j) = x_m$. Let $k \in \mathbb{N}$ be such that $n \leq k \leq m$ and let k also be the largest such value that either u_k is neither 0 nor 1, or else one of x_k, z_k is not in A_R. Then by the above lemma, $\operatorname{dist}_{A'}(x_k, z_k)$ is bounded above by M. We can show this bound M also guarantees a bound for $\operatorname{dist}_{A;A'}(x_k, z_k)$.

Suppose, without loss of generality, that x_k is not in A_R. Let Q be the set of all points in A with $\operatorname{dist}_{A'}$ less than M from $E^{-1}(A_R) \setminus A_R$. Then both x_k and z_k lie in $Q \cap T_{u_k}$. It follows that $\operatorname{diam}_{A;A'}(Q \cap T_{u_k})$ gives a bound to $\operatorname{dist}_{A;A'}(x_k, z_k)$. We may note that $\operatorname{diam}_{A;A'}(Q \cap T_{u_k})$ is finite since $Q \cap T_{u_k}$ is a bounded set in A outside of a neighbourhood of $\{0, 2\pi i\}$.

Suppose, in the other case, we have that u_k is neither 0 nor 1. Then for all $z_k \in T_{u_k}$ we would have that $\operatorname{diam}_{A;A'}(\{z \in T_{u_k} \colon \operatorname{dist}_{A'}(z_k, z) < M\})$ is bounded. This also gives us a bound for $\operatorname{dist}_{A;A'}(x_k, z_k)$. By the expansion with respect to $\operatorname{dist}_{A;A'}$, this further guarantees a bound for $\operatorname{dist}_{A;A'}(x_n, z_n)$. We find similar bounds for points in $\{z \in E^m(C_{\underline{u}}) \colon \operatorname{Re}(z) \leq \min(\operatorname{Re}(X_m))\}$ since this set has bounded diameter with respect to $\operatorname{dist}_{A;A'}$.

It follows then that $X_{n,m}$ is a bounded distance from all points in $E^n(C_{\underline{u}})$ with respect to $\operatorname{dist}_{A;A'}$. Using the again expansion given by Proposition 7.2.4, we may then find escaping points arbitrarily close to any point z in $C_{\underline{u}}$ by pulling back a point in x_n in X_n or $X_{n,m}$ which is a bounded distance from $E^n(z)$. When n increases, $\operatorname{dist}_{A;A'}(z, E^{-n}(x_n) \cap C_{\underline{u}})$ tends to 0 as required. ■

Proof of Lemma 8.0.3. Let a_0 b_0 be in the same strip T_u and let $a_1 := E(a_0)$ and let $b_1 := E(b_0)$ such that $a_1, b_1 \in A_R$. We use from the expansion in Lemma 7.2.2. Let the set X be the union of all the shortest paths between pairs of points in A_R. In a similar way to Lemma 7.4.2 we can see that X lies outside a neighbourhood of $\partial A'$. To be more precise, we may define smooth Jordan curves contained in $\mathbb{C} \setminus A_R$ whose interiors contain $\{0, 2\pi i\}$ and whose collective length is finite. Let J be the union of the image of these curves and let $|J|$ be their interior. Let L be their collective length with respect to $\rho_{A'}$. Let X' be the set of all points $z \in A'$ such that $\mathrm{dist}_{A'}(z, \mathbb{C} \setminus |J|) < 2L$. Then for any path p between two points in A_R such that p is not contained in X', there exists a shorter path p' which is contained in X' and is achieved by replacing appropriate sections of $p \cap |J|$ with sections of ∂J which are at most half as long. By construction, X' lies outside a neighbourhood of $\partial A'$. Therefore the same can be said of X and $E^{-1}(X)$. Let $\lambda > 1$ then be such that $\|\mathrm{D}E(z)\|_{A'}^{A'} > \lambda$ for all $z \in E^{-1}(X)$.

Now let p_1 be the shortest path between a_1 and b_1. Let p_0 be the continuous branch of $E^{-1}(p_1)$ which has an endpoint at b_0. Let its other endpoint be a_0'. We will then have that $p_0 \subset E^{-1}(X)$ and so $\mathrm{dist}_{A'}(a_0', b_0) < \frac{1}{\lambda} \mathrm{dist}_{A'}(a_1, b_1)$. We note that p_1 will not wind around 0 very often (We note that $\rho_{A'}$ is symmetrical about the axis $I := \{z \in \mathbb{C} \colon \mathrm{Re}(z) = 0\}$. If p_1 wound around 0 at least twice then there would be guaranteed to be subpaths of p_1 with endpoints in I and contained entirely to the left of I. We may replace all these with their reflection in I to create some path p_1' with the same length as p_1 and which wound around 0 less than twice). Because the winding of p_1 is bounded, $|\mathrm{Im}(a_0') - \mathrm{Im}(b_0)|$ will also be bounded. This a_0' is some $2\pi i k$ vertical translate of a_0. We note that since $a_0, b_0 \in T_u \cap E^{-1}(A_R)$, which is contained in some horizontal strip of bound vertical height, we have that $|\mathrm{Im}(a_0') - \mathrm{Im}(b_0)|$ will also be bounded, as will $|\mathrm{Im}(a_0') - \mathrm{Im}(a_0)|$.

Since a_0, a_0' are both in $E^{-1}(A_R)$, then $\mathrm{dist}_{A'}(a_0, a_0')$ is bounded by some constant H. Choosing M large enough that $\lambda M + H < M$, we find that when $\mathrm{dist}_{A'}(a_1, b_1) > M$, then $\mathrm{dist}_{A'}(a_0, b_0) < \lambda \,\mathrm{dist}_{A'}(a_1, b_1) + H < \mathrm{dist}_{A'}(a_1, b_1)$, and otherwise $\mathrm{dist}_{A'}(a_0, b_0) < \lambda M + H < M$. ■

8.1 Bounds on Spiralling

In order to prove Lemma 8.0.2, we must first give a description of the geometry of the strips T_u.

Using the following following lemma gives us control over the spiralling of γ_1, and therefore, equivalently, γ_0 and the behaviour of t_u for large negative real values.

8.1.1. Lemma (Bounded Spiralling near 0)**.**
Let h be an entire function such that there exists a continuous injective ray $\gamma\colon (0,\infty) \to \mathbb{C}$ landing at 0 for which h acts as a bijection on $\gamma((0,\infty))$. Let $h'(0) = \lambda$ with $|\lambda| \neq 0, 1$.

By logarithmic change of variables, there exists some ray $t\colon (0,\infty) \to \mathbb{C}$ such that $\gamma = e^t$. There then exist bounds $m_0, c^+, c^- \in \mathbb{R}$ such that for all $z \in t((0,1))$, we have

$$m_0 \operatorname{Re}(z) + c^- < \operatorname{Im}(z) < m_0 \operatorname{Re}(z) + c^+.$$

Proof. Let $h\colon \mathbb{C} \to \mathbb{C}$ be an entire function with $h(0) = 0, h'(0) = \lambda$ such that $|\lambda|$ is neither 0 nor 1. Then by Kœnigs Linearisation Theorem 2.2.1, in some neighbourhood U of 0, we have that h is conjugate to multiplication by λ. By this, we mean there is some holomorphic injective function $\varphi\colon U \to \mathbb{C}$ such that $\varphi \circ h|_U = \lambda\varphi$ and $\varphi'(0) = 1$. Since φ is holomorphic, there will exist some $\alpha > 0$ such that $|z - \varphi(z)| < \alpha|z|^2$ for all $z \in U$.

Let γ' be a ray landing at 0 such that $\lambda\gamma' = \gamma'$ and let t' be a ray such that $e^{t'} = \gamma'$. Taking some $p_0 \in t'$, let p_n be the point on t' such that $e^{p_n} = \lambda^n e^{p_0}$ let P_n be the path in t' from p_n to p_{n+1}. P_{n+1} will then be a translation of P_n by a factor of $p_{n+1} - p_n$, and therefore $P_n = P_0 n(p_1 - p_0)$ where $|\operatorname{Re}(p_1 - p_0)| = |\ln(|\lambda|)| > 0$. Given this, we can then find some $c^+, c^- \in \mathbb{R}$ such that P_0 lies between the diagonal lines

$$D^- := \{ic^- + x(p_1 - p_0)\colon x \in \mathbb{R}\}, D^+ := \{ic^+ + x(p_1 - p_0)\colon x \in \mathbb{R}\}$$

and therefore P_n lies between D^- and D^+ for all $n \in \mathbb{N}$.

Now supposing there exists γ as described in the hypothesis. Then there must also exist γ' as described such that $\varphi(\gamma \cap U) \subset \gamma'$. We can construct this as $\gamma' := \bigcup_{k \in \mathbb{Z}} \lambda^k \varphi(\gamma \cap U)$. Since we know that either $\varphi(\gamma \cap U) \subset \lambda\varphi(\gamma \cap U)$ or $\varphi(\gamma \cap U) \subset \lambda^{-1}\varphi(\gamma \cap U)$, so we know that $\lambda\gamma' = \gamma'$. Let $z \neq 0$ be sufficiently small. Let $p, p' \in \mathbb{C}$ be such that $e^p = z, e^{p'} = \varphi(z)$ and such that $|\operatorname{Im}(p - p')| \leq \pi$. Then we wish to show that as z tends to 0 so $|p - p'|$ will tend to 0 as $|z| \to 0$. This is because there must be a path $\tau \colon [0,1] \to \mathbb{C}$ from p to p' whose image e^τ is the shortest path from z to $\varphi(z)$. The path e^τ will have a length less than $\alpha|z|^2$. Therefore we see that the length of τ will be

$$\int_\tau dz = \int_{e^\tau} \frac{d \ln |z|}{d|z|} dz = \int_{e^\tau} \frac{1}{|z|} dz \leq \frac{\alpha |z|^2}{|z| - \alpha|z|^2}.$$

This length tends to 0 as $|z| \to 0$ and therefore $|p - p'| \to 0$ also. Now there must exist some choice of $r > 0$ such that when $z \in \gamma, |z| < r$ and corresponding $p, p' \in \mathbb{C}$ are defined as before with $e^p = z, e^{p'} = f(z)$, then we have that $|p - p'| < \pi$ by strict inequality. Now let $x > 0$ be such that $\max |\gamma((0,x))| < r$. We may now define t and t' in such a way that $e^t = \gamma, e^{t'} = \gamma'$, and such that for some points $p_a \in t, p'_a \in t'$ with $|\operatorname{Im}(p_a - p'_a)| \leq \pi$, there is some $z_a \in \gamma((0,x))$ with $e^{p_a} = z$ and $e^{p'_a} = \varphi(z)$. We will then have that $|p - p'| < \pi$. As we vary z along $\gamma((0,x))$, let L be the continuous branch of ln defined on $\gamma((0,x))$ such that $L(z_a) = p_a$, and let L' be the continuous branch of $\ln \circ \varphi$ defined on $\gamma((0,x))$ such that $L'(z_a) = p'_a$. If there exist any z_b such that for $p_b := L(z_b)$ and $p'_b := L'(z_b)$, we have $|\operatorname{Im}(p_b - p'_b)| > \pi$. Then by continuity, there must be some similarly defined z_c, p_c, p'_c such that $|\operatorname{Im}(p_c - p'_c)| = \pi$. This choice of z_c, p_c, p'_c, however, contradicts the fact that we must have $|p_c - p'_c| < \pi$.

Therefore, expanding c^-, c^+ if necessary, D^-, D^+ are also lower and upper bounds for $t((0,x))$. Since $t([x,0))$ is bounded, we may similarly expand c^-, c^+ so that D^-, D^+ are lower and upper bounds for $t((0,1))$. Then defining

$m_0 := \frac{\mathrm{Im}(p_1 - p_0)}{\mathrm{Re}(p_1 - p_0)}$, we find the inequality

$$m_0 \operatorname{Re}(z) + c^- < \operatorname{Im}(z) < m_0 \operatorname{Re}(z) + c^+$$

holds as required. ∎

Specifically, we can now derive the following.

8.1.2. Lemma.
There exist bounds $m, c \in \mathbb{R}$ such that for all $u \in \mathbb{N}$ and $l > 0$ exists a strip of the form $\{z \in \mathbb{C} \colon d < \operatorname{Im}(z) < d + ml + c\}$ of height $ml + c$ which contains the set $\{z \in T_u \colon \operatorname{Re}(z) > -l\}$.

Proof. This follows from the fact that $\{z \in T_u \colon \operatorname{Re}(z) > 0\}$ is contained within a strip of bounded height h. The bounds for the imaginary part of $\{z \in T_u \colon -l < \operatorname{Re}(z) < 0\}$ are given by the inequalities in the above lemma for t_u, t_{u+1}, that is, $-m_0 l + c^- < \operatorname{Im}(z) < c^+ + 2\pi$ for some fixed l with $c^+ - c^-$ also fixed. The height required for our strip is therefore $|m_0| l + c^+ + 2\pi i - c^- + h$. ∎

8.2 Speed of Growth

We may then use the following lemma to give us control over the speed of growth for $z \in C_{\underline{u}}$ with sufficiently large real part.

8.2.1. Lemma.
There exists some $D_0 > 0$ and some $R_0 > 0$ such that for $K > R_0$, the following holds. Let $z \in C_{\underline{u}}$ such that when $\operatorname{Re}(z) = \max(0, \min(\operatorname{Re}(C_{\underline{u}}))) + K$. *Then we have*

$$|\operatorname{Re}(E(z))| > \max(0, \min(\operatorname{Re}(E(C_{\underline{u}})))) + \exp(K - D_0) >$$

$$> \max(0, \min(\operatorname{Re}(E(C_{\underline{u}})))) + 2(K - D_0).$$

Proof. Let y_0 be a point in $C_{\underline{u}}$ with minimal real part. We may define the following:

- Let $y_1 \coloneqq E(y_0)$;
- Let $r \coloneqq \max(|y_1|, 2\pi) = 2\pi \exp(\max(\operatorname{Re}(y_0), 0))$;
- Let $z_0 \in C_{\underline{u}}$;
- Let $z_1 \coloneqq E(z_0)$;
- Let $K \coloneqq \operatorname{Re}(z_0) - \max(\operatorname{Re}(y_0), 0)$.

It follows that $|z_1| = r\exp(K)$ and we have that $|z_1 - y_1| \geq r\exp(K) - r$.

By Lemma 8.1.2, we have that

$$|\operatorname{Im}(z_1) - \operatorname{Im}(y_1)| \leq m(r + |\operatorname{Re}(z_1) - \operatorname{Re}(y_1)|) + c.$$

Considering that we also have that

$$|\operatorname{Re}(z_1) - \operatorname{Re}(y_1)| + |\operatorname{Im}(z_1) - \operatorname{Im}(y_1)| \geq |z_1 - y_1|,$$

it follows that

$$|\operatorname{Re}(z_1) - \operatorname{Re}(y_1)| + m|\operatorname{Re}(z_1) - \operatorname{Re}(y_1)| + mr + c \geq |z_1 - y_1|$$

and

$$|\operatorname{Re}(z_1) - \operatorname{Re}(y_1)| \geq \frac{1}{m+1}(|z_1 - y_1| - (mr + c)) \geq$$

$$\geq \frac{1}{m+1}(r\exp(K) - ((m+1)r + c)).$$

Finally, we get that

$$|\operatorname{Re}(z_1)| \geq \frac{1}{m+1}(r\exp(K) - ((m+1)r + c)) - r =$$

$$= r\exp(K - \ln(m+1)) - 2r - \frac{c}{m+1}.$$

Let $D_0 > \ln(m+1)$. We wish to find R_0 large enough that for all $K > R_0$ we have

$$(r\exp(K - \ln(m+1))) - 2r - \frac{c}{m+1} > \exp(K - \ln(m+1)) > \exp(K - D_0).$$

This holds when for all $r \geq 2\pi i$ we have

$$\exp(K - \ln(m+1)) > \frac{2r - \frac{c}{m+1}}{r-1}.$$

Since $\frac{2r - \frac{c}{m+1}}{r-1}$ is bounded, this happens for all $K > R_0$ when R_0 is sufficiently large. We may also choose R_0 large enough (for example, by taking $R_0 > D + \ln(4)$) so that $\exp(K - D_0) > 2(K - D_0)$. ■

The proof of Lemma 8.0.2 then follows immediately from the following.

8.2.2. Lemma.
Let $K > R_0$ and let X be a connected subset of $C_{\underline{u}}$ such that $\tilde{X}$ contains a point of $\mathcal{S}_0$ and

$$\min(\mathrm{Re}(X)) > \max(0, \min(\mathrm{Re}(C_{\underline{u}}))) + K.$$

Then

$$\min(\mathrm{Re}(E(X))) > \max(0, \min(\mathrm{Re}(E(C_{\underline{u}})))) + 2K$$

and $\tilde{E}(X)$ contains a point of $\mathcal{S}_0$.

Proof. We know from the continuity of $\tilde{E}$ that $\tilde{E}(X)$ contains a point of $\mathcal{S}_0$. By Lemma 8.2.1, then for all $z \in X$ we will have

$$|\mathrm{Re}(z)| > \max(0, \min(\mathrm{Re}(E(C_{\underline{u}})))) + 2K.$$

Since $\mathrm{Re}(z)$ is continuous, we will have that $\mathrm{Re}(X)$ is connected. Since $\tilde{E}(X)$

contains a point of $\mathcal{S}_0$, then $\text{Re}(X)$ must be unbounded above. It follows that

$$\min(\text{Re}(E(X))) > \max(0, \min(\text{Re}(E(C_{\underline{u}})))) + 2K.$$

$\blacksquare$

Proof of Lemma 8.0.2. Let X be as in the hypothesis of Lemma 8.0.2 For all points $z \in X$, we have by the above Lemma that $|E^n(z)| > 2^n R_0$ and therefore these points are escaping. $\blacksquare$

This concludes the proof of Theorem 8.0.1. It is now possible to prove our main result.

Proof of Theorem 3.3.2. For exponentially bounded non-singular itineraries $\underline{u}$, by Proposition 6.0.2 $\tilde{C}_{\underline{u}}$ satisfies the conditions in Lemma 6.0.1, from which it follows that $\tilde{C}_{\underline{u}}$ is arclike and all points on $\tilde{C}_{\underline{u}} \cap \overline{\mathcal{S}}$ are terminal.

When $\underline{u}$ is singular, we know by Theorem 8.0.1 that the escaping set is dense in $\tilde{C}_{\underline{u}}$. By Theorem 2.2.5, we know that this escaping set consists entirely of dynamic rays, and by Theorem 2.2.11, we know that this is precisely one dynamic ray. This ray is the branch of some iterated preimage of γ_1. Since γ_1 lands, we know that $\tilde{C}_{(1,1,\ldots)}$ is precisely an arc. Since $\tilde{C}_{\underline{u}}$ eventually maps onto $\tilde{C}_{(1,1,\ldots)}$ by $\tilde{E}$ and in a way which, by Theorem 4.2.3, will always be continuous, then $\tilde{C}_{\underline{u}}$ must also be an arc and therefore arclike continuum. The intersection of $\tilde{C}_{\underline{u}}$ with $\overline{\mathcal{S}}$ will be endpoints of an arc and are therefore terminal points in $\tilde{C}_{\underline{u}}$. $\blacksquare$

Chapter 9

Indecomposability

9.1 Indecomposability of Arclike Continua

9.1.1. Theorem (Indecomposable Arclike Continua with Terminal Points). *Let X be an arclike continuum and let $a \in X$ be a terminal point. X is indecomposable if and only if for all neighbourhoods U_a of a and for all open subsets $M \subset X$ disjoint from U_a, there exist at least two connected components of $X \setminus M$ which intersect U_a.*

Suppose again that X is an arclike continuum and $a, c \in X$ are both terminal points. Then it is sufficient for X to be indecomposable that there exist at least two connected components of $X \setminus M$ whenever M is an open subset of X which is also a neighbourhood of c.

Proof. Suppose there exists some neighbourhood U_a of a and some open subset $M \subset X$ such that there is only one connected component U of $X \setminus M$ which intersects U_a. Then it is possible to take $\overline{U}$ and $\overline{X \setminus U}$ as a decomposition of X. Since U is connected by definition, it follows that $\overline{U}$ is a continuum. since $\overline{U}$ cannot contain M, it must be a proper subcontinuum of X. Similarly, $\overline{X \setminus U}$ cannot intersect the interior of U_a, so it must be a proper subset of X.

To show this is a proper decomposition, it is therefore necessary to prove that $\overline{X \setminus U}$ is connected. Suppose the complement of U has more than one connected component. We may label such a pair of such components as C_0, C_1. Then by the boundary bumping theorem 2.3.6, both $\overline{C}_0$ and $\overline{C}_1$ will intersect $\overline{U}$. We will therefore have that $\overline{C}_0 \cup \overline{U}$ and $\overline{C}_1 \cup \overline{U}$ are both subcontinua of X containing a. However, we also have that neither $\overline{C}_0 \cup \overline{U} \subset \overline{C}_1 \cup \overline{U}$ nor $\overline{C}_1 \cup \overline{U} \subset \overline{C}_0 \cup \overline{U}$. This contradicts the fact that a is terminal. It follows that $\overline{X \setminus U}$ is connected and $\overline{U} \cup \overline{X \setminus U} = X$ is a proper decomposition of X.

Suppose now that all appropriately chosen U_a, M satisfy the hypothesis. Given that a is terminal then for every $\epsilon > 0$, there exists an ϵ-map g_ϵ for which $g_\epsilon(a) = 0$. Then we may choose U_a and M to have diameter less than 2ϵ with M containing $g_\epsilon^{-1}(1)$. Let $b \in U_a$ be disconnected from a by M (by this, we mean it lies in a different component of $X \setminus M$). Any subcontinuum of X containing both a and b must also intersect M. If X is decomposed into two subcontinua C_0, C_1, then at least one of them must intersect M and contain either a or b. This subcontinuum C_j must be 2ϵ dense in X, by which we mean all points in X are within 2ϵ of C_j, since this is true for all ϵ, then it must be the case that one of the subcontinua C_j is ϵ dense for all $\epsilon > 0$ and so must be X itself. This will not be a proper subcontinuum and hence X is indecomposable.

If c is a second terminal point, then the proof follows similarly, except that g_ϵ can be chosen such that $g_\epsilon(c) = 1$ so that M can always be chosen to be a neighbourhood of c. ■

In particular, we may immediately derive the following conditions for arclikeness:

9.1.2. Corollary (Indecomposability of Arclike Continua with Dense Rays)**.** *Suppose for some arclike continuum X, there exists a* dense ray $\gamma\colon (0, \infty) \to$ *X such that $\gamma(x)$ tends to some terminal point a as $x \to \infty$ (by γ being a* dense ray*, we mean that $X = \overline{\gamma}$). Then it is sufficient for X to be indecomposable that for any neighbourhood U_a of a and any point $x \in (0, \infty)$, there exists* *some $y \in (0, x)$ with $\gamma(y) \in U_a$.*

Suppose some arclike continuum X has two terminal points $a, b \in X$. If for every neighbourhood $U_a, U_b \subset X$ of a, b respectively there exists some arc in X with both endpoints in U_a which intersects U_b, then X is indecomposable.

Proof. Suppose X is an arclike continuum with two disjoint open subsets $U_a, M \subset X$ and there is some path $P \subset X$ which intersects M and has endpoints p_0, p_1 in U_a. Then we claim there exist at least two connected components of $X \setminus M$ which intersect U_a.

Let p_x be some point in $P \cap M$, let P_0 be the subpath in P from p_0 to p_x and P_1 the subpath from p_x to p_1. Then let $\epsilon > 0$ be less than the distance from $P_0 \setminus M$ to $P_1 \setminus M$, and less than the distance from p_x to ∂M. Then let g_ϵ be an ϵ-map of X.

Let P_0' be a connected component of $P_0 \setminus M$ and P_1' be a connected component of $P_1 \setminus M$. Let $I_0 := g_\epsilon(P_0')$ and $I_1 := g_\epsilon(P_1')$. These are disjoint intervals in $[0, 1]$. Let P_x' be the connected component of $P \setminus g_\epsilon^{-1}(I_0 \cup I_1)$ which contains p_x. Then by continuity of g_ϵ, we have that $g_\epsilon(P_x')$ must connect I_0 to I_1. Therefore $g_\epsilon(p_x)$ must lie between I_0 and I_1. Without loss of generality we have that $g_\epsilon(p_0) < g_\epsilon(p_x) < g_\epsilon(p_1)$.

Let $I_0' := [0, g_\epsilon(p_x)]$ and $I_1' := [g_\epsilon(p_x), 1]$. Then for $j \in \{0, 1\}$, we have that $g^{-1}(I_j')$ contains some connected component containing p_0 and intersecting U_a. Furthermore, since x is more than ϵ from $X \setminus M$, then we have that $\partial g^{-1}(I_j') \subset M$. Therefore any connected component of $g^{-1}(I_j') \setminus M$ must also be a connected component of $X \setminus M$. Hence there are at least two components of $X \setminus M$ which intersect U_a.

Suppose the first hypothesis holds, that is, X is arclike and contains a dense ray γ and a terminal point a; for any $x \in (0, \infty)$ and neighbourhood U_a of a there exists some $y \in (0, x)$ such that $\gamma(y) \in U_a$. Then for any open set $M \subset X$, we can find some $x \in (0, \infty)$ such that $\gamma(x) \in M$. For any U_a, we may also choose $y \in (0, x)$ such that $\gamma(y) \in U_a$. Then we may take $\gamma([y, \infty])$ as our path. As argued above, there then exist two connected components of $X \setminus M$ which intersect U_a. Therefore X satisfies the first hypothesis of Theorem 9.1.1 so that X is indecomposable. Similarly, when X satisfies the second hypothesis of this corollary, X also satisfies the second hypothesis of Theorem 9.1.1. ■

We recall that for itinerary sets $\tilde{C}_{\underline{u}}$, the points on the circle of addresses $\overline{\mathcal{S}}$ are terminal. We note that if a terminal point a of a continuum X is contained in a subcontinuum C, then a is also a terminal point in C. In particular, for a ray $\gamma_{\underline{s}}$ with address $\underline{s}$, then $\underline{s}$ is a terminal point of $\tilde{C}_{\underline{s}}$ and also of $\gamma_{\underline{s}}$. We may isolate these terminal points in the following way.

9.1.3. Lemma.
Let $\gamma_{\underline{s}}$ be a dynamic ray with non-singular itinerary $\underline{u}$. Then there exists an infinite set $N_{\underline{s}} \subset \mathbb{N}$ such that for $n \in N_{\underline{s}}$, we have that $E^n(\underline{s})$ is the only point in $E^n(\tilde{C}_{\underline{u}}) \cap \overline{T^+ \cap \mathcal{S}}$.

Proof. Suppose $\underline{s}$ has non-binary itinerary $\underline{u}$. Let $n \in \mathbb{N}$ be such that u_n is neither 0 nor 1. Then by Theorem 2.2.11, we have that $E^n(\underline{s})$ is a unique point in $E^n(\tilde{C}_{\underline{u}}) \cap \overline{\mathcal{S}}$. This point then lies in $E^n(\tilde{C}_{\underline{u}}) \cap T^+ \cap \overline{\mathcal{S}}$. Since $\underline{u}$ is non-binary, there are infinitely many such n. Similarly, when $\underline{s}$ has non-singular binary augmented itinerary $\underline{u}^*$, then by Lemma 5.0.4, there is some N sufficiently large that when $n > N$ and $\chi_n \in \{-, +\}$, then $E^n(\underline{s})$ is the unique point in $E^n(\tilde{C}_{\underline{u}}) \cap \overline{T^+ \cap \mathcal{S}}$. Since $\underline{u}^*$ is non-singular, there will again exist infinitely many n such that $\chi_n \in \{-, +\}$. ■

In this way, it is possible to give the following criteria for closures of dynamic rays to be indecomposable by determining when a ray accumulates on its own address.

9.1.4. Lemma.
Let γ be a dynamic ray with address $\underline{s}$ and non-singular itinerary $\underline{u}$ and let $\overline{\gamma}$ be the closure of γ in $\tilde{\mathbb{C}}$. Let $n \in N_{\underline{s}}$. Then $\overline{\gamma}$ is an indecomposable continuum if

and only if for all $x, K > 0$, there exists some $y \in (0, x)$ such that $E^n(\gamma(y)) \in T^+$ and $|E^n(\gamma(y))| > K$.

Proof. Let $n \in N_{\underline{s}}$ and let $U_{\underline{s}}$ be a neighbourhood of $\underline{s}$ in $\tilde{\mathbb{C}}$. Then the set $(T^+ \cap E^n(C_{\underline{u}})) \setminus E^n(U_a)$ must be bounded in $\mathbb{C}$, otherwise $E^n(C_{\underline{u}})$ would accumulate on some other point in $\overline{\mathcal{S} \cap T^+}$. Then when $E^n(z) \in C_{\underline{u}} \cap T^+$ and $|E^n(z)| > K$ for sufficiently large K, we would have $E^n(z) \in E^n(U_{\underline{s}} \cap C_{\underline{u}})$ and therefore $z \in U_{\underline{s}}$.

Now suppose for all $x \in (0, \infty)$ there exists some $y \in (0, x)$ such that $E^n(\gamma(y)) \in T^+$ and $|E^n(\gamma(y))| > K$. Then as above, assuming K is sufficiently large, we have that $\gamma(y) \in U_{\underline{s}}$. Since $\overline{\gamma}$ is a nondegenerate subcontinuum of an arclike continuum $\tilde{C}_{\underline{u}}$, then $\overline{\gamma}$ is also arclike. The above corollary therefore holds and $\overline{\gamma}$ is indecomposable.

If there exists some $x \in (0, \infty)$ and some $K > 0$ such that there is no $y \in (0, x)$ with $E^n(\gamma(y)) \in T^+$ and $|E^n(\gamma(y))| > K$, then there exists some neighbourhood $U_{\underline{s}}$ of $\underline{s}$ such that $\gamma((0, x))$ does not intersect $U_{\underline{s}}$. There then exists some $a \in (0, \infty)$ such that the arc $\overline{\gamma((a, \infty))}$ in $\overline{\gamma}$ contains $\underline{s}$ and is contained in $U_{\underline{s}}$. The continua $\overline{\gamma((a, \infty))}$ and $\overline{\gamma((0, a))}$ will intersect only at the point $\gamma(a)$, and will therefore be a decomposition of $\overline{\gamma}$, showing that $\overline{\gamma}$ is decomposable. ■

9.2 Folding Properties

We find that the behaviour of points in $C_{\underline{u}}$ can be determined by considering when the orbits of those points are contained in A_R. We find that the behaviour of points which stay in A_R is well controlled. By this, we mean that for $x, y \in C_{\underline{u}}$ such that $E(x) \in A_R$ and $E(y) \in A_R$, we find that when $\text{Re}(y) - \text{Re}(x) > 0$ is large, then $\text{Re}(E(y)) - \text{Re}(E(x)) > 0$ is also large. Points which are sufficiently large and whose orbits stay in A_R will be escaping points and will be in some forward tail of a dynamic ray. We find then that it is only when the orbit of a point leaves A_R that it belongs to a part of $C_{\underline{u}}$ which is topologically interesting. To be more precise, we can describe this regular behaviour as follows.

9.2.1. Lemma (Regular Behaviour of Points in A_R)**.**
There is some sufficiently large R_1 *such that for every exponentially bounded non-singular itinerary* $\underline{u}$*, there is some collection of at most two forward tails* $\gamma_j([x_j,\infty))$ *with* $x_j \in (0,\infty)$ *such that the following holds.*

Let $N \in \mathbb{N} \cup \{\infty\}$*. Let* $z \in C_{\underline{u}}$ *with*

$$\operatorname{Re}(z) > \max(\min(\operatorname{Re}(C_{\underline{u}})), 0) + R_1,$$

and with $E^n(z) \in A_R$ *for all* $n \in \mathbb{N}$ *with* $n < N$*. Then*

$$\operatorname{Re}(E^n(z)) > \min \operatorname{Re}(E^n(\gamma_j([x_j,\infty))))$$

for all $n < N$*. If* $N = \infty$*, then* z *is contained in one of these forward tails. Furthermore, these tails are* uniformly escaping *in the sense that*

$$\min \operatorname{Re}(E^n(\gamma_j([x_j,\infty)))) - \min \operatorname{Re}(E^n(C_{\underline{u}}))$$

tends to infinity as n *increases.*

Proof. Let R_0 be as described in Lemma 8.2.2. Then for $\gamma_j \subset C_{\underline{u}}$, let x_j be the greatest value such that $\gamma_j((x_j)) = \max(\min(\operatorname{Re}(C_{\underline{u}})), 0) + R_0$. Then $\gamma_j([x_j,\infty))$ is *uniformly escaping* in the sense described above and the orbit of all points in $\gamma_j([x_j,\infty))$ is contained in A_R.

Let $x, y \in C_{\underline{u}}$ such that the orbits of x, y are contained in A_R. Then we can show there exists some D such that if $\operatorname{Re}(x) + D \le \operatorname{Re}(y)$, then we must have that $\operatorname{Re}(E(x)) + D \le \operatorname{Re}(E(y))$. By Lemma 8.1.2, we know that $E(x), E(y)$ are contained within a horizontal strip of bounded height H. Supposing $\operatorname{Re}(x) + D \le \operatorname{Re}(y)$, then we have

$$|E(y)|-|E(x)| = 2\pi\exp(\operatorname{Re}(y))-2\pi\exp(\operatorname{Re}(x)) \ge 2\pi\exp(D-R)-2\pi\exp(-R)$$

and since $E(x), E(y)$ are contained within a horizontal strip we have

$$|\operatorname{Re}(E(y))| - |\operatorname{Re}(E(x))| \geq 2\pi \exp(D - R) - 2\pi \exp(-R) - H.$$

Choosing D large enough that $2\pi \exp(D - R) - 2\pi \exp(-R) - H > D$, we will have that $|\operatorname{Re}(E(y))| - |\operatorname{Re}(E(x))| > D$. Additionally, we choose D large enough that $D > R$. Then since $E(y) \in A_R$, we have that $\operatorname{Re}(E(y))$ must be positive and $\operatorname{Re}(E(x)) + D < \operatorname{Re}(E(y))$ as required.

Now let R_1 be such that $R_1 > R_0 + D$. Suppose there were some $z \in C_{\underline{u}}$ whose orbit is contained in A_R and with $\operatorname{Re}(z) > \max(\min(\operatorname{Re}(C_{\underline{u}})), 0) + R_1$ which is not contained in one of the forward tails $\gamma_j([x_j, \infty))$. Then we have that $\operatorname{Re}(E^n(\gamma_j(x_j))) + D < E^n(z)$ for all $n \in \mathbb{N}$. Since they are all contained in a horizontal strip, it follows that there exists a point in $E^n(\gamma_j([x_j, \infty)))$ with the same real part as $E^n(z)$ so that the two are a bounded distance apart. The euclidean distance from $E^n(z)$ to $E^n(\gamma_j([x_j, \infty)))$ is therefore less than some H for all $n \in \mathbb{N}$.

On the other hand, considering Lemma 7.4.1, we have that the distance $\operatorname{dist}_{A;A'}(E^n(z), E^n(\gamma_j([x_j, \infty))))$ tends to infinity. By Lemma 7.3.3, if $E^n(z)$ and $E^n(\gamma_j([x_j, \infty)))$ lie in the same connected component of T^- or T^+ for large n, then the euclidean distance between $E^n(z)$ and $E^n(\gamma_j([x_j, \infty)))$ is large. This would contradict the bound of H we have established. Now there will exist infinitely many n such that $E^n(z)$ is contained in the same connected component of T^+ or T^- as the image some forward tail $E^n(\gamma_j([x_j, \infty)))$. This can be seen easily in the non-binary case, when u_n is neither 0 nor 1, then $E^n(C_{\underline{u}})$ intersects precisely one component of T^+.

In the binary case, let $\gamma_a([x_a, \infty)), \gamma_b([x_b, \infty))$ be the two forward tails defined in $C_{\underline{u}}$. Let $N \in \mathbb{N}$ be such that $u_n \in \{0, 1\}$ for $n \geq N$. Then for $n \geq N$, Lemma 5.0.4 implies that $E^n(\gamma_a([x_a, \infty)))$ is contained in one component of T^+ or T^- and $E^n(\gamma_b([x_b, \infty)))$ is contained in a different component of T^- or T^+. We know that $E^n(C_{\underline{u}})$ intersects at most two unbounded components of T^+ or T^-. Therefore, as required, $E^n(z)$ is contained in the same connected component of T^+ or T^- as the image of some forward tail $E^n(\gamma_j([x_j, \infty)))$ for infinitely many n.

Such a z therefore cannot then exist. It follows that these forward tails must contain all the points z with $\text{Re}(z) > \max(\min(\text{Re}(C_{\underline{u}})), 0) + R_1$ whose orbit is contained in A_R. ■

We now require a lemma on the uniformly escaping properties of dynamic ray tails. This fact is immediate from the construction of the dynamic rays in [9], but for completeness we give here a justification.

9.2.2. Lemma.
Let γ be a dynamic ray with non-singular itinerary $\underline{u}$ and let $x > 0$. Then the forward tail $\gamma([x, \infty))$ is uniformly escaping *in the sense that*

$$\min \text{Re}(E^n(\gamma([x, \infty)))) - \min \text{Re}(E^n(C_{\underline{u}}))$$

tends to infinity as n increases.

Proof. Let $x \in (0, \infty)$ and $y \in (0, x)$. Since $\gamma(x)$ and $\gamma(y)$ are both escaping points, there exists some n sufficiently large that the orbits of $E^n(\gamma(x))$ and $E^n(\gamma(y))$ are contained in A_R and both have real part greater than R_1. We may also choose n sufficiently large that the distance between $E^n(\gamma(x))$ and $E^n(\gamma(y))$ is also large. We note that $E^n(\gamma(x))$ and $E^n(\gamma(y))$ are both contained within a strip of bounded height H. Therefore we may have that $|\text{Re}(E^n(\gamma(x))) - \text{Re}(E^n(\gamma(y)))| > R_1 + R$ by choosing n large enough.

Suppose then that $\text{Re}(E^n(\gamma(y))) > \text{Re}(E^n(\gamma(x)))$. Then

$$\text{Re}(E^n(\gamma(y))) > \max(\min \text{Re}(E^n(C_{\underline{u}})), 0) + R_1$$

and by Lemma 9.2.1, it follows that $E^n(\gamma(y))$ must be contained in some uniformly escaping forward tail $\gamma_j([x_j, \infty))$. This forward tail must be a forward tail of $E^n(\gamma)$. There will be a corresponding forward tail of γ which contains $\gamma(y)$, and therefore also contains $\gamma([x, \infty))$ and is uniformly escaping.

Supposing $\mathrm{Re}(E^n(\gamma(x))) > \mathrm{Re}(E^n(\gamma(y)))$. Then

$$\mathrm{Re}(E^n(\gamma(x))) > \max(\min \mathrm{Re}(E^n(C_{\underline{u}})), 0) + R_1$$

and we will have again that $\gamma(x)$ is contained in a uniformly escaping tail. It follows that $\gamma([x, \infty))$ is uniformly escaping. ■

We may now give a definition of the sets which exhibit a certain kind of "folding back" type behaviour.

9.2.3. Definition (Defining R_1, Q_1 and $Q_{\underline{s},n,m}$.)**.**
Recall that $E^2(\mathbb{C} \setminus A_R)$ is contained within T_1. Let R_1 be large enough that it satisfies Lemma 9.2.1 and also $R_1 > \max \mathrm{Re}(E^2(\mathbb{C} \setminus A_R))$. We note that we will later on require that R_1 be sufficiently large for other purposes. We will say for the moment merely that it is sufficiently large.

Let $Q_1 := \{z \in T_1 \cap A_R\colon \mathrm{Re} < R_0\}$. Let $\gamma_{\underline{s}}$ be a dynamic ray with address $\underline{s}$. Let $n \in N_{\underline{s}}$ with and $m \in \mathbb{N}$ with $m > n$ and let $\underline{u}$ be the itinerary of $\gamma_{\underline{s}}$. We may define the set $Q_{\underline{s},n,m}$ to be the set of points $z \in \gamma_{\underline{s}}$ such that the following hold:

- $E^n(z) \in T^+$;
- $\mathrm{Re}(E^n(z)) > \max(\min \mathrm{Re}(C_{\underline{u}}), 0) + R_1$;
- $E^m(z) \in Q_1$;
- $E^j \in \mathbb{C} \setminus Q_1$ for all $n \leq j < m$.

We note that Q_1 is defined in such a way that if a point $z \in \mathbb{C} \setminus A_R$ has $E(z) \in A_R$, then $E^2(z) \in Q_1$. All points whose orbits leave and re-enter A_R will pass through Q_1 at some point. In this way we find that the sets $Q_{\underline{s},n,m}$ describe the points $z \in \gamma_{\underline{s}}$ whose orbits are large at $E^n(z)$ but are "folded back" at $E^m(z)$. We find that the topology of $\overline{\gamma_{\underline{s}}}$ can then be determined in some way by the behaviour of points in $\gamma_{\underline{s}}$ whose orbit at some point intersects Q_1. By studying the sets $Q_{\underline{s},n,m}$ we may find information about the accumulation sets of $\gamma_{\underline{s}}$.

9.2.4. Theorem.
Let $\gamma_{\underline{s}}$ be a dynamic ray with address $\underline{s}$ and non-singular itinerary $\underline{u}$. For all $n \in N_{\underline{s}}$, the set $\bigcap_{M\in\mathbb{N}} \overline{\bigcup_{m\geq M} Q_{\underline{s},n,m}}$ is contained in the accumulation set of $\gamma_{\underline{s}}$ and is also contained within $\gamma_{\underline{s}} \cup \{\underline{s}\}$. The set $\bigcup_{n\in N_{\underline{s}}} \bigcap_{M>n} \overline{\bigcup_{m\geq M} Q_{\underline{s},n,m}}$ is precisely the intersection of $\gamma_{\underline{s}} \cup \{\underline{s}\}$ and the accumulation set of $\gamma_{\underline{s}}$.

Proof. We show first that $\bigcap_{M\in\mathbb{N}} \overline{\bigcup_{m\geq M} Q_{\underline{s},n,m}}$ is contained in the accumulation set of $\gamma_{\underline{s}}$. Since every forward tail of $\gamma_{\underline{s}}$ escapes uniformly, we know that for every $x \in (0,\infty)$, then for sufficiently large M and for all $m \geq M$ we have that $E^m(\gamma_{\underline{s}}([x,\infty))) \subset \mathbb{C} \setminus Q_1$. Therefore $\bigcup_{m\geq M} Q_{\underline{s},n,m} \subset \gamma_{\underline{s}}((0,x))$ for all n and sufficiently large $M > n$. It follows that $\bigcap_{M\in\mathbb{N}} \overline{\bigcup_{m\geq M} Q_{\underline{s},n,m}}$ is contained in $\bigcap_{x>0} \overline{\gamma_{\underline{s}}((0,x))}$, the accumulation set of $\gamma_{\underline{s}}$.

Supposing now that for some $n \in N_{\underline{s}}$, there is an infinite strictly increasing sequence $\underline{m}$ of integers m_j such that $Q_{\underline{s},n,m_j}$ is non-empty for all $j \in \mathbb{N}$. For each j, we have that for a point $z \in Q_{\underline{s},n,m_j}$, the orbit of z cannot be contained in A_R for iterates between n and m_j, otherwise by Lemma 9.2.1, $\mathrm{Re}(E^{m_j}(z))$ would be larger than R_0. The orbit of z can only re-enter A_R once between n and m_j. The iterate after re-entering A_R, it must necessarily visit Q_1. This, however, can only be done at the m_jth iterate. The orbit of z between n and m_j then has the following behaviour: as k increases, $E^k(z)$ is in A_R until some value. When k is larger than that value, then $E^k(z)$ is in $\mathbb{C} \setminus A_R$ until either $k = m_j$ or $k = m_j - 1$. When $k = m_j$, then $E^k(z) \in Q_1$.

We may now find some way to control how long the orbit of a point in $Q_{\underline{s},n,m_j}$ stays in A_R. Since $\underline{u}$ is non-singular, there are infinitely many values m' such that $E^{m'}(\mathbb{C}_{\underline{u}}) \subset A_R$. We may choose the sequence $\underline{m}$ such that there is such a m' between n and m_0. Let m'_j be the greatest value less than m_j such that $E^{m'_j}(C_{\underline{u}}) \subset A_R$. We may note that m'_j tends to infinity as j increases. We then have that $E^m(Q_{\underline{s},n,m_j}) \subset A_R$ for all $m \in \mathbb{N}$ with $n \leq m \leq m'_j$.

Suppose $\underline{u}$ is binary type with $\gamma_{\underline{t}}$ as the second ray contained in $C_{\underline{u}}$. Then by Lemma 8.2.2, there exist forward tails $\gamma_{\underline{s}}([x^n_{\underline{s}}, \infty))$ and $\gamma_{\underline{t}}([x^n_{\underline{t}}, \infty))$ such that

$$\min \operatorname{Re}(E^n(\gamma_{\underline{s}}([x^n_{\underline{s}}, \infty)))) = \min \operatorname{Re}(E^n(\gamma_{\underline{t}}([x^n_{\underline{t}}, \infty)))) =$$

$$= \max(\min \operatorname{Re}(E^n(C_{\underline{u}})), 0) + R_0.$$

Then as in Lemma 9.2.1, we can find that $\min \operatorname{Re}(E^{m'_j}(Q_{\underline{s},n,m_j}))$ is greater than both $\min \operatorname{Re}(E^{m'_j}(\gamma_{\underline{s}}([x^n_{\underline{s}}, \infty))))$ and $\min \operatorname{Re}(E^{m'_j}(\gamma_{\underline{t}}([x^n_{\underline{t}}, \infty))))$ and so all points in $E^{m'_j}(Q_{\underline{s},n,m_j})$ have a bounded euclidean distance H from both $E^{m'_j}(\gamma_{\underline{s}}([x^n_{\underline{s}}, \infty)))$ and $E^{m'_j}(\gamma_{\underline{t}}([x^n_{\underline{t}}, \infty)))$.

A point in $E^{m'_j}(Q_{\underline{s},n,m_j})$ will be in the same component of T^+ or T^- as precisely one of $E^{m'_j}(\gamma_{\underline{s}}([x^n_{\underline{s}}, \infty)))$ or $E^{m'_j}(\gamma_{\underline{t}}([x^n_{\underline{t}}, \infty)))$. By Lemma 7.3.3, we have that either

$$\operatorname{dist}_{A;A'}(E^{m'_j}(Q_{\underline{s},n,m_j}), E^{m'_j}(\gamma_{\underline{s}}([x^n_{\underline{s}}, \infty))))$$

or

$$\operatorname{dist}_{A;A'}(E^{m'_j}(Q_{\underline{s},n,m_j}), E^{m'_j}(\gamma_{\underline{t}}([x^n_{\underline{t}}, \infty))))$$

is bounded above by some constant H' depending only on H.

If j is arbitrarily large and $\operatorname{dist}_{A;A'}(E^{m'_j}(Q_{\underline{s},n,m_j}), E^{m'_j}(\gamma_{\underline{t}}([x^n_{\underline{t}}, \infty)))) < H'$, then by Lemma 7.4.1, $\operatorname{dist}_{A;A'}(E^n(Q_{\underline{s},n,m_j}), E^n(\gamma_{\underline{t}}([x_{\underline{t}}, \infty))))$ will be arbitrarily small. However, we have that $E^n(\gamma_{\underline{t}}([x^n_{\underline{t}}, \infty))) \subset T^-$ and $E^n(Q_{\underline{s},n,m_j}) \subset T^+$ with real parts of both bounded below by R_0. These sets are therefore separated by at least some constant bound and cannot be arbitrarily close. Therefore when j is large we have that

$$\operatorname{dist}_{A;A'}(E^{m'_j}(Q_{\underline{s},n,m_j}), E^{m'_j}(\gamma_{\underline{s}}([x^n_{\underline{s}}, \infty)))) < H'$$

and as j increases $\operatorname{dist}_{A;A'}(E^n(Q_{\underline{s},n,m_j}), E^n(\gamma_{\underline{s}}([x^n_{\underline{s}}, \infty))))$ tends to 0 and therefore $\operatorname{dist}_{A;A'}(Q_{\underline{s},n,m_j}, \gamma_{\underline{s}}([x^n_{\underline{s}}, \infty)))$ also tends to 0.

Supposing now $\underline{u}$ is non-binary. Let $\gamma_{\underline{s}}([x^n_{\underline{s}}, \infty))$ be the forward tail such that $\min \operatorname{Re}(E^n(\gamma_{\underline{s}}([x^n_{\underline{s}}, \infty)))) = \max(\min \operatorname{Re}(E^n(C_{\underline{u}}), 0)) + R_0$. For sufficiently large j, let m'_j be the largest value $m'_j < m_j$ such that $E^{m'_j}(C_{\underline{u}}) \subset A_R$. Let $m''_j \leq m'_j$ be the largest such value that $u_{m''_j}$ is not 0 or 1. As before, we have that $E^m(Q_{\underline{s},n,m_j}) \subset A_R$ for all $m \in \mathbb{N}$ with $n \leq m \leq m''_j$. Therefore the euclidean distance between $E^{m''_j}(Q_{\underline{s},n,m_j})$ and $E^{m''_j}(\gamma_{\underline{s}}([x_{\underline{s}}, \infty)))$ is bounded by H. We may also note that both of these sets are in T^+, and so $\operatorname{dist}_{A;A'}(E^{m''_j}(Q_{\underline{s},n,m_j}), E^{m''_j}(\gamma_{\underline{s}}([x^n_{\underline{s}}, \infty))))$ is also bounded. Since m''_j tends to ∞ as j increases, so $\operatorname{dist}_{A;A'}(Q_{\underline{s},n,m_j}, \gamma_{\underline{s}}([x^n_{\underline{s}}, \infty)))$ tends to 0 in the non-binary case also.

Now, by definition, for every point $z \in \bigcap_{M\in\mathbb{N}} \overline{\bigcup_{m\geq M} Q_{\underline{s},n,m}}$, there is some sequence $\underline{z}$ of $z_j \in \mathbb{C}$ which tends to z, where for each j there is some $j' > j$ such that $z_j \in Q_{\underline{s},n,m'_j}$. If $\lim_{j\to\infty} \operatorname{Re}(z_j) = \infty$, then $z \in \mathcal{S}$, and since $E^n(\underline{s})$ is the only point in $E^n(\tilde{C}_{\underline{u}}) \cap \overline{T}^+ \cap \mathcal{S}$, then $E^n(z_j)$ tends to $E^n(z) = E^n(\underline{s})$ and $z = \underline{s}$. If $\lim_{j\to\infty} \operatorname{Re}(z_j)$ is finite, then z must lie on $\gamma_{\underline{s}}([x^n_{\underline{s}}, \infty))$. We therefore have that $\bigcap_{M>n} \overline{\bigcup_{m\geq M} Q_{\underline{s},n,m}}$ is contained in the intersection of $\gamma_{\underline{s}} \cup \{\underline{s}\}$ and the accumulation set of $\gamma_{\underline{s}}$.

Now we will prove that every point on $\gamma_{\underline{s}} \cup \{\underline{s}\}$ which is also on the accumulation set of $\gamma_{\underline{s}}$ is also in the set $\bigcup_{n\in N_{\underline{s}}} \bigcap_{M>n} \overline{\bigcup_{m\geq M} Q_{\underline{s},n,m}}$. Suppose $z \in \gamma_{\underline{s}}$ was also in the accumulation set of $\gamma_{\underline{s}}$. Let x be such that $\gamma_{\underline{s}}(x) = z$. Then by Lemma 9.2.2, there exists some $n \in N_{\underline{s}}$ such that

$$\min \operatorname{Re}(E^n(\gamma_{\underline{s}}([x, \infty)))) > \max(\min \operatorname{Re}(E^n(C_{\underline{u}})), 0) + R_1.$$

Since $E^n(z)$ is connected to $E^n(\underline{s})$ by a forward tail contained in T^+, we have that $E^n(z) \in T^+$. There is now some $x' \in (0, x)$ such that

$$\min \operatorname{Re}(E^n(\gamma_{\underline{s}}([x', \infty)))) \leq \max(\min \operatorname{Re}(E^n(C_{\underline{u}})), 0) + R_0.$$

Suppose now $y \in (0, x')$ such that $\gamma_{\underline{s}}(y)$ is within some arbitrarily small euclidean distance $\delta > 0$ to z. Then, when δ is sufficiently small, $E^n(\gamma_{\underline{s}}(y))$ will stay within a small bounded distance of $E^m(z)$ until $m > M$ for arbitrarily large M. However, as in Lemma 9.2.1, the orbit of $\gamma_{\underline{s}}(y)$ cannot stay in A_R indefinitely. Since $\underline{u}$ is non-binary the orbit of $\gamma_{\underline{s}}(y)$ cannot then stay in $\mathbb{C} \setminus A_R$ indefinitely and must at some point intersect Q_1. If the orbit of z remains in A_R, this intersection with Q_1 happens after an arbitrarily long time, depending on how small δ is. We note that the orbit of $E^n(z)$ remains in A_R. Therefore, when δ is sufficiently small, $\gamma_{\underline{s}}(y) \in Q_{\underline{s},n,m}$ for n and some $m \in \mathbb{N}$. Additionally, for any $M \in \mathbb{N}$, we can choose δ sufficiently small that $m > M$. For any sequence $\underline{y}$ tending to 0 with $\gamma_{\underline{s}}(y_j)$ tending to z as j increases, and for all M, there is some subsequence $\underline{y}'$ with $\gamma_{\underline{s}}(y'_j)$ tending to z such that for every $j \in \mathbb{N}$, we have $\gamma_{\underline{s}}(y'_j) \in Q_{\underline{s},n,m}$ for some $m > M$. It follows that $z \in \bigcap_{M>n} \overline{\bigcup_{m \geq M} Q_{\underline{s},n,m}}$.

Similarly, suppose $\underline{s}$ is in the accumulation set of $\gamma_{\underline{s}}$. Let $n \in N_{\underline{s}}$. We then have that $E^n(\gamma_{\underline{s}})$ accumulates on $E^n(\underline{s})$. Let $\gamma_{\underline{s}}([x, \infty))$ again be the forward tail in Lemma 9.2.1 such that all points in $\gamma_{\underline{s}}$ with orbits contained in A_R lie on $\gamma_{\underline{s}}([x, \infty))$. For any $M \in \mathbb{N}$, there will exist a sequence $\underline{y}$ of $y_j \in (0, x)$ such that $E^M(\gamma_{\underline{s}}(y_j))$ tends to $E^M(\underline{s})$ as j increases. We may choose $\underline{y}$ such that $\min_{j \in \mathbb{N}} \mathrm{Re}(E^M(\gamma_{\underline{s}}(y_j)))$ is sufficiently large that for all $m \in \mathbb{N}$ with $m < M$, we have that $\mathrm{Re}(E^m(\gamma_{\underline{s}}(y_j)))$ is also large. We may then have that $E^m(\gamma_{\underline{s}}(y_j)) \in A_R$ for $n \leq m \leq M$. By Lemma 9.2.1, we have that $E^m(\gamma_{\underline{s}}(y_j)) \in \mathbb{C} \setminus A_R$ for some $m > M$, which implies again that $\gamma_{\underline{s}}(y_j) \in Q_{\underline{s},n,m'}$ for some $m' > m > M$. In this way, we see again that $\underline{s} \in \bigcap_{M>n} \overline{\bigcup_{m \geq M} Q_{\underline{s},n,m}}$.

In either case, all points in $\gamma_{\underline{s}} \cup \{\underline{s}\}$ and in the accumulation set of $\gamma_{\underline{s}}$ are also in $\bigcup_{n \in N_{\underline{s}}} \bigcap_{M>n} \overline{\bigcup_{m \geq M} Q_{\underline{s},n,m}}$ as required. ■

It is now possible to use properties of $Q_{\underline{s},n,m}$ to study topological properties of the closure of $\overline{\gamma}_{\underline{s}}$.

9.2.5. Definition (Indecomposability Beyond a Point)**.**
Recall that for a dynamic ray γ and some $x \in (0, \infty]$, we say γ is *indecomposable beyond* x if $\overline{\gamma((0,x))}$ is indecomposable and $\overline{\gamma((0,x))} \cap \gamma((x,\infty)) = \emptyset$.

We note that γ being indecomposable past ∞ is equivalent to $\overline{\gamma}$ being indecomposable.

9.2.6. Remark.
If γ is indecomposable beyond x, then $\gamma((0,x,))$ is contained in the accumulation set of γ. In other words, the accumulation set of γ is precisely $\overline{\gamma((0,x))}$.

Proof. Suppose to the contrary, there exists some open interval $(a,b) \subset (0,x)$ such that (a,b) contained no points of the accumulation set of γ. Then $\overline{\gamma((0,x))}$ decomposes into the proper subcontinua $\overline{\gamma((0,b))}$ and $\overline{\gamma((b,x))}$. ■

9.2.7. Lemma.
Let $\gamma_{\underline{s}}$ be a dynamic ray with address $\underline{s}$ and non-singular itinerary $\underline{u}$. Then $\overline{\gamma_{\underline{s}}}$ is indecomposable if and only if for some $n \in N_{\underline{s}}$, we have that

$$\limsup_{m>n} \max \mathrm{Re}(Q_{\underline{s}}, n, m) = \infty.$$

The ray $\gamma_{\underline{s}}$ is indecomposable beyond some $x \in (0,\infty)$ if and only if there exists some $n \in N_{\underline{s}}$ and infinitely many m such that $Q_{\underline{s},n,m}$ is non-empty and

$$\limsup_{m>n} \max \mathrm{Re}(Q_{\underline{s}}, n, m) < \infty.$$

Proof. For $n \in N_{\underline{s}}$, we note that $\underline{s}$ is in the accumulation set of $\gamma_{\underline{s}}$ if and only if $E^n(Q_{\underline{s},n,m})$ accumulates on $E^n(\underline{s})$. This happens if and only if

$$\limsup_{m>n} \max \mathrm{Re}(E^n(Q_{\underline{s},n,m})) = \infty,$$

which in turn happens if and only if

$$\limsup_{m>n} \max \mathrm{Re}(Q_{\underline{s},n,m}) = \infty.$$

By Corollary 9.1.2, we have that $\overline{\gamma_{\underline{s}}}$ is indecomposable if and only if $\underline{s}$ is in the accumulation set of $\gamma_{\underline{s}}$.

Suppose now there exists some $n \in N_{\underline{s}}$ such that there are infinitely many m such that $Q_{\underline{s},n,m}$ is non-empty and $\limsup_{m>n} \max \operatorname{Re}(Q_{\underline{s}}, n, m) < \infty$. Then the intersection of $\gamma_{\underline{s}}$ and the accumulation set of $\gamma_{\underline{s}}$ is non-empty and bounded. Let $x \in (0, \infty)$ be the greatest such value that $\gamma_{\underline{s}}(x)$ is in this intersection. Then we find that $\overline{\gamma_{\underline{s}}}([0, x))$ is an arclike continuum. Furthermore, $\gamma_{\underline{s}}(x)$ is a terminal point of this continuum. If it were not, we could find two subcontinua $U, V \subset \overline{\gamma_{\underline{s}}((0, x))}$ with $\gamma_{\underline{s}}(x) \in U, \gamma_{\underline{s}}(x) \in V$ such that neither $U \subset V$ nor $V \subset U$. Then we would find that there would exist two subcontinua $U', V' \subset \overline{\gamma_{\underline{s}}}$ defined as

$$U' := U \cup \gamma_{\underline{s}}((x, \infty)) \cup \{\underline{s}\}, V' := V \cup \gamma_{\underline{s}}((x, \infty)) \cup \{\underline{s}\}.$$

We would have that neither $U' \subset V'$ nor $V' \subset U'$, contradicting that $\underline{s}$ is a terminal point. Since $\gamma_{\underline{s}}(x)$ is both terminal in $\overline{\gamma_{\underline{s}}((0, x))}$ and in the accumulation set of $\gamma_{\underline{s}}$, it follows from Corollary 9.1.2 that $\overline{\gamma_{\underline{s}}}((0, x))$ is indecomposable.

Conversely, if $\gamma_{\underline{s}}$ is indecomposable beyond some $x \in (0, \infty)$, then $\gamma_{\underline{s}}(x)$ is again a terminal point of $\overline{\gamma_{\underline{s}}}((0, x))$. By Corollary 9.1.2, $\gamma_{\underline{s}}(x)$ must be in the accumulation set of $\gamma_{\underline{s}}$. There must then be some $n \in N_{\underline{s}}$ such that $\gamma_{\underline{s}}(x) \in \bigcap_{M \in \mathbb{N}} \overline{\bigcup_{m \geq M} Q_{\underline{s},n,m}}$. There are therefore infinitely many m such that $Q_{\underline{s},n,m}$ is non-empty. If we had $\limsup_{m>n} \max \operatorname{Re}(Q_{\underline{s}}, n, m) = \infty$, then $\overline{\gamma_{\underline{s}}}$ would be indecomposable. We could not have $\overline{\gamma_{\underline{s}}((0, x))} \cap \gamma((x, \infty)) = \emptyset$ otherwise $\overline{\gamma_{\underline{s}}((0, x))}, \overline{\gamma((x, \infty))}$ would be a decomposition. It follows that we must have that $\limsup_{m>n} \max \operatorname{Re}(Q_{\underline{s},n,m}) < \infty$. ∎

When considering the topology of the closure of two rays $\gamma_{\underline{s}}, \gamma_{\underline{t}}$ with the same itinerary $\underline{u}$, we may use the following theorem to determine the topology of $\tilde{C}_{\underline{u}}$. We may deduce the topology of $\tilde{C}_{\underline{u}}$ by considering individually the topology of $\overline{\gamma_{\underline{s}}}$ and $\overline{\gamma_{\underline{t}}}$.

9.2.8. Theorem.
Let $\underline{u}$ be a non-singular binary type itinerary containing the two rays $\gamma_{\underline{s}}, \gamma_{\underline{t}}$. Then the accumulation set of $\gamma_{\underline{s}}$ is equal to the accumulation set of $\gamma_{\underline{t}}$.

Proof. Let $Q_2 := \{z \in (T_0 \cup T_1) \cap A_R \colon \operatorname{Re}(z) < R_1\}$. Note that by Lemma 7.3.3, we can deduce that $\operatorname{diam}_{A;A'}(Q_2)$ is finite.

For sufficiently large M and for all $m > M$, we have that both $E^m(\gamma_{\underline{s}})$ and $E^m(\gamma_{\underline{t}})$ intersect Q_2 while $\min \operatorname{Re}(C_{\underline{u}}) < 0$. Now suppose z is in the accumulation set of $\gamma_{\underline{s}}$ so that there exists a decreasing sequence $\underline{x}$ tending to 0 such that $\gamma_{\underline{s}}(x_j)$ tends to 0 as j increases. By Lemma 9.2.1, we know that when j is sufficiently large (so that x_j is sufficiently small) the orbit of $\gamma_{\underline{s}}(x_j)$ will intersect Q_2 at some point. Since $\gamma_{\underline{s}}(x_j)$ is escaping, we also know there is then some maximum value m_j such that $E^{m_j}(\gamma_{\underline{s}}(x_j)) \in Q_2$. By removing a finite number of points if necessary, we may assume that x_0 is small enough that m_0 exists and is greater than M. By Lemma 9.2.1 again, we know that for arbitrarily large m, there is some $x \in (0, \infty)$ such that $E^m(\gamma_{\underline{s}}([x, \infty)))$ contains all points whose orbit consists entirely of points with real part greater than R_1. If $x_j < x$, then $m_j > m$. In this way we can see that m_j tends to infinity.

Now for every m_j there is some $y_j \in (0, \infty)$ such that $E^{m_j}(\gamma_{\underline{t}}(y_j)) \in Q_2$. Let $\underline{y}$ be such a sequence. By Lemma 9.2.2, we have that for any $y \in (0, \infty)$ there exists some sufficiently large m_j that the orbit of $E^{m_j}(\gamma_{\underline{t}}([y, \infty)))$ has minimum real part greater than R_1. It follows that $y_j < y$ for sufficiently large j. The sequence $\underline{y}$ therefore tends to 0.

Since $\operatorname{diam}_{A;A'}(Q_2)$ is finite, we have that $\operatorname{dist}_{A;A'}(E^{m_j}(\gamma_{\underline{s}}(x_j)), E^{m_j}(\gamma_{\underline{t}}(y_j)))$ is also finite. Then by Lemma 7.4.1, we have that $\operatorname{dist}_{A;A'}(\gamma_{\underline{s}}(x_j), \gamma_{\underline{t}}(y_j))$ tends to 0 as j increases. If $z \in \mathbb{C}$, then it follows that

$$\lim_{j \to \infty} \gamma_{\underline{s}}(x_j) = \lim_{j \to \infty} \gamma_{\underline{t}}(y_j) = z.$$

If $z \in \mathcal{S}$, then either $z = \underline{s}$ or $z = \underline{t}$. We may assume for the moment that $z = \underline{s}$. Now we note that $\gamma_{\underline{t}}(y_j)$ must also accumulate on $\mathcal{S}$. For some $n \in N_{\underline{s}}$, we have that $E^n(\underline{s})$ is the only point of $C_{\underline{u}} \cap \mathcal{S}$ in T^+. Then $E^n(\gamma_{\underline{t}}(y_j))$ will also be eventually contained in T^+ and the only point it can accumulate on is $E^n(\underline{s})$. So $E^n(\gamma_{\underline{t}}(y_j))$ tends to $E^n(\underline{s})$ and therefore $\gamma_{\underline{t}}(y_j)$ tends to the point

$\underline{s} = z$. The argument follows similarly when $z = \underline{t}$. Therefore every point on the accumulation set of $\gamma_{\underline{s}}$ is also on the accumulation set of $\gamma_{\underline{t}}$, and every point on the accumulation set of $\gamma_{\underline{t}}$ is also on the accumulation set of $\gamma_{\underline{s}}$. ■

9.2.9. Corollary.

Let $\underline{u}$ be a non-singular binary type itinerary containing two rays $\gamma_{\underline{s}}, \gamma_{\underline{t}}$. Then $\tilde{C}_{\underline{u}}$ is indecomposable precisely when both $\overline{\gamma_{\underline{s}}}$ and $\overline{\gamma_{\underline{t}}}$ are indecomposable. Furthermore, for $x_{\underline{s}}, x_{\underline{t}} \in (0, \infty]$, when $\gamma_{\underline{s}}$ is indecomposable beyond $x_{\underline{s}}$ and $\gamma_{\underline{t}}$ is indecomposable beyond $x_{\underline{t}}$, then $\overline{\gamma_{\underline{s}}((x_{\underline{s}}, \infty)} \cup \overline{\gamma_{\underline{t}}((x_{\underline{t}}, \infty)}$ is indecomposable.

Proof. If $\gamma_{\underline{s}}$ is indecomposable beyond $x_{\underline{s}}$ and $\gamma_{\underline{t}}$ is indecomposable beyond $x_{\underline{t}}$, then we have that the accumulation set of $\gamma_{\underline{s}}$ is $\overline{\gamma_{\underline{s}}((0, x_{\underline{s}}))}$ and the accumulation set of $\gamma_{\underline{t}}$ is $\overline{\gamma_{\underline{t}}((0, x_{\underline{t}}))}$. If, for example, there were some $y \in (0, x_{\underline{s}})$ such that $\overline{\gamma_{\underline{s}}((0, x_{\underline{s}}))} \neq \overline{\gamma_{\underline{s}}((0, y))}$, then $\overline{\gamma_{\underline{s}}((0, y))}$ and $\overline{\gamma_{\underline{s}}((y, x_{\underline{s}}))}$ would be a decomposition of $\overline{\gamma_{\underline{s}}((0, x_{\underline{s}}))}$. Because of this, $\overline{\gamma_{\underline{s}}((0, x_{\underline{s}}))}$ must be the accumulation set of $\gamma_{\underline{s}}$. We therefore have that $\overline{\gamma_{\underline{s}}((0, x_{\underline{s}}))} = \overline{\gamma_{\underline{t}}((0, x_{\underline{t}}))}$, and so $\overline{\gamma_{\underline{s}}((0, x_{\underline{s}})} \cup \overline{\gamma_{\underline{s}}((0, x_{\underline{s}})}$ is indecomposable.

In particular, when $\overline{\gamma_{\underline{s}}}$ and $\overline{\gamma_{\underline{t}}}$ are indecomposable, then $\tilde{C}_{\underline{u}} = \overline{\gamma_{\underline{s}}} \cup \overline{\gamma_{\underline{s}}}$ is indecomposable. Suppose one of $\overline{\gamma_{\underline{s}}}$ or $\overline{\gamma_{\underline{t}}}$ was decomposable, without loss of generality let $\overline{\gamma_{\underline{s}}}$ be decomposable. Then there is some forward tail $\gamma_{\underline{s}}([x, \infty))$ which is not in the accumulation set of $\gamma_{\underline{s}}$ or $\gamma_{\underline{t}}$. We therefore have that $\gamma_{\underline{s}}((x, \infty))$ and $\overline{\gamma_{\underline{s}}((0, x))} \cup \overline{\gamma_{\underline{t}}}$ are disjoint.

Since the accumulation sets of $\gamma_{\underline{s}}$ and $\gamma_{\underline{t}}$ are non-empty and equal, we have that $\overline{\gamma_{\underline{s}}((0, x))} \cup \overline{\gamma_{\underline{t}}}$ is connected. Since it is closed in $\tilde{C}_{\underline{u}}$, it is also a continuum. Since $\overline{\gamma_{\underline{s}}((0, x))} \cup \overline{\gamma_{\underline{t}}}$ does not contain $\gamma_{\underline{s}}((x, \infty))$, it is a proper subcontinuum of $\tilde{C}_{\underline{u}}$. It is easy to see, in a similar way, that $\overline{\gamma_{\underline{s}}((x, \infty))}$ is also a proper subcontinuum of $\tilde{C}_{\underline{u}}$, and that $\overline{\gamma_{\underline{s}}(x, \infty)} \cup (\overline{\gamma_{\underline{s}}((0, x))} \cup \overline{\gamma_{\underline{t}}}) = \tilde{C}_{\underline{u}}$. Therefore $\tilde{C}_{\underline{u}}$ is decomposable when one of $\overline{\gamma_{\underline{s}}}$ or $\overline{\gamma_{\underline{t}}}$ is decomposable. ■

9.3 Indecomposability Conditions

It remains now to find conditions for indecomposability of $\overline{\gamma_{\underline{s}}}$ in terms of the augmented itinerary of address $\underline{s}$. To do this, we will study the bounds on the possible itineraries of points in $Q_{\underline{s},n,m}$ for arbitrary $\underline{s}$. We recall that the general behaviour of the orbit of a point $z \in Q_{\underline{s},n,m}$ is that $E^n(z)$ is mapped into A_R for some number of iterates, into $\mathbb{C} \setminus A_R$ for some number of iterates, and then into Q_1 after one or two iterates. We may first determine how long this orbit between the nth and mth iterate stays in $\mathbb{C} \setminus A_R$ compared to how long it stays in A_R.

For the rest of this section, when $Q_{\underline{s},n,m}$ is non-empty, let $m' < m$ be the first value greater than n such that $E^{m'}(z) \in \mathbb{C} \setminus A_R$. We may then establish the following lemma on the growth of some $z \in Q_{\underline{s},n,m}$.

9.3.1. Lemma.
There exists a constant D_1 such that the following holds. Let $z \in Q_{\underline{s},n,m}$ have non-singular itinerary $\underline{u}$ and be such that

$$K := \operatorname{Re}(E^n(z)) - \min(\max \operatorname{Re}(E^n(C_{\underline{u}})), 0) > R_1.$$

Then it follows that

$$-\exp^{m'-n}(\operatorname{Re}(E^n(z)) + D_1) < \operatorname{Re}(E^{m'}(z)) < -\exp^{m'-n}(K - D_1).$$

Proof. Let D_0 be as described in Lemma 8.2.1 and let $D_1 > D_0$. We first aim to prove the upper bound of the inequality, that is

$$\operatorname{Re}(E^{m'}(z)) < -\exp^{m'-n}(K - D_1).$$

For this, we will use Lemma 8.2.1. Let $n' \in \mathbb{N}$ with $n \le n' < m'$, then $|\operatorname{Re}(E^{n'}(z))| - \min(\max \operatorname{Re}(E^n(C_{\underline{u}})), 0)$ is large, and since $E^{n'}(z) \in A_R$, then

$\mathrm{Re}(E^{n'}(z))$ is positive. To be more precise, we will write

$$K_{n'} \coloneqq \mathrm{Re}(E^{n'}(z)) - \min(\max \mathrm{Re}(E^{n}(C_{\underline{u}})), 0),$$

noting here that $K_n = K$. Then Lemma 8.2.1 tells us that

$$|\,\mathrm{Re}(E^{n'+1}(z))| > \max(0, \min(\mathrm{Re}(E^{n'+1}(C_{\underline{u}})))) + \exp(K_{n'} - D_0).$$

When $n' < m' - 1$, then $\mathrm{Re}(E^{n'+1}(z)) = |\,\mathrm{Re}(E^{n'+1}(z))|$ and we find that $K_{n'+1} > \exp(K_{n'} - D_0)$. We note also that $K_{n'} > R_1$ for all n' within our range with K_n' increasing as n' increases. Since $E^{m'}(z) \in \mathbb{C} \setminus A_R$, then we note that $\mathrm{Re}(E^{m'}(z))$ is large and negative with $\mathrm{Re}(E^{m'}(z)) < -\exp(K_{m'-1} - D_0)$. We see then that $\mathrm{Re}(E^{m'}(z))$ can be calculated to be less than something of the form

$$-\exp(\exp(...(\exp(K_n - D_0) - D_0)...) - D_0).$$

We will make this more precise later on. For now, we will define our D_1 such that this nested exponential can be in some way condensed.

Increasing R_1 if necessary, there exists some $D_1 > D_0$ such that for $K > R_1$, we have that $\exp(K - D_0) - D_1 > \exp(K - D_1)$. Rearanging this inequality, we find that such a D_1 may be found when

$$\frac{D_1}{\exp(K - D_1)} + 1 < \frac{D_1}{\exp(R_1 - D_1)} + 1 < \exp(D_1 - D_0)$$

is satisfied. This holds when, for example, $R_1 + 4 > D_1 + 2 > D_0$.

We may now the define maps which we use to describe the growth of $K_{n'}$ and make precise the above nested exponential. Let $F_0 \colon K \mapsto \exp(K) - D_0$ and $F_1 \colon K \mapsto \exp(K) - D_1$. We note that for $K > R_1$, we have that

$$F_0 \circ F_0(K) > F_1 \circ F_0(K) > \exp \circ F_1(K)$$

and $F_1(K - D_0) > \exp(K - D_1)$. It follows that for $n < n' < m'$, we have

$$K_{n'} > \exp \circ F_0^{n'-n-1}(K_n - D_0) > \exp^{n'-n-1} \circ F_1(K_n - D_0) > \exp^{n'-n}(K_n - D_1).$$

Similarly, we have that $\mathrm{Re}(E^{m'}(z)) < -\exp^{m'-n}(K_n - D_1)$.

For the lower bound of $\mathrm{Re}(E^{m'}(z))$, we note that for a point $z_0 \in \mathbb{C}$ with $z_{n'} := E^{n'}(z_0)$, we find that

$$|\mathrm{Re}(z_{n'+1})| \leq 2\pi \exp(\mathrm{Re}(z_{n'})) = \exp(\mathrm{Re}(z_{n'}) + \ln(2\pi)).$$

By a similar argument we may find some D_1 and appropriately large R_1 such that for $K > 0$, we have $\exp(K + \ln(2\pi)) + D_1 < \exp(K + D_1)$. This is satisfied when

$$\frac{D_1}{\exp(K + \ln(2\pi))} + 1 < \frac{D_1}{2\pi} < \exp(D_1 - \ln(2\pi)),$$

which holds, for example, when $D_1 > \ln(2\pi) + 2$. Given that $\mathrm{Re}(z_{n'}) > R_1$ for $n \leq n' < m'$, we may again find that $\mathrm{Re}(z_{m'}) > -\exp^{m'-n}(\mathrm{Re}(z_n) + D_1)$. ∎

We may then establish the following lemma on the behaviour of a point $z \in Q_{\underline{s},n,m}$ which establishes how long it stays near $2\pi i$ after $m' - n$ steps of growth.

9.3.2. Lemma.
There exists a constant $D_2 < R_1 - 1$ such that the following holds. Let $z \in Q_{\underline{s},n,m}$ have non-singular itinerary $\underline{u}$ and be such that

$$K := \mathrm{Re}(E^n(z)) - \min(\max \mathrm{Re}(E^n(C_{\underline{u}})), 0) > R_1.$$

Then there exists some $m' \in \mathbb{N}$ such that n, m', m satisfy

$$\exp^{m'-n}(K - D_2) < m - m' < \exp^{m'-n}(\mathrm{Re}(E^n(z)) + D_2)$$

with $u_{m'+1} = 0$ *and* $u_{m''} = 1$ *for* $m' + 1 < m'' \leq m$.

Proof. We note that $|E'(0)| = |E'(2\pi i)| = 2\pi$. We may assume here that R is sufficiently large that within the components of $\mathbb{C} \setminus A_R$ containing $\{0, 2\pi i\}$, we have that $|E'(z)|$ is bounded below by some value greater than one. There will then exist some $M_+ > M_- > 1$ such that when $|z| < \frac{1}{R}$, then

$$M_-|z| < |E(z) - 2\pi i| < M_+|z|$$

and when $|z - 2\pi i| < \frac{1}{R}$, then

$$M_-|z - 2\pi i| < |E(z) - 2\pi i| < M_+|z - 2\pi i|.$$

When $x > 0$ is sufficiently large and $|z| = \exp(-x)$, then as long as the orbit of z stays in $\mathbb{C} \setminus A_R$ for the first $j - 1$ iterates, we have that

$$M_-^j \exp(-x) < |E^j(z) - 2\pi i| < M_+^j \exp(-x).$$

Let j be the first iterate that $E^j(z) \in A_R$. Then $M_-^{j-1} \exp(-x) < \frac{1}{R}$ and $M_+^j \exp(-x) > \frac{1}{R}$, or equivalently, $j < \frac{x - \ln(R)}{\ln(M_-)} + 1$ and $j > \frac{x - \ln(R)}{\ln(M_+)}$.

Now by the previous lemma, we have

$$\exp(-\exp^{m'}(K - D_1)) < |E^{m'+1}(z)| < \exp(-\exp^{m'}(\mathrm{Re}(E^n(z)) + D_1)),$$

so we may determine a range of m by the following inequality:

$$\frac{\exp^{m'}(K - D_1) - \ln(R)}{\ln(M_+)} < m - m' < \frac{\exp^{m'}(\mathrm{Re}(E^n(z)) + D_1) - \ln(R)}{\ln(M_-)} + 1.$$

If we again ensure that R_1 is sufficiently larger than D_1, there is some $D_2 > D_1$ such that when $K > R_1$, we have that

$$\exp^{m'}(K - D_2) < \frac{\exp^{m'}(K - D_1) - \ln(R)}{\ln(M_+)}$$

and

$$\exp^{m'}(\mathrm{Re}(E^n(z)) + D_2) > \frac{\exp^{m'}(\mathrm{Re}(E^n(z)) + D_1) - \ln(R)}{\ln(M_-)} + 1.$$

Then with this D_2, we have $\exp^{m'}(K-D_2) < m-m' < \exp^{m'}(\mathrm{Re}(E^n(z))+D_2)$ as required. We may note that R_1 may be initially chosen large enough that $m - m' > 2$ is guaranteed and $D_2 < R_1 - 1$.

We may also note that $|E^{m'+1}(z)| < \frac{1}{R}$, and when $m' + 1 < m'' \leq m$, then $|E^{m''}(z) - 2\pi i| < \frac{1}{R}$, and so $u_{m'+1} = 0$ and $u_{m''} = 1$. ∎

These bounds establish the necessary behaviour of points in $Q_{\underline{s},n,m}$ when it is non-empty. In order to determine for which values of n and m the set $Q_{\underline{s},n,m}$ is non-empty, we use the following lemma. This will tell us when $E^m(C_{\underline{u}})$ intersects Q_1 and when there is some sufficiently large point in $E^n(C_{\underline{u}})$ which maps into this intersection.

9.3.3. Lemma.
There exists some $D_3 > 0$ such that the following holds for all $K > 0$:

Let $\underline{u}$ be such that there exists some n with $|u_n| > \exp^n(K + D_3)$. Then there is no $z \in C_{\underline{u}}$ such that $\mathrm{Re}(z) < K$.

Let $\underline{u}$ be such that $|u_n| < \exp^n(K)$. Then there exists some $z \in C_{\underline{u}}$ such that $\mathrm{Re}(z) < K + D_3$. Furthermore, if $X \in \mathbb{C}$ is a connected unbounded set with $\inf_{z\in X} |z| < \exp^n(K + D_3)$, then there exists a point $z \in \mathbb{C}$ with itinerary agreeing with $\underline{u}$ up to u_{n-1} such that $\mathrm{Re}(z) < K + D_3$ and $E^n(z) \in X$.

Proof. Let $K > 0$ and let $z_0 \in \mathbb{C}$ have real part less than K. We will write $z_n := E^n(z_0)$. For $n > 0$, we have that $|z_n| < \exp^n(K + D_1)$. On the other hand, for large u_n we have that T_{u_n} is separated from 0 by at least some linear function on $|u_n|$. So if D_3 is sufficiently larger than D_1, then we can have that $|u_n| > \exp^n(K+D_3)$ implies T_{u_n} is separated from 0 by at least $\exp^n(K+D_1)$, and so z_n cannot visit T_{u_n}. This value of K is therefore a lower bound for the real part of $C_{\underline{u}}$.

Suppose now $\underline{u}$ is such that $|u_n| < \exp^n(K)$, and let D_3 be large. Let $T_{K,0} := \{z \in T_{u_0} \colon a_0 \le \mathrm{Re}(z) \le b_0\}$ where $a_0 := K$ and $b_0 := K + D_3$. We then plan to define $T_{K,j+1} := \{z \in T_{u_{j+1}} \colon a_j < \mathrm{Re}(z) < b_j\}$ in such a way that $T_{K,j+1} \subset E(T_{K,j})$ for all $j \in \mathbb{N}$. We note here that $E(T_{K,j})$ is an annulus with inner radius $\exp(a_j + \ln(2\pi))$ and outer radius $\exp(b_j + \ln(2\pi))$. Let $a_{j+1} = \exp(a_j + \ln(2\pi))$. In this way $T_{K,j+1}$ is guaranteed to lie outside the inner circle of the annulus $E(T_{K,j})$. We may also note here that, using the value of D_1 from Lemma 9.3.1, we have $a_j < \exp^j(K + D_1)$.

We now define H_{j+1} to be the infimum of the positive real values of the set $\{z \in T_{u_{j+1}} \colon |z| > \exp(b_j + \ln(2\pi))\}$. We wish to define b_j in such a way that $a_j < b_j < H_j$ is guaranteed for all $j > 0$. When we have $a_j < b_j < H_j$ for all $j > 0$. Then it follows that $T_{K,j+1}$ lies within the outer circle of $E(T_{K,j})$, so that $T_{K,j+1} \subset E(T_{K,j})$ as required. Taking the value c from Lemma 8.1.2, we know that

$$H_{j+1} \ge \max(\exp(b_j + \ln(2\pi)) - 2\pi|u_{j+1}| + c, 0) \ge$$

$$\ge \max(\exp(b_j + \ln(2\pi)) - \exp^{j+1}(K + \ln(2\pi)) + c, 0).$$

As long as we have $\exp(b_j) > 2(\exp^{j+1}(K + \ln(2\pi)) + c)$, then it follows that $H_{j+1} \ge \frac{1}{2}\exp(b_j + \ln(2\pi)) < \exp(b_j)$. We then define b_j as

$$b_{j+1} := \exp(b_j) := \exp^{j+1}(K + D_4).$$

Setting $D_4 > D_1$, then $b_{j+1} = \exp^{j+1}(K + D_4) > \exp^{j+1}(K + D_1) > a_{j+1}$. It remains to note that when D_4 is large enough, then

$$\exp(b_j) = \exp^{j+1}(K + D_4) > 2(\exp^{j+1}(K + \ln(2\pi)) + c)$$

holds for all $j \in \mathbb{N}$. So b_j may be defined in this way and $T_{K,j}$ will satisfy our desired properties.

For all $N \in \mathbb{N}$, let $T^*_{K,N}$ be the set which contains all the points $z \in T_{K,0}$

such that $E^j(z) \in T_{K,j}$ for all $0 \le j \le N$. Then $T^*_{K,N}$ is non-empty for each N and $T^*_{K,N+1} \subset T^*_{K,N}$. We have that $\bigcup_{N\in\mathbb{N}} \overline{T^*_{K,N}}$ then is non-empty. A point $z \in \bigcup_{N\in\mathbb{N}} \overline{T^*_{K,N}}$ will either be contained in all $T^*_{K,N}$, and have itinerary $\underline{u}$, or else will be on some $\overline{T^*_{K,N}} \setminus T^*_{K,N}$, so that $E^N(z)$ lies on a preimage of γ_0. Suppose $E^{N+1}(z) \in \gamma_0$, then $\underline{u}$ will be singular, in particular, for all $n > N+1$ we would have that $T_{K,n}$ intersects γ_1. In this case, there would exist some $z \in T^*_{K,n} \in T_{K,0}$ for which $E^n(z \in \gamma_0)$ has itinerary $\underline{u}$. In either case, there exists a point in $C_{\underline{u}}$ with real part less than $K + D_3$.

To prove the last part, note that if X has $\inf_{z\in X} |z| < \exp^n(K + D_3)$, then $E(T_{K,n-1})$ intersects X. There then exists a point in $z \in T^*_{K,n-1}$ such that $E^n(z) \in X$. All points in $T^*_{K,n-1}$ have itineraries agreeing with $\underline{u}$ up to u_n. ∎

It is now possible to state a set of conditions on $\underline{s}, n, m$ which must necessarily hold for $(Q_{\underline{s},n,m})$ to be non-empty and large.

9.3.4. Lemma (Necessary Conditions)**.**
Let $\underline{s}$ be a non-singular address with itinerary $\underline{u}$. Let $n \in N_{\underline{s}}$ and $m > n$. Then in order for $Q_{\underline{s},n,m}$ to be non-empty, it is necessary that there exists some $m' \in \mathbb{N}$ such that the following hold:

- $n < m' < m$;
- $u_{m'+1} = 0$;
- *For* $m' + 1 < j \le m$ *we have* $u_j = 1$;
- $E^m(C_{\underline{u}})$ *intersects* Q_1;
- $m - m' > \exp^{m'-n}(R_1 - D_2)$;
- *If $\underline{u}$ is a binary sequence, it is also necessary that* $E^{m'}(\underline{s}) \in T^+$.

Let $K > R_1$. It is necessary for $\sup \operatorname{Re}(Q_{\underline{s}}, n, m) > K$ *that*

$$m - m' > \exp^{m'-n}(K - \min \operatorname{Re}(E^n(C_{\underline{u}})) - D_2),$$

and it is necessary for $\sup \mathrm{Re}(Q_{\underline{s}}, n, m) < K$ *that*

$$m - m' < \exp^{m'-n}(K + D_2).$$

Proof. Suppose $Q_{\underline{s},n,m}$ is non-empty. By Lemma 9.3.2, we immediately have that there will indeed exist some m' such that:

- $n < m' < m$;
- $u_{m'+1} = 0$;
- For $m' + 1 < m'' \leq m$, we have $u_{m''} = 1$;
- $E^m(C_{\underline{u}})$ intersects Q_1.

If $\sup \mathrm{Re}(Q_{\underline{s}}, n, m) > K$, then we have that there is some $z \in Q_{\underline{s},n,m}$ with $\mathrm{Re}(E^n(z)) > K$. It follows from Lemma 9.3.2 that

$$m - m' > \exp^{m'-n}(K - \min \mathrm{Re}(E^n(C_{\underline{u}})) - D_2).$$

Given that when $z \in Q_{\underline{s},n,m}$ we have $\mathrm{Re}(E^n(z)) > \min \mathrm{Re}(E^n(C_{\underline{u}})) + R_1$, it follows that $m - m' > \exp^{m'-n}(R_1 - D_2)$ is necessary in order for $Q_{\underline{s},n,m}$ to be non-empty.

If $\sup \mathrm{Re}(Q_{\underline{s},n,m}) < K$, then we have that there is some $z \in Q_{\underline{s},n,m}$ with $\mathrm{Re}(E^n(z)) < K$. It follows that we have that $m - m' < \exp^{m'-n}(K + D_2)$.

Suppose $\underline{u}$ is binary. Then for $n \leq n' < m'$, we have $E^{n'}(Q_{\underline{s},n,m})$ is in the same component of T^+ or T^- as $E^{n'}(\underline{s})$. We can show this is the case by applying Lemma 9.2.1 to find two forward tails, $E^n(\gamma_{\underline{s}}([x_{\underline{s}}, \infty))) \subset T^+ \cap E^n(C_{\underline{u}})$ and $E^n(\gamma_{\underline{t}}([x_{\underline{t}}, \infty))) \subset T^- \cap E^n(C_{\underline{u}})$, both with minimal real part R_0. These tails will have the property that for each $z \in Q_{\underline{s},n,m}$, then $E^{n'}(z)$ must be in the same component as one of either $E^{n'}(\gamma_{\underline{s}}([x_{\underline{s}}, \infty)))$ or $E^{n'}(\gamma_{\underline{t}}([x_{\underline{t}}, \infty)))$, and $Rea(E^{n'}(z))$ will be greater than the minimal part of both of these tails. Therefore $E^{n'}(Q_{\underline{s},n,m})$ will be within a bounded $\mathrm{dist}_{A;A'}$ of at least one of the forward tails $E^{n'}(\gamma_{\underline{s}}([x_{\underline{s}},\infty)))$ or $E^{n'}(\gamma_{\underline{t}}([x_{\underline{t}},\infty)))$, which we note are forward tails of the rays in $E^{n'}(C_{\underline{u}})$. If $\text{dist}_{A;A'}(E^{n'}(Q_{\underline{s},n,m}), E^{n'}(\gamma_{\underline{t}}([x_{\underline{t}},\infty))))$ is bounded, then $\text{dist}_{A;A'}(E^{n}(Q_{\underline{s},n,m}), E^{n}(\gamma_{\underline{t}}([x_{\underline{t}},\infty))))$ is also bounded. We may choose R_1 such that $\text{dist}_{A;A'}(\{z \in T^+ \colon \text{Re}(z) > R_1\}, T^-)$ is greater than this bound. Since we can have that $E^n(\gamma_{\underline{t}}([x_{\underline{t}},\infty))) \subset T^-$, then it follows that it is impossible for $E^{n'}(\gamma_{\underline{t}}([x_{\underline{t}},\infty)))$ and $E^{n'}(Q_{\underline{s},n,m})$ to be contained in the same component of T^+ or T^-. Therefore $E^{n'}(Q_{\underline{s},n,m})$ must be contained in the same component of T^+ or T^- as $E^{n'}(\gamma_{\underline{s}}([x_{\underline{s}},\infty)))$ and therefore the same component as $E^{n'}(\underline{s})$.

We note that $E^{m'-1}(Q_{\underline{s}}, n, m)$ must lie in the unbounded component of $E^{-1}(A_R) \cap \tilde{T}_{u_{m'-1}}$. The unbounded component of $E^{-1}(A_R) \cap \tilde{T}_{u_{m'-1}}$ must lie in the same component of $T^- \cup \tilde{T}_{u_{m'-1}}$ or $T^+ \cup \tilde{T}_{u_{m'-1}}$ as $E^{m'-1}((\infty))$. We note that $u_{m'} \in \{0,1\}$, and so the components of $T^- \cup \tilde{T}_{u_{m'-1}} \cap \mathcal{S}$ and $T^+ \cup \tilde{T}_{u_{m'-1}} \cap \mathcal{S}$ take as their boundaries some consecutive pair of the addresses $(u_{m'}, 0, 1, 1, ...), (u_{m'}, 1, 1, ...), (u_{m'}+1, 0, 1, ...)$. The preimage of (∞) must be an intermediate address of the form $(u_{m'} + \frac{1}{2}, \infty)$. We see here that the component containing the preimage of (∞) is precisely

$$(u_{m'} + \frac{1}{2}, \infty) \in ((u_{m'}, 1, 1, ...), (u_{m'}+1, 0, 1, ...)).$$

This component will contain $E^{m'-1}(\underline{s})$ precisely when

$$E^{m'}(\underline{s}) \in ((1, 1, ...), (0, 1, 1, ...)) \subset T^+.$$

■

We may similarly state a set of conditions on $\underline{s}, n, m$ which are sufficient to for $(Q_{\underline{s},n,m})$ to be non-empty and conditions which are sufficient for points in $(Q_{\underline{s},n,m})$ to be large.

9.3.5. Lemma (Sufficient Conditions)**.**
Let $\underline{s}, n, m, m'$ *be such that the necessary conditions in Lemma 9.3.4 hold. There is some* $k_0 \in \mathbb{N}$ *and* $D_4 > 0$ *such that if the following conditions also hold, then* $Q_{\underline{s},n,m}$ *is non-empty:*

- $E^{m+k_0}(C_{\underline{u}}) \in Q_1$*;*
- *For* $m < j \le m + k_0$ *we have* $u_j = 1$*;*
- *If* $\underline{u}$ *is a non-binary sequence, there exists some* $n' \ge n$ *with* $n' < m'$ *such that* $u_{n'}$ *is neither* 0 *nor* 1*;*
- *There exists some* $K \ge R_1$ *such that* $m - m' > \exp^{m'-n}(K + D_4)$ *and such that for* $j \in \mathbb{N}$ *with* $j > n$ *we have* $|u_j| < \exp^{j-n}(K)$*.*

Furthermore, it is sufficient for $\sup \operatorname{Re}(Q_{\underline{s}}, n, m) \ge K$ *that*

$$m - m' \ge \exp^{m'-n}(K + D_2).$$

It is sufficient for $\sup \operatorname{Re}(Q_{\underline{s}}, n, m) \le K$ *that*

$$m - m' \le \exp^{m'-n}(K - \min(\max \operatorname{Re}(E^n(C_{\underline{u}})), 0) - D_2).$$

Proof. Let $z \in \mathbb{C} \setminus \{2\pi i\}$ be such that $|z - 2\pi i| < \frac{1}{R}$. We may choose z to be near enough to $2\pi i$ that $\operatorname{dist}_{A'}(z, A_R) > \operatorname{diam}_{A;A'}(Q_1)$. Since $2\pi i$ is a repelling fixed point, there exists some value of k_0 such that $E^{k_0}(z) \in \overline{Q_1}$.

Now suppose there exists some address $\underline{s}$ with itinerary $\underline{u}$, some $K \ge R_1$, and some $n < m' < m$ such that the above conditions for $Q_{\underline{s},n,m}$ to be non-empty are satisfied.

If $E^{m+k_0}(C_{\underline{u}})$ intersects Q_1, then there exists a path P between $E^{k_0}(z)$ and $E^{m+k_0}(C_{\underline{u}}))$ which is homotopic to a path in T_1 and has length less than $\operatorname{diam}_{A;A'}(Q_1)$ (this is measured with respect to the metric $\operatorname{dist}_{A'}$). Let P' be the branch of $E^{-k_0}(P)$ which has an endpoint at z. This is precisely the branch for which for every $j \in \mathbb{N}$ with $j \le k$, we have that both endpoints of $E^j(P')$ are in T_1. We recall that for $m < j \leq m + k_0$, we have $u_j = 1$. It follows that P' has an endpoint on $E^m(C_{\underline{u}})$. We have that

$$\mathrm{dist}_{A'}(z, E^m(C_{\underline{u}})) < \mathrm{diam}_{A:A'}(Q_1) < \mathrm{dist}_{A;A'}(z, A_R),$$

and so the euclidean distance from $2\pi i$ to $E^m(C_{\underline{u}})$ is less than $\frac{1}{R}$.

We use again the constants M_-, M_+ from Lemma 9.3.2. Let $z_0 \in C_{\underline{u}}$ with $z_j := E^j(z)$. We may choose z_0 such that $|z_m - 2\pi i|$ is slightly larger than $\frac{1}{R}$. In this way, $z_m \in Q_1$ and $z_{m-1} \in \mathbb{C} \setminus Q_1$. Then for $m' + 1 < j < m$, we have

$$\frac{1}{RM_+^{m-j}} < |z_j - 2\pi i| < \frac{1}{RM_-^{m-j}} < \frac{1}{R},$$

$$\frac{1}{RM_+^{m-m'-1}} < |z_{m'+1}| < \frac{1}{RM_-^{m-m'-1}} < \frac{1}{R},$$

and so we have

$$(m - m') \ln(M_-) + \ln(R) < -\mathrm{Re}(z_{m'}) < (m - m') \ln(M_+) + \ln(R).$$

It follows from the bounds in the hypothesis on $(m - m')$ that

$$|z_{m'}| \geq |\mathrm{Re}(z_{m'})| > \exp^{m'-n}(K + D_4 + \ln(\ln(M_-))) + \ln(R).$$

We note that since $|u_j| < \exp^{j-n}(K)$ for $j > n$, then it follows from Lemma 9.3.3 that there exists some $z \in E^n(C_{\underline{u}})$ with real part less than $K + D_3$. Suppose $\mathrm{Re}(z_n) \leq K + D_3 + R_1$. Then $|z_{m'}| \leq \exp^{m'-n}(K + D_3 + R_1 + D_1)$. If we set D_4 to be sufficiently larger than $D_3 + R_1 + D_1$, then we can guarantee that $|z_{m'}| > \exp^{m'-n}(K + D_4 + \ln(\ln(M_-))) + \ln(R) > \exp^{m'-n}(K + D_3 + R_1 + D_1)$ so that $\mathrm{Re}(z_n) > \min \mathrm{Re}(E^n(C_{\underline{u}})) + R_1$.

It remains to show that $z_n \in T^+$. Suppose first that $\underline{u}$ is binary and $E^{m'}(\underline{s}) \in T^+$. Then as in the proof of Lemma 9.3.5, we have that the preimage of the unbounded component of $E^{-1}(\mathbb{C} \setminus A_R) \cup T_{u_{m'-1}}$ is contained in the same component of T^- or T^+ as $E^{m-1}(\underline{s})$. It follows again that there is a forward tail $\gamma_{\underline{s}}([x,\infty))$ such that $E^n(\gamma_{\underline{s}}([x,\infty))) \in T^+$ with minimal real part R_0. The minimal real part of $E^{m'-1}(\gamma_{\underline{s}}([x,\infty)))$ will be less than that of $z_{m'-1}$ so that $\operatorname{dist}_{A;A'}(z_{m'-1}, E^{m'-1}(\gamma_{\underline{s}}([x,\infty))))$ is bounded by H'. It follows that $\operatorname{dist}_{A;A'}(z_n, E^n(\gamma_{\underline{s}}((x,\infty))))$ is also bounded by H'. So long as R_1 is chosen sufficiently large that $\operatorname{dist}_{A;A'}(\{z \in T^+ \colon \operatorname{Re}(z) > R_1\}, T^-) > H'$, then we have $\operatorname{dist}_{A;A'}(z_n, T^-) < H'$ and so $z_n \in T^+$.

Similarly, suppose $\underline{u}$ is non-binary and there exists some n' between n and m' such that $u_{n'}$ is neither 0 nor 1. Then both $E^{n'}(\underline{s}) \in T^+$ and $z_{n'} \in T^+$. It follows by the same argument that $z_n \in T^+$.

As in Lemma 9.3.4, if we have that if $\sup \operatorname{Re}(Q_{\underline{s},n,m}) > K$, then it follows that $m - m' > \exp^{m'-n}(K - \min \operatorname{Re}(E^n(C_{\underline{u}})) - D_2)$. Therefore, in order to have $\sup \operatorname{Re}(Q_{\underline{s},n,m}) \leq K$, it is sufficient that

$$m - m' \leq \exp^{m'-n}(K - \min \operatorname{Re}(E^n(C_{\underline{u}})) - D_2).$$

Similarly, if we have that if $\sup \operatorname{Re}(Q_{\underline{s},n,m}) < K$, then it follows that we have $m - m' < \exp^{m'-n}(K + D_2)$. In order to have $\sup \operatorname{Re}(Q_{\underline{s},n,m}) \geq K$, it is sufficient that $m - m' \geq \exp^{m'-n}(K + D_2)$. ■

It is now possible to prove our main theorems on the indecomposability of $\gamma_{\underline{s}}$ in terms of the address $\underline{s}$.

9.3.6. Theorem.
Let $\underline{s}$ be an address with non-singular binary itinerary $\underline{u}$ and with augmented itinerary $\underline{u}^*$. The set $\overline{\gamma_{\underline{s}}}$ is indecomposable if and only if for all $k \in \mathbb{Z}$, there exist infinitely many $m' \in \mathbb{N}$ with $m' \geq k$ such that the following conditions hold:

- $\chi_{m'} \in \{-,+\}$;
- $u_{m'+1} = 0$;
- $u_j = 1$ *for all* $j > m' + 1$ *with* $j < \exp^{m'-k}(1)$.

We have that $\gamma_{\underline{s}}$ is indecomposable beyond some $t \in (0, \infty)$ if and only if the above conditions are satisfied for some, but not all, $k \in \mathbb{Z}$.

Proof. Suppose some $k \in \mathbb{Z}$ exists such that there are infinitely many m' which satisfy the above conditions for k. Let $n \in N_{\underline{s}}$ be sufficiently large that $\exp^{n-k}(1) > R_1 + D_4 + 1$ and $u_j \in \{0, 1\}$ for all $j \geq n$. Let $m' \in \mathbb{N}$ with $m' > n$ and which satisfies the above conditions for k. Let $m := \lfloor \exp^{m'-k}(1) \rfloor - k_0$. Then $m > \exp^{m'-n}(R_1 + D_4 + 1) - 1 - k_0$ and if m' is sufficiently large, then $m - m' > \exp^{m'-n}(R_1 + D_4)$. We then have that $u_j = 1$ for $m+1 < j \leq m+k_0$. Since $u_{m+k_0} = 1$ and $E^{m+k_0}(C_{\underline{u}})$ is connected and intersects both T^- and T^+, we know that the minimum real part of $E^{m+k_0}(C_{\underline{u}})$ is less than 0 and it therefore intersects Q_1. Since $\chi_{m'} \in \{-, +\}$ we know that $E^{m'}(\underline{s}) \in T^+$.

Let $K := R_1$. Then the fact that $|u_j| < \exp^{j-n}(K)$ for $n < j$ follows from the fact $\underline{u}$ is binary. As before, we have $m - m' > \exp^{m'-n}(K + D_4)$.

This is sufficient to satisfy the conditions in Lemma 9.3.5 that $Q_{\underline{s},n,m}$ is non-empty. Since m is defined uniquely for each sufficiently large m', we have that there are infinitely many m such that $Q_{\underline{s},n,m}$ is non-empty.

Now let K be arbitrarily large. There then exists then some $k \in \mathbb{Z}$ such that $E^{n-k}(1) > K + D_2 + 1$ for some $n \in N_{\underline{s}}$. Suppose there are infinitely many $m' > k$ which satisfy the above properties for k. For each such m' let $m := \lfloor \exp^{m'-n}(K + D_2 + 1) \rfloor - k_0$. Then for such m we have

$$m - m' \geq \exp^{m'-n}(K + D_2 + 1) - 1 - k_0 - m'.$$

For sufficiently large m', we have again that

$$m - m' \geq \exp^{m'-n}(K + D_2 + 1) - 1 - k_0 - m' > \exp^{m'-n}(K + D_2).$$

By Lemma 9.3.5, for infinitely many m we have $\sup \operatorname{Re}(E^n(Q_{\underline{s},n,m})) > K$. If infinitely many m' satisfy the above conditions for all $k \in \mathbb{Z}$, then

$$\limsup_{m>n} \max \operatorname{Re}(Q_{\underline{s},n,m}) = \infty,$$

and by Lemma 9.2.7, $\overline{\gamma_{\underline{s}}}$ is indecomposable.

We note now that if the above conditions hold for k, they also hold for $k+1$. If they hold for some but not all k, then there must be some minimal k such that they hold. Suppose now k is such a minimal value and $n \in N_{\underline{s}}$ is chosen as above so that $Q_{\underline{s},n,m}$ is non-empty for infinitely many m. Let now $K := E^{n-k+1}(1)$. Then for all but finitely many appropriately chosen $m', m \in \mathbb{N}$, we have that $m - m' < \exp m' - n(K)$. It follows from Lemma 9.3.5 that $\sup \operatorname{Re}(Q_{\underline{s}}, n, m) \leq K + \min(\max \operatorname{Re}(E^n(C_{\underline{u}})), 0) + D_2$ for all but finitely many m. Since there are only finitely many exceptions and each $Q_{\underline{s},n,m}$ is always bounded we have that $\limsup_{m>n} \max \operatorname{Re}(Q_{\underline{s},n,m})$ is finite. From Lemma 9.2.7, it follows that $\gamma_{\underline{s}}$ is indecomposable beyond some $t \in (0, \infty)$.

Suppose, finally, that there exists no such k satisfying the above conditions for infinitely many m'. Then for all $n \in N_{\underline{s}}$, we may choose some k such that $E^{n-k}(1) < (R_1 - D_2)$. Then there are only finitely many m' such that there exists some m such that we have $\chi_{m'} \in \{-, +\}$, $u_{m'+1} = 0$ and $u_j = 1$ for $m'+1 < j \leq m$ where $m - m' \geq E^{n-m'}(R_1 - D_2)$. Since $\underline{s}$ is chosen to be non-singular, there are only finitely many such m associated with each m'. These are the only m' and m which satisfy the necessary conditions of Lemma 9.3.4. There are therefore only finitely many m such that $Q_{\underline{s},n,m}$ is non-empty. This holds for all $n \in N_{\underline{s}}$, and so by Lemma 9.2.7 $\gamma_{\underline{s}}$ is not indecomposable beyond any $t \in (0, \infty]$. ■

9.3.7. Remark.
For a binary type itinerary $\underline{u}$, the topological behaviour of the rays contained in $\tilde{C}_{\underline{u}}$ may be chosen independently. That is, each ray may be chosen independently to have an indecomposable closure, be indecomposable beyond some point $t \in (0, \infty)$, or not indecomposable beyond any point. To see this,

take m' to be sufficiently large. Whenever $\chi_{m'} \in \{-,+\}$ for one of these addresses, then for the other address we have $\chi_{m'} = 0$. In fact, it follows from Lemma 5.0.3 that for one of these rays, $\chi_{m'} \in \{-,+\}$ will be satisfied precisely when $m'+1$ is the kth 0 in $\underline{u}$ and k is even. For the other ray, $\chi_{m'} \in \{-,+\}$ is satisfied precisely when such a k is odd. The topology of $\tilde{C}_{\underline{u}}$ is then determined by the number of 1s following the odd and even 0s.

9.3.8. Theorem.
Let $\underline{s}$ have an exponentially bounded non-binary itinerary $\underline{u}$. The set $\overline{\gamma_{\underline{s}}}$ is indecomposable if and only if for all $k \in \mathbb{Z}$ there exist infinitely many $m' \in \mathbb{N}$ with $m' \geq k$ such that the following conditions hold:

- $u_{m'+1} = 0$;
- $u_j = 1$ *for all* $j > m'+1$ *with* $j < \exp^{m'-k}(1)$;
- *Let* $M := \lfloor \exp^{m'-k}(1) \rfloor$, *then for all* $j > M$ *we have* $|u_j| \leq \exp^{j-M}(1)$.

We have that $\gamma_{\underline{s}}$ is indecomposable beyond some $t \in (0,\infty)$ if and only if the above conditions are satisfied for some, but not all, $k \in \mathbb{Z}$.

Proof. This proof follows similarly to the proof of Theorem 9.3.6, except that we must determine for which values of m we have that $E^m(C_{\underline{u}})$ intersects Q_1.

Suppose $M \in \mathbb{N}$ is such that $|u_j| \leq \exp^{j-M}(1)$ for $j > M$. Then by Lemma 9.3.3, we have that $\min \operatorname{Re}(E^M(C_{\underline{u}})) < 1 + D_3$. By a similar argument to the the one establishing k_0 in Lemma 9.3.5, we may find some k_1 such that if $u_j = 1$ for $M - k_1 \leq j \leq M$, then $E^{M-k_1}(C_{\underline{u}})$ intersects Q_1. That is, let Q_* be the set $\{z \in T_1 \cap A_R \colon \operatorname{Re}(z) < 1 + D_3\}$. Let $z \in \mathbb{C} \setminus \{2\pi i\}$ be such that $|z - 2\pi i| < \frac{1}{R}$ and $\operatorname{dist}_{A'}(z, \mathbb{C} \setminus A_R) > \operatorname{diam}_{A;A'}(Q_*)$. Then let k_1 be such that $E^{k_1}(z) \in \overline{Q_1}$. By the same argument as before, we have that $\operatorname{dist}_{A'}(z, E^{M-k_1}(C_{\underline{u}})) > \operatorname{diam}_{A;A'}(Q_*)$, and so $E^{M-k_1}(C_{\underline{u}})$ intersects Q_1.

Suppose conversely that $M \in \mathbb{N}$ and there is some $j > M$ such that $|u_j| > \exp^{j-M}(1)$. We may choose that k_1 is also sufficiently large that

$\exp^{k_1}(1) > R_1 + \max \operatorname{Re}(\mathbb{C} \setminus A_R) + D_3$. By Lemma 9.3.3, we have that $\min \operatorname{Re}(E^{M+k_1}(C_{\underline{u}})) > R_1 + \max \operatorname{Re}(\mathbb{C} \setminus A_R)$, and so $E^{M+k_1}(C_{\underline{u}})$ does not intersect Q_1.

Suppose again there is some k such that the above conditions hold for infinitely many m'. There are then infinitely many $M \in \mathbb{N}$ such that for $j > M$, we have $|u_j| \leq \exp^{j-M}(1)$. We choose such an M sufficiently large that $\exp^{M-k}(1) > R_1 + D_4 + 2$ and let $n \in N_{\underline{s}}$ be such that $n > M$. Let $K := \exp^{n-M}(R_1 + D_4 + 1)$. Then we have that $|u_j| \leq \exp^{j-n}(K)$ for all $j \geq n$. We also have that there are infinitely many m' with associated m such that $u_{m'+1} = 0$ and $u_j = 1$ for $m' + 1 < j \leq m$ with

$$m - m' > \exp^{m'-n}(K + 1) - k_1 - k_0 - 1$$

and such that $E^m(C_{\underline{u}})$ intersects Q_1. When m' is sufficiently large, we have that $m - m' > \exp^{m'-n}(K)$. For all sufficiently large m', there exists some $n' \geq n$ with $n' < m'$ such that $u_{n'}$ is neither 0 nor 1. Therefore when infinitely many m' satisfy the above conditions for some k, we have that there are infinitely many m such that $Q_{\underline{s},n,m}$ is non-empty.

It follows from a similar argument as before that for all $K \geq R_1$ and $n \in N_{\underline{s}}$, there is some k such that $E^{n-k}(1) > K + D_2 + 1$. When there are infinitely many m' which satisfy the above conditions for k, it follows as before that this implies there are infinitely many m such that $\sup \operatorname{Re}(E^n(Q_{\underline{s},n,m})) > K$. When all $k \in \mathbb{Z}$ satisfy these equations, then $\limsup_{m>n} \max \operatorname{Re}(Q_{\underline{s},n,m}) = \infty$, and by Lemma 9.2.7 again, $\overline{\gamma_{\underline{s}}}$ is indecomposable.

Similarly, suppose there is some minimal k satisfying the above conditions. There are then only finitely many m' satisfying these conditions for $k - 1$. This means there are only finitely many pairs m', m such that $u_{m'+1} = 0$ and $u_j = 1$ for $m' + 1 < j \leq m$ with $E^m(C_{\underline{u}})$ intersecting Q_1 and with $m > \exp^{m'-k}(1) + k_1$. We may fix some $n \in N_{\underline{s}}$ with $n > k$ such that $Q_{\underline{s},n,m}$ is non-empty for infinitely many m. Let $K := \exp^{n-k+1}(1)$. Then there are only finitely many appropriately chosen m', m such that we have

$$m - m' \geq \exp^{m'-n}(K) + k_1.$$

For sufficiently large m' we have $\exp^{m'-n}(K) + k_1 + m' < \exp^{m'-n}(K+1)$. It follows that there are only finitely many m', m such that

$$m - m' \geq \exp^{m'-n}(K+1).$$

There are therefore only finitely many m such that

$$\sup \mathrm{Re}(Q_{\underline{s},n,m}) \leq K + \min(\max \mathrm{Re}(E^n(C_{\underline{u}})), 0) + D_2 + 1.$$

It follows again that $\limsup_{m>n} \max \mathrm{Re}(Q_{\underline{s},n,m})$ is finite. From Lemma 9.2.7, it follows that $\gamma_{\underline{s}}$ is indecomposable beyond some point $t \in (0, \infty)$.

Suppose the conditions are satisfied for no $k \in \mathbb{Z}$. Let $n \in N_{\underline{s}}$ and $k := n$ so that we have $E^{n-k}(1) < (R_1 - D_2)$. There are only finitely many m', m such that $u_{m'+1} = 0$ and $u_j = 1$ for $m'+1 < j \leq m$, with $E^m(C_{\underline{u}})$ intersecting Q_1, and with $m > \exp^{m'-k}(1) + k_1$. There are therefore only finitely many m, m' such that $m - m' > \exp^{m'-n}(R_1 - D_2)$. These are the only m, m' which satisfy the necessary conditions of Lemma 9.3.4 so that $Q_{\underline{s},n,m}$ is non-empty. For each $n \in N_{\underline{s}}$, there are therefore only finitely many m such that $Q_{\underline{s},n,m}$ is non-empty. By Lemma 9.2.7, $\gamma_{\underline{s}}$ is then not indecomposable beyond any $t \in (0, \infty]$. ■

9.4 Results in Terms of Addresses

We may note that the conditions on the itineraries of rays for Theorem 9.3.6 and Theorem 9.3.8 also hold when applied to the addresses of rays. Before we restate these, we first note the conditions for an address $\underline{s}$ to have a binary or singular type itinerary.

9.4.1. Lemma.
An address $\underline{s}$ has singular itinerary if and only if there is some $N \in \mathbb{N}$ such that for $n > N$, we have that $s_n = 1$.

An address $\underline{s}$ has binary type itinerary if and only if there is some $N \in \mathbb{N}$ such that for $n > N$ we have that $s_n \in \{0, 1, 2\}$ and:

- *If $s_n = 0$, then either $s_{n+1} \in \{1, 2\}$ or s_n is followed by a sequence of the form $0, 2$ or $0, 1, ..., 1, 2$ or $0, 1, 1, ...$;*
- *If $s_n = 2$, then $s_{n+1} = 0$ and either $s_{n+2} = 0$ or s_{n+1} is followed by a sequence of the form $1, ..., 1, 0$.*

Proof. Suppose $\underline{s}$ is such that there is some $N \in \mathbb{N}$ such that for $n > N$, we have that $s_n = 1$. Then $\underline{s}$ clearly has singular itinerary. Let $\underline{s}$ have a singular itinerary $\underline{u}$. Let N be such that $u_n = 1$ for all $n > N$. It follows that since $\sigma^n(\underline{s}) \in [(1, 0, 1, 1, ...), (2, 0, 1, 1, ...))$ for $n > N$, then $s_n \in \{1, 2\}$. However, when $s_n = 2$, then $s_{n+1} = 0$, which cannot be the case.

Suppose $\underline{s}$ is such that the above conditions for having binary type itinerary are satisfied for some N. It can be seen that for all $n > N$ that $\sigma^n(\underline{s}) \in [(0, 0, 1, 1, ...), (2, 0, 1, 1, ...))$ is also satisfied. The itinerary of $\underline{s}$ must then be binary type.

Suppose $\underline{s}$ has binary type itinerary $\underline{u}$. There then exists some N such that for $n > N$, we have $\sigma^n(\underline{s}) \in [(0, 0, 1, 1, ...), (2, 0, 1, 1, ...))$. It can be easily seen that the above conditions then necessarily hold for $n > N$. ■

We may now restate Theorem 9.3.6 as follows.

9.4.2. Corollary.
Let $\underline{s}$ be an address with non-singular binary itinerary. The set $\overline{\gamma_{\underline{s}}}$ is indecomposable if and only if for all $k \in \mathbb{Z}$ there exist infinitely many $m' \in \mathbb{N}$ with $m' \geq k$ such that the following conditions hold:

- $s_{m'} \in \{0, 2\}$;

- $s_{m'+1} = 0$;
- $s_j = 1$ *for all* $j > m' + 1$ *with* $j < \exp^{m'-k}(1)$.

We have that $\gamma_{\underline{s}}$ *is indecomposable beyond some* $t \in (0, \infty)$ *if and only if the above conditions are satisfied for some, but not all,* $k \in \mathbb{Z}$.

Proof. Let $\underline{s}$ have non-singular binary augmented itinerary $\underline{u}^*$ and let $N \in \mathbb{N}$ be such that for $n > N$, we have $\sigma^n(\underline{s}) \in [(0, 0, 1, 1, ...), (2, 0, 1, 1, ...))$. Let $m' > N$ and $n > m' + 1$ be such that $u_{m'+1} = 0$ and for $m' + 1 < j \leq n$ we have $u_j = 1$. It follows that $s_{m'+1} = 0$, and for $m' + 1 < j \leq n - 1$, we have $s_j = 1$. Similarly, let $m' > N$ and $n > m' + 1$ be such that $s_{m'+1} = 0$ and for $m' + 1 < j \leq n$ we have $s_j = 1$. It follows that $u_{m'+1} = 0$ and for $m' + 1 < j \leq n - 1$ we have $u_j = 1$. When $s_{m'} = 0$, then $\chi_{m'} = -$; when $s_{m'} = 2$, then $\chi_{m'} = +$; when $s_{m'} = 1$, then $\chi_{m'} = 0$.

Let $k \in \mathbb{Z}$ and suppose m' is sufficiently large. Suppose these conditions hold for $\underline{u}^*$:

- $\chi_{m'} \in \{-, +\}$;
- $u_{m'+1} = 0$;
- $u_j = 1$ for all $j > m' + 1$ with $j < \exp^{m'-k}(1) + 1$.

Then the following conditions also hold for $\underline{s}$:

- $s_{m'} \in \{0, 2\}$;
- $s_{m'+1} = 0$;
- $s_j = 1$ for all $j > m' + 1$ with $j < \exp^{m'-k}(1)$.

If the conditions in Theorem 9.3.6 on $\underline{u}^*$ hold for k, the conditions on $\underline{s}$ in this corollary hold for $k + 1$. By a similar argument, we find that when the conditions on $\underline{s}$ in this corollary hold for k, the conditions in Theorem 9.3.6 on $\underline{u}^*$ hold for $k + 1$.

When $\underline{u}^*$ is such that the conditions in Theorem 9.3.6 are satisfied for all k, then the conditions on $\underline{s}$ are also satisfied for all k. When $\underline{u}^*$ is such that the conditions in Theorem 9.3.6 are satisfied only for some k, then the conditions on $\underline{s}$ are also satisfied only for some k. When $\underline{u}^*$ is such that the conditions in Theorem 9.3.6 are not satisfied for any k, then the conditions on $\underline{s}$ are not satisfied for any k. ■

9.4.3. Corollary.
Let $\underline{s}$ have an exponentially bounded non-binary itinerary. The set $\overline{\gamma_{\underline{s}}}$ is indecomposable if and only if for all $k \in \mathbb{Z}$ there exist infinitely many $m' \in \mathbb{N}$ with $m' \geq k$ such that the following conditions hold:

- $s_{m'+1} = 0$;
- $s_j = 1$ *for all* $j > m' + 1$ *with* $j < \exp^{m'-k}(1)$;
- *Let* $M := \lfloor \exp^{m'-k}(1) \rfloor$. *Then for all* $j > M$ *we have* $|s_j| \leq \exp^{j-M}(1)$.

We have that $\gamma_{\underline{s}}$ is indecomposable beyond some $t \in (0, \infty)$ if and only if the above conditions are satisfied for some, but not all, $k \in \mathbb{Z}$.

Proof. This follows from Theorem 9.3.8 by the same argument as before, except we also note the fact that $|s_j|$ differs from $|u_j|$ by at most 1. Taking $M := \lfloor \exp^{m'-k}(1) \rfloor$, if $|s_j| \leq \exp^{j-M}(1)$, then $|u_j| \leq \exp^{j-M}(1) + 1$. Taking $M' := \lfloor \exp^{m'-(k+1)}(1) \rfloor$, then if m' is sufficiently large with respect to k, then $|u_j| \leq \exp^{j-M'}(1) + 1$ follows. We have again that when when the conditions on $\underline{s}$ in this corollary hold for k, the conditions in Theorem 9.3.8 on $\underline{u}^*$ hold for $k + 1$.

By a similar argument, when the conditions on $\underline{s}$ in this corollary hold for k, the conditions in Theorem 9.3.8 on $\underline{u}^*$ hold for $k+1$. The proof concludes as before. ■

9.5 Generalising to the Riemann Sphere

The results in Theorem 9.3.6 and Theorem 9.3.8 also apply when taking the closure of $\gamma_{\underline{s}}$ in the Riemann sphere. In this section, for a set $X \in \mathbb{C}$, we will define $\tilde{X}$ to be its closure in $\tilde{\mathbb{C}}$, we define $\hat{X}$ to be its closure in $\hat{\mathbb{C}}$, and we define $\overline{X}$ to be its closure in $\mathbb{C}$.

We note that when $\underline{s}$ has non-binary itinerary, then $\tilde{\gamma}_{\underline{s}}$ and $\hat{\gamma}_{\underline{s}}$ both add only one point at infinity (by this, we mean that each of these closures is the one point compactification of the closure of $\overline{\gamma}_{\underline{s}}$). These closures are homeomorphic to each other. On the other hand, when $\underline{s}$ has binary type itinerary, $\tilde{\gamma}_{\underline{s}} \setminus \overline{\gamma}_{\underline{s}}$ contains up to two points while $\hat{\gamma}_{\underline{s}} \setminus \overline{\gamma}_{\underline{s}}$ contains just the point at infinity. In the case where $\tilde{\gamma}_{\underline{s}} \setminus \overline{\gamma}_{\underline{s}}$ contains two points, then $\hat{\gamma}_{\underline{s}}$ is homeomorphic to $\tilde{\gamma}_{\underline{s}}$ with the two points at infinity *identified*.

9.5.1. Definition (Identifying Points)**.**
Let X be a topological space and let a, b be two separate points in X. By *identifying* X by the two points a, b, we mean to define a topological space $X_{a,b} := (X \setminus \{a, b\}) \cup \{c\}$ where a set $U \in X_{a,b}$ is open precisely when:

- $U \subset X \setminus \{a, b\}$ and U is open in X;
- $c \in U$ and there exists an open set $U' \subset X$ which contains a and b such that $U = (U' \setminus \{a, b\}) \cup \{c\}$.

In this respect, when $\tilde{\gamma}_{\underline{s}}$ has two points a, b at infinity, then $\hat{\gamma}_{\underline{s}}$ is precisely the identifying of a and b in $\tilde{\gamma}_{\underline{s}}$ setting $c = \infty \in \hat{\mathbb{C}}$. The neighbourhoods of ∞ in $\hat{\gamma}_{\underline{s}}$ are precisely the neighbourhoods of $\{a, b\}$ in $\tilde{\gamma}_{\underline{s}}$.

We find then that the following holds for identified spaces:

9.5.2. Theorem.
Let X be a continuum and let $X_{a,b}$ be a continuum constructed by identifying a and b in X. Then X is indecomposable precisely when $X_{a,b}$ is indecomposable.

Proof. Suppose $X_{a,b}$ was decomposable into two proper subcontinua C_x and

C_y such that $C_x \cup C_y = X_{a,b}$. At least one of C_x and C_y will contain the identified point c. Without loss of generality, suppose $c \in C_x$. Let $C'_x \coloneqq C_x \setminus c$, then by the boundary bumping theorem, the boundary in $X_{a,b}$ of all connected components of C'_x is precisely c. It follows that the boundary in X of every connected component of C'_x is either a, b or $\{a, b\}$. Let $C''_x \coloneqq C'_x \cup \{a, b\}$, this is then a closed proper subset of X containing at most two components. Similarly, for C_y, we can find some equivalent C''_y which is closed proper subset of X with at most two components. We will find that $C''_x \cup C''_y = X$, and so X can be decomposed into at most four proper subcontinua.

If X can be decomposed into finitely many proper subcontinua, it can be decomposed into two subcontinua and X will be decomposable. It follows that when X is indecomposable, then $X_{a,b}$ is also indecomposable.

Suppose X was decomposable into two proper subcontinua C_x and C_y. Without loss of generality we have that C_x intersects $\{a, b\}$. We may define $C'_x \subset X_{a,b}$ to be $C_x \setminus \{a, b\}$. This will be a proper subset of $X_{a,b} \setminus \{c\}$ unless $C_x = X \setminus \{a\}$ or $C_x = X \setminus \{b\}$, however in this case C_x would not be closed. The boundary in $X_{a,b}$ of each component of C'_x will be precisely c. So $C''_x \coloneqq C'_x \cup \{c\}$ will be a closed connected proper subset of $X_{a,b}$. Similarly we may define C''_y to be a proper subcontinuum of $X_{a,b}$, noting that if C_y does not intersect $\{a, b\}$, then we may simply set $C''_y \coloneqq C_y$. We then find that $C''_x \cup C''_y = X_{a,b}$ is a decomposition of $X_{a,b}$. When $X_{a,b}$ is indecomposable, then X must also be indecomposable. ■

The following corollary is then immediate.

9.5.3. Corollary.

Theorem 9.3.6, Theorem 9.3.8, Corollary 9.4.2 and Corollary 9.4.3 all still hold when replacing the closure in $\tilde{\mathbb{C}}$ with the closure in $\hat{\mathbb{C}}$.

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