ON SOME APPLICATIONS OF SOBOLEV FLOWS OF SDES WITH UNBOUNDED DRIFT COEFFICIENTS

OLIVIER MENOUKEU PAMEN

African Institute for Mathematical Sciences, Ghana
University of Ghana, Ghana
Institute for Financial and Actuarial Mathematics, Department of Mathematical Sciences, University of Liverpool, L69 7ZL, United Kingdom

Abstract. We study two applications of spatial Sobolev smoothness of stochastic flows of unique strong solution to stochastic differential equations (SDEs) with irregular drift coefficients. First, we analyse the stochastic transport equation assuming that the drift coefficient is Borel measurable, with spatial linear growth and show that the above equation has a unique Sobolev differentiable weak coefficient for all $t \in [0,T]$ for T small enough. Second, we consider the Kolmogorov equation and obtain a representation of the spatial derivative of its solution v. The latter result is obtained via the martingale representation theorem given in (Elliott and Kohlmann, 1988) and generalises the results in (Elworthy and Li, 1994; Menoukeu-Pamen et al., 2013).

1. introduction

This paper aims at studying two applications of Sobolev regularity of flows of strong solution to the following SDE

$$\mathrm{d}X_t = b(t,X_t)\mathrm{d}t + \mathrm{d}B_t, \ \ 0 \le t \le T, \quad X_0 = x \in \mathbb{R}^d, \tag{1.1}$$

where the drift coefficient $b : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a Borel measurable function satisfying spatial linear growth condition and B_t is a $d-$dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. It is known that when b is Lipschitz and satisfies linear growth, then there exists a unique strong solution to the SDE (1.1). Zvonkin (Zvonkin, 1974) shows that if $d = 1$ and b is bounded and measurable, the SDE (1.1) has a unique strong solution. This result was generalised in (Veretennikov, 1979) to the multidimensional case. In the one dimensional case, the authors in (Engelbert and Schmidt, 1989, 1991) show that there exists a unique strong solution to the SDE (1.1) when b is time homogeneous and of spatial linear growth. The time dependent drift coefficient was considered in (Nilssen, 2012), where Malliavin smoothness and Sobolev differentiability of the flows of the solution were also obtained for small time interval. The work (Menoukeu-Pamen et al., 2013) uses Malliavin calculus and white noise analysis to construct a unique strong Malliavin differentiable solution to the SDE (1.1) in multi-dimension. In the recent work (Menoukeu-Pamen and Mohammed, 2016), the authors extended the previous results to the case of irregular and possibly unbounded drift coefficients. In fact, they show that when the drift coefficient b has linear growth with respect to the space variable, there exists a unique strong Malliavin smooth solution to the SDE (1.1). In addition, the solution has a Sobolev differentiable flows. The above results significantly extend the existing ones. Note that when b is only bounded and measurable, the Sobolev smoothness of the unique stochastic flows of (1.1) was obtained in (Mohammed et al., 2015).

E-mail address: Menoukeu@liv.ac.uk.

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In this paper, we use the existence and uniqueness of the Sobolev differentiable stochastic flows to the SDE (1.1) to show Malliavin differentiability of the unique weak solution $u(t,x)$ to the following (Stratonovich) stochastic transport equation

$$\begin{cases} \mathrm{d}_t u(t,x) + (b(t,x)\cdot Du(t,x))\mathrm{d}t + \sum_{i=1}^d e_i \cdot Du(t,x) \circ \mathrm{d}B_t^i = 0 \\ u(0,x) = u_0(x), \end{cases} \tag{1.2}$$

with $\{e_i\}_{i=1,\dots,d}$ a canonical basis of $\mathbb{R}^d, b : [0,1]\times\mathbb{R}^d \to \mathbb{R}^d$ a specified measurable vector field satisfying spatial linear growth condition and $u_0 : \mathbb{R}^d \to \mathbb{R}$ a specified initial data. The driving noise B_t is a d-dimensional Brownian motion with respect to the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{\{0\le t\le T\}}, P)$ where $(\mathcal{F}_t)_{\{0\le t\le T\}}$ is the P-augmented filtration generated by B_t. The stochastic integration is understood in the Stratonovich sense.

Note that when b is not Lipschitz, the associated deterministic transport may have singularities in the sense that the derivatives of the solution may be discontinuous or even blow up; see for example (Fredrizzi and Flandoli, 2013; Flandoli, 2011) and references therein. Hence, the noise may have a "regulatization" effect on the deterministic transport equation.

When the initial data and the vector field b are sufficiently smooth, it was shown in (Kunita, 1990) that there exists an explicit strong solution $u(t,x) = u_0(\phi_{s,t}(x))$ to (1.2). Here $\phi_{s,t}(x)$ is the flow generated by the strong solutions $\{X_t^x\}_{t\ge 0}$ of the following SDE

$$X_t^{s,x} = x + \int_s^t b(u, X_u^{s,x})\mathrm{d}u + B_t - B_s,\ \ s,t \in \mathbb{R} \text{ and } x \in \mathbb{R}^d, \tag{1.3}$$

Assuming that the drift satisfies an integrability conditions,the authors in (Fredrizzi and Flandoli, 2013) show that there exists a unique Sobolev differentiable weak solution to the SPDE (1.2). Their approach is based on the analysis of the associated stochastic flow of characteristics. Similar results were obtained in (Mohammed et al., 2015) assuming that drift coefficient b is only bounded and measurable. Moreover, the unique weak solution turns out to be Malliavin differentiable. The method employed in (Mohammed et al., 2015) is based on Malliavin calculus. Hence, our results extend the above ones to the case of drift coefficients with linear growth. Note that since the drift coefficient is unbounded, the proof uses the fact that the test functions are of compact support.

The other application of the Sobolev regularity of the stochastic flows of the SDE (1.1) we are interested to, pertains to spatial differentiability of the solution to the following Kolmogorov equation

$$\begin{cases} \frac{\partial}{\partial t} v(t,x) = \sum_{j=1}^d b_j(t,x) \frac{\partial}{\partial x_j} v(t,x) + \frac{1}{2}\sum_{j,i=1}^d \frac{\partial^2}{\partial x_i \partial x_j} v(t,x) \\ v(0,x) = \Phi(x), \end{cases} \tag{1.4}$$

where the drift coefficient is given as above and $\Phi \in C_b(\mathbb{R}^d)$. When $\Phi \in C_b^1(\mathbb{R}^d)$ and b is smooth, it was shown in (Elworthy and Li, 1994) that the solution is differentiable with respect to the initial x and a probabilistic representation is given. The latter representation is known as the Bismut-Elworthy-Li formula. Assuming that $\Phi \in C_b(\mathbb{R}^d)$ and b is bounded and measurable, the above result was generalised in (Menoukeu-Pamen et al., 2013). The proof is based on the Malliavin differentiability of the solution to the associated SDE, the chain rule and the duality formula of the Malliavin derivative as well as the Clark-Ocone formula. When b is smooth similar result based on the duality formula for Malliavin derivatives was also derived in (Fournier et al., 1999). The proof is based on the flow property of the solution to the SDE (1.3) and on the martingale representation theorem given in (Elliott and Kohlmann, 1988) rather than the chain rule and duality for Malliavin derivative.

The paper is organised as follows: Section 2 considers the application to the (Stratonovich) stochastic transport equation whereas Section 3 is devoted to the application to the stochastic representation of the spatial derivative of the solution to the Kolmogorov equation.

2. Application to stochastic transport equation

Let us consider the (Stratonovich) stochastic transport equation (1.2). In this section, we establish existence of a unique weak Sobolev differentiable solution in space $u(t,x)$ to the stochastic transport equation (1.2). In addition, we show that the solution is Malliavin differentiable.

A weak solution to the SPDE (1.2) is to be understood in the following sense (see also (Mohammed and Scheutzow, 1998) or (Fredrizzi and Flandoli, 2013)).

Definition 2.1. *Let $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be a Borel measurable function with spatial linear growth and $u_0 \in L^\infty(\mathbb{R}^d)$. We call a weak solution to the SPDE (1.2), a measurable process $u : \Omega \times [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfying the following:*

(1) *The function $u(t,\cdot)$ is weakly differentiable a.s. for every t and*
$$\sup_{0\le t\le 1} E\Big[\int_{\mathbb{R}^d} \chi_{B(\omega)}|Du(t,x)|^4 \mathrm{d}x\Big] < \infty \text{ for } B(\omega) \text{ almost surely compact set.}$$
Here $Du(t,x)$ represents the spatial weak derivative of $u(t,x)$.

(2) *For every $\varphi \in C_0^\infty(\mathbb{R}^d)$ the process $\int_{\mathbb{R}^d} u(t,x)\varphi(x)\mathrm{d}x$ is progressively measurable and has a continuous adapted modification satisfying*

$$\int_{\mathbb{R}^d} u(t,x)\varphi(x)\mathrm{d}x = \int_{\mathbb{R}^d} u_0(x)\varphi(x)\mathrm{d}x - \int_0^t \int_{\mathbb{R}^d} b(s,x)\cdot Du(s,x)\varphi(x)\mathrm{d}x\mathrm{d}s$$
$$+ \sum_{i=1}^d \int_0^t \Big(\int_{\mathbb{R}^d} u(s,x) D_i\varphi(x)\mathrm{d}x\Big)\mathrm{d}B_s^i + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} u(s,x)\Delta\varphi(x)\mathrm{d}x\mathrm{d}s. \quad (2.1)$$

Consider the autonomous SDE
$$X_t^x = x + \int_0^t b(X_u^x)\mathrm{d}u + B_t,\ t\in\mathbb{R},\ x\in\mathbb{R}^d, \quad (2.2)$$
where $b : \mathbb{R}^d \to \mathbb{R}^d$ is Borel measurable and has linear growth. Let $\phi_t : \mathbb{R}^d \to \mathbb{R}^d$, $t \in \mathbb{R}$ be the Sobolev differentiable flow of the homeomorphisms associated to the SDE (2.2) (see for example (Menoukeu-Pamen and Mohammed, 2016)) and let ϕ_t^{-1} be its inverse.

Next, we define a class of weighted Sobolev spaces. Let $\mathfrak{p} : \mathbb{R}^d \to (0,\infty)$ be a Borel measurable function such that
$$\int_{\mathbb{R}^d} e^{|x|^2}\mathfrak{p}(x)\mathrm{d}x < \infty. \quad (2.3)$$
Let $L^p(\mathbb{R}^d,\mathfrak{p})$ be the Banach space of all Borel measurable functions $u = (u_1,\dots,u_d) : \mathbb{R}^d \to \mathbb{R}^d$ satisfying
$$\int_{\mathbb{R}^d} |u(x)|^p \mathfrak{p}(x)\mathrm{d}x < \infty \quad (2.4)$$
and equipped with the norm
$$\|u\|_{L^p(\mathbb{R}^d,\mathfrak{p})} := \Big[\int_{\mathbb{R}^d} |u(x)|^p\mathfrak{p}(x)\mathrm{d}x\Big]^{\frac{1}{p}}. \quad (2.5)$$
In addition, let $W^{1,p}(\mathbb{R}^d,\mathfrak{p})$ be the space of functions $u \in L^p(\mathbb{R}^d,\mathfrak{p})$ with weak partial derivatives $\nabla_j u \in L^p(\mathbb{R}^d,\mathfrak{p})$ for $j = 1,\dots,d$. Let us define the norm $\|u\|_{1,p,\mathfrak{p}}$ of each $u \in W^{1,p}(\mathbb{R}^d,\mathfrak{p})$ by
$$\|u\|_{1,p,\mathfrak{p}} := \|u\|_{L^p(\mathbb{R}^d,\mathfrak{p})} + \sum\nolimits_{i,j=1}^d \|\nabla_j u_i\|_{L^p(\mathbb{R}^d,\mathfrak{p})}. \quad (2.6)$$
Equipped with the norm (2.6), the space $W^{1,p}(\mathbb{R}^d,\mathfrak{p})$ is a Banach space provided that the weight $\mathfrak{p}$ satisfied the known A_p-condition.

The following theorem constitutes the main result of this section.

Theorem 2.2. *Assume that b is Borel measurable with spatial linear growth. Assume that $u_0 \in C_b^1(\mathbb{R}^d)$. Then the SPDE (1.2) has a unique Malliavin differentiable weak solution. In addition, for every $t\in[0,T]$ and $p>1$, $u(t,\cdot) \in W^{1,p}(\mathbb{R}^d,\mathfrak{p})$ a.s. for T small enough.*

Proof. We will first prove the uniqueness and then the existence.

Uniqueness: Let u be a weak solution to the SPDE (1.2) such that $u \in W^{1,p}(\mathbb{R}^d,\mathfrak{p})$ and
$$\sup_{0\le t\le 1} E\Big[\int_{\mathbb{R}^d} \chi_{B(\omega)}|Du(t,x)|^4\mathrm{d}x\Big] < \infty \text{ for } B(\omega) \text{ almost surely compact set.}$$

Since $u_0 \in C_b^1(\mathbb{R}^d)$, in order to show uniqueness, it is sufficient to show that

$$u(t,x) = u_0(\phi_t^{-1}(x)) \text{ a.e.}$$

This is due to the following claim

Claim 1: For all bounded random variable Z and all smooth functions with compact support φ we have

$$E\Big[Z\int_{\mathbb{R}^d}\varphi(x)u(t,\phi_t(x))\mathrm{d}x\Big] = E\Big[Z\int_{\mathbb{R}^d}\varphi(x)u_0(x)\mathrm{d}x\Big]. \tag{2.7}$$

Proof of Claim 1 Consider the locally integrable function $u(t,\cdot)$ on $\mathbb{R}^d$ and let u_n defined by

$$u_n(t,x) := \int_{\mathbb{R}^d} u(t,y)\rho_n(x-y)\mathrm{d}y, \tag{2.8}$$

where ρ_n are the usual mollifiers. Then $u_n(t,\cdot)$ is smooth and satisfies

$$u_n(t,x) = u_{0,n}(x) - \int_0^t (b\cdot Du)_n(s,x)\mathrm{d}s - \int_0^t (Du)_n(s,x)\circ \mathrm{d}B_s. \tag{2.9}$$

Hence, the Itô-Ventzell formula (see (Kunita, 1990)) yields

$$u_n(t,\phi_t(x)) = u_{0,n}(x) - \int_0^t \Big((Du)_n(s,\phi_s(x))\cdot b(s,\phi_s(x)) - (b\cdot Du)_n(s,\phi_s(x))\Big)\mathrm{d}s. \tag{2.10}$$

Let Z be a bounded random variable and φ be a smooth function with compact support $K\subset\mathbb{R}^d$. Let $\mathcal{B}$ be an open and bounded subset of $\mathbb{R}$ such that $K\subset\mathcal{B}$. Then we have

$$\begin{aligned}
&E\Big[Z\int_{\mathbb{R}^d}\varphi(x)u_n(t,\phi_t(x))\mathrm{d}x\Big]\\
=&E\Big[Z\int_{\mathbb{R}^d}\varphi(x)u_{0,n}(x)\mathrm{d}x\Big]\\
&-E\Big[Z\int_0^t\int_{\mathbb{R}^d}\varphi(x)\Big((Du)_n(s,\phi_s(x))\cdot b(s,\phi_s(x)) - (b\cdot Du)_n(s,\phi_s(x))\Big)\mathrm{d}x\mathrm{d}s\Big]\\
=&E\Big[Z\int_{\mathcal{B}}\varphi(x)u_{0,n}(x)\mathrm{d}x\Big]\\
&-E\Big[Z\int_0^t\int_{\mathcal{B}}\varphi(x)\Big((Du)_n(s,\phi_s(x))\cdot b(s,\phi_s(x)) - (b\cdot Du)_n(s,\phi_s(x))\Big)\mathrm{d}x\mathrm{d}s\Big].
\end{aligned} \tag{2.11}$$

We know from (Menoukeu-Pamen and Mohammed, 2016, Theorem 3.4) that the random diffeomorphisms $\phi_t^{-1}(\cdot),\phi_t(\cdot)\in W^{1,p}_{loc}(\mathbb{R}^d)$ a.e. Thus applying (Hajlasz, 1993, Theorem 2) to $\phi_t^{-1}(\cdot)$ to the bounded and open set $\mathcal{B}$ yields

$$\begin{aligned}
&E\Big[Z\int_0^t\int_{\mathcal{B}}\varphi(x)\Big((Du)_n(s,\phi_s(x))\cdot b(s,\phi_s(x)) - (b\cdot Du)_n(s,\phi_s(x))\Big)\mathrm{d}x\mathrm{d}s\Big]\\
=&E\Big[Z\int_0^t\int_{\phi_s(\mathcal{B})}\varphi(\phi_s^{-1}(x))\Big((Du)_n(s,x)\cdot b(s,x) - (b\cdot Du)_n(s,x)\Big)|\det(J\phi_s^{-1}(x))|\mathrm{d}x\mathrm{d}s\Big]\\
=&E\Big[Z\int_0^t\int_{\phi_s(\mathcal{B})}\varphi(\phi_s^{-1}(x))(Du)_n(s,x)\cdot b(s,x)|\det(J\phi_s^{-1}(x))|\mathrm{d}x\mathrm{d}s\Big]\\
&-E\Big[Z\int_0^t\int_{\phi_s(\mathcal{B})}\varphi(\phi_s^{-1}(x))(b\cdot Du)_n(s,x)|\det(J\phi_s^{-1}(x))|\mathrm{d}x\mathrm{d}s\Big]\\
=&I_1^n + I_2^n
\end{aligned} \tag{2.12}$$

The boundedness of $\mathcal{B}$ implies that there exists $N\in\mathbb{N}$ such that $\mathcal{B}\subset\bar{\mathcal{B}}\subset\mathcal{B}_N=[-N,N]^d$. It follows from (2.8) that

$$\|(Du)_n\|_{L^2(\phi_s(\mathcal{B}))} \le \|Du\|_{L^2(\phi_s(\mathcal{B}_N))} \tag{2.13}$$

Now, define

$$\tilde{b}(t,z) = \frac{b(t,z)}{1+|z|}, t \in [0,T],\ z \in \mathbb{R}^d.$$

Using Hölder inequality, Fubini's Theorem and (2.13), it follows that there exists a positive constant C which may change from line to line and such that

$$\begin{aligned}
I_1^n =& E\Big[Z\int_0^t\int_{\phi_s(\mathcal{B})}\varphi(\phi_s^{-1}(x))(Du)_n(s,x)\cdot b(s,x)|\det(J\phi_s^{-1}(x))|\mathrm{d}x\mathrm{d}s\Big]\\
\leq& CE\Big[\int_0^t\Big(\int_{\mathbb{R}^d}\Big(\chi_{\phi_s(\mathcal{B})}\varphi(\phi_s^{-1}(x))b(s,x)|\det(J\phi_s^{-1}(x))|\Big)^2\mathrm{d}x\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^d}\Big(\chi_{\phi_s(\mathcal{B})}(Du)_n^2(s,x)\mathrm{d}x\Big)^{\frac{1}{2}}\mathrm{d}s\Big]\\
\leq& CE\Big[\int_0^t\Big(\int_{\mathbb{R}^d}\Big(\chi_{\phi_s(\mathcal{B})}\varphi(\phi_s^{-1}(x))(1+|z|)\frac{b(t,x)}{1+|x|}|\det(J\phi_s^{-1}(x))|\Big)^2\mathrm{d}x\Big)^{\frac{1}{2}}\\
&\times\Big(\int_{\mathbb{R}^d}\chi_{\phi_s(\mathcal{B}_N)}|Du(s,x)|^2\mathrm{d}x\Big)^{\frac{1}{2}}\mathrm{d}s\Big]\\
\leq& C\Big\{\int_0^t E\Big[\int_{\mathbb{R}^d}\Big(\chi_{\phi_s(\mathcal{B})}\varphi(\phi_s^{-1}(x))(1+|x|)\tilde{b}(t,x)|\det(J\phi_s^{-1}(x))|\Big)^2\mathrm{d}x\Big]\mathrm{d}s\\
&+\int_0^t E\Big[\int\mathbb{R}^d\chi_{\phi_s(\mathcal{B}_N)}|Du(s,x)|^2\mathrm{d}x\Big]\mathrm{d}s\Big\}\\
\leq& C_{\|\tilde{b}\|_\infty,\varphi}\Big\{\int_0^t\Big(\int_{\mathbb{R}^d}E\Big[\chi_{\phi_s(\mathcal{B})}|\det(J\phi_s^{-1}(x))|^9\Big]\mathrm{d}x\Big)^{\frac{1}{6}}\Big(\int_{\mathbb{R}^d}E\Big[\chi_{\phi_s(\mathcal{B})}(1+|x|)^3\Big]\mathrm{d}x\Big)^{\frac{1}{3}}\mathrm{d}s\\
&+\int_0^t E\Big[\int_{\mathbb{R}^d}\chi_{\phi_s(\mathcal{B}_N)}|Du(s,x)|^2\mathrm{d}x\Big]\mathrm{d}s\Big\}\\
\leq& C_{\|\tilde{b}\|_\infty,\varphi}\Big\{\int_0^t\int_{\mathbb{R}^d}E\Big[\chi_{\phi_s(\mathcal{B})}|\det(J\phi_s^{-1}(x))|^9\Big]\mathrm{d}x\mathrm{d}s+\int_0^t\Big(\int_{\mathbb{R}^d}E\Big[\chi_{\phi_s(\mathcal{B})}(1+|x|)^3\Big]\mathrm{d}x\Big)^{\frac{2}{5}}\mathrm{d}s\\
&+\int_0^t E\Big[\int_{\mathbb{R}^d}\chi_{\phi_s(\mathcal{B}_N)}|Du(s,x)|^2\mathrm{d}x\Big]\mathrm{d}s\Big\}.
\end{aligned} \tag{2.14}$$

Using Girsanov transform, the property normal density and integration by part formula, one can show as in (Mohammed et al., 2015, Theorem 20) that

$$\int_0^t\Big(\int_{\mathbb{R}^d}(1+|x|^3)E\Big[\chi_{\phi_s(\mathcal{B}_N)}(x)\Big]\mathrm{d}x\Big)^{\frac{2}{5}}\mathrm{d}s<\infty.$$

We now focus on the term

$$\int_0^t\int_{\mathbb{R}^d}E\Big[\chi_{\phi_s(\mathcal{B})}|\det(J\phi_s^{-1}(x))|^9\Big]\mathrm{d}x\mathrm{d}s$$

Repeated use of Hölder inequality and (Menoukeu-Pamen and Mohammed, 2016, Proposition 4.11), we have for T small enough

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}^d} E\Big[\chi_{\phi_s(\mathcal{B})}|\det(J\phi_s^{-1}(x))|^9\Big]\mathrm{d}x\mathrm{d}s\\
\leq&\int_0^t\int_{\mathbb{R}^d} E\Big[\chi_{\phi_s(\mathcal{B}_N)}\Big]^{\frac{1}{2}}E\Big[|\det(J\phi_s^{-1}(x))|^{18}\Big]^{\frac{1}{2}}\mathrm{d}x\mathrm{d}s\\
\leq&C\int_0^t\int_{\mathbb{R}^d} E\Big[\chi_{\phi_s(\mathcal{B}_N)}\Big]^{\frac{1}{2}}\exp\Big(C_1k^2T(1+|x|^2)\Big)\exp\Big(C_2^2d^2T^{1/2}\|\tilde{b}\|_\infty^2|x|^2\Big)\mathrm{d}x\mathrm{d}s\\
\leq&C\int_0^t\int_{\mathbb{R}^d} P\Big(\phi_s^{-1}(x)\in\mathcal{B}_N\Big)^{\frac{1}{2}}\exp\Big(C_1k^2T(1+|x|^2)\Big)\exp\Big(C_2^2d^2T^{1/2}\|\tilde{b}\|_\infty^2|x|^2\Big)\mathrm{d}x\mathrm{d}s\\
\leq&C\int_0^t\int_{\mathbb{R}^d} P\Big(x+B_s\in\mathcal{B}_N\Big)^{\frac{1}{8}}\exp\Big(C_1k^2T(1+|x|^2)\Big)\exp\Big(C_2^2d^2T^{1/2}\|\tilde{b}\|_\infty^2|x|^2\Big)\mathrm{d}x\mathrm{d}s\\
\leq&C\int_0^t\int_{\mathbb{R}^d} P\Big(x+B_s\in[-N,N]^d\Big)^{\frac{1}{8}}\exp\Big(C_1k^2T(1+|x|^2)\Big)\exp\Big(C_2^2d^2T^{1/2}\|\tilde{b}\|_\infty^2|x|^2\Big)\mathrm{d}x\mathrm{d}s\\
\leq&C\int_0^t\Big(\int_0^\infty\Big(1-\Phi(\frac{x-N}{\sqrt{s}})\Big)^{\frac{1}{8}}\exp\Big(C_1k^2T(1+x^2)\Big)\exp\Big(C_2^2d^2T^{1/2}\|\tilde{b}\|_\infty^2|x|^2\Big)\mathrm{d}x\Big)^d\mathrm{d}s \qquad (2.15)
\end{aligned}
$$

where the fourth inequality follows from the Beněs Theorem and the Hölder inequality for T small. The last inequality comes from the symmetry of the normal distribution, the other integrands and the fact that $(1+\prod_{i=1}^n a_i)\leq\prod_{i=1}^n(1+a_i)$ for $a_i>0, i=1,\cdots,n$.

On the other hand, we have that

$$1-\Phi(x)\leq\frac{1}{2\pi x}\exp\big(\frac{-x^2}{2}\big) \text{ for all } x>0.$$

Thus there exist a constant C that might change from line to line and depending on N and d such that

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}^d} E\Big[\chi_{\phi_s(\mathcal{B})}|\det(J\phi_s^{-1}(x))|^9\Big]\mathrm{d}x\mathrm{d}s\\
\leq&C\int_0^t\Big(\int_0^\infty\Big(\frac{\sqrt{s}}{2\pi(x-N)}\exp\{-\frac{(x-N)^2}{2s}\}\Big)^{\frac{1}{8}}\exp\{C_1k^2Tx^2\}\exp\Big(C_2^2d^2T^{1/2}\|\tilde{b}\|_\infty^2x^2\Big)\mathrm{d}x\Big)^d\mathrm{d}s\\
\leq&C\Big\{\int_0^t\Big(\int_0^N\Big(\frac{\sqrt{s}}{2\pi(x-N)}\exp\{-\frac{(x-N)^2}{2s}\}\Big)^{\frac{1}{8}}\exp\{C_1k^2T^{1/2}x^2\}\exp\Big(C_2^2d^2T^{1/2}\|\tilde{b}\|_\infty^2x^2\Big)\mathrm{d}x\Big)^d\mathrm{d}s\\
&+\int_0^t\Big(\int_N^\infty\Big(\frac{\sqrt{s}}{2\pi(x-N)}\exp\{-\frac{(x-N)^2}{2s}\}\Big)^{\frac{1}{8}}\exp\{C_1k^2T^{1/2}x^2\}\exp\Big(C_2^2d^2T^{1/2}\|\tilde{b}\|_\infty^2x^2\Big)\mathrm{d}x\Big)^d\mathrm{d}s\Big\}\\
\leq&C_N\Big\{1+\int_0^t\Big(\int_N^\infty\Big(\frac{\sqrt{s}}{2\pi(x-N)}\exp\Big\{-(\frac{1}{2s}-8C_1k^2T^{1/2}-8C_2^2d^2k^2T^{1/2})(x-N)^2\\
&+(8C_1k^2T^{1/2}+8C_2^2d^2k^2T^{1/2})(x-N)\Big\}\Big)^{\frac{1}{8}}\mathrm{d}x\Big)^d\mathrm{d}s\Big\}\\
\leq&C_N\Big\{1+\int_0^t\Big(\int_0^\infty\Big(\frac{\sqrt{s}}{2\pi x}\exp\Big\{-(\frac{1}{2s}-(8C_1k^2+8C_2^2d^2k^2)T^{1/2})x^2+(8C_1k^2+8C_2^2d^2k^2)T^{1/2}x\Big\}\Big)^{\frac{1}{8}}\mathrm{d}x\Big)^d\mathrm{d}s\Big\}\\
\leq&C_N<\infty. \qquad (2.16)
\end{aligned}
$$

Notice that for T small enough (for example $T^{1/2}<\frac{1}{8C_1k^2+8C_2^2d^2k^2}$), we have that s is small enough and the coefficient of x^2 in the exponential is negative.

From the existence result, we have that $(Du)(s,x)\in L^p_{\text{loc}}(\mathbb{R}^d)$ and $(Du)_n(s,x)\to(Du)(s,x)$ in $L^p_{\text{loc}}(\mathbb{R}^d)$ for all $p>0$, thus from (2.14) and the dominated convergence Theorem, we have

$$\lim_{n\to\infty}I_1^n=E\Big[Z\int_0^t\int_{\phi_s(K)}\varphi(\phi_s^{-1}(x))(Du)(s,x)\cdot b(s,x)|\det(J\phi_s^{-1}(x))|\mathrm{d}x\mathrm{d}s\Big].$$

Similar arguments yield

$$\lim_{n\to\infty} I_2^n = -E\Big[Z\int_0^t \int_{\phi_s(K)} \varphi(\phi_s^{-1}(x))(b\cdot Du)(s,x)|\det(J\phi_s^{-1}(x))|\mathrm{d}x\mathrm{d}s\Big].$$

Furthermore, we have

$$\lim_{n\to\infty} E\Big[Z\int_K \varphi(x)u_n(t,\phi_t(x))\mathrm{d}x\Big] = E\Big[Z\int_K \varphi(x)u_n(t,\phi_t(x))\mathrm{d}x\Big]$$

and

$$\lim_{n\to\infty} E\Big[Z\int_K \varphi(x)u_{0,n}(x)\mathrm{d}x\Big] = E\Big[Z\int_K \varphi(x)u_0(t,x)\mathrm{d}x\Big].$$

Combining all the above yields

$$E\Big[Z\int_{\mathbb{R}^d} \varphi(x)u(t,\phi_t(x))\mathrm{d}x\Big] = E\Big[Z\int_{\mathbb{R}^d} \varphi(x)u_0(x)\mathrm{d}x\Big] \tag{2.17}$$

and the claim is proved. □

It follows from *Claim 1* that $u(t,\phi_t(x)) = u_0(x),\ P\times \mathrm{d}x$-a.e. The Lusin condition on bounded open subset (see (Hajlasz, 1993, Theorem 2)) of $\phi_t^{-1}(\cdot)$ together with the continuity of u in time yield to the existence of Ω_0 with $P(\Omega_0)=1$ such that for all $\omega\in\Omega_0$

$$u(t,x) = u_0(\phi_t^{-1}(x))\ \mathrm{d}x\text{-a.e. and uniformly in } t.$$

Existence: Consider a sequence of smooth functions $b_n : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ with compact support and such that $b_n(t,x)\to b(t,x)$ $\mathrm{d}t\times\mathrm{d}x$-a.e. and there is a positive k with

$$\sup_{n\geq 1}|b_n(t,x)| \leq k(1+|x|),$$

for all $(t,x)\in[0,T]\times\mathbb{R}^d$. Then the stochastic transport equation (1.2) has a unique strong solution u_n with b replaced by $b_n,\ n\geq 1$ (see (Kunita, 1990)). The solution u_n is given by $u_n(t,x) = u_0(\phi_{n,t}^{-1}(x))$ where $\phi_{n,t}^{-1}(x)$ is the inverse stochastic flow associated to the SDE (2.2) when b is replaced by $b_n,\ n\geq 1$. It follows that for every $\varphi\in C_0^\infty(\mathbb{R}^d)$, we have

$$\begin{aligned}\int_{\mathbb{R}^d} u_n(t,x)\varphi(x)\mathrm{d}x &= \int_{\mathbb{R}^d} u_0(x)\varphi(x)\mathrm{d}x - \int_0^t\int_{\mathbb{R}^d} b_n(s,x)\cdot Du_n(s,x)\varphi(x)\mathrm{d}x\mathrm{d}s\\ &\quad + \sum_{i=1}^d \int_0^t \Big(\int_{\mathbb{R}^d} u_n(s,x)D_i\varphi(x)\mathrm{d}x\Big)\mathrm{d}B_s^i + \frac{1}{2}\int_0^t\int_{\mathbb{R}^d} u_n(s,x)\Delta\varphi(x)\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{2.18}$$

Next, we show that the process defined by $u(t,x) := u_0(\phi_t^{-1}(x))$ is a weak solution to the transport equation by letting n go to infinity.

We will show convergence in $L^2(P)$ of each term of (2.18).

$$\begin{aligned}&\Big\|\int_{\mathbb{R}^d} u_n(t,x)\varphi(x)\mathrm{d}x - \int_{\mathbb{R}^d} u(t,x)\varphi(x)\mathrm{d}x\Big\|_{L^2(P)}\\ &\leq \int_{\mathbb{R}^d}\Big\|u_n(t,x)-u(t,x)\Big\|_{L^2(P)}|\varphi(x)|\mathrm{d}x\\ &\leq \int_{\mathbb{R}^d}\Big\|u_0(\phi_{n,t}^{-1}(x)) - u_0(\phi_t^{-1}(x))\Big\|_{L^2(P)}|\varphi(x)|\mathrm{d}x\\ &\leq \int_{\mathbb{R}^d}\|Du_0\|_\infty\Big\|\phi_{n,t}^{-1}(x)-\phi_t^{-1}(x)\Big\|_{L^2(P)}|\varphi(x)|\mathrm{d}x.\end{aligned} \tag{2.19}$$

Since $u_0\in C_b^1(\mathbb{R}^d)$, we have that Du_0 is bounded and $\phi_{n,t}^{-1}$ converges to ϕ_t^{-1} in $L^2(P)$ (by (Menoukeu-Pamen and Mohammed, 2016, Theorem 4.13)), thus it follows from the dominated convergence theorem that $\int_{\mathbb{R}^d} u_n(t,x)\varphi(x)\mathrm{d}x$ converges to $\int_{\mathbb{R}^d} u(t,x)\varphi(x)\mathrm{d}x$.

Similarly

$$\int_0^t \int_{\mathbb{R}^d} u_n(s,x)\Delta\varphi(x)\mathrm{d}x\mathrm{d}s \text{ converges in } L^2(P) \text{ to } \int_0^t \int_{\mathbb{R}^d} u(s,x)\Delta\varphi(x)\mathrm{d}x\mathrm{d}s$$

Using Itô isometry, the boundedness of u_0 and the dominated convergence theorem

$$\sum_{i=1}^d \int_0^t \Big(\int_{\mathbb{R}^d} u_n(s,x)D_i\varphi(x)\mathrm{d}x\Big)\mathrm{d}B_s^i \text{ converges in } L^2(P) \text{ to } \sum_{i=1}^d \int_0^t \Big(\int_{\mathbb{R}^d} u(s,x)D_i\varphi(x)\mathrm{d}x\Big)\mathrm{d}B_s^i.$$

Finally, let rewrite

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} b_n(s,x)\cdot Du_n(s,x)\varphi(x)\mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{R}^d} b(s,x)\cdot Du(s,x)\varphi(x)\mathrm{d}x\mathrm{d}s\\
=&\int_0^t \int_{\mathbb{R}^d} b_n(s,x)\cdot Du_n(s,x)\varphi(x)\mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{R}^d} b(s,x)\cdot Du_n(s,x)\varphi(x)\mathrm{d}x\mathrm{d}s\\
&+\int_0^t \int_{\mathbb{R}^d} b(s,x)\cdot Du_n(s,x)\varphi(x)\mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{R}^d} b(s,x)\cdot Du(s,x)\varphi(x)\mathrm{d}x\mathrm{d}s\\
=&\int_0^t \int_{\mathbb{R}^d} b_n(s,x)\cdot Du_0(\phi_{n,s}^{-1}(x))D\phi_{n,s}^{-1}(x)\varphi(x)\mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{R}^d} b(s,x)\cdot Du_0(\phi_{n,s}^{-1}(x))D\phi_{n,s}^{-1}(x)\varphi(x)\mathrm{d}x\mathrm{d}s\\
&+\int_0^t \int_{\mathbb{R}^d} b(s,x)\cdot Du_0(\phi_{n,s}^{-1}(x))\phi_{n,s}^{-1}(x)\varphi(x)\mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{R}^d} b(s,x)\cdot Du_0(\phi_s^{-1}(x))D\phi_s^{-1}(x)\varphi(x)\mathrm{d}x\mathrm{d}s\\
=&\int_0^t \int_{\mathbb{R}^d} b_n(s,x)\cdot Du_0(\phi_{n,s}^{-1}(x))D\phi_{n,s}^{-1}(x)\varphi(x)\mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{R}^d} b(s,x)\cdot Du_0(\phi_{n,s}^{-1}(x))D\phi_{n,s}^{-1}(x)\varphi(x)\mathrm{d}x\mathrm{d}s\\
&+\int_0^t \int_{\mathbb{R}^d} b(s,x)\cdot Du_0(\phi_{n,s}^{-1}(x))\phi_{n,s}^{-1}(x)\varphi(x)\mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{R}^d} b(s,x)\cdot Du_0(\phi_t^{-1}(x))D\phi_{n,s}^{-1}(x)\varphi(x)\mathrm{d}x\mathrm{d}s\\
&+\int_0^t \int_{\mathbb{R}^d} b(s,x)\cdot Du_0(\phi_s^{-1}(x))D\phi_{n,s}^{-1}(x)\varphi(x)\mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{R}^d} b(s,x)\cdot Du_0(\phi_t^{-1}(x))D\phi_s^{-1}(x)\varphi(x)\mathrm{d}x\mathrm{d}s\\
=&I_{n,1}+I_{n,2}+I_{n,3}.
\end{aligned} \tag{2.20}$$

Using Hölder inequality and Fubini's theorem, we have

$$\begin{aligned}
E[I_{n,1}^2] =&E\Big[\Big(\int_0^t \int_{\mathbb{R}^d}(b_n(s,x)-b(s,x))\cdot Du_0(\phi_{n,s}^{-1}(x))D\phi_{n,s}^{-1}(x)\varphi(x)\mathrm{d}x\mathrm{d}s\Big)^2\Big]\\
\le& t\|\varphi\|_{L^1(\mathbb{R}^d)}\int_0^t \int_{\mathbb{R}^d}|b_n(s,x)-b(s,x)|^2(E[|Du_0(\phi_{n,s}^{-1}(x))|^4])^{\frac12}(E[|D\phi_{n,s}^{-1}(x)|^4])^{\frac12}|\varphi(x)|\mathrm{d}x\mathrm{d}s\\
\le& t\|\varphi\|_{L^1(\mathbb{R}^d)}\|Du_0\|_\infty^2 \sup_n \sup_{0\le s\le t, x\in K}(E[|D\phi_{n,s}^{-1}(x)|^4])^{\frac12}\int_0^t \int_{\mathbb{R}^d}|b_n(s,x)-b(s,x)|^2|\varphi(x)|\mathrm{d}x\mathrm{d}s,
\end{aligned}$$

where, the last inequality follows from (Menoukeu-Pamen and Mohammed, 2016, Proposition 4.11). Since φ is of compact support, the dominated convergence theorem yields $I_{n,1}$ converges strongly to 0 in $L^2(P)$ as n goes to infinity.

Using once more Hölder inequality, we get

$$\begin{aligned}
E[I_{n,2}^2] =&E\Big[\Big(\int_0^t \int_{\mathbb{R}^d} b(s,x)\cdot(Du_0(\phi_{n,s}^{-1}(x))-Du_0(\phi_s^{-1}(x)))D\phi_{n,s}^{-1}(x)\varphi(x)\mathrm{d}x\mathrm{d}s\Big)^2\Big]\\
\le& t\|\varphi\|_{L^1(\mathbb{R}^d)}\int_0^t \int_{\mathbb{R}^d}|b(s,x)|^2(E[|Du_0(\phi_{n,s}^{-1}(x))-Du_0(\phi_s^{-1}(x))|^4])^{\frac12}(E[|D\phi_{n,s}^{-1}(x)|^4])^{\frac12}|\varphi(x)|\mathrm{d}x\mathrm{d}s\\
\le& t\|\varphi\|_{L^1(\mathbb{R}^d)}\|\tilde b\|_\infty^2 \sup_n \sup_{0\le s\le t, x\in K}(E[(1+|x|^2)|D\phi_{n,s}^{-1}(x)|^4])^{\frac12}\\
&\times\int_0^t \int_{\mathbb{R}^d}(E[|Du_0(\phi_{n,s}^{-1}(x))-Du_0(\phi_s^{-1}(x))|^4])^{\frac12}|\varphi(x)|\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

Using the boundedness and continuity of Du_0, it follows from the dominated convergence theorem that $I_{n,2}$ converges strongly to 0 in $L^2(P)$ as n goes to infinity.

Now let $Z \in L^2(P)$ then

$$E[I_{n,3}Z] = \int_0^t E\Big[\int_{\mathbb{R}^d} b(s,x)\cdot Du_0(\phi_s^{-1}(x))(D\phi_{n,s}^{-1}(x) - D\phi_s^{-1}(x))\varphi(x)Z\mathrm{d}x\Big]\mathrm{d}s$$
$$= \int_0^t E\Big[\int_{\mathbb{R}^d} \tilde{b}(s,x)\cdot Du_0(\phi_s^{-1}(x))(D\phi_{n,s}^{-1}(x) - D\phi_s^{-1}(x))(1+|x|)\varphi(x)Z\mathrm{d}x\Big]\mathrm{d}s$$

Since φ is of compact support and $Du_0, \tilde{b}$ and $x \mapsto (1+|x|)\varphi(x)$ are bounded and $D\phi_{n,s}^{-1}$ converges weakly to $D\phi_s^{-1}$ (see (Menoukeu-Pamen and Mohammed, 2016, Proof of Proposition 3.8))), it follows that $I_{n,3}$ converges weakly to 0. Combining the above, we get that

$\int_0^t \int_{\mathbb{R}^d} b_n(s,x)\cdot Du_n(s,x)\varphi(x)\mathrm{d}x\mathrm{d}s$ converges weakly in $L^2(P)$ to $\int_0^t \int_{\mathbb{R}^d} b(s,x)\cdot Du(s,x)\varphi(x)\mathrm{d}x\mathrm{d}s$.

Note that since the other term converges strongly in $L^2(P)$ it follows that

$$\int_0^t \int_{\mathbb{R}^d} b_n(s,x)\cdot Du_n(s,x)\varphi(x)\mathrm{d}x\mathrm{d}s \text{ converges strongly in } L^2(P).$$

Since the strong and the weak limit coincide, we get

$\int_0^t \int_{\mathbb{R}^d} b_n(s,x)\cdot Du_n(s,x)\varphi(x)\mathrm{d}x\mathrm{d}s$ converges strongly in $L^2(P)$ to $\int_0^t \int_{\mathbb{R}^d} b(s,x)\cdot Du(s,x)\varphi(x)\mathrm{d}x\mathrm{d}s$.

We conclude that $u(t,x) = u_0(\phi_s^{-1}(x))$ is a weak solution to the transport equation (1.2).

Finally, since $\phi_s^{-1}(x)$ is Malliavin differentiable (confer (Menoukeu-Pamen and Mohammed, 2016, Theorem 3.1)), using the chain rule for Malliavin derivatives, it follows that $u(t,x)$ has a version which is Malliavin differentiable.

We know from (Menoukeu-Pamen and Mohammed, 2016, Theorem 3.4) that $\phi_{t,s}(\cdot)$ and $\phi_{t,s}^{-1}(\cdot) \in L^2(\Omega, W^{1,p}(\mathbb{R}^d;\mathfrak{p}))$ for all $s,t \in \mathbb{R}$ and $p \in (1,\infty)$; since $u_0 \in C_b^1(\mathbb{R}^d)$ it follows that $u(t,x) = u_0(\phi_s^{-1}(x)) \in W^{1,p}(\mathbb{R}^d;\mathfrak{p})$ a.s. □

3. Application to Bismut-Elworthy-Li formula with unbounded coefficients

In this section, we study the stochastic representation of the spatial derivative of the solution to the following Kolmogorov equation

$$\frac{\partial}{\partial t}v(t,x) = \sum_{j=1}^d b_j(t,x)\frac{\partial}{\partial x_j}v(t,x) + \frac{1}{2}\sum_{j,i=1}^d \frac{\partial^2}{\partial x_i \partial x_j}v(t,x), \tag{3.1}$$

with initial condition $v(0,x) = \Phi(x)$.

Assuming that Φ is continuous and b is bounded, it was shown (see (Veretennikov, 1981)) $v(t,x)$ given by

$$v(t,x) = E[\Phi(X_t^x)] \tag{3.2}$$

is a solution to (3.1). In addition, $v(t,x)$ is the unique solution belonging to the space $\cap_{p>1} W_{loc}^{(1,2),p}([0,1)\times\mathbb{R}^d) \cap C([0,1]\times\mathbb{R}^d)$, where $W_{loc}^{(1,2),p}([0,1)\times\mathbb{R}^d)$ is the space of functions that are once weakly differentiable in time and twice weakly differentiable in space, with derivatives which are locally integrable to the p-th power.

Our objective is to use Malliavin and Sobolev differentiability of the solution to the SDE (1.3) to give a representation of $\frac{\partial}{\partial x}v$ that is independent on the derivative of Φ. The following theorem is the main result of this section

Theorem 3.1. *Suppose $\Phi \in C_b(\mathbb{R}^d)$ and let U be an open and bounded subset of $\mathbb{R}^d$. Suppose in addition that b in* (3.1) *is of spatial linear growth. Then the derivative of the solution to* (3.1) *can be represented as follows*

$$\frac{\partial}{\partial x}v(t,x) = E\Big[\Phi(X_t^x)t^{-1}\Big[\int_0^t \Big[\frac{\partial}{\partial x}X_s^x\Big]^*\mathrm{d}B_s\Big]\Big]^* \tag{3.3}$$

for almost all $x \in U$ and all $t \in (0,1]$, with * denoting the transposition.

Remark 3.2. *Theorem 3.1 generalises (Menoukeu-Pamen et al., 2013, Theorem 4.6) to the case of linear growth drift coefficient. Note that in the latter case, the representation* (3.2) *holds as a limit of a sequence as shown in the proof.*

Proof. The proof does not use the duality formula for Malliavin calculus but rather the version of the martingale representation theorem in (Elliott and Kohlmann, 1988). In addition, since the drift coefficient is unbounded, we use the fact that the test function is of compact support; we give details below.

We show the result for $\Phi \in C_b^2(\mathbb{R}^d)$. The case $\Phi \in C_b(\mathbb{R}^d)$ follows by an approximation argument. Let us consider a sequence of smooth and compactly supported functions $b_n : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ such that $b_n(t,x) \to b(t,x)$ $\mathrm{d}t \times \mathrm{d}x$-a.e. Assume that there exist $k > 0$ such that

$$\sup_{n\geq 1} |b_n(t,x)| \leq k(1+|x|),$$

for all $(t,x) \in [0,T] \times \mathbb{R}^d$. Denote by $\{X_{\cdot}^{n,s,x}\}_{n=1}^{\infty}$ the sequence of solutions of (1.3) corresponding to $b = b_n,\ n \geq 1$. we know that (see (Menoukeu-Pamen and Mohammed, 2016, Theorem 4.13)) $\{X_t^{n,s,x}\}_{n=1}^{\infty}$ converges to $X_t^{s,x}$ in $L^2(P;\mathbb{R}^d)$ for all t and x. Let

$$v_n(t,x) = E[\Phi(X_t^{n,x})]$$

be the unique solution to (3.1) with b replaced by b_n. Then since $\{X_t^{n,s,x}\}_{n=1}^{\infty}$ converges to $X_t^{s,x}$ in $L^2(P;\mathbb{R}^d)$, one can use the mean value theorem to show that $v_n(t,x) \to v(t,x)$ for each t and x.

Consider the times $0 < s < t \leq 1$. Then by the flow property

$$X_t^{n,0,x} = X_t^{n,s,\cdot}(X_s^{n,0,x}) \text{ a.s.}$$

and the chain rule yields

$$\frac{\partial}{\partial x} X_t^{n,0,x} = \frac{\partial}{\partial y} X_t^{n,s,y} \cdot \frac{\partial}{\partial x} X_s^{n,0,x} \text{ a.s.},$$

where $\cdot$ denotes matrix multiplication for the Jacobian derivatives and we have

$$y := X_s^{n,0,x}, \quad \frac{\partial}{\partial y} X_t^{n,s,y} := \frac{\partial}{\partial x} X_t^{n,s,x}\Big|_{x=y}.$$

Hence

$$\frac{\partial}{\partial x} X_t^{n,0,x} = t^{-1} \int_0^t \frac{\partial}{\partial y} X_t^{n,s,y} \cdot \frac{\partial}{\partial x} X_s^{n,0,x} \mathrm{d}s. \tag{3.4}$$

Interchanging integration and differentiation, using the martingale representation Theorem (see (Elliott and Kohlmann, 1988, Theorem 1)) and the Itô product rule, we have

$$\begin{aligned}
\frac{\partial}{\partial x} v_n(t,x) &= E[\Phi'(X_t^{n,x}) \frac{\partial}{\partial x} X_t^{n,x}] \\
&= E\Big[t^{-1} \int_0^t \Phi'(X_t^{n,x}) \frac{\partial}{\partial y} X_t^{n,s,y} \cdot \frac{\partial}{\partial x} X_s^{n,0,x} \mathrm{d}s\Big] \\
&= E\Big[t^{-1} \int_0^t E_{s,y}\Big[\Phi'(X_t^{n,x}) \frac{\partial}{\partial y} X_t^{n,s,y}\Big] \cdot \frac{\partial}{\partial x} X_s^{n,0,x} \mathrm{d}s\Big] \\
&= t^{-1} E\Big[\int_0^t E_{s,y}\Big[\Phi'(X_t^{n,x}) \frac{\partial}{\partial y} X_t^{n,s,y}\Big]^* \mathrm{d}B_s \Big[\int_0^t \Big[\frac{\partial}{\partial x} X_s^{n,0,x}\Big]^* \mathrm{d}B_s\Big]\Big] \\
&= t^{-1} E\Big[\Big(\Phi(X_t^{n,x}) + E[\Phi(X_t^{n,x})]\Big) \Big[\int_0^t \Big[\frac{\partial}{\partial x} X_s^{n,0,x}\Big]^* \mathrm{d}B_s\Big]\Big] \\
&= E\Big[\Phi(X_t^{n,x}) t^{-1} \Big[\int_0^t \Big[\frac{\partial}{\partial x} X_s^{n,x}\Big]^* \mathrm{d}B_s\Big]\Big]^*.
\end{aligned}$$

Let $\varphi \in C_0^{\infty}(\mathbb{R}^d)$.

Claim: We claim that

$$\int_{\mathbb{R}^d} \frac{\partial}{\partial x} \varphi(x) v(t,x) \mathrm{d}x = - \int_{\mathbb{R}^d} \varphi(x) E\Big[\Phi(X_t^x) t^{-1} \Big[\int_0^t \Big[\frac{\partial}{\partial x} X_s^x\Big]^* \mathrm{d}B_s\Big]\Big]^* \mathrm{d}x. \tag{3.5}$$

Proof. It follows from the dominated convergence and the choice of the subsequence that

$$\int_{\mathbb{R}^d}\frac{\partial}{\partial x}\varphi(x)v(t,x)\mathrm{d}x = -\lim_{n\to\infty}\int_{\mathbb{R}^d}\varphi(x)E\Big[\Phi(X_t^{n,x})t^{-1}\Big[\int_0^t\Big[\frac{\partial}{\partial x}X_s^{n,x}\Big]^*\mathrm{d}B_s\Big]\Big]^*\mathrm{d}x$$
$$= -\lim_{n\to\infty}I_n^1 - \lim_{n\to\infty}I_n^2,$$

where

$$I_n^1 = \int_{\mathbb{R}^d}\varphi(x)E\Big[\left(\Phi(X_t^{n,x})-\Phi(X_t^x)\right)t^{-1}\Big[\int_0^t\Big[\frac{\partial}{\partial x}X_s^{n,x}\Big]^*\mathrm{d}B_s\Big]\Big]^*\mathrm{d}x,$$
$$I_n^2 = \int_{\mathbb{R}^d}\varphi(x)E\Big[\Phi(X_t^x)t^{-1}\Big[\int_0^t\Big[\frac{\partial}{\partial x}X_s^{n,x}\Big]^*\mathrm{d}B_s\Big]\Big]^*\mathrm{d}x.$$

Let U be the compact support of φ. Using Cauchy inequality, Itô's isometry, and (Menoukeu-Pamen and Mohammed, 2016, Proposition 4.11), we have the following bound for the first term

$$i)_n \le \int_{\mathbb{R}^d}|\varphi(x)|\|\frac{\partial}{\partial x}\Phi\|_\infty\|X_t^{n,x}-X_t^x\|_{L^2(\Omega;\mathbb{R}^d)}t^{-1/2}\Big(\sup_{k\ge1,s\in[0,1]}\sup_{x\in U}E\Big[\|\frac{\partial}{\partial x}X_s^{k,x}\|^2_{\mathbb{R}^{d\times d}}\Big]\Big)^{1/2}\mathrm{d}x$$

The right hand side of the above expression goes to zero as n tends to infinity by the choice of the sequence and Lebesque dominated convergence theorem.

We know from (Menoukeu-Pamen and Mohammed, 2016, Theorem 3.4) that the SDE (1.3) has a Sobolev differentiable flow. Using once more the martingale representation theorem (see for (Elliott and Kohlmann, 1988, Theorem 1)), the product rule and the property of stochastic integral, we have

$$I_n^2 = \int_{\mathbb{R}^d}\varphi(x)E\Big[\Phi(X_t^x)t^{-1}\Big[\int_0^t\Big[\frac{\partial}{\partial x}X_s^{n,x}\Big]^*\mathrm{d}B_s\Big]\Big]^*\mathrm{d}x$$
$$= \int_{\mathbb{R}^d}t^{-1}\varphi(x)E\Big[\Big(E\Big[\Phi(X_t^x)\Big]+\int_0^t E_{s,y}\Big[\Phi'(X_t^{n,x})\frac{\partial}{\partial y}X_t^{n,s,y}\Big]^*\mathrm{d}B_s\Big)\Big[\int_0^t\Big[\frac{\partial}{\partial x}X_s^{n,x}\Big]^*\mathrm{d}B_s\Big]\Big]^*\mathrm{d}x$$
$$= t^{-1}\int_0^t\int_{\mathbb{R}^d}\varphi(x)E\Big[E\Big[\nabla_y\Phi(X_t^{s,y})\Big|\mathcal{F}_s\Big]\frac{\partial}{\partial x}X_s^{n,x}\Big]\mathrm{d}x\mathrm{d}s. \tag{3.6}$$

Using (Menoukeu-Pamen and Mohammed, 2016, Proposition 4.11), one can check that $\varphi(\cdot)E\Big[\nabla_y\Phi(X_t^{s,y})\Big|\mathcal{F}_s\Big] = \varphi(\cdot)E_{s,y}\Big[\Phi'(X_t^{n,x})\frac{\partial}{\partial y}X_t^{n,s,y}\Big]$ belongs to $L^2(\mathbb{R}^d\times\Omega)$. Hence, it follows from the weak convergence of $\frac{\partial}{\partial x}X_s^{n,x}$ (for a subsequence)(see (Menoukeu-Pamen and Mohammed, 2016, Lemma 4.12)), that the function

$$g_n(s) = \int_{\mathbb{R}^d}\varphi(x)E\Big[E\Big[\nabla_y\Phi(X_t^{s,y})\Big|\mathcal{F}_s\Big]\frac{\partial}{\partial x}X_s^{n,x}\Big]\mathrm{d}x$$

converges to

$$\int_{\mathbb{R}^d}\varphi(x)E\Big[E\Big[\nabla_y\Phi(X_t^{s,y})\Big|\mathcal{F}_s\Big]\frac{\partial}{\partial x}X_s^{x}\Big]\mathrm{d}x$$

for each s. We have the next bound for g_n

$$|g_n(s)| \le \int_{\mathbb{R}^d}|\varphi(x)|\|\nabla_y\Phi(X_t^{s,y})\|_{L^2(\Omega;\mathbb{R}^d)}\Big\|\frac{\partial}{\partial x}X_s^{n,x}\Big\|_{L^2(\Omega;\mathbb{R}^d)}\mathrm{d}x$$
$$\le \sup_{y\in U}\sup_{k\ge1,r\in[0,1]}\|\nabla_y\Phi(X_t^{r,y})\|_{L^2(\Omega;\mathbb{R}^d)}\Big\|\frac{\partial}{\partial y}X_r^{k,y}\Big\|_{L^2(\Omega;\mathbb{R}^d)}\int_{\mathbb{R}^d}|\varphi(x)|\mathrm{d}x < \infty.$$

The Lebesgue dominated convergence theorem yields

$$\lim_{n\to\infty}I_n^2 = \int_0^t\int_{\mathbb{R}^d}\varphi(x)E\Big[E\Big[\nabla_y\Phi(X_t^{s,y})\Big|\mathcal{F}_s\Big]\frac{\partial}{\partial x}X_s^{n,x}\Big]\mathrm{d}x\mathrm{d}s.$$

The equality (3.5) follows by reversing equations (3.6) with $\frac{\partial}{\partial x}X_s^x$ replaced by $\frac{\partial}{\partial x}X_s^{n,x}$. □

The proof of proposition is completed by using integration by part. □

Remark 3.3. *Once can also prove in a similar way that the following equation holds:*

$$\frac{\partial}{\partial x} v(t,x) = E\Big[\Phi(X_t^x)\Big[\int_0^t a(s)\Big[\frac{\partial}{\partial x}X_s^x\Big]\mathrm{d}B_s\Big]\Big]^* \tag{3.7}$$

for almost all $x \in U$ *and all* $t \in (0,1]$, *where* a *is a bounded measurable function satisfying* $\int_0^t a(s)\,\mathrm{d}s = 1$.

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