Birationally Rigid Complete Intersections

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Abstract

This thesis studies the birational geometry of Fano complete intersections of index one with simple singularities, in particular constructing families which are birationally superrigid. There are two cases considered, Fano complete intersections of codimension two and Fano complete intersections of high codimension. In each case the construction of the birationally superrigid families allows estimates of the codimension of the locus of non-birationally superrigid varieties.

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Contents

Introduction

This thesis deals with birational (super)rigidity of (singular) Fano complete intersections of index one. The origins of birational (super)rigidity lie in the papers of Fano [22, 23] on algebraic three-folds. Unfortunately all the proofs of his main results contain errors but the papers had useful ideas. The study of three-folds, in which the question of determining when a three-fold is rational or not had yet to be solved. The case of curves and surfaces is solved by differential geometric invariants. For curves this is determined by the genus of the curve and for surfaces by Castelnuovo's rationality criterion [76]. The ideas of Fano were used in the breakthrough paper of Iskovskih and Manin [33]. The paper proves a nonsingular quartic hypersurface $V_4 \subset \mathbb{P}^4$ is non rational. This gives a counter example to the Lüroth problem (using [45]).

Birational (super)rigidity was first used as a rigorous concept in the paper [56]. But in this thesis we use the modern definition, given in Chapter two. A survey of this subject can be found in [5]. Be aware that different authors use different definitions of birational rigidity [1, 9, 12, 26], but they all have the same geometric implication, that is, the uniqueness of the structure as a Mori-Fano fibre space.

The first variety to be shown to be birationally rigid was the non-singular quartic in [33]. Birational rigidity has since been shown in families of non-singular Fano hypersurfaces [13, 14, 15, 57], non-singular Fano complete intersections [10, 59, 67, 66, 65, 80], and other non-singular varieties see [9, 61, 34, 62].

In considering Fano varieties with singularities, birational rigidity was shown in the case of the quartic threefold with an elementary singularity in [54, 12, 47, 78]. The connectedness principle of Shokurov and Kóllar [77, 38, 42] improved the method of proving birational (super)rigidity for singular Fano varieties [6, 7, 8, 63, 17, 69, 71].

For the case of Fano hypersurfaces of index one $V_d \subset \mathbb{P}^d$, i.e. $\deg V_d = d$ with $d = 4, 5, 6, 7, 8$ were shown to be birationally (super)rigid in [9]. Generic non-singular

Fano hypersurfaces of index one, in that they satisfy regularity conditions to be discussed in chapter 2, are shown to be birationally superrigid in [57]. In this thesis we make no reference to weighted complete intersections, for which there are many results in that direction [35, 51, 50, 52]

The results that show certain families of non-singular Fano complete intersections of index one don't give bounds on the codimension of the set of non-birationally (super)rigid varieties in their natural parameter space as they only prove the birational (super)rigidity of non-singular varieties satisfying certain conditions. To obtain a bound, singular varieties satisfying certain other conditions have to be shown to be birationally (super)rigid.

The application of regularity conditions to singular Fano varieties first occurred in [17]. This paper dealt with the case of singular Fano hypersurfaces with certain singularities. This paper gave a bound on the codimension of the set of non-birationally rigid varieties. The next result in this programme was for singular Fano complete intersections of index one and codimension two under some regularity conditions, which is given in Chapter 3. The case of high codimension is given in Chapter 4 and is the main result of this thesis.

The first chapter is a review of results and terminology with all results from basic algebraic geometry covered in the books [18, 25, 29, 30, 32, 48, 73, 74, 75]. The starting tool for studying birational (super)rigidity is studying mobile linear systems, that is, free from fixed components. The first chapter also contains results on complete intersections and intersection theory, which are required later.

The second chapter contains the general methods and techniques, which are required to show birational (super)rigidity. This starts by defining the threshold of canonical adjunction, which is the modern way to define birational (super)rigidity. The method of maximal singularities is discussed next. This is the method used to prove that a variety is birationally (super)rigid. The method is split up into distinct steps, the first being the resolution of geometric valuations, the second the oriented graph, and finally the Noether-Fano inequality. The method of maximal singularities reduces the proof of birational (super)rigidity to excluding all centres of maximal singularities. The way of exclusion can be split up into two different ways, the linear method and the quadratic method. The linear method uses inversion of adjunction to reduce to a variety of small dimension. The quadratic method uses that the self intersection of a mobile linear system has high multiplicity along the maximal singularity if the mobile linear system contains this maximal singularity.

Finally, the chapter ends with the specific techniques needed for complete intersections with singularities. The method is applicable to complete intersections that satisfy some criteria called regularity conditions and a condition on the singularities, that is, we only allow correct multiquadratic singularities. Then hypertangent divisors are introduced, using these divisors we can complete the main part of the proof of birational (super)rigidity.

Chapter 3 contains the case of the Fano complete intersection of index one and codimension two. That is, the proof of Theorem 3.0.1. The theorem proves that if $V \subset \mathbb{P}^{M+2}$, with $M \geqslant 13$, is a Fano complete intersection of index one and codimension two, which is given by polynomials f_1 and f_2, which satisfies certain regularity conditions then V is birationally superrigid. The theorem also gives a bound on the codimension of the set of non-birationally rigid varieties in the natural parameter space given in Chapter 3.

There are two different methods given for proof of birational superrigidity of this theorem: the original proof, and a new proof, which is then generalised in Chapter 4. These methods are based on the results stated in Chapter 2. The rest of the chapter deals with calculating the bound on the codimension of the set of non-birationally rigid varieties. The first task is to look at when a pair of polynomials gives an irreducible factorial complete intersection, then when the polynomials satisfy the regularity conditions. This is done by an induction argument using the projection method and the last step of the induction requires special attention. The last task is to check when a pair gives the correct singularities, that is, when correct multiquadratic singularities occur.

Chapter 4 contains the main result of the thesis, that is, Theorem 4.0.1. This theorem proves for Fano complete intersections of index one and high codimension, k, $V \subset \mathbb{P}^{M+k}$, defined by polynomials $f_1, \ldots, f_{M+k}$. which satisfies certain regularity conditions then V is birationally superrigid. The theorem also gives a bound on the codimension of the set of non-birationally rigid varieties in the natural parameter space given in Chapter 4.

The proof of birational superrigidity is a generalisation of the codimension two example using the $4n^2$ inequality for complete intersection singularities. The difficulty of the proof is in the second part, finding the bound on the codimension of the set of non-birationally rigid varieties. It is proved by the projection method only, but a number of reductions are required first. This reduces to looking at the case of a non-singular point on the complete intersection with all degrees equal, i.e. V is complete intersection of k hypersurfaces all of degree d. This part of the proof is purely a combinatorial problem. The next part is using analytic techniques to solve this combinatorial problem.

We now explain the system of enumeration and cross-references. All theorems, lemmas, corollaries, definitions, remarks and examples are labelled together. A reference to Definition 2.4.2, refers to Chapter 2, Section 4. A reference to Section 2.1 refers to Chapter 2, Section 1 and similarly, a reference to Subsection 2.1.3 refers to Chapter 2, Section 1 and Subsection 3.

Finally, all the diagrams were produced using IPE, http://ipe.otfried.org/.

Chapter 1

Background

In this chapter we will review results needed in the later chapters for the method of maximal singularities. We assume the reader is familiar with some introductory algebraic geometry found in the following [29, 32, 73, 74]. For an introduction to the modern theory see any of the following [18, 25, 30, 48, 75]. It is assumed throughout that we are working over the complex numbers $\mathbb{C}$ and a variety is defined to be an irreducible projective variety unless otherwise stated.

1.1 Algebraic Varieties and Rational Maps

In this section we recall the definition of a complete intersection and rational connectedness. Then we consider when two varieties are birational, and finally go through the example of the blow up of a variety X along a subvariety $Y \subset X$.

Definition 1.1.1. A subvariety $V \subset \mathbb{P}^N$ of codimension k is called a complete intersection if and only if the homogeneous ideal I of V in the graded ring $S = \mathbb{C}[z_0, \dots, z_N]$ is generated by k homogeneous polynomials $f_1, \dots, f_k$.

Lemma 1.1.2. *Let $V \subset \mathbb{P}^N$ be a complete intersection. Then for all $l \geqslant 0$ the natural map*

$$H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(l)) \to H^0(V, \mathcal{O}_V(l))$$

is surjective.

Definition 1.1.3. A non-singular projective variety is said to be *rationally connected* if any of its two points $x_1, x_2 \in X$ in general position can be joined by an irreducible rational curve, that is, there exists a morphism $f : \mathbb{P}^1 \to X$ such that

$f(t_1) = x_1$ and $f(t_2) = x_2$ for some $t_1, t_2 \in \mathbb{P}^1$.

Example 1.1.4. The projective space $\mathbb{P}^N$ is rationally connected.

Example 1.1.5. Let $Q \subset \mathbb{P}^3$ be a non-singular quadric surface. Then Q is rationally connected.

Definition 1.1.6. Two varieties X and Y are said to be *birational* if one of the equivalent definitions hold:

- there exist isomorphic Zariski open subsets $U \cong V$ with $U \subset X$ and $V \subset Y$;
- the two varieties X and Y have isomorphic function fields $\mathbb{C}(X) \cong \mathbb{C}(Y)$;
- there exist invertible rational maps $X \dashrightarrow Y$ and $Y \dashrightarrow X$ such that the composition is an isomorphism on some Zariski open subset of each variety.

Definition 1.1.7. A variety X is said to be *rational* if it is birational to some projective space $\mathbb{P}^n$.

The simplest birational maps are constructed by blowing up some subvariety of codimension at least two.

Definition 1.1.8. Let $X = \operatorname{Spec} A$ be an affine scheme and

$$Y = \{f_1 = \ldots = f_k = 0\} \subset X$$

be a closed subscheme. The *blow up of X along Y*, denoted by $\mathrm{Bl_Y X}$, is the closure in $\mathbb{P}^{k-1}_A$ of the graph of the morphism

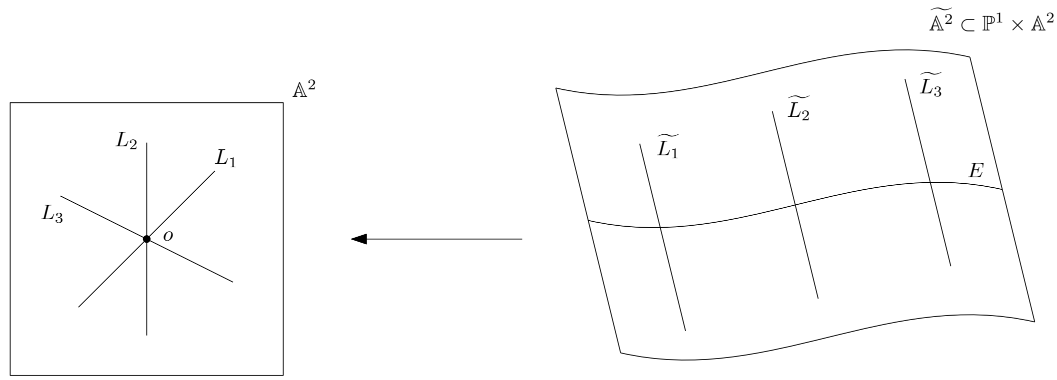
$$\alpha : X \setminus Y \to \mathbb{P}^{k-1}_A,$$

where the map is given by $y_i \mapsto f_i$ if $\mathbb{P}^{k-1}_A$ is defined by $A[y_1, \ldots, y_k]$. The *exceptional divisor* is defined to be $(\mathrm{Bl_Y X} \setminus \alpha(\mathrm{X} \setminus \mathrm{Y}))$.

Remark 1.1.9. The general case of blow ups could be achieved by the above definition and gluing affine schemes, but it is usually done instead by the **Proj** construction in [30]. The only explicit blow ups required in this thesis is the case of the projective space blown up at a point. The map from the blow up to the variety X is a morphism with an exceptional divisor.

Example 1.1.10. Let us construct $\mathbb{P}^n$ blown up at a point, first by considering $\mathbb{A}^n$ blown up at the origin, with $o = \{x_1 = \ldots = x_n = 0\}$. The blow up $\widetilde{\mathbb{A}^n} \subset$

$\mathbb{P}^{n-1} \times \mathbb{A}^n$ is defined by the kernel of the ring homomorphism, which can be given by $\{y_i x_j = y_j x_i | i, j = 1, \ldots n\}$. Clearly, away from o, there is an isomorphism and the closure contains the divisor $E \cong \mathbb{P}^{n-1} \times \{o\}$, which gets contracted to the point o.



Definition 1.1.11. Let $Z \subset X = \operatorname{Spec} A$ be a subvariety. The *strict transform of* Z is defined to be the closure of the image of $Z \setminus (Z \cap Y)$ under the morphism α, which should agree with the blow up of Z along $Z \cap Y$.

Remark 1.1.12. It can be shown that the exceptional divisor of X blown up at a point p is the projectivised tangent cone, $\mathbb{P}(T_p(X))$, of X at p. In particular if $X \subset \mathbb{P}^n$ is a hypersurface with X defined by $f(x_0 \ldots, x_n) = 0$ at p, then the tangent cone is given by the lowest degree homogeneous component of f.

Theorem 1.1.13. *Let X be a projective variety. There exists a non-singular variety Y and a birational morphism $\varphi : Y \to X$, which is a composition of blow ups of non-singular subvarieties. This map is called a resolution of singularities.*

Proof. See [31].

1.2 Divisors and Linear Systems

This section includes definitions and results about divisors on algebraic varieties. The section starts with elementary definitions about divisors and standard results about when a variety is factorial. Then we define Fano varieties and consider when a complete intersection is a Fano variety. The sections ends with the definition of discrepancies and gives two examples for which the discrepancy can be calculated. Throughout this section we assume that the variety X is normal.

Definition 1.2.1. Let X be an irreducible variety. Let $\operatorname{Div}(X)$ be the free group generated by all irreducible subvarieties of codimension 1. A *Weil divisor D* is an element of $\mathrm{Div}(X)$, i.e. it can be written as $\sum_i a_i D_i$ with $a_i \in \mathbb{Z}$ and D_i are distinct irreducible subvarieties of codimension 1. The D_i's are called *prime* divisors and if $a_i \geqslant 0$ for all i then D is said to be *effective*. The set $\cup_i D_i$ is said to be the *support* of D.

Remark 1.2.2. For every prime divisor D we can find an open affine subset $U \subset X$ such that U is non-singular and that $D \cap U$ is defined by a local equation π. Then for any regular function $f \neq 0$ on U, there exists an integer $k \geqslant 0$ such that $f \in (\pi^k)$ and $f \notin (\pi^{k+1})$. The number k is now denoted by $\nu_D(f)$. If X is irreducible, then any function $f \in \mathbb{C}(X)$ can be written in the form $\frac{g}{h}$ with g, h regular on U. If $f \neq 0$ we set $\nu_D(f) = \nu_D(g) - \nu_D(h)$.

Definition 1.2.3. A *principal Weil divisor* is a divisor D such that

$$D = \sum \nu_C(f) C,$$

for some $f \in \mathbb{C}$ and the sum is taken over all prime divisors C.

Definition 1.2.4. Two divisors D and D' are said to be *linearly equivalent* if $D - D'$ is a principal divisor. The group of all divisors modulo linear equivalence is called the *divisor class group* denoted by $\mathrm{Cl}\, X$.

Definition 1.2.5. A *Cartier Divisor* on X is a pair (U_i, f_i) with U_i an open cover of X, and rational functions f_i such that the f_i are not identically 0, and $\frac{f_i}{f_j}$ and $\frac{f_j}{f_i}$ are both regular on $U_i \cap U_j$. The pair (U_i, f_i) defines the same divisor as (V_i, g_i) if $\frac{f_i}{g_j}$ and $\frac{g_j}{f_i}$ are regular on $U_i \cap V_j$. A *principal Cartier divisor* is a Cartier divisor (f, U_i) if $f_i = f$ for every open set. The *support* of a Cartier divisor is the closed subset, which in each U_i consists of points at which f_i is either not regular, or equal to 0.

Definition 1.2.6. The *Picard group* of X, denoted by $\mathrm{Pic}(X)$, is the group of classes of Cartier divisors with respect to linear equivalence.

Definition 1.2.7. A $\mathbb{Q}$-*divisor* is a $\mathbb{Q}$-linear combination of integral divisors. A variety X is said to have $\mathbb{Q}$-*factorial singularities* if every Weil divisor on X is $\mathbb{Q}$-Cartier.

Definition 1.2.8. A divisor D on X is *very ample* if there exists a closed embedding $X \subset \mathbb{P}^N$ of X into some projective space such that $\mathcal{O}_X(D) = \mathcal{O}_{\mathbb{P}^N}(1)|_X$, i.e. D is the hyperplane section under this embedding. A divisor D on X is *ample* if mD is very ample for some $m > 0$.

Theorem 1.2.9. *Let D be a Cartier divisor on a projective variety X. Then D is ample on X if and only if for every irreducible subvariety $Y \subset X$ we have*

$$\left(D^{\dim Y} \cdot Y\right) > 0,$$

where $r = \dim Y$.

Proof. See [49].

Definition 1.2.10. A variety X is *factorial* if all its local rings $\mathcal{O}_x$ are unique factorisation domains.

Theorem 1.2.11. *If X is a factorial variety, then there is a natural isomorphism*

$$\operatorname{Cl} X \cong \operatorname{Pic} X.$$

Proof. A well-known fact, see [30].

Example 1.2.12. Non-singular varieties are factorial.

Definition 1.2.13. A variety X has only *complete intersection singularities* if all its local rings $\mathcal{O}_x$ are complete intersection rings.

Remark 1.2.14. Assume a variety $X = \bigcup_{i \in I} U_i$ has a cover by open affine subsets such that each subset U_i is a complete intersection. Then X has only complete intersection singularities.

Theorem 1.2.15. *Let X be a variety with only complete intersection singularities and assume the following inequality*

$$\operatorname{codim}(\operatorname{Sing} X) \geqslant 4$$

holds. Then X is factorial.

Proof. First done in [27, XI.Cor.3.14], a modern proof is given in [3].

Definition 1.2.16. If $p \in V \subset \mathbb{A}^n$ is a hypersurface given locally around p as the zero locus of a regular function f, we can choose local coordinates $z = (z_1, \ldots, z_n)$ on V in a neighbourhood of p and expand f around p, writing

$$f(z) = f_0 + f_1(z) + f_2(z) + \ldots$$

with $f_i(z)$ homogeneous of degree i. We say V has *multiplicity* m at p if $f_0 = f_1(z) = \ldots = f_{m-1}(z) = 0$ and $f_m \neq 0$. We write $\operatorname{mult}_p V$ for the multiplicity of V at p.

Remark 1.2.17. In the above definition the projective variety $\{f_m = 0\} \subset \mathbb{P}^{n-1}$ is called the projectivised tangent cone of V at p. To extend this definition of multiplicity to an arbitrary scheme we construct the projectivised tangent cone and define the multiplicity to be the degree of the projectivised tangent cone. For full details see [19, Section 1.3.8].

Definition 1.2.18. Let Z and Y be irreducible subvarieties of V. The *multiplicity of Z along Y* is defined to be

$$\operatorname{mult}_Z Y = \min\{\operatorname{mult}_p Y | p \in Z\}.$$

Example 1.2.19. Let X be a projective variety. Consider the blow up $\pi : \widetilde{X} \to X$ at a non-singular point o. Let E be the exceptional divisor. We have the equality

$$\operatorname{Pic}(\widetilde{X}) = \operatorname{Pic}(X) \oplus \mathbb{Z}E.$$

For a divisor D, $\pi^*(D) = D^+ + mE$ where $m = \operatorname{mult}_o D$ and D^+ is the strict transform of D on $\widetilde{X}$.

Definition 1.2.20. Let X be a non-singular variety of dimension n. Then the *canonical divisor* K_X is the Weil divisor associated to a non-zero rational differential n-form on X.

Proposition 1.2.21. *(Adjunction formula) Let $Y \subset X$ be an irreducible closed subvariety of codimension 1, and X a non-singular variety. Then*

$$K_Y = (K_X + Y)|_Y.$$

Proof. See [30, II, 8.20].

Remark 1.2.22. The adjunction formula can be generalised to singular varieties. For a full treatment see [39]. For all (singular) varieties considered in this thesis the same adjunction formula is true.

Definition 1.2.23. A non singular projective variety X is called a *Fano variety* if its anticanonical divisor $-K_X$ is ample. If a normal projective variety X has singular points, and some positive multiple $-nK_X$, $n \in \mathbb{N}$, of an anticanonical Weil divisor $-K_X$ is an ample Cartier divisor, then X is called a *singular Fano variety.* Assume now on a possibly singular Fano variety X that $\operatorname{Pic} X = \mathbb{Z}H$, so that $K_X = -rH$. We denote r as the index of the Fano variety, if $r = 1$ we say X is a *primitive Fano variety.*

Theorem 1.2.24. *A non-singular Fano variety is rationally connected.*

Proof. See [37].

Example 1.2.25. Let $V \subset \mathbb{P}^N$ be a non-singular complete intersection of codimension k given by equations of degree $(d_1, \ldots, d_k)$. The canonical divisor is

$$K_V = \left(\sum_{i=1}^{k} d_k - N - 1 \right) H_V,$$

by the adjunction formula, where H_V is a hyperplane section of V. In particular if $\sum_{i=1}^{k} d_k = N$ we have

$$K_V = -H_V.$$

Example 1.2.26. If $V \subset \mathbb{P}^N$ is a non-singular complete intersection of codimension k given by equations of degree $(d_1, \ldots, d_k)$ with $d_1 + \ldots + d_k = N$ then

$$\left((-K_V)^{\dim Y} \cdot Y\right) = \left(H^{\dim Y} \cdot Y\right) = \deg Y > 0,$$

so that $-K_V$ is ample and V is a Fano variety. Moreover a complete intersection is a Fano variety if and only if $d_1 + \ldots + d_k < N + 1$.

Definition 1.2.27. Let D be a divisor on X. We define $|D|$ to be the set of effective divisors linearly equivalent to D. This set $|D|$ has the structure of a projective space. A projective subspace $\Sigma \subset |D|$ is called a *linear system* on X. We say a linear system Σ is *mobile* if it contains no fixed components.

Definition 1.2.28. Let X be a normal variety, assume mK_X is Cartier for some $m \in \mathbb{Z}_{>0}$ and let $f : Y \to X$ be a birational morphism with Y normal and $E \subset Y$ an irreducible divisor with $e \in E$ a general point. Let $\{y_1, \ldots, y_n\}$ be local coordinates with E defined by y_1 near e. If G is a local generator of $\mathcal{O}_X(mK_X)$ at $f(e)$ then

$$f^*(G) = y_1^{c(E,X)} \cdot s \cdot (dy_1 \wedge \ldots dy_n)^{\otimes m},$$

where s is a unit and $c(E, X) \in \mathbb{Z}$. The *discrepancy of* E is defined to be $a(E, X) = \frac{1}{m}\, c(E, X)$.

Remark 1.2.29. The discrepancy is independent of the birational model of Y, and if the E_i are the f exceptional divisors (see [41]) this gives

$$mK_Y \sim f^*(mK_X) + \sum_i (m \cdot a(E, X))E_i.$$

Example 1.2.30. Let X be a non-singular variety, $B \subset X$ a non-singular subvariety of codimension $k \geqslant 2$, let $\sigma : X^+ \to X$ the blow up of X along B so that $E = \sigma^{-1}(B)$ is the exceptional divisor. Then

$$K_{X^+} = \sigma^* K_X + (k-1)E.$$

Example 1.2.31. Let $V \subset \mathbb{P}^N$ be a hypersurface with an isolated singular point B of multiplicity m. Let $\sigma : V^+ \to V$ be the blow up of V at B and $E = \sigma^{-1}(B)$ the exceptional divisor. Then

$$K_{V^+} = \sigma^* K_V + (N-1-m)E.$$

1.3 Intersection Theory

The later chapters make use of intersection theory. For further discussion see [30, Appendix A] and for a full treatment see [24]. In this section we include definitions and results required in this thesis.

Definition 1.3.1. A *k-cycle* on a projective variety X is a $\mathbb{Z}$-linear combination of irreducible subvarieties of dimension k, the *free Abelian group of k-cycles*, with the natural operation of addition, is denoted by $Z_k X$.

Remark 1.3.2. To define the intersection number between two cycles let us start by looking at the case when $A, B \subset X$ are two subvarieties of a smooth variety such that every irreducible component C of the intersection $A \cap B$ has codimension $\operatorname{codim} C = \operatorname{codim} A + \operatorname{codim} B$. For each component there is a positive integer $m_C(A,B)$ called the intersection multiplicity of A and B along C, which can be defined by Serre's formula [19, Theorem 2.7] or an alternative definition given in [24, Chapter 7]. Then we define $(A \cdot B) = \sum_C m_C(A,B)$ over all irreducible components $C \in A \cap B$. This definition can then be extended when two cycles don't intersect properly using rational equivalence see [24, Chapter 1].

Definition 1.3.3. Let D be a zero cycle on a projective variety X, that is, $D = \sum_i a_i x_i$. We define the *degree* map

$$\deg Z_0 X \to \mathbb{Z},$$

sending D to $\sum_i a_i$. The image of D is called the *degree* of C.

Definition 1.3.4. Two k-cycles Y_1 and Y_2 are said to be *numerically equivalent* if, for any cycle C of codimension k, we have

$$\deg(Y_1 \cdot C) = \deg(Y_2 \cdot C).$$

Definition 1.3.5. Let X be a projective variety. The ring of numerical Chow cycles $A_k X$ is the ring of k-cycles on X modulo numerical equivalence of the intersection product. We also use the notation $A^k X$ to denote the ring of cycles of codimension k.

Definition 1.3.6. Set $A^1_{\mathbb{R}}X = A^1 X \otimes \mathbb{R}$. The *psuedo effective cone* $A^1_+ X \subset A^1_{\mathbb{R}} X$ is the closure of the cone spanned by the classes of effective $\mathbb{R}$ divisors. A divisor D is *psuedo effective* if D belongs to the cone $A^1_+ X$.

Theorem 1.3.7. *Let $X \subset \mathbb{P}^N$ be a non-singular subvariety of dimension n. If $n \geqslant \frac{N}{2} + 1$ the restriction determines an isomorphism*

$$\operatorname{Pic} \mathbb{P}^N \to \operatorname{Pic} X.$$

Proof. See [43, Section 3.2].

Corollary 1.3.8. *Let $X \subset \mathbb{P}^N$ be a non-singular subvariety of dimension n. Then*

$$Z_k(X) \cong Z_k(\mathbb{P}^N) \cong \mathbb{Z}H^k$$

for $k \geqslant \frac{N}{2} + 1$, where H^k is the k-th intersection of a hyperplane section on $\mathbb{P}^N$.

Definition 1.3.9. Let X be a complete variety or scheme. A Cartier divisor D on X is nef if

$$(D \cdot C) \geqslant 0$$

for all irreducible curves $C \subset X$.

Definition 1.3.10. Let $f : X \to Z$ be a proper morphism and X irreducible. A Cartier divisor is f*-nef* if

$$(D \cdot C) \geqslant 0$$

for all irreducible curves C contained in a closed fibre of f.

Definition 1.3.11. Let X be a proper and irreducible variety over a field. Let D be a divisor on X. We say that D is *big* if there is a constant $\epsilon > 0$ such that

$$h^0(X, \mathcal{O}_X(mD)) \geqslant \epsilon \cdot m^{\dim X}$$

for all sufficiently large m.

Definition 1.3.12. Let $f : X \to Z$ be a proper morphism and X irreducible. Let D be a divisor on X. We say D is *f-big* if D is big on the fibre of f over the generic point of $f(X)$.

Theorem 1.3.13. *Let D be a nef divisor on an irreducible projective variety X of dimension n. Then D is big if and only if its top self-intersection is strictly positive, i.e. $(D)^n > 0$.*

Proof. See [43, Theorem 2.2.16.].

Example 1.3.14. Let $f : X \to \mathbb{P}^2$ be the blow up of the projective plane at two distinct points p and q. Then we have $\mathrm{PicX} = \mathbb{Z}H \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2$, where E_1 and E_2 are the two exceptional divisors. Let $D = dH - rE$, where d and r are positive integers, be a divisor on X with $E = E_1 + E_2$. If $d \geqslant 2r$ the D is nef. If $d^2 > 2r^2$ then D is big. We also see that all divisors are f-big and f-nef.

Example 1.3.15. Let $g : X \to \mathbb{P}^1$ be the composition of f defined above and $\pi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ the projection from the point p. If $d \geqslant 2r$ then the divisor D is g-nef. If $d > 2r$ then D is g-big.

Let V be an irreducible projective variety, $B \subset V$ be an irreducible cycle of codimension at least 2 and $B \not\subset \mathrm{Sing}V$. Let $\sigma : V^+ \to V$ be the blow-up of V along B and $E = \sigma^{-1}(B)$ the exceptional divisor.

Let X and Y be two different prime Weil divisors on V, and let X^+ and Y^+ be their strict transforms on V^+.

Lemma 1.3.16. *Assume* $\mathrm{codim}B \geqslant 3$. *Then*

$$X^+ \circ Y^+ = (X \circ Y)^+ + Z,$$

where $\circ$ *is the scheme-theoretic intersections and* $\mathrm{Supp}Z \subset E$. *In addition*

$$\mathrm{mult}_B(X \circ Y) = (\mathrm{mult}_B(X))(\mathrm{mult}_B(Y)) + \deg Z.$$

Lemma 1.3.17. *Assume* $\mathrm{codim}B = 2$. *Then*

$$X^+ \circ Y^+ = Z + Z_1$$

where $\operatorname{Supp} Z \subset E$ *and* $\operatorname{Supp} \sigma(Z_1)$ *does not contain* B, *and*

$$X \circ Y = [(\mathrm{mult}_B(X))(\mathrm{mult}_B(Y)) + \deg Z]B + \sigma_* Z_1.$$

Proof. See [19, 24].

Chapter 2

Methods and Techniques

This chapter contains general methods for proving birational superrigidity most of which are contained in [53], with the last three sections detailing the more specific techniques required for the cases of complete intersections we will use in the next two chapters. The chapter starts by defining birational superrigidity, then the method of maximal singularities, which is the method which gives a sufficient condition for a variety to be birationally superrigid. The method required the construction of a certain oriented graph, and the resolution of discrete valuations. Three techniques take up the remainder of the chapter namely $4n^2$ inequality; inversion of adjunction; hypertangent linear systems.

2.1 Birational Superrigidity

In this section we will define the property of being birationally superrigid and give some consequences of this property.

Definition 2.1.1. Let V be a rationally connected projective variety. Let $A^1V = \operatorname{Pic} V$ be the Picard group and $A^1_+V \subset A^1V \otimes \mathbb{R}$ the cone of pseudo effective classes. The *threshold of canonical adjunction* of a divisor D on the variety V is the number

$$c(D, V) = \sup\{\varepsilon \in \mathbb{Q}_+ | D + \varepsilon K_V \in A^1_+V\}.$$

If Σ is a non-empty linear system on V, then we set $c(\Sigma, V) = c(D, V)$, where $D \in \Sigma$ is an arbitrary divisor. This is well defined as the threshold of canonical adjunction is independent of linear equivalence of divisors.

Definition 2.1.2. For a mobile linear system Σ on a variety V, define the *virtual threshold of canonical adjunction* by

$$c_{\text{virt}}(\Sigma) = \inf_{V^{\#} \to V} \{c(\Sigma^{\#}, V^{\#})\},$$

where the infimum is taken over all birational morphisms $V^{\#} \to V$. Here $V^{\#}$ a smooth projective model of $\mathbb{C}(V)$ and $\Sigma^{\#}$ is the strict transform of Σ on $V^{\#}$.

Example 2.1.3. Let $\pi : \mathbb{P}^N \dashrightarrow \mathbb{P}^M$ be the projection from a $(N - M - 1)$ dimensional plane P and $\Lambda \subset |nH_{\mathbb{P}^M}|$ a mobile linear system of hypersurfaces of degree n and Σ its pullback on $\mathbb{P}^N$, so that, $c(\Sigma, \mathbb{P}^N) = \frac{n}{N+1}$. Let $\sigma : \mathbb{P}^+ \to \mathbb{P}^M$ be the blow-up of P and Σ^+ the strict transform of the linear system Σ, then σ has rationally connected fibres so that $c(\Sigma^+, \mathbb{P}^+) = 0$, see [16, Chapter 4].

Proposition 2.1.4. *Let V be a variety and assume for all mobile linear systems Σ that $c_{\text{virt}}(\Sigma) > 0$. If $\rho : V \dashrightarrow S$ is a rational dominant map, such that for a general point $s \in S$, $\rho^{-1}(s)$ is rationally connected then* $\dim S = 0$*. This implies S is a point and this is the trivial fibration.*

Proof. This is clear, let $Y = \rho^{-1}(Q)$ where Q is a very ample divisor on S and this proves the proposition. Q.E.D.

Proposition 2.1.5. *Let $\pi : V \to S$ be a rationally connected fibre space. Assume each mobile linear system Σ with $c_{virt}(\Sigma) > 0$ is the pullback of a system on the base, that is, $\Sigma = \pi^*\Lambda$ with Λ some mobile linear system on S. Let Bir be any birational map*

$$\begin{array}{ccc} V & \overset{Bir}{\dashrightarrow} & V^{\#} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi^{\#}} \\ S & & S^{\#} \end{array}$$

with $\pi^{\#} : V^{\#} \to S^{\#}$ a rationally connected fibre space. Then there exists a rational dominant map $\rho : S \to S^{\#}$ making the diagram commute.

Proof. Similar to the proof of the previous proposition. Q.E.D.

Definition 2.1.6. (i) A variety V is said to be *birationally superrigid* if for any mobile linear system Σ on V the following equality holds:

$$c_{\text{virt}}(\Sigma) = c(\Sigma, V).$$

(ii) A variety V is said to be *birationally rigid* if for any mobile linear system Σ on V there exists a birational self-map $\chi \in \operatorname{Bir} V$ such that the following equality holds

$$c_{\mathrm{virt}}(\Sigma) = c(\chi_* \Sigma, V).$$

Proposition 2.1.7. *Let V be a primitive Fano variety, that is, $\operatorname{Pic}(V) = \mathbb{Z}H$, X a Fano variety with $\mathbb{Q}$-factorial terminal singularities (see [40, 72]) and Picard number one, and $\chi : V \dashrightarrow X$ a birational map. Assume V is birationally superrigid. Then χ is a biregular isomorphism.*

Proof. Let

$$\psi : Y \xrightarrow{\varphi} V \overset{\chi}{\dashrightarrow} X$$

be the resolution of the singularities of χ so that $\psi : Y \to X$ is a birational morphism with Y non-singular and

$$\operatorname{Pic} Y = \mathbb{Z}\varphi^* K_V \oplus \bigoplus_{i \in I} \mathbb{Z} E_i,$$

where $\{E_i \mid i \in I\}$ is the set of all φ-exceptional divisors. Also that

$$\operatorname{Pic} Y \otimes \mathbb{Q} = \mathbb{Q}\psi^* K_X \oplus \bigoplus_{j \in J} \mathbb{Q} E_j^X,$$

where $\{E_j^X \mid j \in J\}$ is the set of all ψ-exceptional divisors. We get

$$K_Y = \varphi^* K_V + \sum_{i \in I} a_i E_i = \psi^* K_X + \sum_{j \in J} a_j^X E_j^X, \tag{2.1}$$

with $a_i \in \mathbb{Z}$ and $a_i \geqslant 1$ and $a_j^X \in \mathbb{Q}$, $a_j^X > 0$.

Let $\Sigma^X = |-mK_X|$, $m >> 0$, be a very ample linear system so that $c(\Sigma^X, X) = m$. Take its strict transform $\Sigma \subset |-nK_V|$ and clearly $c(\Sigma, V) = n$. By assumption that V is birationally superrigid we get $n = m$. The strict transforms of both linear systems agree on Y so there exists positive integers b_i, $i \in I$ such that

$$-n\psi^* K_X = -n\varphi^* K_V - \sum_{i \in I} b_i E_i.$$

Comparing this to (2.1) we get that

$$0 = \sum_{i \in I} \left(\frac{b_i}{n} - a_i \right) E_i + \sum_{j \in J} a_j^X E_j^X.$$

All the divisors E_j^X are φ-exceptional and $\{E_i \mid i \in I\} = \{E_j^X \mid j \in J\}$ otherwise rank $(\operatorname{Pic} X) \geqslant 2$. So χ is an isomorphism in codimension one, that is, it does not contract any divisors. Then $\Sigma = |-nK_V|$ and χ induces an isomorphism of the linear systems Σ^X and Σ. This gives that χ and χ^{-1} are defined everywhere and hence an isomorphism. Q.E.D.

Corollary 2.1.8. *Under the assumptions of Proposition 2.1.7, the groups of birational and biregular self-maps of V coincide,* $\operatorname{Bir} V = \operatorname{Aut} V$.

2.2 Maximal Singularities

In this section we will show that if a variety is not birationally superrigid then it must have a maximal singularity, and we will relate this to log pairs.

Definition 2.2.1. A *geometric discrete valuation*, ν, on a variety V is a discrete valuation such that there exists a birational morphism $\widetilde{V} \to V$ and an exceptional divisor $E \subset \widetilde{V}$ such that $\nu = \operatorname{ord}_E$.

Definition 2.2.2. A log pair (V, D) with V a variety and D a $\mathbb{Q}$-divisor is

- *non canonical* if there exists a geometric discrete valuation ν_E such that $\nu_E(D) > a(E, V)$;
- *non log canonical* if there exists a geometric discrete valuation ν_E such that $\nu_E(D) > a(E, V) + 1$.

Here $a(E, V)$ is the discrepancy introduced in Definition 1.3.8.

Definition 2.2.3. A *maximal singularity* is defined to be a non canonical geometric discrete valuation.

Proposition 2.2.4. *If V is not birationally superrigid then there exists a mobile linear system Σ, $D \in \Sigma$ a general element, such that the log pair*

$$\left(V, \frac{1}{n}D\right)$$

is non canonical, where $n = c(\Sigma, V)$.

Proof. If V is not birationally superrigid then there exists a linear system Σ satisfying the inequality

$$c_{\text{virt}}(\Sigma, V) < c(\Sigma, V).$$

By definition there exists a birational morphism $\varphi : V^+ \to V$ for which $c(\Sigma^+, V^+) < c(\Sigma, V)$ holds, with Σ^+ the strict transform of the system Σ. There exists an exceptional divisor $E \subset V^+$ contracted by the map φ (if not then φ is an isomorphism in codimension one, which would imply $c(\Sigma^+, V^+) = c(\Sigma, V)$). Every such divisor determines a discrete valuation $\mathrm{ord}_E(\_)$ on the field of rational functions $\mathbb{C}(V)$. This valuation is independent of the model V^+ chosen in the following way: let $\varphi^* : V^* \to V$ be another birational morphism, which is an isomorphism at the general point of E via the map $(\varphi^*)^{-1} \circ \varphi : V^+ \dashrightarrow V^*$ so that $E^* = (\varphi^*)^{-1} \circ \varphi(E)$ is an exceptional divisor of the morphism φ^*, then $\mathrm{ord}^*_E(\_) = \mathrm{ord}_E(\_)$. The irreducible subvariety $\varphi(E) \subset V$ is called the centre of the discrete valuation $\mathrm{ord}_E(\_)$ and also does not depend on the model chosen, see [39, Definition 2.1].

A divisor D is given on a variety V by local equations. We obtain the multiplicity $\nu_E(D) \in \mathbb{Z}_{\geqslant 0}$ of an effective divisor by applying the valuation $\mathrm{ord}_E(\_)$. Let $\mathcal{E}$ be the set of exceptional divisors of the morphism φ, and D^+ the strict transform of D on V^+, then we get

$$\varphi^* D = D^+ + \sum_{E \in \mathcal{E}} \nu_E(D) E. \tag{2.2}$$

Comparing the canonical classes we get

$$K_{V^+} = \varphi^* K_V + \sum_{E \in \mathcal{E}} a(E) E, \tag{2.3}$$

where $a(E) \geqslant 1$ is the discrepancy of the geometric valuation E which is also independent of the model V^+. By assumption $n = c(\Sigma) > 0$ and $c(\Sigma^+, V^+) < c(\Sigma, V)$ we obtain

$$D^+ + nK_{V^+} \notin A^1_+ V.$$

Then from (2.2) and (2.3) we get that

$$D^+ + nK_{V^+} = \varphi^*(D + nK_V) - \sum_{E \in \mathcal{E}} (\nu_E(D) - na(E)) E,$$

is not psuedoeffective. Since $D + nK_V$ is pseudoeffective, its pullback is also. This implies that for some exceptional divisor E with $\nu_E(D) - na(E) > 0$, which is what is required to prove the pair is non canonical. Q.E.D.

Definition 2.2.5. An irreducible subvariety $Y \subset V$ of codimension $\geqslant 2$ is called a *maximal subvariety* of the linear system Σ if the inequality

$$\mathrm{mult}_Y \Sigma > n(\mathrm{codim}\, Y - 1)$$

holds, where $\operatorname{mult}_Y \Sigma = \operatorname{mult}_Y D$ for a general divisor $D \in \Sigma$.

Definition 2.2.6. If $B \subset V$ is the centre of a maximal singularity and if B is not a maximal subvariety, then it is called an *infinitely near maximal singularity.*

2.3 Resolution of Discrete Valuation

In this section we will resolve a geometric discrete valuation given by an exceptional divisor E over X. This is done constructing a birational morphism $\widetilde{X} \to X$ where E_K is a prime Weil divisor in $\widetilde{X}$, E_K is not contained in $\operatorname{Sing} \widetilde{X}$ and $\nu_{E_K} = \operatorname{ord}_{E_K}(\_) = \operatorname{ord}_E(\_)$. Let E be an exceptional divisor lying over V, that is, V has a birational model V^+ such that

$$\phi : V^+ \dashrightarrow V,$$

which contracts E. Let $B = \phi(E) \subset V$ be the centre of the exceptional divisor, with $B \not\subset \operatorname{Sing} X$. Assume that B has codimension at least two in V. Let

$$\sigma_B : V(B) \to V,$$

be the blow-up of the subvariety B and $E(B) = \sigma_B^{-1}(B)$ the corresponding exceptional divisor.

Proposition 2.3.1. *The following alternative hold:*

- *either the birational map $(\sigma_B^{-1} \circ \phi) : V^+ \dashrightarrow V(B)$ is an isomorphism in a neighbourhood of the generic point of the divisor E and then $(\sigma_B^{-1} \circ \phi)(E) = E(B)$*
- *or $B^+ = (\sigma_B^{-1} \circ \phi)(E)$ is an irreducible subvariety of codimension $\geqslant 2$ and $B^+ \subset E(B)$ and $\sigma_B(B^+) = B$.*

Assume the second case of the above proposition occurs then we blow up $V(B)$ along B^+ and denote this morphism σ_{B^+}. Consider the composition morphism $\sigma_{B^+} \circ \sigma_B : V(B^+) \to V$ and check which case of the above proposition holds for this morphism. This can be iterated until the first case of the proposition at some step and we stop. We obtain a sequence of blow-ups:

$$\begin{array}{rccc} \varphi_{i,i-1} : & V_i & \to & V_{i-1} \\ & \cup & & \cup \\ & E_i & \to & B_{i-1} \end{array}$$

for $i = 1, 2, \ldots$, with $V_0 = V$, and $B_0 = \text{centre}(E, V)$, B_j is the centre of E on V_j, $E_i = \varphi_{i,i-1}^{-1}(B_{i-1})$ is the exceptional divisor, and B_{i-1} the centre of the blow-up $\varphi_{i,i-1}$. For $i > j$ set

$$\varphi_{i,j} = \varphi_{j+1,j} \circ \ldots \circ \varphi_{i,i+1} : V_i \to V_j,$$

and $\varphi_{i,i} = \text{id}_{V_i}$ also by Proposition 2.3.1, $\varphi_{i,j}(B_i) = B_j$.

Proposition. 2.3.2. *The sequence of blow-ups terminates, that is, for some* $K \geqslant 1$ *the first case of Proposition 2.3.1. occurs, that is,* $(\sigma_{K,0}^{-1} \circ \phi)(E) = E_K$.

Proof. The discrepancies of the exceptional divisors increase. This means $a(E_i, V) \geqslant i$ but $a(E_i, V) \leqslant a(E)$ as the centre of E is contained in E_i. Q.E.D.

Definition 2.3.3. For an irreducible subvariety $Y \subset V_j$ the strict transform on V_i, provided it is well defined, is denoted by Y^i, that is, adding the upper index i. This can be extended linearly to effective algebraic cycles when it is well defined.

2.4 Oriented Graph

The oriented graph contains combinatorial data associated to the resolution of the discrete valuation of the previous section.

Definition 2.4.1. Let E be an exceptional divisor lying over V as in Section 2.3. Let $\{E_1, \ldots, E_K\}$ be the set of exceptional divisors obtained from the resolution of the discrete valuation E using Propositions 2.3.1 and 2.3.2. This set $\{E_1, \ldots, E_K\}$ of exceptional divisors has the structure of an *oriented graph* in the following way:

- Vertices $\{1, \ldots, K\}$.
- Arrows $i \to j$, if $i > j$ and $B_{i-1} \subset E_j^{i-1}$.
- Paths $p_{i,j} = \#$ paths from i to j, for $i > j$ and $p_{i,i} = 1$.

Example 2.4.2.

$$\begin{tikzcd}
 & & & 6 \arrow[d] \arrow[ddll] \\
 & & & 5 \arrow[d] \arrow[dl] \\
1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l]
\end{tikzcd}$$

$$p_{6,1} = p_{6,2} = p_{5,2} + 1 = p_{4,2} + p_{3,2} + 1 = 1 + 1 + 1 = 3$$

The graph structure describes the strict transforms of the exceptional divisors, in that

$$E^i_j = \varphi^*_{i,j} E_j - \sum_{j \leftarrow k \leqslant i} \varphi^*_{i,k} E_k,$$

with $\varphi_{i,j}$ defined in the previous section.

Proposition 2.4.3. *The following decomposition holds*

$$\varphi^*_{i,j} E_j = \sum_{k=j}^{i} p_{k,j} E^i_k.$$

.

Proof. This is given by induction on $i \geqslant j$. If $i = j$ there is nothing to prove. If $i = j + 1$ then

$$\varphi^*_{j+1,j} E_j = p_{j,j} E^{j+1}_j + p_{j+1,j} E^{j+1}_{j+1} = E^{j+1}_j + E_{j+1},$$

is true as $B_j \subset E_j$ and E_j is non-singular at the generic point of B_j. If $i \geqslant j + 2$ then

$$\begin{aligned}
\varphi^*_{i,j} E_j &= \varphi^*_{i,i-1}(\varphi^*_{i-1,j} E_j) \\
&= \varphi^*_{i,i-1} \sum_{k=j}^{i-1} p_{k,j} E^{i-1}_k \\
&= \sum_{k=j}^{i-1} p_{k,j} E^i_k + \sum_{B_{i-1} \subset E^{i-1}_k} p_{k,j} E_i.
\end{aligned}$$

The last sum is taken over all k such that there exists an arrow $i \to k$ from the oriented graph. The following equality proves the proposition

$$p_{i,j} = \sum_{i \to k} p_{k,j}.$$

The invariants $p_{i,j}$ obtained from the oriented graph give explicit descriptions for multiplicities and discrepancies. Let Σ^j be the strict transform of the linear system Σ on V_j also set $\nu_j = \mathrm{mult}_{B_{j-1}} \Sigma^{j-1}$ and $\beta_j = \mathrm{codim} B_{j-1} - 1$ to obtain

$$\nu_{E_K}(\Sigma) = \nu_E(\Sigma) = \sum_{i=1}^{K} p_{K,i} \nu_i, \qquad a(E) = \sum_{i=1}^{K} p_{K,i} \beta_i.$$

For convenience write $p_i = p_{K,i}$ to obtain the Noether-Fano inequality

$$\sum_{i=1}^{K} p_i \nu_i > n \sum_{i=1}^{K} p_i \beta_i. \tag{2.4}$$

2.5 $4n^2$ Inequalities

The following section contains three results: Proposition 2.5.1, Theorem 2.5.9, and Theorem 2.5.15. These will be required to exclude maximal singularities whose centres have codimension at least three.

2.5.1 A local inequality for a surface

Let $o \in S$ be a germ of a non-singular surface, $o \in C$ a non-singular curve and Σ a mobile linear system on S. Let $Z = (D_1 \circ D_2)$ be the self-intersection of the linear system Σ, that is, an effective 0-cycle. The situation is local. We may assume the support of the cycle Z is at the point o, that is,

$$\deg Z = (D_1 \circ D_2)_o.$$

Proposition 2.5.1. *Assume that for some real number $a < 1$ the pair*

$$\left(S, \frac{1}{n}\Sigma + aC\right) \tag{2.5}$$

is not log canonical, where $n > 0$ is a positive real number. Then the following estimate holds:

$$\deg Z > 4(1-a)n^2.$$

Proof. Let the sequence of blow-ups

$$\varphi_{i,i-1} : S_i \to S_{i-1},$$

where $i = 1, \dots, N$ be the resolution of the non-log canonical singularity of the pair (2.5) with $S_0 = S$ and $N = K$ in the notation of the previous section. The centre of the blow-up $\varphi_{i,i-1}$ is given by the point $s_{i-1} \in S_{i-1}$ with the exceptional divisor

$$E_i = \varphi_{i,i-1}^{-1}(s_{i-1}) \subset S_i,$$

with $s_0 = o$, the blown up points lying over each other and $s_i \in E_i$. The last exceptional divisor E_N realises the non-log canonical singularity of (2.5), that is,

$$\nu_E\left(\frac{1}{n}\Sigma + aC\right) > a(E) + 1.$$

This gives the log Noether-Fano inequality

$$\sum_{i=1}^{N} p_i\nu_i + an\nu_E(C) > n\left(\sum_{i=1}^{N} p_i + 1\right)$$

$$\sum_{i=1}^{N} p_i\nu_i + an\sum_{s_{i-1}\in C^{i-1}} p_i > n\left(\sum_{i=1}^{N} p_i + 1\right),$$

with the usual notations $\nu_i = \mathrm{mult}_{s_{i-1}}\Sigma^{i-1}$. Assume that

$$s_{i-1} \in C^{i-1}$$

for $i = 1, \dots, k \leqslant N$, then the log Noether-Fano inequality becomes

$$\sum_{i=1}^{N} p_i\nu_i > n\left(\sum_{i=1}^{k}(1-a)p_i + \sum_{i=k+1}^{N} p_i + 1\right). \tag{2.6}$$

The proofs of the following two lemmas are omitted since they are obvious.

Lemma 2.5.2. *The following inequality holds*

$$\deg Z \geqslant \sum_{i=1}^{N} \nu_i^2.$$

Lemma 2.5.3. *For each $i \in \{1, \dots, N-1\}$ the following estimate hold*

$$\nu_i \geqslant \sum_{j\to i} \nu_j.$$

Lemma 2.5.4. *The following estimate is true:*

$$\sum_{i=1}^{N} \nu_i^2 > \frac{\Delta^2}{q} n^2,$$

where

$$\Delta = 1 + (1-a)\sum_{i=1}^{k} p_i + \sum_{i=k+1}^{N} p_i$$

and $q = \sum_{i=1}^{N} p_i^2$.

Proof. As (2.6) is given by

$$\sum_{i=1}^{N} p_i\nu_i > n\Delta,$$

squaring both sides and diving by q to obtain

$$\frac{(\sum_{i=1}^{N} p_i \nu_i)^2}{\sum_{i=1}^{N} p_i^2} > \frac{\Delta^2}{q} n^2.$$

The lemma is now obvious. Q.E.D.

Lemma 2.5.5. *There are no arrows $j \to i$, where $k \geqslant j+1 > i \geqslant 1$.*

Proof. This is a consequence of C being a nonsingular curve, which implies $E_i^{j-1} \cap C^{j-1} = \emptyset$ for $j+1 > i$. Q.E.D.

Lemma 2.5.6. *The following inequality holds:*

$$\Delta^2 \geqslant 4(1-a)q.$$

Proof. The proof is by induction on N and k. First assume $N = 1$ then $p_1 = 1$ and

$$\Delta^2 = (2-a)^2 \geqslant 4(1-a) = 4(1-a)q,$$

holds. Now assume $N > 1$ and assume by induction is true for $N-1$. The inequality can be reduced to showing the positivity of the quadratic form of the variable a:

$$a^2 \left(\sum_{i=1}^{k} p_i \right)^2 + 2a \left(2q - \left(\sum_{i=1}^{k} p_i \right) \left(\sum_{i=1}^{N} p_i + 1 \right) \right) + \left(\left(\sum_{i=1}^{N} p_i + 1 \right)^2 - 4q \right) \tag{2.7}$$

on the interval $a \leqslant 1$. This is achieved by showing the minimum is positive. The minimum of (2.7) is given by the formula

$$\left(\sum_{i=1}^{k} p_i \right) \left(\sum_{i=1}^{N} p_i + 1 \right) - \sum_{i=1}^{N} p_i^2. \tag{2.8}$$

up to some positive factor. The centres of the resolution are points, which lie over each other. If there is an arrow $j+1 \to i$ with $j > i$ then there also exists an arrow $j \to i$. Assume $k = 1$ and there are l arrows to the vertex 1, that is, $2 \to 1, \ldots, l+1 \to 1$ and $l+2 \not\to 1$. Then $p_1 = p_2 + p_3 + \ldots + p_{l+1}$ and the expression (2.8) becomes

$$\left(\sum_{i=2}^{l+1} p_i \right) \left(\sum_{i=l+2}^{N} p_i + 1 \right) - \sum_{i=2}^{N} p_i^2,$$

by induction on N this is positive and completes the case for $k = 1$. Before completing this proof we need another result.

Lemma 2.5.7. *For any $i \in \{1, \ldots, N\}$ the following inequality holds:*

$$p_i \leqslant \sum_{j \geqslant i+2} p_j + 1,$$

if the set $\{j \geqslant i+2\}$, then the sum is assumed to be zero.

Proof. This is a purely combinatorial fact proven by decreasing induction on N. If $i = N$ or $i = N-1$ then the result is true. Now

$$\begin{aligned} p_i - \sum_{j \geqslant i+2} p_j &= \sum_{j \to i} p_j - \sum_{j \geqslant i+2} p_j & (2.9) \\ &= p_{i+1} + \sum_{j \to i, j \geqslant i+2} p_j - \sum_{j \geqslant i+2} p_j & (2.10) \\ &= p_{i+1} - \sum_{j \not\to i, j \geqslant i+2} p_j. & (2.11) \end{aligned}$$

Set $\{j \mid j \to i\} = \{i+1, \ldots, i+l\}$. If $l = 1$, the above equality, by the induction hypothesis, becomes

$$p_i - \sum_{j \geqslant i+2} p_j = -p_{i+2} + 1,$$

which is what is required. If $l \geqslant 2$ the induction hypothesis implies

$$p_{i+1} = \ldots = p_{i+l} \leqslant \sum_{j \geqslant i+l+2} p_j + 1.$$

Therefore

$$p_i - \sum_{j \geqslant i+2} p_j \leqslant \sum_{j \geqslant i+l+2} p_j + 1 - \sum_{j \geqslant i+l+1} p_j,$$

which completes the proof of the lemma.

Let $k \geqslant 2$, then

$$p_1 = p_2 = \ldots = p_{k-1} \leqslant \sum_{i=k+1}^{N} p_j + 1,$$

which means the expression (2.8) is bounded below by

$$\left(\sum_{i=2}^{k} p_i \right) \left(\sum_{i=1}^{N} p_i + 1 \right) - \sum_{i=2}^{N} p_i^2.$$

The induction hypothesis on N completes the proof. Q.E.D.

Remark 2.5.8. This follows the proof given in [53, Chapter 2, Proposition 4.1]. An alternative proof is given in [12].

2.5.2 The $4n^2$ inequality

Theorem 2.5.9. *Let E be an infinitely near maximal singularity of Σ on the variety V, in particular it has centre B with codimension $\geqslant 3$. Consider the self-intersection $Z = (D_1 \circ D_2)$ of the linear system Σ and assume $n = c(\Sigma) > 0$ is the threshold of canonical adjunction and the Noether-Fano inequality holds. The following estimate*

$$\mathrm{mult}_B Z > 4n^2$$

holds.

Proof. Divide the resolution $\varphi_{i,i-1} : V_i \to V_{i-1}$ into the lower part $i \in \{1, \dots L\}$, with $L \leqslant K$ corresponding to i such that $\mathrm{codim} B_{i-1} \geqslant 3$ and the upper part $i \in \{L+1, \dots K\}$ corresponding to i such that $\mathrm{codim} B_{i-1} = 2$, it may occur that $L = K$ and the upper part is empty.

Let $D_1, D_2 \in \Sigma$ be two different generic divisors and define a sequence of codimension 2 cycles on V_i inductively as

$$D_1^i \circ D_2^i = (D_1^{i-1} \circ D_2^{i-1})^i + Z_i$$

with $Z_i \subset E_i$, so that for $i \leqslant L$ we have the cycle decomposition

$$D_1^i \circ D_2^i = Z_0^i + Z_1^i + \ldots + Z_{i-1}^i + Z_i.$$

For $j > i$ and $j \leqslant L$ set

$$m_{i,j} = \mathrm{mult}_{B_{j-1}}(Z_i^{j-1})$$

and $d_i = \deg Z_i$. The following inequality

$$\nu_i^2 + d_i = m_{0,i} + \ldots + m_{i-1,i}$$

holds for $1 \leqslant i \leqslant L$. Also

$$d_L \geqslant \sum_{i=L+1}^{K} \nu_i^2 \deg[(\varphi_{i-1,L})_* B_{i-1}] \geqslant \sum_{i=L+1}^{K} \nu_i^2.$$

Lemma 2.5.10. If $m_{i,j} > 0$ then $j \to i$.

Proof. If $m_{i,j} > 0$ then some component of Z_i^{j-1} contains B_{j-1} and $Z_i^{j-1} \subset E_i^{j-1}$, which gives the arrow.

Lemma 2.5.11. If for any $i \geqslant 1$ and $j \leqslant L$ then $m_{i,j} \leqslant d_i$.

Proof. The centres B_* are non-singular at their generic points and the maps $\varphi_{a,b} : B_a \to B_b$ are surjective, hence multiplicities can be counted at the generic points. Also multiplicities are non-increasing, so it reduces to the case of a hypersurface in projective space.

Proposition 2.5.12. The following inequality

$$\sum_{i=1}^{L} p_i m_{0,i} \geqslant \sum_{i=1}^{L} p_i \nu_i^2 + p_L \sum_{i=L+1}^{K} \nu_i^2.$$

holds.

Proof. We have the following inequality for $1 \leqslant i \leqslant L$:

$$p_i(\nu_i^2 + d_i) = p_i(m_{0,i} + \ldots + m_{i-1,i}),$$

taking the sum we have

$$\sum_{i=1}^{L} p_i \nu_i^2 + \sum_{i=1}^{L} p_i d_i = \sum_{i=1}^{L} p_i m_{0,i} + \sum_{i=2, j+1 \leqslant i}^{L} p_i m_{j,i}.$$

The above two lemmas give the following inequalities

$$\begin{aligned} \sum_{i=2, j+1 \leqslant i}^{L} p_i m_{j,i} = \sum_{i=2, m_{j,i} \neq 0}^{L} p_i m_{j,i} \leqslant \sum_{i=2, i \to j}^{L} p_i d_j \\ \leqslant \sum_{j=1}^{L-1} p_j d_j, \end{aligned}$$

which completes the proof of the proposition. Q.E.D.

Corollary 2.5.13. Set $m = m_{0,1} = \mathrm{mult}_B(D_1 \circ D_2)$, then the following holds

$$m \left(\sum_{i=1}^{L} p_i \right) \geqslant \sum_{i=1}^{L} p_i \nu_i^2 + p_L \sum_{i=L+1}^{K} \nu_i^2. \tag{2.12}$$

Corollary 2.5.14. The following inequality holds

$$m \left(\sum_{i=1}^{L} p_i \right) \geqslant \sum_{i=1}^{K} p_i \nu_i^2.$$

Proof of Theorem 2.5.9. Recall The Noether-Fano inequality (equation (2.4))

$$\sum_{i=1}^{K} p_i \nu_i > n \sum_{i=1}^{K} p_i \beta_i.$$

The right hand side of (2.12), considered as a quadratic form in ν_i, is greater than or equal to the case $\nu_K = \ldots = \nu_1 = \nu$, as they satisfies the inequalities $\nu_1 \geqslant \nu_2 \geqslant \ldots \geqslant \nu_K$. The Noether-Fano inequality restricted to this line is

$$\nu > n \frac{\sum_{i=1}^{K} p_i \beta_i}{\sum_{i=1}^{K} p_i},$$

so that

$$m \left(\sum_{i=1}^{L} p_i \right) > \left(\sum_{i=1}^{K} p_i \right) \left(n^2 \frac{(\sum_{i=1}^{K} p_i \beta_i)^2}{(\sum_{i=1}^{K} p_i)^2} \right).$$

Set

$$\Sigma_l = \sum_{i=1}^{L} p_i, \qquad \Sigma_u = \sum_{i=L+1}^{K} p_i$$

and $\beta_i \geqslant 2$ for $i \leqslant L$ to obtain

$$m > \frac{(2\Sigma_l + \Sigma_u)^2}{\Sigma_l(\Sigma_l + \Sigma_u)} n^2.$$

The proof is complete as $m = \mathrm{mult}_B Z$ and the right hand side is not smaller than $4n^2$. Q.E.D.

2.5.3 The $4n^2$ inequality for complete intersection singularities

This is a recent result from the paper [69]. It is an important result that is required in the proof of Theorem 4.0.1.

Theorem 2.5.15. *Let (V, o) be a germ of a generic complete intersection singularity of codimension l and type $\underline{\mu} = (\mu_1, \ldots, \mu_l)$ where*

$$\mathrm{dim} V = M \geqslant l + \mu_1 + \ldots + \mu_l + 3$$

and the generic condition is explained below. Let Σ be a mobile linear system on V. Assume that for some positive $n \in \mathbb{Q}$ the pair $(V, \frac{1}{n}\Sigma)$ is not canonical at the point o but canonical outside this point. Then the self-intersection $Z = (D_1 \circ D_2)$ of the system Σ satisfies the inequality

$$\mathrm{mult}_o Z > 4n^2 \mathrm{mult}_o V.$$

The germ (V, o) is given by a system of l analytic equations

$$\begin{aligned} 0 =& q_{1,\mu_1} + q_{1,\mu_1+1} + \ldots \\ & \ldots \\ 0 =& q_{l,\mu_l} + q_{1,\mu_l+1} + \ldots \end{aligned}$$

in $\mathbb{C}^{M+l}$. Set

$$\text{mult}_o V = \mu = \mu_1 \cdots \mu_l,$$

the multiplicity of the point o assuming general position of the polynomials q_{i,μ_i} explained below. Also set

$$|\underline{\mu}| = \mu_1 + \ldots + \mu_l.$$

By assumption $M \geqslant l + |\underline{\mu}| + 3$. Let P be a linear subspace in $\mathbb{C}^{M+l}$ of dimension $2l + |\underline{\mu}| + 3$ and the intersection $V \cap P$ as V_P.

Definition 2.5.16. We say that the complete intersection singularity (V, o) is *generic* if for any linear subspace P of dimension $2l + |\underline{\mu}| + 3$, the singularity $o \in V_P$ is an isolated singularity, $\dim V_P = l + |\underline{\mu}| + 3$ and the blow-up of the point

$$\varphi_P : V_P^+ \to V_P$$

is non-singular in the neighbourhood of the exceptional divisor $Q_P = \varphi_P^{-1}(o)$, which is a non-singular complete intersection

$$Q_P = \{q_{1,\mu_1} = \ldots = q_{l,\mu_l} = 0\} \subset \mathbb{P}^{2l+|\underline{\mu}|+2}$$

of codimension l and type $\mu = (\mu_1, \ldots, \mu_l)$.

Remark 2.5.17. From now on assume that (V, o) is generic.

Proof of Theorem 2.5.15. Now reduce to the isolated singular point case, that is, let P be a general linear subspace of dimension $2l + |\underline{\mu}| + 3$ and $\Sigma_P = \Sigma|_P$ the restriction of Σ onto P. By inversion of adjunction, Theorem 2.6.1, the pair

$$\left(V_P, \frac{1}{n}\Sigma_P\right)$$

is not canonical (when P is no-trivial then it is also not log canonical). Let

$$Z_P = Z|_P = (Z \circ V_P)$$

be the self intersection of the linear system Σ_P and $\text{mult}_o Z = \text{mult}_o Z_P$. Therefore we can assume $M = l + |\underline{\mu}| + 3$ and the singularity o is isolated.

Now restrict to a linear subspace, that is, let $\Pi \ni o$ be a general linear subspace of dimension $|\underline{\mu}| + 3$ and V_Π is the intersection $V \cap \Pi$. Clearly $o \in V_\Pi \subset \Pi \cong \mathbb{C}^{|\underline{\mu}|+3}$ is an isolated complete intersection singularity of codimension l. Let

$$\varphi_\Pi : V_\Pi^+ \to V_\Pi$$

be the blow-up of the point o with $Q_\Pi = \varphi_\Pi^{-1}(o)$ the exceptional divisor and $Q_\Pi \subset \mathbb{P}^{|\underline{\mu}|+2}$ is a non-singular complete intersection of type $|\underline{\mu}|$ and codimension l.

Notice that by the adjunction formula we have the discrepancy $a(Q_\Pi, V_\Pi) = 2$.

Let $D \in \Sigma$ be a general divisor and $D^+ \in \Sigma^+$ its strict transform on V^+ then

$$D^+ \sim -\nu Q,$$

for some positive integer ν and recall that we only consider the local situation of (V, o) being a germ.

If $\nu > 2n$, then

$$\operatorname{mult}_o Z \geqslant \nu^2 \mu \geqslant 4n^2 \mu$$

and the required inequality holds. This means from now on assume

$$\nu \leqslant 2n.$$

Restrict the divisors to the linear subspace Π, that is, set $D_\Pi = D|_\Pi$ so that $D_\Pi \sim -\nu Q_\Pi$. By inversion of adjunction, the pair

$$(V_\Pi, \frac{1}{n} D_\Pi)$$

is not log canonical at the point o, moreover it is not canonical. So for some exceptional divisor E_Π over V_Π the Noether-Fano inequality

$$\operatorname{ord}_{E_\Pi} \Sigma_\Pi > n a(E_\Pi, V_\Pi)$$

holds. As $\nu \leqslant 2n$ and $a(Q_\Pi, V_\Pi) = 2$ then $E_\Pi \neq Q_\Pi$, by the above Noether-Fano inequality replacing E_Π by Q_Π to get a contradiction, so that E_Π is a non log canonical singularity of the pair

$$\left(V_\Pi^+, \frac{1}{n} D_\Pi^+ + \frac{\nu - 2n}{n} Q_\Pi \right).$$

Denote $\Delta_\Pi \subset Q_\Pi$ the centre of E_Π on V_Π^+, which is an irreducible subvariety in Q_Π.

Proposition 2.5.18. *If* $\operatorname{codim}(\Delta_\Pi \subset Q_\Pi) = 1$*, then the estimate*

$$\operatorname{mult}_o Z \geqslant 8n^2\mu$$

holds.

Proof. First notice that $\operatorname{mult}_o Z = \operatorname{mult}_o Z_\Pi$ since the linear space $o \in \Pi$ is generic. Then by Proposition 2.5.1 we obtain

$$\operatorname{mult}_o Z_\Pi \geqslant \nu^2\mu + 4\left(3 - \frac{\nu}{n}\right)n^2\mu = 8n^2\mu + \mu(2n-\nu)^2,$$

which completes the proof.

Therefore we can assume $\operatorname{codim}(\Delta_\Pi \subset Q_\Pi) \geqslant 2$ and $\dim \Delta \geqslant 2l$.

Recall we have the Noether-Fano inequality

$$\begin{aligned}\operatorname{ord}_{E_\Pi} \Sigma_\Pi &> na(E_\Pi, V_\Pi)\\ &> n(a(E_\Pi, V_\Pi^+) + a(Q_\Pi, V_\Pi)\operatorname{ord}_{E_\Pi} Q_\Pi).\end{aligned}$$

As $o \in \Pi$ is generic then $\operatorname{ord}_{E_\Pi} \Sigma_\Pi = \operatorname{ord}_E \Sigma$ and $\operatorname{ord}_{E_\Pi} Q_\Pi = \operatorname{ord}_E Q$. Recall that $a(Q_\Pi, V_\Pi) = 2$ and $a(E_\Pi, V_\Pi^+) \geqslant a(E, V^+)$, which is shown using the adjunction formula, to obtain the inequality

$$\operatorname{ord}_E \Sigma > n(a(E, V^+) + 2\operatorname{ord}_E Q),$$

the Noether-Fano type inequality.

Consider the resolution of the singularity E with $V_1 = V^+$, $E_1 = Q$, $B_0 = o$ and $B_1 = \Delta$, so that E_K defines the discrete valuation ord_E and all the constructions in section 2.3 will work for blow-ups $\varphi_{i,i-1}$ with $i \geqslant 2$ as V_i are non-singular at the generic point of B_i. As in the $4n^2$ inequality, the graph associated with this resolution will be split into the lower part $\{1, 2, \ldots, L\}$ and the upper part $\{L+1, \ldots, K\}$. The Noether-Fano type inequality becomes

$$\sum_{i=1}^{K} p_i\nu_i > n\left(2p_1 + \sum_{i=2}^{K} p_i\beta_i\right),$$

with $\beta_i = \operatorname{codim}(B_{i-1} \subset V_{i-1})$. As $\nu_1 \leqslant 2n$ we may assume $\nu_K > n$ (replacing, if required, E_K be a lower singularity E_j for some $j < K$). The following result is required.

Proposition 2.5.19. *Let* $Y \subset \mathbb{P}^N$ *be a complete intersection of codimension* $l \geqslant 1$*,* $S \subset Y$ *an irreducible variety of codimension* $a \geqslant 1$ *and* $B \subset Y$ *an irreducible* *subvariety of codimension al, where the estimate $N \geqslant (l+1)(a+1)$ is satisfied, then the inequality*

$$\operatorname{mult}_B S \leqslant m$$

holds, where $m \geqslant 1$ is defined by the condition $S \sim mH_Y^a$ and $H_Y \in A^1Y$ is the class of hyperplane sections of Y.

Proof. The case $l = 1$ was given in [60]. The case for arbitrary l is done in [79]. Q.E.D.

Applying Proposition 2.5.19 to a divisor in the linear system $\Sigma^1|_Q$ and $\dim B_1 = \dim \Delta \geqslant 2l$ we get $\nu_1 \geqslant \nu_2$. From the resolution, we get the inequalities

$$\nu_2 \geqslant \nu_3 \geqslant \ldots \geqslant \nu_K$$

Use the technique of counting multiplicities, let $D_1, D_2 \in \Sigma$ be general divisors and

$$Z = (D_1 \circ D_2)$$

the scheme-theoretic intersection, the self intersection of the linear system Σ. For $i \geqslant 1$ there is the sequence of codimension two cycles on V_i given by

$$D_1^i \circ D_2^i = (D_1^{i-1} \circ D_2^{i-1})^i + Z_i,$$

where Z_i is supported on E_i so can be seen as an effective divisor on E_i. Thus for any $i \leqslant L$ we get

$$D_1^i \circ D_2^i = Z_0^i + Z_1^i + \ldots + Z_{i-1}^i + Z_i.$$

For any $j > i, j \leqslant L$ set

$$m_{i,j} = \operatorname{mult}_{B_{j-1}} Z_i^{j-1},$$

and $d_i = \deg Z_i$

Remark 2.5.20. This is the same construction used in the original $4n^2$ inequality, the only modification needed now is on the first exceptional divisor $E_1 = Q$ we have the relation

$$Z_1 \sim d_1 H_Q,$$

where H_Q is the class of a hyperplane section on the complete intersection $Q \subset \mathbb{P}^{4l+2}$. Following the same construction as the $4n^2$ inequality we obtain the system of equalities

$$\begin{aligned} \mu(\nu_1^2 + d_1) &= m_{0,1}, \\ \nu_2^2 + d_2 &= m_{0,2} + m_{1,2}, \\ &\dots \\ \nu_i^2 + d_i &= m_{0,i} + \dots + m_{i-1,i}, \\ &\dots \end{aligned}$$

$i = 2, 3, \dots, L$ where the estimate

$$d_L \geqslant \sum_{i=L+1}^{K} \nu_i^2$$

holds.

Proposition 2.5.21. *The following inequalities*

- $d_1 \geqslant m_{1,2}$,

- $m_{0,1} \geqslant \mu m_{0,2}$,

hold.

Proof. The first inequality follows from $Z_i \sim d_1 H_Q$ and Proposition 2.5.19 with $\dim B_1 \geqslant 2l$. To show the second inequality we have the numerical equivalence

$$\begin{aligned} Z^1 \circ E_1 &\sim \frac{1}{\mu} \deg(Z^1 \circ E_1) H_Q^2 \\ &\sim \frac{1}{\mu} m_{0,1} H_Q^2. \end{aligned}$$

Applying Proposition 2.5.19 to the cycle $Z^1 \circ E_1 = Z^1 \circ Q$ we obtain the inequality

$$m_{0,2} = \operatorname{mult}_{B_1} Z_0^1 \leqslant \operatorname{mult}_\Delta (Z^1 \circ Q) \leqslant \frac{1}{\mu} m_{0,1},$$

which completes the proof of the proposition. Q.E.D.

Moreover $m_{0,1} \geqslant \mu m_{0,i}$ for $i \geqslant 3$ as $m_{0,2} \geqslant m_{0,3} \geqslant \dots \geqslant m_{0,L}$. Now set

$$m_{i,j}^* = \mu m i, j$$

for $(i, j) \neq (0, 1)$ and $m_{0,1}^* = m_{0,1}$. Also set

$$d_i^* = \mu d_i$$

for $i = 1, \ldots, L$. We obtain the following inequalities

$$\begin{aligned}
\mu\nu_1^2 + d_1^* &= m_{0,1}^*, \\
\mu\nu_2^2 + d_2^* &= m_{0,2}^* + m_{1,2}^*, \\
&\ldots \\
\mu\nu_i^2 + d_i^* &= m_{0,i}^* + \ldots + m_{i-1,i}^*, \\
&\ldots
\end{aligned}$$

$i = 2, 3, \ldots, L$ where the estimate

$$d_L^* \geqslant \mu \sum_{i=L+1}^{K} \nu_i^2$$

holds. The integers $m_{i,j}^*$ and d_i^* satisfy the same properties as the integers $m_{i,j}$ and d_i in the $4n^2$ inequality. Repeating that argument we obtain the inequality

$$\left(\sum_{i=1}^{L} p_i\right) \operatorname{mult}_o Z \geqslant \mu \sum_{i=1}^{K} p_i \nu_i^2.$$

Repeating that argument gives the estimate

$$\operatorname{mult}_o Z \geqslant 4\mu n^2.$$

This completes the proof of the theorem. Q.E.D.

2.6 Inversion of Adjunction

In this section we state results required for the *linear method* of excluding maximal singularities. The methods will be used when restricting a variety $V \subset \mathbb{P}^N$ onto some linear subspace of $\mathbb{P}^N$, which is required in Theorem 2.5.15 and in the later chapters.

Theorem 2.6.1. *Let $o \in V$ be a germ of a $\mathbb{Q}$-factorial terminal variety, D an effective $\mathbb{Q}$-divisor, the support of which contains the point o. Let $R \subset V$ be an irreducible subvariety of codimension one, $R \not\subset \operatorname{Supp} D$, and R is a Cartier divisor. Assume that the pair (V, D) is not canonical at the point o but it is canonical outside this point, that is, the point o is an isolated centre of non-canonical singularities of that pair. Then the pair $(R, D_R = D|_R)$ is not log canonical at the point o.*

Proof. Let $D = \sum_{i \in I} d_i D_i$ be the sum of the irreducible components, so that $d_i \in \mathbb{Q}_{>0}$ for all $i \in I$. The pair is canonical outside the point o so for all geometric valuations E^* the inequality $\nu_E(D) \leqslant a(E^*)$ holds. Taking a general irreducible subvariety $B \subset \operatorname{Supp} D_i$ for some i of codimension one then $a(E) = 1$, but $\nu_E(D_i) \geqslant 1$, which implies $d_i \leqslant 1$ for all $i \in I$. The condition to be non canonical at o is a strict inequality so that D can be replaced by $\frac{1}{1+\varepsilon}D$ for some small enough $\varepsilon \in \mathbb{Q}_{>0}$, so we can assume $d_i < 1$ for all $i \in I$.

Let $\varphi : V^+ \to V$ be the resolution of singularities of the pair $(V, D + R)$ to get the canonical divisor

$$K_{V^+} = \varphi^*(K_V + D + R) + \sum_{j \in J} e_j E_j - \sum_{i \in I} d_i D_i^+ - R^+, \tag{2.13}$$

where E_j are φ exceptional divisors and D_i^+, R^+ are the strict transforms of the divisors D_i, R on V^+.

Now by definition $e_j = a(E_j) - \operatorname{ord}_{E_j} \varphi^* D - \operatorname{ord}_{E_j} \varphi^* R$. Clearly for some subset $J' \subset J$ we have

$$\varphi^{-1}(o) = \bigcup_{j \in J'} E_j.$$

Then as $o \in R$ we have for all $j \in J'$, $\operatorname{ord}_{E_j} \varphi^* R \geqslant 1$. Also (X, D) is not canonical only at o then

$$\operatorname{ord}_{E_j} \varphi^* D > a(E_j)$$

for some $j \in J'$. This gives

$$\operatorname{ord}_{E_j} \varphi^* D > e_j + \operatorname{ord}_{E_j} \varphi^* D + \operatorname{ord}_{E_j} \varphi^* R,$$

which gives $e_j < -1$ for some $j \in J'$.

By the connectedness principle, Theorem 2.6.2 below, as R^+ has coefficient -1 in (2.13) and there is some index l such that $e_l < -1$ then

$$E_l \cap R^+ \neq \emptyset.$$

By the adjunction formula and (2.13) we have

$$K_{R^+} = (K_{V^+} + R^+)|_{R^+} = \varphi^*(K_V + D + R)|_{R^+} + \sum_{j \in J} e_j E_j|_{R^+} - \sum_{i \in I} d_i D_i^+|_{R^+}$$

The adjunction formula also gives $K_R = (K_V + R)|_R$, by setting $\varphi_R : R^+ \to R$ the restriction of φ to R^+, we get

$$K_{R^+} = (K_{V^+} + R^+)|_{R^+} = \varphi_R^*(K_R + D|_R) + \sum_{j \in J} e_j E_j|_{R^+} - \sum_{i \in I} d_i D_i^+|_{R^+}.$$

Also $e_l < -1$, which is the coefficient of $E_l|_{R^+}$, which proves the theorem. Q.E.D.

Theorem 2.6.2. *Let V, S be normal varieties and $h : V \to S$ a proper morphism with connected fibres and $D = \sum d_i D_i$ a $\mathbb{Q}$-divisor on V. Assume that D is effective and the class $-(K_V + D)$ is h-nef and h-big. Let*

$$f : Y \xrightarrow{g} V \xrightarrow{h} S$$

be a resolution of singularities of the pair (V, D). Set

$$K_Y = g^*(K_V + D) + \sum e_i E_i.$$

The support of the $\mathbb{Q}$-divisor $\sum_{e_i \leqslant -1} e_i E_i$ is connected in a neighbourhood of any fibre of the morphism f.

Proof. See [42, Theorem 17.4].

Proposition 2.6.3. *Assume that the pair (V, D) is not canonical at the point o, which is an isolated centre of a non-canonical singularity of this pair with o a terminal singularity. Assume also that for some integer $k \geqslant 1$ the inequality*

$$\nu_E(D) + k \leqslant \delta \tag{2.14}$$

holds, where $\delta = a(E, V)$ is the discrepancy of E. Then the pair (V^+, D^+), where V^+ is the blow-up of the point o on V given by the morphism φ, E the exceptional divisor and D^+ the strict transform of D on V, is not log canonical and there is a non-log canonical singularity $\widetilde{E} \subset \widetilde{V}$ of that pair with

$$\mathrm{centre}(\widetilde{E}, V^+) \subset E,$$

which is of dimension $\geqslant k$.

Proof. Assume $V \subset \mathbb{P}^N$ is projectively embedded and consider a generic linear subspace $P \subset \mathbb{P}^N$ of codimension k containing the point o. Let $\Lambda_P = |H - P|$ be the linear system of hyperplanes containing P and $\Lambda = \Lambda_P|_V = |H_V - P_V|$ the corresponding linear system of sections on the variety V. Let $\epsilon > 0$ be a sufficiently small rational number of the form $\frac{1}{K}$ and $\{H_i | i \in I\} \subset \Lambda$ a set of generic divisors with $I = \{1, 2, \ldots, Kk + 1\}$. Set

$$R = D + \sum_{i \in I} \epsilon H_i,$$

and R^+ the strict transform of R on V^+. The pair (V^+, D^+) is not log canonical, see the arguments used in [64]. The centre of any of its non-log canonical singularities is contained in E. Being non-log canonical is a strict inequality so we are able to slightly decrease the coefficients of D and (2.14) becomes a strict inequality, i.e. we may assume $\nu_E(D) + k < \delta$.

Consider the pair (V^+, R^+), then every non-log canonical singularity of the pair (V^+, D^+) is a non-log canonical singularity of the former pair. There is one additional non-log canonical singularity given by $B_P = (V \cap P)^+$, this is shown by assuming B is centre of a non-log canonical singularity of (V^+, R^+), there are the following two cases

- $B_P \not\subset B$, then for any exceptional divisor $E(B)$ lying over B we have $\nu_{E(B)}(H_i^+) = 0$ for all $i \in I$ and every centre is a centre for (V^+, D^+) or,

- $B_P = B$, which excludes the case $B_P \supset B$, then let E^* be the exceptional divisor by the blow up of $B_P \subset V^+$ so that $a(E^*) = k - 1$, which gives the non log canonical singularity and excludes the other case.

By the strict version of (2.14), the class $-(K_{V^+} + R^+)$ is obviously φ-nef and φ-big so applying Theorem 2.6.2 with

$$f : Y \xrightarrow{g} V^+ \xrightarrow{\varphi} V$$

and R^+ as the effective divisor, then the union of the centres of non-log canonical singularities of the pair (V^+, R^+) on V^+ is connected. The linear subspace, P, being generic, is only possible if B_P intersects some centre of a non-log canonical singularity of the pair (V^+, D^+) of dimension at least k. Q.E.D.

2.7 Multi-quadratic Singularities

Let us describe some conditions for the singularities of a complete intersection that guarantee its factoriality. For any k-tuple $\underline{d} = (d_1, \ldots, d_k)$ set $M = |\underline{d}| - k$ where $|d| = d_1 + \cdots + d_k$ and let

$$\mathcal{P}(\underline{d}) = \prod_{i=1}^{k} \mathcal{P}_{d_i, M+k+1}$$

be the space of k-tuples of homogeneous polynomials of degree $d_1, \ldots, d_k$, respectively, on the complex projective space $\mathbb{P} = \mathbb{P}^{M+k}$. Here the symbol $\mathcal{P}_{a,N}$ stands for the linear space of homogeneous polynomials of degree a in N variables which are naturally interpreted as polynomials on $\mathbb{P}^{N-1}$. We write $\underline{f} = (f_1, \dots, f_k) \in \mathcal{P}(\underline{d})$ for an element of the space $\mathcal{P}(\underline{d})$. We set also

$$\mathcal{P}_{\mathrm{fact}}(\underline{d}) \subset \mathcal{P}(\underline{d})$$

to be the set of k-uples $\underline{f} = (f_1, \dots, f_k)$ such that the zero set

$$V(\underline{f}) = \{f_1 = \cdots = f_k = 0\} \subset \mathbb{P}$$

is an irreducible, reduced and factorial complete intersection of codimension k. Note that for any $\underline{f} \in \mathcal{P}_{\mathrm{fact}}(\underline{d})$ the projective variety $V(\underline{f})$ is a primitive Fano variety of index 1, that is,

$$\mathrm{Cl}\, V(\underline{f}) = \mathrm{Pic}\, V(\underline{f}) = \mathbb{Z}H,$$

where H is the class of a hyperplane section (this is by the Lefschetz theorem), and $K_{V(\underline{f})} = -H$.

Take an arbitrary k-uple $\underline{f} \in \mathcal{P}(\underline{d})$, the zero set $V = V(\underline{f})$, which is an irreducible reduced complete intersection of codimension k. Let $o \in V$ be a point. Fix a system of affine coordinates $(z_1, \dots, z_{M+k})$ on an affine chart $\mathbb{C}^{M+k} \subset \mathbb{P}$ with the origin at the point o. Write the corresponding dehomogenized polynomials (denoted by the same symbols) in the form

$$\begin{aligned} f_1 &= q_{1,1} + q_{1,2} + \cdots + q_{1,d_1}, \\ &\dots \\ f_k &= q_{k,1} + q_{k,2} + \cdots + q_{k,d_k}, \end{aligned}$$

where $q_{i,j}$ is a homogeneous polynomial in z_* of degree j. For a general point $o \in V$

$$\dim\langle q_{1,1}, \dots, q_{k,1}\rangle = k,$$

that is, $o \in V$ is non-singular. Assume now that $\dim\langle q_{1,1}, \dots, q_{k,1}\rangle \leqslant k - 1$, that is to say, $o \in V$ is a singular point.

Definition 2.7.1. The singularity $o \in V$ is a *correct multi-quadratic singularity of type* 2^l, where $l \in \{1, \dots, k\}$, if the following conditions are satisfied:

- $\dim\langle q_{1,1}, \dots, q_{k,1}\rangle = k - l$,
- for a general linear subspace $P \subset \mathbb{P}$ of dimension $\max\{2k+2, k+3l+3\}$, containing the point o, the intersection $V_P = V \cap P$ has an isolated singularity at the point o,

- for the blow up $\varphi_P\colon V_P^+ \to V_P$ of the point o, the exceptional divisor $Q_P = \varphi^{-1}(o)$ is a non-singular complete intersection of type 2^l in $\max\{k+l+1, 4l+2\}$-dimensional projective space.

Note that by Definition 2.7.1, the codimension of the singular locus of V near a correct multi-quadratic singularity is at least $2k+2$.

Now let us discuss the conditions of Definition 2.7.1 in more detail. Assuming there is a subset $I \subset \{1, \dots, k\}$ such that $|I| = k-l$ and the linear forms $q_{i,1}$, $i \in I$, are linearly independent:

$$\langle q_{1,1}, \dots, q_{k,1} \rangle = \langle q_{i,1} \mid i \in I \rangle.$$

By the genericity of P, the restrictions $q_{i,1}|_P$, $i \in I$ remain linearly independent, so that the zero set

$$V_{P,I} = \{f_i|_P = 0 \mid i \in I\}$$

near the point o is a non-singular complete intersection of codimension $k-l$. Let

$$\varphi_{P,I}\colon V_{P,I}^+ \to V_{P,I}$$

be the blow up of the point $o \in V_{P,I}$ with the exceptional divisor $E_{P,I} = \varphi_{P,I}^{-1}(o)$ being the $\max\{k+l+1, 4l+2\}$-dimensional projective space. Now we can consider the blow up φ_P as the restriction of the blow up $\varphi_{P,I}$ onto V_P, that is, V_P^+ is the strict transform of V_P on $V_{P,I}^+$. In terms of this presentation, the exceptional divisor $Q_P \subset E_{P,I}$ is given by the set of l equations

$$q_{i,2}|_{E_{P,I}} = 0, \quad i \in \{1, \dots, k\} \setminus I.$$

Definition 2.7.1 requires Q_P to be a non-singular complete intersection of type 2^l in $E_{P,I}$.

Definition 2.7.2. We say that an irreducible reduced complete intersection $v = V(\underline{f})$ has at most correct multi-quadratic singularities if every point $o \in V$ is either non-singular or a correct multi-quadratic singularity of type 2^l for some $l \in \{1, \dots, k\}$.

The set of k-uples $\underline{f} \in \mathcal{P}(\underline{d})$ such that $V(\underline{f})$ satisfies Definition 2.7.2 is denoted $\mathcal{P}_{\mathrm{mq}}(\underline{d})$. The subset $\mathcal{P}_{\mathrm{mq}}(\underline{d}) \subset \mathcal{P}(\underline{d})$ is obviously Zariski open. If $\underline{f} \in \mathcal{P}_{\mathrm{mq}}(\underline{d})$ we have

$$\operatorname{codim}(\operatorname{Sing} V(\underline{f}) \subset V(\underline{f})) \geqslant 2k+2 \geqslant 6,$$

and by Grothendieck's theorem on parafactoriality of local rings (see [3]), the complete intersection $V(\underline{f})$ is a factorial variety. Therefore, $\mathcal{P}_{\mathrm{mq}}(\underline{d}) \subset \mathcal{P}_{\mathrm{fact}}(\underline{d})$.

2.8 Regularity Conditions

We keep the coordinate notations of Section 2.7 at a point $o \in V$. For brevity and uniformity we treat the non-singular case $o \notin \mathrm{Sing}\, V$ as a multi-quadratic case of type 2^l for $l = 0$. Let us place the homogeneous polynomials

$$q_{i,1}, i \in I, \quad q_{i,j}, j \geqslant 2,$$

in the *standard order*, corresponding to the lexicographic order of pairs (i,j): (i_1, j_1) precedes (i_2, j_2), if $j_1 < j_2$ or $j_1 = j_2$ but $i_1 < i_2$. Thus we obtain a sequence

$$h_1, h_2, \ldots, h_{M+k-l} \tag{2.15}$$

of $M + k - l$ homogeneous polynomials in z_* of non-decreasing degrees: $\deg h_{e+1} \geqslant \deg h_e$.

Definition 2.8.1. The point $o \in V$ is N_l *regular*, where N_l is an integer function with the variable l, or shortened to *regular* if the sequence of polynomials that is obtained from (2.15) by taking the first $N_l + k$ members is a regular sequence.

In plain words, Definition 2.8.1 requires that the set of common zeros of the polynomials $h_e(z)$ in the sequence obtained from (2.15) by taking the first $N_l + k$ members, is of the correct codimension. Since the polynomials $h_1, h_2, \ldots, h_{M+k-l}$ are homogeneous, we may consider them as polynomials on the projective space $\mathbb{P}^{M+k-1}$ in the homogeneous coordinates $(z_1 : \ldots : z_{M+k})$ and so understand the regularity in the projective setting.

2.9 Hypertangent Divisors

In order to exclude the maximal singularity E, we need the construction of *hypertangent linear systems*. It is well known and has been published many times (see [59] or [53, Chapter 3] or the most recent application [71]), but some minor modifications are needed to cover the multi-quadratic case, so we give this construction here. We fix a point o and use the notations of Section 2.7 and work in the affine chart $\mathbb{C}^{M+k}$ of the space $\mathbb{P}$ with the coordinates $z_1, \ldots, z_{M+k}$; the point $o \in V$ is the origin. Let $j \geqslant 2$ be an integer. Recall that for some $l \in \{0, 1, \ldots, k\}$ and a subset $I \subset \{1, \ldots, k\}$, such that $|I| = k - l$, the linear forms $q_{i,1}$, $i \in I$ are linearly independent, whereas the other forms $q_{i,1}$, $i \notin I$, are their linear combinations. Denote by

$$f_{i,\alpha} = q_{i,1} + \cdots + q_{i,\alpha}$$

the truncated i-th equation in the tuple $\underline{f}$.

Definition 2.9.1. The linear system

$$\Lambda(j)=\left\{\left(\sum_{i\in I}q_{i,1}s_{i,j-1}+\sum_{i=1}^{k}\sum_{\alpha=2}^{d_i-1}f_{i,\alpha}s_{i,j-\alpha}\right)\bigg|_V=0\right\},$$

where $s_{i,j-\alpha}$ independently run through the set of homogeneous polynomials of degree $j-\alpha$ in the variables $z_1,\dots,z_{M+k}$ (if $j-\alpha<0$, then $s_{j-\alpha}=0$), is called the *j-th hypertangent system at the point o.*

For uniformity of notations, we write $\Lambda(1)$ for the *tangent linear system*:

$$\Lambda(1)=\left\{\left(\sum_{i\in I}q_{i,1}s_{i,0}\right)\bigg|_V=0\right\}.$$

The Zariski tangent space $\{q_{i,1}=0\,|\,i\in I\}$ will be written as T. We set $c(1)=k-l$ and for $j\geqslant 2$

$$c(j)=k-l+\sharp\{(i,\alpha)\,|\,i=1,\dots,k,\ 1\leqslant\alpha\leqslant\min\{j,d_i-1\}\,\}.$$

Further, set $m(j)=c(j)-c(j-1)$, where $c(0)=0$, and for $j=1,\dots,d_k-1$ take $m(j)$ general divisors

$$D_{j,1},\dots,D_{j,m(j)}$$

in the linear system $\Lambda(j)$. Putting them into the standard order, corresponding to the lexicographic order of the pairs (j,α) (with $(j_1,\alpha_1)<(j_2,\alpha_2)$ if $\alpha_1<\alpha_2$ or $\alpha_1=\alpha_2$ and $j_1<j_2$), we obtain a sequence

$$R_1,\dots,R_{M-l}$$

of effective divisors on V.

Example 2.9.2. Let $k=5$ and $\underline{d}=(2,2,3,4,4)$ so that V is defined at $o\in V$ in affine coordinates by

$$\begin{aligned}
f_1&=q_{1,1}+q_{1,2},\\
f_2&=q_{2,1}+q_{2,2},\\
f_3&=q_{3,1}+q_{3,2}+q_{3,3},\\
f_4&=q_{4,1}+q_{4,2}+q_{4,3}+q_{4,4},\\
f_5&=q_{5,1}+q_{5,2}+q_{5,3}+q_{5,4}.
\end{aligned}$$

j	1	2	3	4
$c(j)$	5	8	10	10
$m(j)$	5	3	2	0

For the case $l = 0$ we have

This means we have 10 divisors $D_{1,1}, D_{1,2}, \dots D_{3,1}, D_{3,2}$, which are ordered lexicographically to be $D_1, \dots D_{10}$.

If now $l = 1$ we have

j	1	2	3	4
$c(j)$	4	7	9	9
$m(j)$	4	3	2	0

This means we have 9 divisors, which are the same divisors as in the case $l = 0$ and then omitting the divisor $D_{1,5}$. Notice that $m(j)$ for $j \geqslant 2$ is independent of the value of l and the initial term is just $m(1) = k - l$.

Set $N_l \leqslant M - l$ as the number of divisors defined by the regularity conditions. In what follows, we will really use only the divisors $R_1, \dots, R_{N_l}$, but it is convenient to keep the entire sequence.

Proposition 2.9.3. *The equality*

$$\operatorname{codim}_o \left(\left(\bigcap_{j=1}^{N_l} |R_j| \right) \subset V \right) = N_l$$

holds, where $|R_j|$ *stands for the support of* R_j.

Proof. Since

$$f_{i,\alpha}|_V = (-q_{i,\alpha+1} + \dots)|_V \tag{2.16}$$

for $1 \leqslant \alpha \leqslant d_i - 1$, where the dots stand for higher order terms in z_*, the codimension of the base locus of the tangent linear system $\Lambda(1)$ near the point o is equal to $(k - l)$ and the hypertangent linear system $\Lambda(j)$, $j \geqslant 2$ is equal to

$$(k - l) + \operatorname{codim}(\{q_{i,\alpha}|_T = 0 \,|\, 1 \leqslant i \leqslant k,\, 1 \leqslant \alpha \leqslant 1 + \min\{j, d_i - 1\}\} \subset T).$$

Therefore, for a general choice of hypertangent divisors R_*, the equality

$$\operatorname{codim}_o \left(\left(\bigcap_{j=1}^{i} |R_j| \right) \subset V \right) = i$$

follows from the regularity of the subsequence

$$h_1, \ldots, h_i$$

of the sequence (2.15. Now our claim follows immediately from the regularity condition, Definition 2.8.1. Q.E.D.

For a hypertangent divisor $R_i = D_{j,\alpha}$, where $j \in \{1, \ldots, d_k - 1\}$ and $\alpha \in \{1, \ldots, m(j)\}$, the number

$$\beta_{l,i} = \beta(R_i) = \frac{j+1}{j}$$

is its *slope*.

Set $\varphi \colon V^+ \to V$ to be the blow up of the point o with $Q = \varphi^{-1}(o)$ the exceptional divisor. The symbol R_i^+ means the strict transform of R_i on V^+.

Proposition 2.9.4. (i) $R_i^+ \sim j\varphi^* H - \gamma_i Q$, *where* $\gamma_i \geqslant j+1$.

(ii) *For any irreducible subvariety* $Y \subset V$ *of codimension* $\geqslant 2$ *such that* $Y \not\subset |R_i|$ *the algebraic cycle* $(Y \circ R_i)$ *of the scheme-theoretic intersection satisfies the inequality*

$$\frac{\mathrm{mult}_o}{\deg}(Y \circ R_i) \geqslant \beta_{l,i} \frac{\mathrm{mult}_o}{\deg} Y.$$

(Here the symbol $\mathrm{mult}_o / \deg$ means, as usual, the ratio of the multiplicity at o to the degree in $\mathbb{P}$.)

Proof. (i) follows from (2.16), (ii) follows from (i). Q.E.D.

2.9.1 The non-singular case

Assume that $B \not\subset \mathrm{Sing}\, V$ and $\mathrm{codim}\, B \subset V \geqslant 3$. We want to show that this case of B being the centre of a maximal singularity is impossible by obtaining a contradiction. We write N for N_0 and β_i for $\beta_{0,i}$ for simplicity of notations.

By Theorem 2.5.9 the $4n^2$-inequality is satisfied:

$$\mathrm{mult}_B Z > 4n^2,$$

recalling that Z is the self-intersection of the mobile system $\Sigma \subset |nH|$. Take a point $o \in B$ of general position, $o \notin \mathrm{Sing}\, V$, and let Y_2 be an irreducible component of Z with the maximal value of the ratio $\mathrm{mult}_o / \deg$. Then

$$\frac{\mathrm{mult}_o}{\deg} Y_2 > \frac{4}{d},$$

with $d = d_1 \cdots d_k$. Take general hypertangent divisors $R_1, \dots, R_M$ from Definition 2.9.2. The first k of them, $R_1, \dots, R_k$, are actually tangent divisors and we know that

$$\operatorname{codim}_o((|R_1| \cap \cdots \cap |R_k|) \subset V) = k.$$

We construct a sequence of irreducible subvarieties

$$Y_2, \dots, Y_k,$$

such that $\operatorname{codim} Y_i \subset V = i$. Without loss of generality we can assume $Y_2 \not\subset |R_1|$ and let $R_1 \circ Y_2$ be the scheme-theoretic intersection. Define $Y_3 \in R_1 \circ Y_2$ to be an irreducible component with maximal value of the ratio $\operatorname{mult}_o / \deg$ then

$$\frac{\operatorname{mult}_o}{\deg} Y_3 > \frac{2^3}{d}.$$

Repeat this procedure for $Y_i \circ R_{i-1}$ for $i = 2, \dots, k-1$ to obtain

$$\frac{\operatorname{mult}_o}{\deg} Y_k > \frac{2^k}{d}.$$

Lemma 2.9.5. $Y_k \not\subset |R_{k-1}|$.

Proof. Assume the converse: $Y_k \subset |R_{k-1}|$. The hypertangent divisors being general, this implies that

$$Y_k \subset \{q_{1,1}|_V = \cdots = q_{k,1}|_V = 0\}.$$

However, as $\operatorname{codim}(\operatorname{Sing} V \subset V) \geqslant 2k + 2$, we can take the section V_P of V by a generic linear subspace $P \subset \mathbb{P}$ of dimension $3k + 1$, which is a $(2k + 1)$-dimensional non-singular complete intersection in $\mathbb{P}^{3k+1}$. By Lefschetz, the scheme-theoretic intersection of codimension k on V_P

$$(\{q_{1,1}|_{V_P} = 0\} \circ \cdots \circ \{q_{k,1}|_{V_P} = 0\})$$

must be irreducible and reduced. Therefore, the scheme-theoretic intersection of codimension k on V

$$(\{q_{1,1}|_V = 0\} \circ \cdots \circ \{q_{k,1}|_V = 0\})$$

is irreducible and reduced. By the regularity condition, this irreducible subvariety has multiplicity precisely 2^k at the point o and the degree d. Therefore, it cannot be equal to Y_k. We obtained a contradiction proving the lemma. Q.E.D.

By the last lemma, proceed in the same way as before. Take $Y_{k+1} \in R_{k-1} \circ Y_k$ to the irreducible component of maximal ratio $\mathrm{mult}_o / \deg$ to obtain a subvariety $Y_{k+1} \subset V$ of codimension $k+1$ satisfying the inequality

$$\frac{\mathrm{mult}_o}{\deg} Y_{k+1} > \frac{2^{k+1}}{d}.$$

After that, still following the arguments of [59, Section 2], we use the hypertangent divisors $R_{k+2}, \ldots, R_N$ to obtain a sequence of irreducible subvarieties $Y_{k+2}, \ldots, Y_N$ of codimension $\operatorname{codim}(Y_i \subset V) = i$, such that Y_i is a component of the algebraic cycle $(Y_{i-1} \circ R_i)$ of scheme-theoretic intersection of Y_{i-1} and R_i (the regularity condition and genericity of hypertangent divisors in their linear systems guarantee that $Y_{i-1} \not\subset |R_i|$) with the maximal value of the ratio $\mathrm{mult}_o / \deg$. Therefore,

$$\frac{\mathrm{mult}_o}{\deg} Y_i \geqslant \beta_i \frac{\mathrm{mult}_o}{\deg} Y_{i-1}.$$

The last subvariety Y_N is positive-dimensional and satisfies the estimate

$$\frac{\mathrm{mult}_o}{\deg} Y_N > \gamma = \frac{2^{k+1}}{d} \cdot \prod_{i=k+2}^{N} \beta_i. \tag{2.17}$$

Remark 2.9.6. The divisors R_k and R_{k+1} are omitted from the calculations. This is due to the fact of the remaining divisors $R_k, \ldots, R_N$, they have the greatest ratio β_i and Y_{k+1} could be contained in at most two of $|R_i|$ for $i = k, \ldots, N$.

Equation (2.17) can be rewritten as

$$\frac{\mathrm{mult}_o}{\deg} Y_N > \frac{4}{3} (\beta(0))^{-1},$$

where

$$\beta(0) = \prod_{i=N+1}^{M} \beta_i.$$

This follows from

$$\prod_{i=k+2}^{M} \beta_i \geqslant \left(\frac{4}{3} \cdots \frac{d_1}{d_1 - 1} \right) \prod_{i=2}^{k} \left(\frac{3}{2} \cdots \frac{d_i}{d_i - 1} \right)$$

and

$$\prod_{i=k+2}^{N} \beta_i = \prod_{i=k+2}^{M} \beta_i (\beta(0))^{-1}.$$

2.9.2 The multi-quadratic case

Assume now that B is contained in the closure of the locus of multi-quadratic points of type 2^l but not in the closure of the locus of multi-quadratic points of type 2^j for $j \geqslant l+1$. In other words, a general point $o \in B$ is a singular multi-quadratic point of type 2^l. Let us fix this point.

Proposition 2.9.7. *The self-intersection of the mobile linear system $\Sigma \subset |nH|$, Z, satisfies the following inequality:*

$$\operatorname{mult}_o Z > 2^{l+2} n^2.$$

Proof. This is the $4n^2$-inequality for complete intersection singularities, see Theorem 2.5.15. Q.E.D.

Remark 2.9.8. The condition for a point $o \in V$ to be a correct multi-quadratic singularity (see Definition 2.7.1) is in fact much stronger than that required in Theorem 2.2.15.

Now let us exclude the multi-quadratic case.

Assume first that $1 \leqslant l \leqslant k-2$. Let

$$R_1, \dots, R_{k-l}$$

be general tangent divisors. Since by the regularity condition

$$\operatorname{codim}_o \left(\left(\bigcap_{i=1}^{k-l} |R_i| \right) \subset V \right) = k - l,$$

we may argue as in the non-singular case, and construct a sequence of irreducible subvarieties

$$Y_2, \dots, Y_{k-l}$$

of codimension $\operatorname{codim}(Y_i \subset V) = i$, where Y_2 is an irreducible component of the cycle Z with the maximal value of $\operatorname{mult}_o / \deg$, and Y_{i+1} is an irreducible component of $(Y_i \circ R_{i-1})$ with the same property. Obviously,

$$\frac{\operatorname{mult}_o}{\deg} Y_{k-l} > \frac{2^k}{d}.$$

By Lefschetz, the scheme-theoretic intersection

$$(R_1 \circ R_2 \circ \cdots \circ R_{k-l})$$

is irreducible and reduced: we make this conclusion, intersecting that cycle with the section V_P of V by a generic linear subspace P of dimension $3k+1$, exactly as in the proof of Lemma 2.9.5 (in fact, in order to apply Lefschetz, we could take a subspace P of a smaller dimension here). We conclude that $Y_{k-l} \not\subset |R_{k-l-1}|$ and construct an irreducible subvariety Y_{k-l+1}, satisfying the inequality

$$\frac{\mathrm{mult}_o}{\deg} Y_{k-l+1} > \frac{2^{k+1}}{d}.$$

After that we argue exactly as in the non-singular case, producing a sequence of irreducible subvarieties $Y_{k-l+2}, \ldots, Y_N$, the last one of which satisfies the estimate

$$\frac{\mathrm{mult}_o}{\deg} Y_N > \gamma_l = \frac{4}{3}\beta(l)^{-1},$$

where

$$\beta(l) = \prod_{i=N_l+1}^{M-l} \beta_{l,i}. \tag{2.18}$$

Finally, assume that $l \in \{k-1, k\}$. In this case the subvariety Y_2 (an irreducible component of the self-intersection Z with the maximal value of $\mathrm{mult}_o / \deg$) satisfies the inequality

$$\frac{\mathrm{mult}_o}{\deg} Y_2 > \frac{2^{k+1}}{d}$$

by Proposition 2.9.4. In this case we omit the part of our arguments, which deals with tangent divisors, and proceed straight to the second part, repeating the arguments for the case $l \leqslant k-2$ word for word.

Chapter 3

Complete Intersection of Codimension Two

This chapter contains the proof of birational superrigidity for a family of Fano complete intersections of codimension two and index one. This family includes complete intersections with certain singularities allowed, which makes it possible to estimate the codimension of the subset of non-rigid varieties in the parameter space of the family. This chapter contains results obtained by two different methods. The first method is the original method which does not use the $4n^2$ inequality for complete intersection singularities. The second method is a new method which does use the $4n^2$ inequality for complete intersection singularities. The exclusion of maximal subvarieties and maximal singularities contained entirely in the non-singular locus work for both methods. Then the exclusion of maximal singularities in the singular locus can be accomplished by either one of the two different methods. In this chapter the conditions on the multi-quadratic singularities are weakened. The regularity conditions are defined by the function N_l and using these regularity conditions allows us to show if a variety satisfies them, it is birationally superrigid. The regularity conditions allows us to obtain an estimate of the codimension of the subset of non-rigid varieties in the parameter space of this family.

Introduction

3.0.1 Statement of the main result

Birational (super)rigidity is known for almost all families of non-singular Fano complete intersections of index one in the projective space, see [65, 66, 67]. Here we prove it for possibly singular complete intersections of codimension two. In this chapter, the symbol $\mathbb{P}$ stands for the complex projective space $\mathbb{P}^{M+2}$, where $M \geqslant 13$ (see the proof of Lemma 3.1.4). Fix two integers $d_2 \geqslant d_1 \geqslant 2$, such that $d_1 + d_2 = M + 2$ and consider the space

$$\mathcal{P} = \mathcal{P}_{d_1, M+3} \times \mathcal{P}_{d_2, M+3}$$

of pairs of homogeneous polynomials (f_1, f_2) on $\mathbb{P}$ (that is to say, in $M + 3$ variables $x_0, \dots, x_{M+2}$) of degrees d_1 and d_2, respectively. The symbol $V(f_1, f_2)$ denotes the set of common zeros of f_1 and f_2. The following claim is the main result of this chapter.

Theorem 3.0.1. *There exists a Zariski open subset $\mathcal{P}_{\mathrm{reg}} \subset \mathcal{P}$ such that:*

(i) *for every pair $(f_1, f_2) \in \mathcal{P}_{\mathrm{reg}}$ the closed set $V = V(f_1, f_2)$ is irreducible, reduced and of codimension 2 in $\mathbb{P}$ with the singular locus $\operatorname{Sing} V$ of codimension at least 7 in V, so that V is a factorial projective algebraic variety; the singularities of V are terminal, and V is a primitive Fano variety of index 1 and dimension M;*

(ii) *the estimate*

$$\operatorname{codim}((\mathcal{P} \backslash \mathcal{P}_{\mathrm{reg}}) \subset \mathcal{P}) \geqslant \frac{(M-9)(M-10)}{2} - 1$$

holds;

(iii) *for every pair $(f_1, f_2) \in \mathcal{P}_{\mathrm{reg}}$ the Fano variety $V = V(f_1, f_2)$ is birationally superrigid.*

See Chapter 2 for the definitions of birational rigidity and superrigidity as well as for the standard implications of these properties: Theorem 3.0.1 implies that for every pair $(f_1, f_2) \in \mathcal{P}_{\mathrm{reg}}$ the corresponding Fano complete intersection $V = V(f_1, f_2) \subset \mathbb{P}$ admits no structures of a rationally connected fibre space, that is to say, there exists no rational dominant map $\varphi\colon V \dashrightarrow S$ onto a positive dimensional base S, with $\dim V > \dim S$ such that the general fibre is rationally connected. In particular, V is non-rational. Another well known implication is that the groups of birational and biregular self-maps of V are the same: $\operatorname{Bir} V = \operatorname{Aut} V$.

Now we describe the set $\mathcal{P}_{\mathrm{reg}}$ by explicit conditions (some of them are global but most of them are local) and outline the proof of Theorem 3.0.1.

3.0.2 Regular complete intersections

Definition. A pair of homogeneous polynomials $(f_1, f_2) \in \mathcal{P}$, both non-zero is called a *regular pair* if it satisfies all the conditions (R0.1) to (R3.2) given below.

(R0.1) The polynomial f_1 is irreducible and the hypersurface $\{f_1 = 0\} = F_1$ has at most quadratic singularities of rank 5.

Remark 3.0.2. This condition ensures that F_1 is a factorial variety so that $\mathrm{Cl}\, F_1 \cong \mathrm{Pic}\, F_1$ is generated by the class of a hyperplane section and every effective divisor on F_1 is cut out by a hypersurface in $\mathbb{P}$.

(R0.2) $f_2|_{F_1} \not\equiv 0$ and moreover the closed set $\{f_2|_{F_1} = 0\}$ is irreducible and reduced.

(R0.3) Every point $o \in V = V(f_1, f_2)$ is either

- non-singular,
- a quadratic singularity,
- or a biquadratic singularity.

For each of the three types the local regularity conditions will be stated separately. Given a point $o \in V$, we fix a system of affine coordinates $z_1, \ldots, z_{M+2}$ on an affine subset $o \in \mathbb{A}^{M+2} \subset \mathbb{P}^{M+2}$ with the origin at o, and write down the expansions of the polynomials f_i:

$$\begin{array}{lcccccccccccc} f_1 & = & q_{1,1} & + & q_{1,2} & + & \ldots & + & q_{1,d_1}, & & & & \\ f_2 & = & q_{2,1} & + & q_{2,2} & + & \ldots & + & q_{2,d_1} & + & \ldots & + & q_{2,d_2}, \end{array}$$

where $q_{i,j}$ are homogeneous of degree j. We list the homogeneous polynomials in the *standard order* as follows:

$$q_{1,1}, q_{2,1}, q_{1,2}, q_{2,2}, \ldots, q_{1,d_1}, q_{2,d_1}, \ldots, q_{2,d_2},$$

so that polynomials of smaller degrees precede the polynomials of higher degrees and for $j \leqslant d_1$ the form $q_{1,j}$ precedes $q_{2,j}$.

Every non-singular point $o \in V$ is assumed to satisfy the regularity condition

(R1) the polynomials $q_{i,j}$ in the standard order with the last two of them removed form a regular sequence in $\mathcal{O}_{o,\mathbb{P}}$.

Every quadratic point $o \in V$ is assumed to satisfy a number of regularity conditions. Note that in this case at least one of the linear forms $q_{1,1}, q_{2,1}$ is non-zero and the other one is proportional to it. We denote a non-zero form in the set $\{q_{1,1}, q_{2,1}\}$ by the symbol $q_{*,1}$.

(R2.1) The rank of the quadratic point $o \in V$ is at least 9.

Remark 3.0.3. When we cut V by a general linear subspace $P \subset \mathbb{P}$ of dimension 10, containing the point o, we get a complete intersection $V_P \subset P \cong \mathbb{P}^{10}$ of dimension 8 with the point o an isolated singularity resolved by one blow up $V_P^+ \to V_P$, the exceptional divisor of which, Q_P, is a non-singular 7-dimensional quadric.

Apart from (R2.1), the quadratic point o is assumed to satisfy the condition

(R2.2) the polynomials

$$q_{*,1}, q_{1,2}, q_{2,2}, \ldots, q_{2,d_2}$$

in the standard order with q_{2,d_2} removed, form a regular sequence in $\mathcal{O}_{o,\mathbb{P}}$.

Now let us consider the biquadratic points, that is, the points $o \in V$ for which $q_{1,1} \equiv q_{2,1} \equiv 0$.

(R3.1) For a general linear subspace $P \subset \mathbb{P}$ of dimension 12, containing the point o, the intersection $V_P = V \cap P$ is a complete intersection of codimension 2 in $P = \mathbb{P}^{12}$ with the point $o \in V_P$ an isolated singularity resolved by one blow up $V_P^+ \to V_P$ with the exceptional divisor Q_P, which is a non-singular complete intersection of two quadrics in $\mathbb{P}^{11}$, $\dim Q_P = 9$.

Apart from (R3.1), the biquadratic point o is assumed to satisfy the condition (R3.2) the polynomials

$$q_{1,2}, q_{2,2}, \ldots, q_{2,d_2}$$

form a regular sequence in $\mathcal{O}_{o,\mathbb{P}}$.

The subset $\mathcal{P}_{\text{reg}}$ consists of the pairs (f_1, f_2) such that the conditions (R0.1-R0.3) are satisfied and the conditions (R1), (R2.1) and (R2.2), (R3.1) and (R3.2) are satisfied for every non-singular, quadratic and biquadratic point respectively.

Remark 3.0.4. These are stronger conditions than being correct multiquadratic singularities given in Definition 2.7.1. These conditions are required when using the proof without the $4n^2$ inequality for intersection singularities. In terms of Definition 2.8.1 we require every point to be $N_l = M - 2$ regular.

3.0.3 The structure of the proof of Theorem 3.0.1

By the well known Grothendieck's theorem [3] for every pair $(f_1, f_2) \in \mathcal{P}_{\text{reg}}$ the variety $V(f_1, f_2)$ satisfies the conditions of part (i) of Theorem 3.0.1. Therefore, Theorem 3.0.1 is implied by the following two claims.

Theorem 3.0.5. *The estimate*

$$\operatorname{codim}((\mathcal{P}\backslash\mathcal{P}_{\text{reg}}) \subset \mathcal{P}) \geqslant \frac{1}{2}(M-9)(M-10) - 1$$

holds.

Theorem 3.0.6. *For every pair $(f_1, f_2) \in \mathcal{P}_{\text{reg}}$ the variety $V = V(f_1, f_2)$ is birationally superrigid.*

The two claims are independent of each other and for that reason will be shown separately: Theorem 3.0.5 in Section 3.3 and Theorem 3.0.6 in Sections 3.1 and 3.2.

In order to prove Theorem 3.0.6, we fix a mobile linear system $\Sigma \subset |nH|$ on V, where H is the class of a hyperplane section. All we need to show is that Σ has no maximal singularities. Therefore, we consider the following four options:

- Σ has a maximal subvariety,
- Σ has an infinitely near maximal singularity, the centre of which on V is not contained in the singular locus $\operatorname{Sing} V$,
- Σ has an infinitely near maximal singularity, the centre of which on V is contained in $\operatorname{Sing} V$ but not in the locus of biquadratic points,
- Σ has an infinitely near maximal singularity, the centre of which on V is contained in the locus of biquadratic points.

The first two options are excluded in Section 3.1 (this is fairly straightforward), where we also prove a useful technical claim strengthening the $4n^2$-inequality in the non-singular case. The two remaining options are excluded in Section 3.2 (which is much harder and requires some additional work).

Theorem 3.0.5 is shown in Section 3.3, which completes the proof of Theorem 3.0.1.

3.1 Exclusion of maximal singularities. I. Maximal subvarieties and non-singular points

In this section we exclude maximal subvarieties of the mobile linear system Σ (Subsection 3.1.1) and infinitely near maximal singularities of Σ, the centre of which is not contained in the singular locus of V (Subsection 3.1.2). After that we show an improvement of the $4n^2$-inequality (Subsection 3.1.3), which will be used in Section 3.2 in the cases where the usual $4n^2$-inequality is insufficient.

3.1.1 Exclusion of maximal subvarieties

We start with the following claim.

Proposition 3.1.1. *The linear system Σ has no maximal subvarieties.*

Proof. Assume that $B \subset V$ is a maximal subvariety for Σ. Let us consider first the case $\operatorname{codim}(B \subset V) = 2$. For a general linear subspace $P \subset \mathbb{P}$ of dimension 7 the intersection $V_P = V \cap P$ is a non-singular complete intersection of codimension 2 in $\mathbb{P}^7$, hence for the numerical Chow group of classes of cycles of codimension 2 on V_P we have

$$A^2 V_P = \mathbb{Z} H_P^2,$$

where H_P is the class of a hyperplane section of V_P. Now the standard arguments [53, Chapter 2, Section 2] give the inequality

$$\operatorname{mult}_{B\cap P} \Sigma_P \leqslant n,$$

where Σ_P is the restriction of Σ onto V_P, a mobile subsystem of $|nH_P|$. Therefore, $\operatorname{mult}_B \Sigma \leqslant n$ and B is not a maximal subvariety — a contradiction.

Now let us consider the case $\operatorname{codim}(B \subset V) \geqslant 3$, $B \not\subset \operatorname{Sing} V$. In this case we have the inequality

$$\operatorname{mult}_B Z > 4n^2,$$

where $Z = (D_1 \circ D_2)$ is the self-intersection of the system Σ, $D_i \in \Sigma$ are general divisors. As $\deg Z = n^2 \deg V = n^2 d_1 d_2$, we use the inequality

$$\frac{\operatorname{mult}_o}{\deg} Y \leqslant \frac{4}{d_1 d_2},$$

which holds for any smooth point $o \in V$ and any irreducible subvariety $Y \subset V$ of codimension 2 (see Proposition 3.1.3 below) to obtain a contradiction. Finally, assume that $B \subset \mathrm{Sing}\, V$. In this case $\mathrm{codim}(B \subset V) \geqslant 10$, so that

$$\mathrm{mult}_B\, \Sigma > \delta n,$$

where $\delta \geqslant 7$. Therefore, we have the inequality

$$\mathrm{mult}_B\, Z > 98n^2,$$

which is impossible as for any singular point $o \in V$ and subvariety Y of codimension 2 the inequality

$$\frac{\mathrm{mult}_o}{\deg} Y \leqslant \frac{9}{d_1 d_2}$$

holds, see Propositions 3.2.1 and 3.2.2.

We have excluded all options for B.

Q.E.D. for Proposition 3.1.1.

3.1.2 Exclusion of maximal singularities, the centre of which is not contained in the singular locus.

Our next step is the following

Proposition 3.1.2. *The centre B of maximal singularity E is contained in the singular locus* $\mathrm{Sing}\, V$.

Proof. Assume the converse: $B \not\subset \mathrm{Sing}\, V$. Since B is not a maximal subvariety of Σ, we see that $\mathrm{codim}(B \subset V) \geqslant 3$ and the $4n^2$-inequality holds:

$$\mathrm{mult}_B\, Z > 4n^2. \tag{3.1}$$

Now let us show the opposite inequality.

Proposition 3.1.3. *For any non-singular point $o \in V$ and any irreducible subvariety Y of codimension 2 the inequality*

$$\frac{\mathrm{mult}_o}{\deg} Y \leqslant \frac{4}{d_1 d_2}$$

holds.

Proof. We consider the general case when $d_1 + 2 \leqslant d_2$; the two remaining cases $d_2 = d_1 + 1$ and $d_2 = d_1$ are done later.

This is the method oulined in Section 2.8 and 2.9, with $N_0 = N = M - 2$ with the M divisors

$$D_{1,1}, D_{1,2}, D_{2,1}, D_{2,2}, \ldots, D_{d_1-1,1}, D_{d_1-1,2}, D_{d_1,1}, D_{d_1+1,1}, \ldots D_{d_2-3,1}.$$

Considering the lexicographic order they are denoted by

$$R_1, \ldots, R_M,$$

where R_1 and R_2 are divisors from the tangent linear system. These are used to obtain a surface Y_N satisfying eqaution (2.17)

$$\frac{\mathrm{mult}_o}{\deg} Y_N > \gamma = \frac{2^{k+1}}{d} \cdot \prod_{i=k+2}^{N} \beta_i.$$

Lemma 3.1.4. Assuming that for $Y \ni o$ the claim of Proposition 3.1.3 does not hold, that is,

$$\frac{\mathrm{mult}_o}{\deg} Y > \frac{4}{d},$$

which is the assumption in Section 2.9, then $\gamma \geqslant 1$.

Proof. By direct calculation we have

$$\frac{\mathrm{mult}_o}{\deg} Y_N > \gamma = \frac{2^3}{d_1 d_2} \cdot \frac{3}{2} \cdot \left(\frac{4}{3} \cdot \ldots \cdot \frac{d_1}{d_1 - 1}\right)^2 \cdot \frac{d_1 + 1}{d_1} \cdot \ldots \cdot \frac{d_2 - 2}{d_2 - 3} =$$

$$= \frac{4}{d_2} \cdot \frac{d_2 - 2}{3} \geqslant 1$$

(the last inequality in this sequence holds as $d_2 \geqslant 8$). Therefore, $\mathrm{mult}_o Y_N > \deg Y_N$, which is impossible. Proposition 3.1.3 is shown.

The remaining two cases are now shown to be true.

Case. $d_2 = d_1 + 1$. There is a surface S such that

$$\begin{aligned}
\frac{\mathrm{mult}_o}{\deg} S &\geqslant \left(\frac{\mathrm{mult}_o}{\deg} Y\right) \cdot \frac{2}{1} \cdot \frac{3}{2} \cdot \left(\frac{4}{3} \cdot \frac{5}{4} \cdots \frac{d_1 - 1}{d_1 - 2}\right)^2 \cdot \frac{d_1}{d_1 - 1} \\
&\geqslant \left(\frac{\mathrm{mult}_o}{\deg} Y\right) \cdot \frac{2}{1} \cdot \frac{3}{2} \cdot \left(\frac{d_1 - 1}{3}\right)^2 \cdot \frac{d_1}{d_1 - 1} \\
&\geqslant \left(\frac{\mathrm{mult}_o}{\deg} Y\right) \cdot \frac{d_1(d_1 - 1)}{3} \\
&> \frac{4}{d_1(d_1 + 1)} \cdot \frac{d_1(d_1 - 1)}{3} = \frac{4(d_1 - 1)}{3(d_1 + 1)} \geqslant 1,
\end{aligned}$$

as $d_1 \geqslant 7$.

Case. $d_2 = d_1$. There is a surface S such that

$$\begin{aligned}
\frac{\mathrm{mult}_o}{\deg} S &\geqslant \left(\frac{\mathrm{mult}_o}{\deg} Y\right) \cdot \frac{2}{1} \cdot \frac{3}{2} \cdot \left(\frac{4}{3}\frac{5}{4} \cdots \frac{d_1 - 1}{d_1 - 2}\right)^2 \\
&\geqslant \left(\frac{\mathrm{mult}_o}{\deg} Y\right) \cdot \frac{(d_1 - 1)^2}{3} \\
&> \frac{4}{d_1^2} \cdot \frac{(d_1 - 1)^2}{3} > 1
\end{aligned}$$

as $d_1 \geqslant 8$.

Therefore, the inequality (3.1) is impossible. Proof of Proposition 3.1.2 is complete.

Remark 3.1.5. This can be calculated directly using

$$\frac{\mathrm{mult}_o}{\deg} > \frac{4}{3}(\beta(0))^{-1},$$

where

$$\beta(0) = \prod_{M-1}^{M} \beta_i,$$

as $N = M - 2$.

3.1.3 An improvement of the $4n^2$-inequality

This is the method used before the $4n^2$-inequality for complete intersection singularities was known. This will provide an estimate for $\mathrm{mult}_B Z$ in the singular cases, which will eventually excluded these cases.

Let us consider the following general situation: X is a smooth affine variety, $B \subset X$ a smooth subvariety of codimension at least 3, Σ_X a mobile linear system on X such that

$$\mathrm{mult}_B \Sigma_X = \alpha n \leqslant 2n$$

for some $\alpha \in (1, 2]$ and positive $n \in \mathbb{Q}$, but the pair $\left(X, \frac{1}{n}\Sigma_X\right)$ has a non-canonical singularity with the centre B. In other words, for some birational morphism $\varphi\colon \widetilde{X} \to X$ of smooth varieties and a φ-exceptional divisor $E \subset \widetilde{X}$, such that $\varphi(E) = B$, the Noether-Fano inequality

$$\mathrm{ord}_E \varphi^* \Sigma_X > na(E, X)$$

holds. By the symbol $Z_X = (D_1 \circ D_2)$ we denote the self-intersection of the mobile linear system Σ_X.

Theorem 3.1.6. *The following inequality holds:*

$$\mathrm{mult}_B Z_X > \frac{\alpha^2}{\alpha - 1} n^2$$

Remark 3.1.7. It is easy to see that the minimum of the real function $\frac{t^2}{t-1}$ on the interval (1,2] is attained at $t = 2$, so that the theorem improves the $4n^2$-inequality (Theorem 2.5.9). The proof given below is based on the idea that was first used in [55] and later in several other papers.

Proof. We follow the arguments given to prove Theorem 2.5.9, using the notations of the proof of the $4n^2$-inequality. Repeating those arguments word for word, we

- resolve the singularity E,
- consider the oriented graph Γ of the resolution,
- divide the set of vertices of Γ into the lower part ($\mathrm{codim}\, B_{i-1} \geqslant 3$) and the upper part ($\mathrm{codim}\, B_{i-1} = 2$),
- employ the technique of counting multiplicities.
- use the optimization procedure for the quadratic function $\sum\limits_{i=1}^{K} p_i \nu_i^2$

and obtain the inequality

$$\mathrm{mult}_B Z > \frac{(2\Sigma_l + \Sigma_u)^2}{\Sigma_l(\Sigma_l + \Sigma_u)} n^2.$$

Now set $m = \frac{1}{n^2} \mathrm{mult}_B Z$, so that the equality above can be re-written as

$$(4 - m)\Sigma_l^2 + (4 - m)\Sigma_l\Sigma_u + \Sigma_u^2 < 0.$$

As the elementary multiplicities $\nu_i = \mathrm{mult}_{B_{i-1}} \Sigma_X^{i-1}$ are non-increasing, we get the inequalities

$$\alpha n = \nu_1 \geqslant \nu_2 \geqslant \cdots \geqslant \nu_i \geqslant \nu_{i+1} \geqslant \ldots,$$

so that the Noether-Fano inequality implies the estimate

$$\alpha(\Sigma_l + \Sigma_u) > 2\Sigma_l + \Sigma_u.$$

As $1 < \alpha \leqslant 2$ by assumption, we conclude that

$$\Sigma_u > \frac{2-\alpha}{\alpha-1}\Sigma_l.$$

Now the quadratic function $\gamma(t) = t^2 + (4-m)t + (4-m)$ attains the minimum at $t = \frac{1}{2}(m-4) > 0$ and is negative at $t = 0$. Therefore, if $\gamma(t_0) < 0$ for some

$$t_0 > \frac{2-\alpha}{\alpha-1},$$

then

$$\gamma\left(\frac{2-\alpha}{\alpha-1}\right) = \left(\frac{2-\alpha}{\alpha-1}\right)^2 + (4-m)\left(\frac{2-\alpha}{\alpha-1}\right) + (4-m) < 0,$$

which easily transforms to the required inequality $m > \alpha^2/(\alpha-1)$. Q.E.D. for Theorem 3.1.5.

The following elementary fact will be useful in Section 3.2 when maximal singularities, the centre of which is contained in the singular locus of V, are excluded.

Proposition 3.1.8. *The function of real argument*

$$\beta(t) = \frac{t^3}{t-1}$$

is decreasing for $1 < t \leqslant \frac{3}{2}$ *and increasing for* $t \geqslant \frac{3}{2}$*, so that it attains its minimum on* $(1, \infty)$ *at* $t = \frac{3}{2}$*, which is equal to* $\frac{27}{4}$*.*

Proof. Obvious calculations. Q.E.D.

3.2 Exclusion of maximal singularities. II. Quadratic and biquadratic points.

In this section we exclude infinitely near maximal singularities of the linear system Σ, the centre of which is contained in the singular locus of V using Theorem 3.1.6 in Subsections 3.2.1-3.2.5. We start by using the technique of hypertangent divisors to obtain estimates for the multiplicities $\mathrm{mult}_o\,\Sigma$ and $\mathrm{mult}_o\,Z$, where o is a general point in the centre of the maximal singularity and Z is the self-intersection of the mobile system Σ (Subsection 3.2.1). After that, we consider separately the cases when the centre is contained in the locus of the quadratic singularities (Subsection 3.2.2 and 3.2.3) and biquadratic singularities (Subsections 3.2.4 and 3.2.5). Finally the $4n^2$ inequality for complete intersection singularities will be used to exclude infinitely near maximal singularities in the singular locus (Subsection 3.2.6). We make use of the inversion of adjunction and the connectedness principle explained in the previous chapter.

3.2.1 The technique of hypertangent divisors

Let $o \in \operatorname{Sing} V$ be a singularity (either a quadratic or a biquadratic point), $\sigma \colon V^+ \to V$ its blow up with the exceptional divisor $Q \subset V^+$. We consider σ as the restriction of the blow up $\sigma_{\mathbb{P}} \colon \mathbb{P}^+ \to \mathbb{P}$ of the same point o on the projective space $\mathbb{P}$ with the exceptional divisor $E_{\mathbb{P}} = \sigma_{\mathbb{P}}^{-1}(o)$, so that Q is either a quadric in a hyperplane in $E_{\mathbb{P}} \cong \mathbb{P}^{M+1}$ or a complete intersection of two quadrics in $E_{\mathbb{P}}$. For a generic divisor $D \in \Sigma$ set

$$D^+ \sim \sigma^* D - \nu Q$$

for some $\nu \in \mathbb{Z}_+$; thus $\operatorname{mult}_o D = 2\nu$ in the quadratic and $\operatorname{mult}_o D = 4\nu$ in the biquadratic case. In the singular case Proposition 1.3 has to be replaced by the following facts. Let $Y \subset V$ be an irreducible subvariety.

Proposition 3.2.1. *Assume that* $\operatorname{mult}_o V = 2$.

(i) *If* $\operatorname{codim}(Y \subset V) = 2$, *then the inequality*

$$\frac{\operatorname{mult}_o}{\deg} Y \leqslant \frac{7}{d_1 d_2}$$

holds.

(ii) *If* $\operatorname{codim}(Y \subset V) = 3$, *then the inequality*

$$\frac{\operatorname{mult}_o}{\deg} Y \leqslant \frac{72}{7 d_1 d_2}$$

holds.

(iii) *The inequality* $\nu \leqslant \sqrt{\frac{7}{2}} n$ *holds.*

Similarly, for the biquadratic case we have

Proposition 3.2.2. *Assume that* $\operatorname{mult}_o V = 4$.

(i) *If* $\operatorname{codim}(Y \subset V) = 2$, *then the inequality*

$$\frac{\operatorname{mult}_o}{\deg} Y \leqslant \frac{9}{d_1 d_2}$$

holds.

(ii) *The inequality* $\nu \leqslant \frac{3}{2}n$ *holds.*

Proof of Proposition 3.2.1. The claim (iii) follows from (i): for the self-intersection Z of the mobile system Σ we have the inequality $\mathrm{mult}_o Z \geqslant 2\nu^2$. As $\deg Z = n^2 d_1 d_2$, we get the inequality of part (iii), assuming (i).

In order to show the claim (i), we apply the technique of hypertangent divisors in the same way as in the proof of Proposition 3.1.3, but starting with the second hypertangent divisor and completing the procedure with the hypertangent divisor $D_{d_2-2,1}$ — one more than in the proof of Proposition 3.1.3, so that now we use the hypertangent divisors

$$D_{1,2}, D_{2,1}, D_{2,2}, \dots, D_{d_1-1,1}, D_{d_1-1,2}, D_{d_1,1}, D_{d_1+1,1}, \dots D_{d_2-2,1}.$$

If the claim (i) is not true, we obtain an irreducible surface $S \ni o$, satisfying the inequality

$$\frac{\mathrm{mult}_o}{\deg} S \geqslant \left(\frac{\mathrm{mult}_o}{\deg} Y\right) \cdot \frac{3}{2} \cdot \left(\frac{4}{3} \cdot \dots \cdot \frac{d_1}{d_1-1}\right)^2 \cdot \frac{d_1+1}{d_1} \cdot \dots \cdot \frac{d_2-1}{d_2-2} =$$

$$= \left(\frac{\mathrm{mult}_o}{\deg} Y\right) \cdot \frac{d_1(d_2-1)}{6} > \frac{7(d_2-1)}{6d_2} > 1,$$

which is impossible. The contradiction proves the claim (i).

Finally, to show the claim (ii), we argue in exactly the same way as above, starting with the hypertangent divisors $D_{2,1}, D_{2,2}$ (removing $D_{1,2}$), so that if the claim (ii) does not hold, we obtain an irreducible surface $S \ni o$, satisfying the inequality

$$\frac{\mathrm{mult}_o}{\deg} S > \frac{72}{7d_1d_2} \cdot \left(\frac{4}{3} \cdot \dots \cdot \frac{d_1}{d_1-1}\right)^2 \cdot \frac{d_1+1}{d_1} \cdot \dots \cdot \frac{d_2-1}{d_2-2}.$$

The right hand side simplifies to

$$\frac{72(d_2-1)}{63d_2} \geqslant 1$$

for $d_2 \geqslant 8$, which gives the desired contradiction and completes the proof of Proposition 3.2.1. Q.E.D.

Proof of Proposition 3.2.2. This is very similar to the proof of Proposition 3.2.1. First, we note that part (i) implies part (ii) by way of looking at the multiplicity of the self-intersection Z at the point o. In order to show the claim (i), we use the hypertangent divisors

$$D_{2,1}, D_{2,2}, \dots, D_{d_1-1,1}, D_{d_1-1,2}, D_{d_1,1}, D_{d_1+1,1}, \dots D_{d_2-1,1}.$$

to obtain the required estimate. Q.E.D. for Preposition 3.2.2.

3.2.2 Exclusion of the quadratic case, part I

In this subsection and in the next one, we assume that the centre of the maximal singularity is contained in the singular locus $\operatorname{Sing} V$ but not in the locus of biquadratic points. We will show that this assumption leads to a contradiction. To begin with, fix a general point $o \in V$ in the centre of the maximal singularity.

Let $\Pi \subset P$ be a general 6-plane in a 10-plane in $\mathbb{P}$ through the point o. Denote by V_Π and V_P the intersections $V \cap \Pi$ and $V \cap P$, respectively. By our assumptions about the singularities of V, the varieties V_Π and V_P are non-singular outside o. Let

$$\begin{array}{ccccccc} & V_\Pi^+ & \subset & & V_P^+ & \subset & V^+ & \\ \sigma_\Pi & \downarrow & & \sigma_P & \downarrow & & \downarrow & \sigma \\ & V_\Pi & \subset & & V_P & \subset & V & \end{array} \tag{3.2}$$

be the blow ups of the point o on V_Π, V_P and V. The varieties V_Π^+ and V_P^+ are non-singular. Denote the exceptional divisors of σ_Π, σ_P and σ by Q_Π, Q_P and Q, respectively. The quadrics Q_Π and Q_P are non-singular. The hyperplane sections of V_Π and V_P will be written as H_Π and H_P. Obviously, for a general divisor $D \in \Sigma$ we have

$$D_\Pi^+ \sim nH_\Pi - \nu Q_\Pi, \quad D_P^+ \sim nH_P - \nu Q_P,$$

where $D_\Pi = D|_{V_\Pi}$, $D_P = D|_{V_P}$ and the upper index $+$ means the strict transform. By inversion of adjunction the pairs $(V_\Pi, \frac{1}{n} D_\Pi)$ and $(V_P, \frac{1}{n} D_P)$ are not log canonical at the point o. As by Proposition 3.2.1, (iii) we have $\nu < 2n$, whereas $a(Q_\Pi, V_\Pi) = 2$, the pair

$$\left(V_\Pi^+, \frac{1}{n} D_\Pi^+ + \frac{(\nu - 2n)}{n} Q_\Pi \right) \tag{3.3}$$

is not log canonical, and the centre of any of its non-log canonical singularities is contained in the exceptional quadric Q_Π (see Lemma 4.1 in [53, Chapter 2]). The union of all centres of non-log canonical singularities of the pair (3.3) is a connected closed set by the Connectedness Principle, Theorem 2.6.2, or [42, 77]. Therefore,

- either it is a point,
- or it is a connected 1-cycle,
- or it contains a surface on the quadric Q_Π.

As the union of all centres of non-log canonical singularities of the pair (3.3) is a section of the union of all centres of non-log canonical singularities of the pair

$$\left(V_P^+, \frac{1}{n}D_P^+ + \frac{(\nu - 2n)}{n}Q_P\right) \tag{3.4}$$

by $V_\Pi^+ \cap Q_P$ (which is a section of the non-singular quadric Q_P by a general 4-plane in $\langle Q_P \rangle$), we see that the first option is impossible, as the smooth 7-dimensional quadric Q_P cannot contain a linear subspace of dimension 4. Therefore, we conclude that the pair (3.4) is not log canonical at an irreducible subvariety $\Delta \subset Q_P$ of codimension either 1 or 2.

Proposition 3.2.3. *The case* $\operatorname{codim}(\Delta \subset Q_P) = 1$ *is impossible.*

Proof. Assume that Δ is a divisor on Q_P. Then by Proposition 2.5.1 we have the following estimate for the multiplicity of the self-intersection Z_P of the system $\Sigma_P = \Sigma|_{V_P}$ at the point o:

$$\operatorname{mult}_o Z_P \geqslant 2\nu^2 + 2 \cdot 4 \left(3 - \frac{\nu}{n}\right) n^2$$

(the factor 2 in the second component of the right hand side appears since we have the inequality $\deg \Delta \geqslant 2$), and easy calculations give

$$\operatorname{mult}_o Z = \operatorname{mult}_o Z_P \geqslant 16n^2,$$

which contradicts Proposition 3.2.1, (i). Q.E.D. for Proposition 3.2.3.

Therefore we assume that $\Delta \subset Q_P$ is an irreducible subvariety of codimension 2. That option will be shown to be impossible in the next subsection.

3.2.3 Exclusion of the quadratic case, part II

Our arguments are very similar to those in [53, Chapter 2, Section 4]. Let $D_1, D_2 \in \Sigma$ be general divisors, $Z = (D_1 \circ D_2)$ the self-intersection of the system Σ. We can write

$$((D_1|_{V_P})^+ \circ (D_2|_{V_P})^+) = Z_P^+ + Z_{P,Q}$$

where $Z_{P,Q}$ is an effective divisor on the quadric Q_P. By the standard rules of intersection theory,

$$\operatorname{mult}_o Z = \operatorname{mult}_o Z_P = \deg(Z_P^+ \circ Q_P) = 2\nu^2 + \deg Z_{P,Q}.$$

Let us consider the cases $\deg\Delta = 2$ (when Δ is a section of Q_P by a linear subspace of codimension 2 in $\langle Q_P\rangle$) and $\deg\Delta \geqslant 4$ separately. Set $\alpha = \frac{\nu}{n} < 2$. Note that since $\operatorname{mult}_\Delta \Sigma_P^+ > n$ and $\Sigma_P^+|_{Q_P} \sim \nu H_Q$, where H_Q is the hyperplane section of the quadric Q_P, we have the inequality $\nu > n$, so that $\alpha > 1$. By Theorem 3.1.6,

$$\operatorname{mult}_\Delta(Z_P^+ + Z_{P,Q}) > \frac{\alpha^2}{\alpha - 1} n^2.$$

Assume now that $\deg\Delta \geqslant 4$. By Proposition 3.2.1, (i) we have:

$$4\operatorname{mult}_\Delta Z_P^+ \leqslant \deg(Z_P^+ \circ Q_P) \leqslant 7n^2,$$

so that

$$\operatorname{mult}_\Delta Z_{P,Q} > \left(\frac{\alpha^2}{\alpha-1} - \frac{7}{4}\right) n^2.$$

However, for $l \in \mathbb{Z}_+$ defined by the equivalence

$$Z_{P,Q} \sim lH_Q$$

we have the estimate $l \geqslant \operatorname{mult}_\Delta Z_{P,Q}$, so that

$$\operatorname{mult}_o Z = 2(\nu^2 + l) > 2\left(\alpha^2 + \frac{\alpha^2}{\alpha-1} - \frac{7}{4}\right) n^2.$$

The right hand side simplifies to

$$2\left(\frac{\alpha^3}{\alpha-1} - \frac{7}{4}\right) n^2 \geqslant 10n^2$$

by Proposition 3.1.8. Therefore, we obtained the inequality $\operatorname{mult}_o Z > 10n^2$, which contradicts Proposition 3.2.1, (i). The case $\deg\Delta \geqslant 4$ is now excluded.

From now on, and until the end of this subsection, we assume that $\deg\Delta = 2$, that is, Δ is cut out on Q_P by a linear subspace in $\langle Q_P\rangle$ of codimension 2. By construction, that means that there is a subvariety $\Delta_V \subset Q$ of codimension 2 and degree 2 (that is, Δ_V is cut out on the quadric Q by a linear subspace in $\langle Q\rangle$ of codimension 2), such that the pair

$$\left(V^+, \frac{1}{n}\Sigma^+ + \frac{\nu - 2n}{n} Q\right)$$

is not log canonical at Δ_V and

$$\Delta = \Delta_V \cap V_P^+.$$

Let R be a general hyperplane section of V, such that $R \ni o$ and the strict transform R^+ contains Δ_V. Let $Z_R = (Z \circ R)$ be the self-intersection of the mobile system $\Sigma_R = \Sigma|_R$. Obviously,

$$\operatorname{mult}_o Z_R = \operatorname{mult}_o Z + 2 \operatorname{mult}_{\Delta_V} Z^+.$$

Now set $Z_{P,R} = (Z_P \circ Z_R)$. By generality of both P and R we have the equalities

$$\operatorname{mult}_o Z_{P,R} = \operatorname{mult}_o Z_R, \quad \operatorname{mult}_\Delta Z_P^+ = \operatorname{mult}_{\Delta_V} Z^+.$$

Applying Proposition 3.2.1, (iii) and taking into account the equalities above, we get the estimate

$$\operatorname{mult}_o Z_P + 2 \operatorname{mult}_\Delta Z_P^+ \leqslant \frac{72}{7} n^2. \tag{3.5}$$

On the other hand, Q_P is a non-singular (quadric) hypersurface, so that by [53, Chapter 2, Proposition 2.3] we have the estimate

$$\deg Z_{P,Q} \geqslant 2 \operatorname{mult}_\Delta Z_{P,Q}$$

and for that reason

$$\operatorname{mult}_o Z_P \geqslant 2\nu^2 + 2 \operatorname{mult}_\Delta Z_{P,Q},$$

so that using (3.5) we get:

$$\begin{aligned} \tfrac{72}{7} n^2 &\geqslant 2\nu^2 + 2(\operatorname{mult}_\Delta Z_{P,Q} + \operatorname{mult}_\Delta Z_P^+) \\ &> 2\left(\alpha^2 + \tfrac{\alpha^2}{\alpha-1}\right) n^2 = 2\tfrac{\alpha^3}{\alpha-1} n^2. \end{aligned}$$

Now we apply Proposition 3.1.8 and obtain the inequality $\frac{72}{7} > \frac{27}{2}$, which is false. This contradiction excludes the quadratic case completely.

3.2.4 Exclusion of the biquadratic case, part I.

In this section and in the next one we assume that the centre of the maximal singularity is contained in the locus of biquadratic points. Again, we show that this assumption leads to a contradiction. For a start, we fix a general point $o \in V$ in the centre of the maximal singularity.

Now we take a general 7-plane Π through the point o and a general 12-plane $P \supset \Pi$. The notations V_Π, V_P etc. have the same meaning as in the quadratic case (Subsection 3.2.2), the same applies to the diagram (3.2) and the subsequent introductory arguments. The only difference is that the exceptional divisors Q_Π and Q_P of the blow ups of the point o on V_Π on V_P are now non-singular complete intersections of two quadrics. Instead of Proposition 3.2.1, we use Proposition 3.2.2, (ii) to obtain the inequality $\nu \leqslant \frac{3}{2}n < 2n$ and, once again, to conclude that the pair (3.3) is non-log canonical. Repeating the arguments of Subsection 3.2.2, we obtain the following four options for the union of all centres of non-log canonical singularities of the pair (3.3) in the biquadratic case:

- either it is a point,
- or it is a connected 1-cycle,
- or it is a connected closed set of dimension 2,
- or it contains a divisor on the 4-dimensional complete intersection Q_Π.

Passing over to the pair (3.4) in exactly the same way as we did it in the quadratic case, we see that the first option is impossible as a non-singular 9-fold Q_P can not contain a linear subspace of dimension 5. Therefore, the pair (3.4) is not log canonical at an irreducible subvariety $\Delta \subset Q_P$ of codimension 1,2 or 3. The divisorial case ($\operatorname{codim}(\Delta \subset Q_P) = 1$) is excluded by the arguments of the proof of Proposition 3.2.3 — in fact, we get a stronger estimate in this case:

$$\operatorname{mult}_o Z_P \geqslant 4\nu^2 + 4 \cdot 4 \left(3 - \frac{\nu}{n}\right) n^2$$

(as $\operatorname{mult}_o V_P = 4$ and $\deg \Delta \geqslant 4$), so that

$$\operatorname{mult}_o Z = \operatorname{mult}_o Z_P \geqslant 32n^2,$$

which contradicts Proposition 3.2.2, (i).

The case $\operatorname{codim}(\Delta \subset Q_P) = 2$ is excluded by the arguments of Subsection 3.2.3 as $\deg \Delta \geqslant 4$ and the resulting estimate $\operatorname{mult}_o Z > 10n^2$ contradicts Proposition 3.2.2, (i).

It remains to exclude the last option, when $\operatorname{codim}(\Delta \subset Q_P) = 3$, for which there is no analog in the quadratic case.

3.2.5 Exclusion of the biquadratic case, part II

From now on, and until the end of this section, $\Delta \subset Q_P$ is an irreducible subvariety of codimension 3. Slightly abusing our notations, which should not generate any misunderstanding, we show first the following claim.

Proposition 3.2.4. *Let $Q = G_1 \cap G_2 \subset \mathbb{P}^N$, $N \geqslant 11$, be a non-singular complete intersection of two quadrics G_1 and G_2, $W \subset Q$ an irreducible subvariety of codimension 2 and $\Delta \subset Q$ an irreducible subvariety of codimension 3. Let $l \in \mathbb{Z}_+$ be defined by the relation*

$$W \sim lH_Q^2,$$

where H_Q is the class of a hyperplane section of Q. Then the inequality

$$\operatorname{mult}_\Delta W \leqslant l$$

holds.

Proof. Assume the converse. For a point $p \in Q$ we denote, by the symbol $|H_Q - 2p|$, the pencil of tangent hyperplane sections at that point.

Lemma 3.2.5. *Let Y be an irreducible subvariety of codimension 2, containing the subvariety Δ. For a general point $p \in \Delta$ and any divisor $T \in |H_Q - 2p|$ we have $Y \not\subset T$.*

Proof of the lemma. Assume the converse. Then for general points $p, q \in \Delta$ and some hyperplane sections $T_p \in |H_Q - 2p|$ and $T_q \in |H_Q - 2q|$ we have $Y \subset T_p \cap T_q$, so that $Y = T_p \cap T_q$ is a section of Q by a linear subspace of codimension 2. Since $\operatorname{Sing}(T_p \cap T_q)$ is at most 1-dimensional (see, for instance, [58]) and $\operatorname{codim}(\Delta \subset Q) = 3$, we obtain a contradiction, varying the points p, q. Q.E.D. for the lemma.

We conclude that for a general point $p \in \Delta$ and an *arbitrary* hyperplane section $T_p \in |H_Q - 2p|$ the cycle $W_p = (W \circ T_p)$ is well defined. It is an effective cycle of codimension 3 on Q and 2 on T_p (the latter variety is a complete intersection of two quadrics in $\mathbb{P}^{N-1}$ with at most 0-dimensional singularities). Let $H_p \in \operatorname{Pic} T_p$ be the class of a hyperplane section. Then we can write $W_p \sim lH_p^2$. Set

$$\Delta_p = \Delta \cap T_p.$$

Obviously, for a general point p the closed set Δ_p is of codimension 3 on T_p. For any point $q \in \Delta_p$ the inequality

$$\operatorname{mult}_q W_p > l$$

holds. Besides, by construction $\mathrm{mult}_p W_p > 2l$.

Now let us consider a point $q \in \Delta_p$ of general position. Repeating the proof of Lemma 3.2.5 word for word (and taking into account that the complete intersection of two quadrics T_p has zero-dimensional singularities), we see that for *any* divisor $T_q \in |H_Q - 2q|$ none of the components of the effective cycle W_p is contained in T_q, so that

$$W_{pq} = (W_p \circ T_q)$$

is a well defined effective cycle of codimension 2 on $T_p \cap T_q$, of codimension 3 on T_p and 4 on Q. Since T_q is an arbitrary hyperplane section in the pencil $|H_Q - 2q|$, we can choose it to be the one containing the point p. Now W_{pq} is an effective cycle of codimension 6 on $\mathbb{P}^N$ of degree $\deg W_{pq} = 4l$, satisfying the inequalities

$$\mathrm{mult}_p W_{pq} > 2l \quad \text{and} \quad \mathrm{mult}_q W_{pq} > 2l.$$

Taking a general projection onto $\mathbb{P}^{N-6}$, we conclude that the line $[p,q] \subset \mathbb{P}^N$, joining the points p and q, is contained in the support of the cycle W_{pq}. Therefore, for *any* point $q \in \Delta_p$ we have $[p,q] \subset W$ and so for *any* point $q \in \Delta$ we have $[p,q] \subset W$. Since Δ is not a linear subspace in $\mathbb{P}^N$ (Q cannot contain linear subspaces of dimension $N-5$) and $\dim W = N-4$, we conclude that Δ is a hypersurface in a linear subspace of dimension $N-4$ and W is that linear subspace, which is again impossible. The proof of Proposition 3.2.4 is now complete. Q.E.D.

Now coming back to the biquadratic case and using the notation of that case, we write for general divisors $D_1, D_2 \in \Sigma$:

$$((D_1|_{V_P})^+ \circ (D_2|_{V_P})^+) = Z_P^+ + Z_{P,Q},$$

where again $Z_{P,Q}$ is an effective divisor on the exceptional divisor of the blow up σ_P of the point o, which is a non-singular complete intersection of two quadrics. Again,

$$\mathrm{mult}_o Z = \mathrm{mult}_o Z_P = \deg(Z_P^+ \circ Q_P) = 4\nu^2 + \deg Z_{P,Q}. \tag{3.6}$$

We set $\alpha = \frac{\nu}{n} \leqslant \frac{3}{2}$. By Theorem 3.1.6,

$$\mathrm{mult}_\Delta (Z_P^+ \circ Q_P) + \mathrm{mult}_\Delta Z_{P,Q} > \frac{\alpha^2}{\alpha - 1} n^2.$$

By Proposition 3.2.4,

$$\mathrm{mult}_\Delta (Z_P^+ \circ Q_P) \leqslant \frac{1}{4} \deg(Z_P^+ \circ Q_P) = \mathrm{mult}_o Z_P.$$

As $\deg Q_P = 4$, we also have the estimate

$$\operatorname{mult}_\Delta Z_{P,Q} \leqslant \frac{1}{4} \deg Z_{P,Q},$$

so that

$$\operatorname{mult}_o Z_P + \deg Z_{P,Q} > 4\frac{\alpha^2}{\alpha - 1} n^2.$$

Using (3.6), we get finally:

$$2 \operatorname{mult}_o Z > 4 \left(\alpha^2 + \frac{\alpha^2}{\alpha - 1} \right) n^2 = 4\frac{\alpha^3}{\alpha - 1} n^2.$$

Applying Proposition 3.1.8, we conclude that

$$\operatorname{mult}_o Z > \frac{27}{2} n^2,$$

which contradicts Proposition 3.2.2, (i).

The proof of Theorem 3.0.6 is now complete.

3.2.6 Exclusion using $4n^2$ for complete intersections

This is the method that will be generalised in Chapter 4 and is more efficient for excluding maximal singularities than the ideas in the previous Subsections 3.1.1 through 3.1.5.

Proposition 3.2.6. *Assume that* $\operatorname{mult}_o V = 2$*. If* $\operatorname{codim}(Y \subset V) = 2$*, then the inequality*

$$\frac{\operatorname{mult}_o}{\deg} Y \leqslant \frac{8}{d_1 d_2}$$

holds.

Proposition 3.2.7. *Assume that* $\operatorname{mult}_o V = 4$*. If* $\operatorname{codim}(Y \subset V) = 2$*, then the inequality*

$$\frac{\operatorname{mult}_o}{\deg} Y \leqslant \frac{16}{d_1 d_2}$$

holds.

These are just weaker versions of Proposition 3.2.1 and Proposition 3.2.2. The proof of Theorem 3.0.6 follows from Theorem 2.5.15, which states

$$\operatorname{mult}_o Z > 4n^2 2^l,$$

where $l = 1$ if $\mathrm{mult}_o V$=2, and $l = 2$ if $\mathrm{mult}_o V$=4 and Z is the self intersection of the linear system $\Sigma \subset |nH|$. The following inequality follows

$$\frac{\mathrm{mult}_o}{\deg} Z > \frac{4 \cdot 2^l}{d_1 d_2},$$

which contradicts Prosition 3.2.6 and Proposition 3.2.7, which completes the proof of Theorem 3.0.6.

3.3 Regularity conditions

In this section we will prove Theorem 3.0.2 in several steps. We first notice that

$$\mathrm{codim}((\mathcal{P}\backslash\mathcal{P}_{\mathrm{reg}}) \subset \mathcal{P}) = \ \min_{\{*\in S\}}\{\mathrm{codim}((\mathcal{P}\backslash\mathcal{P}_*) \subset \mathcal{P})\},$$

where $S = \{$(R0.1),(R0.2),$\ldots$,(R3.2)$\}$ and

$$\mathcal{P}_* = \{(f_1, f_2) \in \mathcal{P} \mid \text{the pair satifies the regularity condition } *\}.$$

We first deal with the global conditions (R0.1-R0.3) (Subsection 3.3.1). Then move onto estimating the codimension of the bad set for the condition (R1) (that is, the set of pairs (f_1, f_2) that does not satisfy that condition) and show that the same estimates work for the conditions (R2.2) and (R3.2) (Subsections 3.3.2 and 3.3.3). Lastly, we deal with the conditions (R2.1) and (R3.1) to get our total estimate (Subsection 3.3.4).

3.3.1 Global conditions

We first start by splitting the condition (R0.1) up into two conditions. The first is the irreducibility condition for the hypersurface $\{f_1 = 0\}$; the set of pairs (f_1, f_2) with f_1 irreducible is denoted by $\mathcal{P}_{\mathrm{irred}}$. The second condition is that the hypersurface $\{f_1 = 0\}$ has at most quadratic singularities of rank at least 5; the corresponding subset of $\mathcal{P}$ is denoted by $\mathcal{P}_{\mathrm{qsing}\geqslant 5}$.

Proposition 3.3.1. *The codimension of $\mathcal{P} \setminus \mathcal{P}_{\mathrm{irred}}$ in $\mathcal{P}$ is at least $\frac{M(M+3)}{2}$.*

Proof. This is independent of the choice of f_2, hence it reduces to looking at $f \in \mathcal{P}_{d_1,M+3}$ such that $f = g_1 \cdot g_2$ with $\deg g_1 = a$ and $\deg g_2 = d_1 - a$, $a = 1, 2, \ldots, d_1 - 1$. Then we define

$$\mathcal{F}_i = \mathcal{P}_{i,M+3} \times \mathcal{P}_{d_1-i,M+3}.$$

Obviously, we have

$$\dim \mathcal{P} \setminus \mathcal{P}_{\text{irred}} \leqslant \max\{\dim \mathcal{F}_i \,|\, i = 1, 2, \ldots, d_1 - 1\}.$$

We calculate:

$$\dim \mathcal{F}_i = \binom{i + M + 2}{M + 2} + \binom{d_1 - i + M + 2}{M + 2}.$$

By the assumption $d_1 \leqslant \frac{M}{2} + 1$, we see that this gives the maximum dimension occuring at $i = 1$, or $i = d_1 - 1$ as $\mathcal{F}_i = \mathcal{F}_{d_1 - i}$. Then

$$\dim \mathcal{F}_1 = (M + 3) + \binom{d_1 + M + 1}{M + 2},$$

which immediately produces the estimate of the codimension of $\mathcal{P} \setminus \mathcal{P}_{\text{irred}}$ in $\mathcal{P}$ from below

$$\binom{d_1 + M + 2}{M + 2} - \left((M + 3) + \binom{d_1 + M + 1}{M + 2}\right) = \binom{d_1 + M + 1}{M + 1} - (M + 3).$$

The mininal value occurs at $d_1 = 2$ to get the estimate claimed by our proposition. Q.E.D. for Proposition 3.3.1.

Proposition 3.3.2. *The codimension of $\mathcal{P} \setminus \mathcal{P}_{\text{qsing}\geqslant 5}$ in $\mathcal{P}$ is at least $\binom{M-1}{2} + 1$.*

Proof. This is essentially a calculation about the rank of quadratic forms which has been done in many places, see [17]. Q.E.D.

As $\mathcal{P}_{(R0.1)} = \mathcal{P}_{\text{irred}} \cap \mathcal{P}_{\text{qsing}\geqslant 5}$, we get that the codimension of $\mathcal{P} \setminus \mathcal{P}_{(R0.1)}$ in $\mathcal{P}$ is at least $\binom{M-1}{2} + 1$.

Now we consider $\mathcal{P}_{(R0.2)} \subset \mathcal{P}$ consisting of pairs (f_1, f_2) satisfying the regularity condition (R0.2). We have two cases to consider: the first is if the hypersurfaces contains a common component; the second is if the intersection is non-reduced or reducible. The second case is the only one which needs considering as the first one gives a much higher codimension of the bad set. Fixing f_1 we consider the set $\mathcal{H} \subset \mathcal{P}_{d_2, M+3}$ such that $F_1 \cap F_2$ is reducible or non-reduced.

Proposition 3.3.3. *The codimension of $\mathcal{H}$ in $\mathcal{P}_{d_2, M+3}$ is at least $\binom{M+2}{2} - 2$.*

Proof. Taking into account Remark 0.1, we see that if $f_2 \in \mathcal{H}$, then:

$$f_2|_{F_1} \in \mathcal{P}_{i, M+3}|_{F_1} \times \mathcal{P}_{d_2 - i, M+3}|_{F_1},$$

for some $i = 1, 2, \ldots d_2 - 1$. Arguing similar to the proof of Proposition 3.3.1, we get: the codimension of $\mathcal{H}$ in $\mathcal{P}_{d_2, M+3}$ is greater or equal than

$$\binom{d_2 + M + 2}{d_2} - \left((M + 3) + \binom{d_2 + M + 1}{d_2 - 1} + \binom{d_2 - d_1 + M + 2}{d_2 - d_1}\right)$$

$$= \frac{1}{(M+2)!}\left(\frac{(M+2)(d_2+M+1)!}{d_2} - \frac{(d_2-d_1+M+2)!}{(d_2-d_1)!}\right) - (M+3).$$

Using the substitution $s = d_2 - d_1$, we see that for a fixed s the minimum of the above expression occurs for $d_2 = s+2$ and is equal to

$$\frac{1}{(M+2)!}\left(\frac{(M+2)(s+M+3)!}{d_2} - \frac{(s+M+2)!}{s!}\right) - (M+3).$$

An easy check shows that this is an increasing function of s, so that the minimum occurs at $s = 0$ to give us the required estimate. Q.E.D. for Proposition 3.3.3.

3.3.2 Regularity conditions for smooth points

Recall that a smooth point satisfies the regularity condition (R1) if the homogeneous components $q_{i,j}$ in the standard order, with the last two terms (that is, the two terms of highest degree) removed, form a regular sequence. If $d_1 < d_2$, then we need

$$W = \{q_{1,1} = q_{1,2} = \ldots = q_{1,d_1} = q_{2,1} = q_{2,2} = \ldots = q_{2,d_2-2} = 0\}$$

to be a finite set of surfaces in $\mathbb{A}^{M+2}$. If $d_1 = d_2$, then we need

$$W = \{q_{1,1} = q_{1,2} = \ldots = q_{1,d_1-1} = q_{2,1} = q_{2,2} = \ldots = q_{2,d_2-1} = 0\}$$

to be a finite set of surfaces in $\mathbb{A}^{M+2}$.

The linear forms $q_{1,1}$ and $q_{2,1}$ define the tangent space T_pV at the point p, so in the case $d_2 > d_1$

$$W = \{q_{1,2}|_{T_pV} = \ldots = q_{1,d_1}|_{T_pV} = q_{2,2}|_{T_pV} = \ldots = q_{2,d_2-2}|_{T_pV} = 0\} \subset \mathbb{A}^M$$

and similarly for the case $d_1 = d_2$. Finally as all the terms above are homogeneous we can consider the projective variety defined by the same equations in the projectivized tangent space. Denote this by $\widetilde{W} \subset \mathbb{P}^{M-1}$. We now have redefined the regularity condition under consideration to be $\operatorname{codim}(\widetilde{W} \subset \mathbb{P}^{M-1}) = M-2$, that is, $\widetilde{W}$ is a finite set of curves.

Proposition 3.3.4. *The codimension of $\mathcal{P} \setminus \mathcal{P}_{(R1)}$ in $\mathcal{P}$ is at least*

$$\lambda(M) = \frac{(M-5)(M-6)}{2} - (M+1).$$

Proof. We follow the methods given in [57, 59] to estimate the codimension of the space of varieties, which violate the regularity conditions. The scheme of these methods will be briefly outlined here. Firstly we introduce the necessary definitions.

We say a sequence of polynomials $p_1, p_2, \ldots p_l$ is *k-regular*, with $k \leqslant l$, if the subsequence $p_1, p_2, \ldots p_k$ is regular.

We re-label our polynomials in their standard ordering by $h_1 = q_{1,2}$, $h_2 = q_{2,2}$, etc. Also define deg $h_i = m_i$ to get our sequence $h_1, \ldots h_{M-2}$, with $m_i \leqslant m_{i+1}$ in the space

$$\mathcal{L} = \prod_{i=1}^{M-2} \mathcal{P}_{m_i,M}.$$

We further look at the partial products defined by:

$$\mathcal{L}_k = \prod_{i=1}^{k} \mathcal{P}_{m_i,M}.$$

We also define

$$Y_k(p) = \{(h_*) \in \mathcal{L}_k \mid (h_*) \text{ is a nonregular sequence at the point } p\},$$

emphasizing the choice of fixing the point p as our origin of affine coordinates. We will now consider $k = 1, 2, \ldots, M-2$ and denote

$$Y(p) = \bigcup_{k=1}^{M-2} Y_k(p),$$

the set of sequences, which are not regular at some stage. Clearly, it is sufficient to check that the codimension of Y_k in $\mathcal{L}_k$ is at least $\lambda(M) + M$. Now we outline the two methods of estimating the codimension of the bad set, with the most important cases considered explicitly.

Method 1. We will use this method to get estimates for all cases but the one when the regularity fails at the last stage. This method is given in [57].

Case 1. For a start, let us consider the trivial case $k = 1$. Here

$$Y_1(p) = \{h_1 \equiv 0 \in \mathcal{P}_{2,M}\},$$

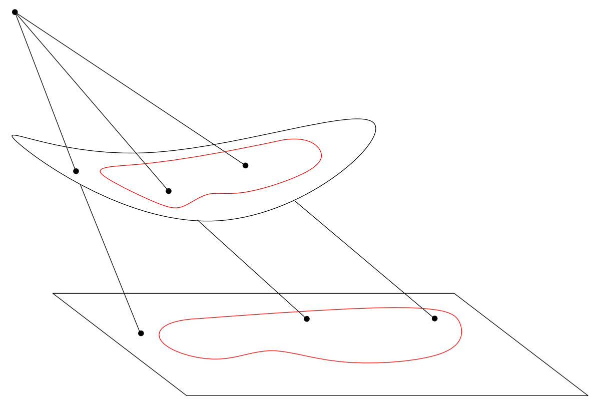
so that

$$\operatorname{codim}(Y_1(x) \subset \mathcal{L}_1) = \dim \mathcal{P}_{2,M} = \binom{M+1}{2}.$$

Case 2. Now assume that $k = 2$. This is the first non-trivial case and all the following cases follow this method. We have that

$$Y_2(p) = \{(h_1, h_2) \in \mathcal{P}_{2,M} \times \mathcal{P}_{2,M} \mid \operatorname{codim}\{h_1 = h_2 = 0\} < 2\}.$$

Now we have $Q = \{h_1 = 0\} = \bigcup Q_i \subset \mathbb{P}^{M-1}$, the decomposition into its irreducible components and we assume that $h_1 \not\equiv 0$. Pick a general point $r \in \mathbb{P}^{M-1}$ not on Q_i and consider the projection from this point to get the map $\pi\colon \mathbb{P}^{M-1} \dashrightarrow \mathbb{P}^{M-2}$, so that restricting this projection onto each Q_i we get a finite map π_{Q_i}, see the figure below.



Now take some $g \in H^0(\mathbb{P}^{M-2}, \mathcal{O}_{\mathbb{P}^{M-2}}(2))$ and look at $\pi^*_{Q_i}(g)$: as the map is finite, we get that $\pi^*_{Q_i}$ is injective. Therefore, for the closed subset

$$W_2 = \pi^* H^0(\mathbb{P}^{M-2}, \mathcal{O}_{\mathbb{P}^{M-2}}(2)) \subset \mathcal{P}_{2,M-1}$$

we have $W_2 \cap Y_2(x) = \{0\}$. Now we know $\dim W_2 = \binom{M}{2}$ so that $\operatorname{codim} Y_2(x) \geqslant \binom{M}{2}$. Therefore in the case $k = 2$ we obtain the estimate

$$\operatorname{codim}(Y_2(p) \subset \mathcal{L}_2) \geqslant \binom{M}{2}.$$

The remaining cases. We follow this method for the other values of $k = 3, \ldots, M-3$; we deal with the case $k = M-2$ separately (and by means of a different technique) later. Using this method we obtain for $k \geqslant 2$ ($k = 1$ is a special case) the inequality

$$\operatorname{codim}(Y_k(p) \subset \mathcal{L}_k) \geqslant \binom{\alpha_k}{\beta_k},$$

where the values of α_k and β_k are listed in the following table (k is changing from 1 to $k = M-3$:

$$\begin{array}{lllllllllll}
\alpha_k: & M+1, & M, & M, & M-1, & M-1, & \cdots & d_2, & d_2, & d_2, & \cdots & d_2; \\
\beta_k: & 2, & 2, & 3, & 3, & 4, & \cdots & d_1, & d_1+1, & d_1+2, & \cdots & d_2-3.
\end{array}$$

If $d_1 = 2$, then the smallest estimate is given by $\binom{M}{2}$, so we assume $d_1 \geqslant 3$ and the smallest estimate is given by $\binom{d_2}{3}$. Now as $d_2 \geqslant \frac{M}{2} + 1$ we get

$$\binom{d_2}{3} \geqslant \frac{M(M+2)(M-2)}{48},$$

which is better than we need.

Method 2. It remains to consider the case $k = M-2$. The previous projection method outlined above in this case does not produce the estimate we need and so we use a different method that was developed in [59]. We fix $Y^* = Y_{M-2}(p)$. Note that for any $(h_*) \in Y^*$ the sequence $h_1, \ldots, h_{M-3}$ is regular.

Assume the sequence (h_*) belongs to Y^*, which means there exists an irreducible component $B \subseteq Z(h_1, \ldots, h_{M-3})$, which is a surface with $h_{M-2}|_B \equiv 0$, where $Z(h_1, \ldots, h_{M-3}) \subset \mathbb{P}^{M-1}$ is the set of common zeros of these polynomials restricted to the projectivized tangent space.

We look at the linear span $\langle B \rangle$ of B and consider all possible values of:

$$b = \operatorname{codim}(\langle B \rangle \subset \mathbb{P}^{M-1}).$$

Now we split Y^* up into the union

$$Y^* = \bigcup_{b=0}^{M-3} Y^*(b),$$

where $Y^*(b)$ is the set of $(M-3)$-uples $(h_*) \in Y^*$ such that for some irreducible curve $B \subseteq Z(h_1, \ldots, h_{M-3})$ such that $\operatorname{codim}\langle B \rangle = b$, the polynomial h_{M-2} vanishes on B.

To begin with, let us consider the case $b = 0$. This means that $\langle B \rangle = \mathbb{P}^{M-1}$. Notice that non-zero linear forms in $z_1, \ldots z_M$, the coordinates on $\mathbb{P}^{M-1}$, do not vanish on B. As h_{M-2} has degree $d_2 - 2$ or $d_2 - 1$, we consider the worst case with the smaller degree, that is, the space:

$$W = \left\{ \prod_{i=1}^{d_2-2} (a_{i,1} z_1 + \ldots + a_{1,M} z_M) \right\} \subset \mathcal{P}_{d_2-2, M-1}.$$

W is a closed set with $\dim W = (M-1)(d_2-2)+1$; as $d_2 \geqslant \frac{M}{2} + 1$ we have $\dim W \geqslant \frac{(M-1)(M-2)}{2} + 1$. As $Y^*(0) \cap W = \{0\}$, we have

$$\operatorname{codim} Y^*(0) \geqslant \frac{(M-2)(M-1)}{2} + 1.$$

Now let us deal with the case $1 \leqslant b < M-3$. We use the technique of good sequences and associated subvarieties, developed and described in detail in [59].

Let us fix some linear subspace $P \subset \mathbb{P}^{M-1}$ of codimension b. Let $Y^*(P)$ be the set of all $(M-2)$-uples $(h_*) \in Y^*(b)$ such that the closed subset $Z(h_1, \ldots, h_{M-3})$ contains an irreducible component B such that $\langle B \rangle = P$ and $h_{M-2}|_B \equiv 0$.

We know [59] that good sequences form an open set in the space of tuples of polynomials and that the number of associated subvarieties is bounded from above by a constant, depending on their degrees. Therefore, we may assume that some $(M-3-b)$ polynomials from the set $(h_1|_P, \ldots, h_{M-3}|_P)$ form a good sequence and B is one of its associated subvarieties. The worst estimate corresponds to the case when the polynomials

$$h_{b+1}|_P, \ldots, h_{M-3}|_P$$

of the *highest* possible degrees form a good sequence, and B is one of its associated subvarieties, and we will assume that this is the case.

So we fix the polynomials $h_{b+1}, \ldots, h_{M-3}$ and estimate the number of independent conditions imposed on the polynomials $h_1, \ldots, h_b, h_{M-2}$ by the requirement that they vanish on B, arguing as in the case $b = 0$. Subtracting the dimension of the Grassmannian of linear subspaces of codimension b in $\mathbb{P}^{M-1}$, we get the estimate

$$\operatorname{codim}(Y^*(b) \subset \mathcal{L}) \geqslant (M-1-b) \cdot \left(\sum_{j=1}^{b} \deg h_j + \deg h_{M-2} - b \right) + 1.$$

Denote the right hand side of this inequality by θ_b.

Proposition 3.3.5. *The following inequality*

$$\theta_b \geqslant \frac{(M-2)(M-1)}{2} + 1 \tag{3.7}$$

holds for all $b = 1, 2, \ldots, M-4$.

Proof. It is easy to check that

$$\gamma_b = \theta_{b+1} - \theta_b = (M-2-b)(\deg h_{b+1} - 1) - \left(\sum_{j=1}^{b} \deg h_j - b + \deg h_{M-2} \right),$$

and since for $b \geqslant 2(d_1 - 1)$ we have $\deg h_{b+1} = \deg h_b + 1$, for these values of b the equality

$$\gamma_b = \gamma_{b-1} + (M-2-b) - 2(\deg h_b - 1)$$

holds. From this equality we can see that the sequence θ_b, where $b = 2(d_1-1), 2d_1 - 1, \ldots, M-4$, has one of the following three types of behaviour:

- either it is non-decreasing,
- or it is first increasing for $b = 2d_1 - 2, \ldots, a$, and then decreasing,

- or it is decreasing.

Below it is checked that θ_{M-4} satisfies the inequality (3.7). Therefore, in order to show (3.7) for $b = 2(d_1 - 1), \dots, M-4$, we only need to show this inequality for $b = 2(d_1 - 1)$, which is a part of the computation that we start now.

Assume that $b = 2l$, where $l = 1, \dots, d_1 - 1$. Here $\theta_b = \omega_1(l)$, where

$$\omega_1(t) = (M - 1 - 2t)(t^2 + t + d_2 - 2) + 1.$$

It is easy to check that $\omega_1'(t) \geqslant 0$ for $1 \leqslant t \leqslant t_1$ for some $t_1 > 1$, and $\omega_1'(t) < 0$ for $t > t_1$, so that the function of real argument $\omega_1(t)$ is first increasing (on the interval $[1, t_1]$) and then decreasing (on $[t_1, \infty)$). It follows that

$$\min\{\theta_{2l} \,|\, l = 1, \dots, d_1 - 1\} = \min\{\theta_2,\, \theta_{2(d_1-1)}\}.$$

Now $\theta_2 = \omega_1(1) = (M-3)d_2 + 1 \geqslant \frac{1}{2}(M+2)(M-3) + 1$, which satisfies (3.7).

Let us consider the second option: for $t = d_1 - 1$ we get

$$\omega_1(d_1 - 1) = (M - 2d_1 + 1)(d_1^2 - 2d_1 + M) + 1.$$

As $2d_1 - 2 \leqslant M - 4$, we get the bound $d_1 \leqslant \frac{M}{2} - 1$. Looking at the derivative of the function

$$\omega_2(t) = (M - 2t + 1)(t^2 - 2t + M) + 1,$$

we conclude that its minimum on the interval $[2, \frac{M}{2} - 1]$ is attained at one of the endpoints, so is equal to the minimum of the two numbers:

$$M(M-3) + 1 \quad \text{and} \quad \frac{3}{4}(M^2 - 4M + 12) + 1.$$

Clearly, both satisfy the inequality (3.7).

In order to complete the proof of our proposition, it remains to consider the case $B = 2l + 1$, where $l = 0, \dots, d_1 - 2$. Here $\theta_b = \omega_3(l)$, where

$$\omega_3(t) = (M - 2 - 2t)(t^2 + 2t + d_2 - 1) + 1.$$

For $d_1 \geqslant 3$ it is easy to check that the function $\omega_3(t)$ behaves similarly to $\omega_1(t)$, first increasing and then decreasing, so that it is sufficient ti show that $\omega_3(0)$ and $\omega_3(d_1 - 2)$ satisfy the estimate (3.7). Indeed,

$$\omega_3(0) = (M-2)(d_2 - 1) + 1$$

satisfies (3.7) as $d_2 \geqslant \frac{M}{2} + 1$, and for $t = d_1 - 2$ we get $\omega_3(d_1 - 2) = \omega_4(d_1)$, where

$$\omega_4(t) = (M - 2t + 2)(t^2 - 3t + M - 1)$$

and easy computations show that (3.7) is satisfied here as well.

Finally, in the case $d_1 = 2$ we get the number

$$\omega_3(0) = (M - 2)(M - 1) + 1.$$

Now the only case to consider is $b = M - 4$. Here we get

$$\operatorname{codim}(Y^*(b) \subset \mathcal{L}) \geqslant \frac{3}{4}(M^2 - 4M + 6) + 1.$$

Proof of Proposition 3.3.5 is complete. Q.E.D.

In order to complete the proof of Proposition 3.3.4, we have to consider the only remaining case $b = M - 3$. Here $\langle B \rangle = \mathbb{P}^2$, which clearly implies $B \subset \mathbb{P}^{M-1}$ itself is a plane. We do an easy dimension count. For a polynomial h to satisfy $h|_B \equiv 0$ with $\deg h = e$ we get a closed algebraic set of polynomials of codimension $\binom{e+2}{2}$ in $\mathcal{P}_{e,M}$. Therefore

$$\operatorname{codim}(Y^*(M-3) \subset \mathcal{L}) \geqslant \sum_{i=1}^{M-2} \binom{m_i + 2}{2} - 3(M - 3).$$

The sum takes the minimum value when $d_1 = d_2$ and then we have the estimate

$$\operatorname{codim}(Y^*(M-3) \subset \mathcal{L}) \geqslant \frac{M(M+4)(M+2)}{24} - 3M + 1.$$

Combining the results of both methods and simple calculation gives the estimate

$$\operatorname{codim}(Y(p) \subset \mathcal{L}) \geqslant \frac{(M-5)(M-6)}{2} + 1.$$

Now Proposition 3.3.4 follows from a standard dimension count argument.

Remark 3.3.6. This is clearly not the tightest bound possible; however, in Proposition 3.3.9 we have a weaker estimate.

3.3.3 Regularity conditions for singular points.

Recall that a point is a quadratic singularity if $q_{1,1}$ and $q_{2,1}$ are proportional and at least one of the terms is non-zero. We say a point is a biquadratic singularity if

$q_{1,1} = q_{2,1} = 0$. The regularity conditions (R2.2) and (R3.2) for both of these cases are similar to the smooth case (R1). The arguments used for smooth points (R1) follow in a similar way for the two cases (R2.2) and (R3.2). For quadratic points we work in $\mathbb{P}^M$, and for biquadratic points we work in $\mathbb{P}^{M+1}$, instead of $\mathbb{P}^{M-1}$ and calculations are almost identical. We obtain larger estimates for the codimension of non-regular sequences given below.

Proposition 3.3.7. *The codimension of $\mathcal{P} \setminus \mathcal{P}_*$ in $\mathcal{P}$ is at least*

$$\lambda(M) = \frac{(M-5)(M-6)}{2} - (M+1).$$

for $*=$(R2.2) and (R3.2).

Proof. We will outline the proof for the quadratic case (R2.2) and the biquadratic case is treated in the same way. Instead of restricting to the tangent space we restrict to the Zariski tangent space $\{q_{i,1} = 0\}$ for which every $q_{i,1}$ is no-zero and work in $\mathbb{P}^M$. We now have one extra polynomial to get our standard ordering to be given by $h_1, \ldots, h_{M-1}$, and our polynomials now belong to $\mathcal{P}_{m_i,M+1}$. For the method 1, case 1 we get the estimate:

$$\operatorname{codim}(Y_1(x) \subset \mathcal{L}_1) = \dim \mathcal{P}_{2,M+1} = \binom{M+2}{2}.$$

The remaining cases follow in the same way with the table given now

$$\begin{array}{cccccccccc} \alpha_k: & M+2, & M+1, & M+1, & M, & \cdots & d_2+1, & d_2+1, & d_2+1, & \cdots & d_2+1; \\ \beta_k: & 2, & 2, & 3, & 3, & \cdots & d_1, & d_1+1, & d_1+2, & \cdots & d_2-2. \end{array}$$

Note that we get an extra term as we have an extra polynomial h_{M-2}. Again if $d_1 = 2$, then the minimum is given by $\binom{M+1}{2}$ and if $d_1 \geqslant 3$, then the minimum is given by $\binom{d_2+1}{3}$. Now when using the method 2 for the last case $k = M-1$, we first get codim $Y^*(0) \geqslant \frac{1}{2}M^2 + 1$, so that in the notations of the proof of Proposition 3.3.5 we have possible values $b = 1, \ldots, M-2$. For $b < M-2$ we consider good sequences and get that:

$$\operatorname{codim}(Y^*(b) \subset \mathcal{L}) \geqslant (M-b) \cdot \left(\sum_{j=1}^{b} \deg h_j + \deg h_{M-1} - b \right) + 1.$$

It follows easily that

$$\operatorname{codim}(Y^*(b) \subset \mathcal{L}) \geqslant (M-1-b) \cdot \left(\sum_{j=1}^{b} \deg h_j + \deg h_{M-2} - b \right) + 1,$$

for $b=1,\ldots,M-3$. For $b=M-2$ we now get

$$\operatorname{codim}(Y^*(M-2)\subset\mathcal{L})\geqslant\sum_{i=1}^{M-1}\binom{m_i+2}{2}-3(M-2),$$

and again see the estimate in the case (R1) works here also. Q.E.D.

We are left with the remaining two cases to consider now, that is, (R2.1) and (R3.1).

Proposition 3.3.8. *The codimension of the set of complete intersections with quadratic singularities of rank at most 8, that is, the set $\mathcal{P}\setminus\mathcal{P}_{(R2.1)}$ in $\mathcal{P}$ is at least $\binom{M-5}{2}+1$.*

Proof. Without loss of generality assume $q_{1,1}\neq 0$ and $q_{2,1}=\lambda q_{1,1}$ with $\lambda\in\mathbb{C}$. The rank of the quadratic point is then given by the rank of the quadratic form $(q_{2,2}-\lambda q_{1,2})$. The result is due now to well known results on the codimension of quadrics of rank at most k (here $k=8$), see, for instance, [17], where a similar computation has been done for Fano hypersurfaces. Q.E.D. for Proposition 3.3.8.

Proposition 3.3.9. *The codimension of the set violating the condition (R3.1), that is the set $\mathcal{P}\setminus\mathcal{P}_{(R3.1)}$ in $\mathcal{P}$ is at least $\binom{M-9}{2}-1$.*

Proof. Here we work with the space

$$\mathcal{Q}=\mathcal{P}_{2,M+2}\times\mathcal{P}_{2,M+2}$$

of pairs of quadratic forms on $\mathbb{P}^{M+1}$ (the latter projective space interpreted as the exceptional divisor of the blow up of a point $o\in\mathbb{P}^{M+2}$). Let $(g_1,g_2)\in\mathcal{Q}$ be a pair of forms. The codimension of the closed set of quadratic forms of rank less than 5 is $\frac{(M-4)(M-3)}{2}$, so removing a closed set of that codimension we may assume that $\operatorname{rk} g_1\geqslant 5$. This means that the quadric $G_1=\{g_1=0\}$ is factorial, $\operatorname{Pic}G_1=\operatorname{Cl}G_1=\mathbb{Z}H_{G_1}$, where H_{G_1} is the class of a hyperplane section. Now for $g_2|_{G_1}$ to be non-reduced or reducible it has to split up into hyperplane sections which gives dimension $2M+4$. This has codimension $\frac{(M+2)(M-1)}{2}$ in $\mathcal{P}_{2,M-2}$. Therefore, removing a closed set of codimension $\frac{(M-4)(M-3)}{2}$, we obtain a set $\mathcal{Q}^*\subset\mathcal{Q}$ of pairs (g_1,g_2) such that the closed set $\{g_1=g_2=0\}$ is an irreducible and reduced complete intersection of codimension 2.

Let us consider the singular set of such a complete intersection, which we denote by $\operatorname{Sing}(g_1,g_2)$. Note that $\operatorname{Sing}(g_1,g_2)$ is the set of the points $p\in\{g_1=g_2=0\}$ where the Jacobian matrix of g_1 and g_2 has linearly dependent rows, that is, there exists some $[\lambda_1 : \lambda_2] \in \mathbb{P}^1$ with $p \in \mathrm{Sing}\{\lambda_1 g_1 + \lambda_2 g_2\}$ (where the symbol $\mathrm{Sing}(g)$ denotes the singular locus of the hypersurface $\{g = 0\}$). Therefore,

$$\mathrm{Sing}(g_1, g_2) \subset \bigcup_{[\lambda_1:\lambda_2]\in\mathbb{P}^1} \mathrm{Sing}\{\lambda_1 g_1 + \lambda_2 g_2\},$$

so that if

$$\mathrm{codim}(\mathrm{Sing}(g_1, g_2) \subset \{g_1 = g_2 = 0\}) \leqslant k, \tag{3.8}$$

then the line joining g_1 and g_2 in $\mathcal{P}_{2,M+2}$ meets the closed set of quadratic forms of rank at most $(k+2)$. We conclude that the set of pairs $(g_1, g_2) \in \mathcal{Q}^*$ satisfying the inequality (3.8), has codimension at least

$$\frac{(M-k+1)(M-k)}{2} - 1$$

in $\mathcal{Q}$. Putting $k = 10$ (and comparing the result with the codimension of the complement $\mathcal{Q} \setminus \mathcal{Q}^*$ obtained at the previous step), we complete the proof. Q.E.D. for Proposition 3.3.9.

Comparing the codimensions of the sets $\mathcal{P} \setminus \mathcal{P}_* \subset \mathcal{P}$ for all regularity conditions and finding the minimum completes the proof of Theorem 3.0.2.

Chapter 4

Complete Intersections of High Codimension

This chapter contains the main result of the thesis, that is, certain families of Fano complete intersections of index one are birationally superrigid and using these families we estimate the codimension of the complement of the birationally rigid varieties. The families satisfy two important conditions, they only have *correct multiquadratic singularities* (see Section 2.7) and the defining polynomials satisfy a *regularity condition* (see Section 2.8). The proof of birational superrigidity relies on the recent result of the $4n^2$ inequality for complete intersection singularities (Theorem 2.5.9 or [69]). The main part of this chapter deals with calculating the estimate of the codimension of the complement of the birationally rigid varities.

Introduction

4.0.1 Complete intersections of index one

Let $k \geqslant 20$ be a fixed integer and use the notation of Definition 1.6.5. For any integral k-uple $\underline{d} = (d_1, \dots, d_k)$, such that $2 \leqslant d_1 \leqslant \dots \leqslant d_k$ set $M = |\underline{d}| - k$, where $|\underline{d}| = d_1 + \dots + d_k$ and let

$$\mathcal{P}(\underline{d}) = \prod_{i=1}^{k} \mathcal{P}_{d_i, M+k+1}$$

be the space of k-uples of homogeneous polynomials of degree $d_1, \dots, d_k$, respectively, on the complex projective space $\mathbb{P} = \mathbb{P}^{M+k}$. Here the symbol $\mathcal{P}_{a,N}$ stands for the linear space of homogeneous polynomials of degree a in N variables, which are naturally interpreted as polynomials on $\mathbb{P}^{N-1}$. We write $\underline{f} = (f_1, \dots, f_k) \in \mathcal{P}(\underline{d})$ for an element of the space $\mathcal{P}(\underline{d})$. We set also

$$\mathcal{P}_{\mathrm{fact}}(\underline{d}) \subset \mathcal{P}(\underline{d})$$

to be the set of k-uples $\underline{f} = (f_1, \dots, f_k)$ such that the zero set

$$V(\underline{f}) = \{f_1 = \cdots = f_k = 0\} \subset \mathbb{P}$$

is an irreducible, reduced and factorial complete intersection of codimension k. Note that for any $\underline{f} \in \mathcal{P}_{\mathrm{fact}}(\underline{d})$ the projective variety $V(\underline{f})$ is a primitive Fano variety of index 1, that is,

$$\mathrm{Cl}\, V(\underline{f}) = \mathrm{Pic}\, V(\underline{f}) = \mathbb{Z}H,$$

where H is the class of a hyperplane section (this is by the Lefschetz theorem), and $K_{V(\underline{f})} = -H$.

Theorem 4.0.1. *Assume that $M \geqslant 8k \log k$. Then there exists a non-empty Zariski open subset $\mathcal{P}_{\mathrm{reg}}(\underline{d}) \subset \mathcal{P}_{\mathrm{fact}}(\underline{d})$, such that:*

(i) *for every $\underline{f} \in \mathcal{P}_{\mathrm{reg}}(\underline{d})$ the variety $V = V(\underline{f})$ is birationally superrigid,*

(ii) *the inequality*

$$\mathrm{codim}((\mathcal{P}(\underline{d}) \setminus \mathcal{P}_{\mathrm{reg}}(\underline{d})) \subset \mathcal{P}(\underline{d})) \geqslant \frac{(M-5k)(M-6k)}{2} \tag{4.1}$$

holds.

We now proceed to the explicit definitions of the Zariski open subsets $\mathcal{P}_{reg}(\underline{d})$ in $\mathcal{P}(\underline{d})$.

4.0.2 Construction of the set $\mathcal{P}_{reg}$

The first condition is that $V = V(\underline{f})$ has only *correct multiquadratic singularities* so that $\mathcal{P}_{reg}(\underline{d}) \subset \mathcal{P}_{mq}(\underline{d}) \subset \mathcal{P}_{fact}(\underline{d})$. The codimension of the set of polynomials, which don't satisfy this condition is estimated.

Theorem 4.0.2. *The following estimate holds:*

$$\mathrm{codim}((\mathcal{P}(\underline{d}) \setminus \mathcal{P}_{\mathrm{mq}}(\underline{d})) \subset \mathcal{P}(\underline{d})) \geqslant \frac{(M-4k+1)(M-4k+2)}{2} - (k-1) \tag{4.2}$$

The second condition is that the t-tuple $\underline{f}$ is N_l regular (Definition 2.8.1), with

$$N_l = \begin{cases} M - [2\log k], & \text{if } l \leqslant [2\log k] \\ M - l, & \text{if } l > [2\log k]. \end{cases}$$

(Here $[\cdot]$ means the integral part of a non-negative real number.)

Definition 4.0.3. The complete intersection $V = V(\underline{f})$, for $\underline{f} \in \mathcal{P}_{\mathrm{mq}}(\underline{d})$ is *regular*, if it is regular at every point $o \in V$, singular, or non-singular. If this is the case, we write $\underline{f} \in \mathcal{P}_{\mathrm{reg}}(\underline{d})$.

Theorem 4.0.4. *Assume that $\underline{f} \in \mathcal{P}_{\mathrm{reg}}(\underline{d})$. Then $V = V(\underline{f})$ is birationally superrigid.*

Theorem 4.0.5. *The following estimate holds:*

$$\operatorname{codim}((\mathcal{P}_{\mathrm{mq}}(\underline{d}) \setminus \mathcal{P}_{\mathrm{reg}}(\underline{d})) \subset \mathcal{P}(\underline{d})) \geqslant \frac{(M-5k)(M-6k)}{2}. \tag{4.3}$$

Proof of Theorem 4.0.1. Since the right hand side of (4.3) is obviously higher than that of (4.2), Theorem 4.0.1 follows immediately from Theorems 4.0.2, 4.0.4 and 4.0.5. Q.E.D.

The rest of this chapter is organized in the following way. In Section 4.1 Theorem 4.0.4 is shown. This is done by the technique of hypertangent divisors (Section 2.9), combined with the $4n^2$-inequality for complete intersection singularities (Theorem 2.5.9). We need to take into consideration the fact that the regularity condition holds, generally speaking, not for the whole sequence ($N_l \neq M - l$ in general), but for a shorter one, so that the resulting estimates are weaker than in [59]. However, we check that they are still sufficient for birational superrigidity. By the way, the biggest deviation from the computations in [59] is for non-singular points.

In Section 4.2 we prove Theorem 4.0.2. This is rather straightforward and done by induction on the codimension k of the complete intersection (here there is no need to assume that $k \geqslant 20$; the case $k = 2$ was done in chapter 3, $k = 1$ in [17].

In Section 4.3 we show Theorem 4.0.5. The computations needed for the proof are difficult, every effort was made to make them as clear and compact as possible. The estimates for the codimension are obtained by the *projection technique* as in Subsection 3.3.2 which was first used in [57].

4.1 Proof of birational rigidity

In this section we prove Theorem 4.0.4. First, in Subsection 4.1.1, we recall the definition of a maximal singularity and prove that the centre of a maximal singularity is of codimension at least 3. In Subsection 4.1.2 we exclude the case when the centre of the maximal singularity is not contained in the singular locus of V. In Subsection 4.1.3 we exclude the case when the centre of the maximal singularity is contained in the locus of multi-quadratic points of type 2^l. Since it follows that a mobile linear system can not have a maximal singularity, the variety V is shown to be birationally superrigid.

4.1.1 Maximal singularities.

As usual, we prove that a variety $V = V(\underline{f})$, where $\underline{f} \in \mathcal{P}_{\mathrm{reg}}(\underline{d})$, is birationally superrigid by assuming the converse and obtaining a contradiction. So fix a tuple $\underline{f} \in \mathcal{P}_{\mathrm{reg}}(\underline{d})$ and the corresponding complete intersection $V = V(\underline{f})$ and assume that V is not birationally superrigid. This implies immediately that for some mobile linear system $\Sigma \subset |nH|$ and some exceptional divisor E over V the Noether-Fano inequality

$$\mathrm{ord}_E \Sigma > n \cdot a(E)$$

is satisfied, where $a(E)$ is the discrepancy of E with respect to V. In other words, E is a *maximal singularity* of Σ (see Section 2.2). Let $B \subset V$ be the centre of E on V, an irreducible subvariety of codimension $\geqslant 2$.

Lemma 4.1.1. $\mathrm{codim}(B \subset V) \geqslant 3$.

Proof. Assume the converse: $\mathrm{codim}(B \subset V) = 2$. Then $B \not\subset \mathrm{Sing}\, V$, so that the inequality

$$\mathrm{mult}_B \Sigma > n$$

holds. Consider the self-intersection $Z = (D_1 \circ D_2)$ of the system Σ, where $D_1, D_2 \in \Sigma$ are general divisors. Obviously, $Z = \beta B + Z_1$, where $\beta > n^2$ and the effective cycle Z_1 of codimension 2 does not contain B as a component.

Let $P \subset \mathbb{P}$ be a general $(2k+1)$-subspace. Since $\mathrm{codim}(\mathrm{Sing}\, V \subset V) \geqslant 2k+2$, the intersection $V_P = V \cap P$ is non-singular. By Lefschetz, the numerical Chow group $A^2 V_P$ of codimension 2 cycles on V_P is $\mathbb{Z} H_P^2$, where H_P is the class of a hyperplane section of V_P. Setting $Z_P = Z|_P$ and $B_P = B|_P$, we obtain the inequality

$$\deg (Z_P - \beta B_P) \geqslant 0.$$

As $B_P \sim mH_P^2$ for some $m \geqslant 1$, this inequality implies that

$$\deg V \cdot (n^2 - m\beta) \geqslant 0,$$

which is impossible. Q.E.D. for the lemma.

Note that if $\mathrm{codim}(B \subset V) \leqslant 2k+1$, then B is not contained in the singular locus $\mathrm{Sing}\, V$ of the complete intersection V.

4.1.2 The non-singular case

In the notations of Subsection 2.9.1, assume that $B \not\subset \mathrm{Sing}\, V$. We want to show that this case is impossible by obtaining a contradiction. Recall Equation (2.17)

$$\frac{\mathrm{mult}_o}{\deg} Y_N > \gamma = \frac{2^{k+1}}{d} \cdot \prod_{i=k+2}^{N} \beta_i.$$

Proposition 4.1.1. *The inequality $\gamma \geqslant 1$ holds.*

Note that this claim provides the contradiction we need and excludes the non-singular case.

Proof. It is more convenient to use the equation

$$\frac{\mathrm{mult}_o}{\deg} Y_N > \frac{4}{3}(\beta(0))^{-1},$$

where

$$\beta(0) = \prod_{i=N+1}^{M} \beta_i. \tag{4.4}$$

and our proposition follows from

Lemma 4.1.2. *The inequality $\beta(0) < \frac{4}{3}$ holds.*

Proof of the lemma. Note first of all that for $j \geqslant N+1$ we have $\beta_j \leqslant 1 + \frac{1}{a}$, where $a = \left[\frac{M}{k}\right]$. Indeed, assume the converse: $\beta_{N+1} > 1 + \frac{1}{a}$. This means that for all $j = 1, \ldots, N$ the homogeneous polynomials $h_{k+1}, \ldots, h_{k+N}$ in the sequence (2.15) are some $q_{i,\alpha}$ with $\alpha < a$. Therefore,

$$N \leqslant \sharp\{q_{i,\alpha} \,|\, 1 \leqslant i \leqslant k,\, 2 \leqslant \alpha \leqslant a-1\}.$$

But the right-hand side of this inequality does not exceed $k \cdot (a-2) < M - k$. So we get:

$$M - [2 \log k] < M - k,$$

which is a contradiction.

We have shown that

$$\beta \leqslant \left(1 + \frac{1}{a}\right)^{[2 \log k]} \leqslant \left(1 + \frac{1}{a}\right)^{a/4}$$

as $M \geqslant 8k \log k$ by assumption. Therefore, $\beta < e^{1/4} < \frac{4}{3}$, as required. Q.E.D. for the lemma.

Proof of Proposition 4.1.1 is complete.

We have shown that the case $B \not\subset \operatorname{Sing} V$ is impossible.

4.1.3 The multi-quadratic case

Now let us exclude the multi-quadratic case and complete the proof of Theorem 4.0.4. Recall from subsection 2.9.2 we have

$$\frac{\mathrm{mult}_o}{\deg} Y_N > \gamma_l = \frac{4}{3} \beta(l)^{-1},$$

where

$$\beta(l) = \prod_{i=N_l+1}^{M-l} \beta_{l,i} \tag{4.5}$$

(recall that $N_l = M - 2[\log k]$ for $l \leqslant 2[\log k]$ and $N_l = M - l$, otherwise). The product (4.5) contains fewer terms than (4.4) and it is easy to see that $\beta_{l,M-l-j} = \beta_{M-j}$ for $j = 0, 1, \ldots, M - l - N_l - 1$. Therefore, $\beta(l) < \beta(0)$ for $l \geqslant 1$ and so $\gamma_l > \gamma > 1$, which gives us the desired contradiction. The multi-quadratic case is excluded.

Q.E.D. for Theorem 4.0.4.

4.2 Irreducible factorial complete intersections

In this section we prove Theorem 4.0.2. In Subsection 4.2.1 we explain the strategy of the proof and show the case of a hypersurface. After that in Subsection 4.2.2 we start the inductive part of the proof, first looking at the easier issue of complete intersections being irreducible and reduced. Finally, in Subsection 4.2.3 we complete the proof considering complete intersections with correct multi-quadratic singularities.

4.2.1 Complete intersections with correct multi-quadratic singularities

Set

$$\mathcal{P}^{\geqslant j} = \prod_{i=j}^{k} \mathcal{P}_{d_i, M+k+1}$$

to be the space of truncated tuples $(f_j, \ldots, f_k)$ and let $\mathcal{P}^{\geqslant j}_{\mathrm{mq}}$ be the set of tuples such that

$$V(f_j, \ldots, f_k) = \{f_j = \ldots = f_k = 0\} \subset \mathbb{P}$$

is an irreducible reduced complete intersection of codimension $k - j + 1$ with at most correct multi-quadratic singularities, in the sense of Definition 2.7.1 where k is replaced by $k - j + 1$. Note that $\mathcal{P}^{\geqslant 1} = \mathcal{P}(\underline{d})$ and $\mathcal{P}^{\geqslant 1}_{mq} = \mathcal{P}_{mq}(\underline{d})$. We will prove Theorem 4.0.2 by decreasing induction on $j = k, k-1, \ldots, 1$ in the following form

$$\operatorname{codim}((\mathcal{P}^{\geqslant j} \setminus \mathcal{P}^{\geqslant j}_{\mathrm{mq}}) \subset \mathcal{P}^{\geqslant j}) \geqslant \frac{(M - 4k + 1)(M - 4k + 2)}{2} - (k - 1). \tag{4.6}$$

The basis of the induction is the case of a hypersurface $V(f_k) \subset \mathbb{P}$ of degree $d_k \geqslant 25$. It is easy to calculate that the closed subset of reducible or non-reduced polynomials of degree d_k has codimension

$$\binom{M + k + d_k - 1}{d_k} - (M + k + 1)$$

in $\mathcal{P}_{d_k, M+k+1}$ (which corresponds to the case when f_k has a linear factor), and the closed subset of polynomials f_k, such that the hypersurface $V(f_k)$ has at least one singular point, which is not a quadratic singularity of rank at least 7, has codimenison

$$\frac{(M + k - 6)(M + k - 5)}{2} + 1$$

in $\mathcal{P}_{d_k, M+k+1}$ (see a similar detailed calculation in [17] for the case of rank at least 5). Therefore, the inequality (4.6) is true for $k = 1$.

Now let us proceed to the inductive argument.

4.2.2 The step of induction: irreducibility

We assume that (4.6) is shown for $j+1$. The task is, for a fixed tuple $(f_{j+1},\dots,f_k)\in\mathcal{P}^{\geqslant j+1}_{mq}$, to estimate the codimension of the set of polynomials $f_j\in\mathcal{P}_{d_j,M+k+1}$ such that $V(f_j,\dots,f_k)$ does not satisfy the required condition, that is, $(f_j,f_{j+1},\dots f_k)\notin\mathcal{P}^{\geqslant j}_{mq}$.

Let us first consider the issue of irreducibility and reducedness. Since by the inductive assumption and the Grothendieck theorem [3], $V(f_{j+1},\dots,f_k)$ is a factorial complete intersection, we have the isomorphism

$$\mathrm{Cl}\,V(f_{j+1},\dots,f_k)\cong\mathrm{Pic}\,V(f_{j+1},\dots,f_k)\cong\mathbb{Z}H,$$

where H is the class of a hyperplane section, and moreover, for every $a\in\mathbb{Z}_+$ the restriction map

$$r_a:\mathrm{H}^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(a))\to\mathrm{H}^0(V_{j+1},\mathcal{O}_{V_{j+1}}(a))$$

is surjective (where for simplicity of notation we write V_{j+1} for $V(f_{j+1},\dots,f_k)$). For $a<d_{j+1}$ it is also injective, and for $a=d_{j+1}$ we have

$$\dim\mathrm{Ker}\,r_a=\#\{i\in\{j+1,\dots,k\}\mid d_i=d_{j+1}\}.$$

Now easy calculations show that the set of polynomials $f_j\in\mathcal{P}_{d_j,M+k+1}$, such that $V(f_j,f_{j+1},\dots,f_k)$ is either reducible or non-reduced, is of codimension at least

$$\binom{M+k+d_j-1}{d_j}-(M+k+1)-(k-j)$$

(again, this corresponds to the case when the divisor $\{f_j|_{V_{j+1}}=0\}$ has a hyperplane section of V_{j+1} as a component). This estimate is higher (and, in fact, much higher) than what we need so we may assume that $V(f_j,f_{j+1},\dots,f_k)$ is irreducible and reduced.

Finally, we need to consider the condition for the singularities of the complete intersection $V(f_j,f_{j+1},\dots,f_k)$ to be multi-quadratic. In order to avoid cumbersome formulae, we will consider the final case $j=1$ only, when the estimate is the weakest. For higher values of j the arguments are identical, just the indices and dimensions need to be adjusted appropriately.

4.2.3 Multi-quadratic singularities.

Fix a point $o\in\mathbb{P}$ and consider a tuple $(f_1,\dots,f_k)\in\mathcal{P}^{\geqslant 1}$ with $o\in V=V(f_1,\dots,f_k)$. Fix a system of affine coordinates $(z_1,\dots,z_{M+k})$ on an affine chart $\mathbb{C}^{M+k}\subset\mathbb{P}$ with the origin at the point o. Write the corresponding dehomogenized polynomials (denoted by the same symbols) in the form

$$f_1 = q_{1,1} + q_{1,2} + \cdots + q_{1,d_1},$$
$$\cdots$$
$$f_k = q_{k,1} + q_{k,2} + \cdots + q_{k,d_k},$$

where $q_{i,j}$ is a homogeneous polynomial in z_* of degree j. Assume that

$$\dim\langle q_{1,1}, \ldots, q_{k,1}\rangle = k - l,$$

with $l \geqslant 0$. Let $I \subset \{1, \ldots, k\}$ be a subset with $|I| = k - l$ such that the linear forms $\{q_{i,1} \mid i \in I\}$ are linearly independent. Set $\Pi \subset \mathbb{C}^{M+k}$ to be the subspace

$$\Pi = \{q_{i,1} = 0 \mid i \in I\} \cong \mathbb{C}^{M+l}.$$

By assumption, for every $j \in J = \{1, \ldots, k\} \setminus I$ there are (uniquely determined) constants $\beta_{j,i}$, $i \in I$, such that

$$q_{j,1} = \sum_{i \in I} \beta_{j,i} q_{i,1}.$$

Set for every $j \in J$

$$q^*_{j,2} = \left(q_{j,2} - \sum_{i \in I} \beta_{j,i} q_{i,2} \right) \Bigg|_{\Pi}.$$

The following statement translates the condition for the point o to be a correct multi-quadratic singularity into the language of properties of the quadratic forms $q^*_{j,2}$ introduced above.

Proposition 4.2.1. *Assume that for a general subspace $\Theta \subset \mathbb{P}(\Pi)$ of dimension*

$$\max\{k + l + 1, 4l + 2\}$$

the set of quadratic equations

$$\left\{ q^*_{j,2}|_{\Theta} = 0 \mid j \in J \right\}$$

defines a non-singular complete intersection of type 2^l. Then $o \in V$ is a correct multi-quadratic singularity of type 2^l.

Proof. Indeed, it is easy to see that the germ $o \in V$ is analytically equivalent to the closed set in Π defined by l equations

$$0 = q^*_{j,2} + \ldots, \quad j \in J,$$

where the dots stand for higher order terms. The rest is obvious. Q.E.D.

Remark 4.2.2. In the notations of Definition 2.7.1, the exceptional divisor Q_P is precisely the complete intersection of l quadrics $\{q^*_{j,2}|_\Theta = 0\}$, $j \in J$, in the $\max\{k+l+1, 4l+2\}$-dimensional space Θ. Proposition 4.2.1 gives a sufficient condition for the point o to be a correct multi-quadratic singularity. Now we use this criterion to estimate the codimension of the set of tuples violating the conditions of Definition 2.7.1 at the point $o \in V$.

Definition 4.2.3. We say that an l-uple $(q^*_{j,2}|_\Theta \mid j \in J)$ is *correct*, if its zero set in $\mathbb{P}(\Pi)$ is an irreducible reduced complete intersection Q_Π satisfying the inequality

$$\operatorname{codim}(\operatorname{Sing} Q_\Pi \subset Q_\Pi) \geqslant b = \max\{k+l+1, 4l+2\}.$$

Corollary 4.2.4. *Assume that the l-uple $(q^*_{j,2}|_\Theta \mid j \in J)$ is correct. Then $o \in V$ is a correct multi-quadratic singularity of type 2^l.*

Since in the subsequent arguments (up to the end of this subsection) only the quadratic forms $q_{i,2}$ will be involved, we may assume without loss of generality that

$$J = \{1, \dots, l\}$$

and $I = \{l+1, \dots, k\}$. Fixing the forms $q_{i,2}$ for $i \in I$, we work with the l-uples

$$(q^*_{j,2} \,|\, j = 1, \dots, l) \in \mathcal{P}^{\times l}_{2,M+l}.$$

Theorem 4.0.2 is obviously implied by the following proposition.

Proposition 4.2.5. *The codimension of the closed set $\mathcal{X} \subset \mathcal{P}^{\times l}_{2,M+l}$ of incorrect l-uples is at least*

$$\frac{(M+3-b)(M+4-b)}{2} - (l-1).$$

(Recall that $b = \max\{k+l+1, 4l+2\}$, see Definition 4.2.3.)

Proof. Elementary computations show that the codimension of the closed subset $\mathcal{X}_* \subset \mathcal{P}^{\times l}_{2,M+l}$ of l-uples, such that their zero set is not an irreducible reduced complete intersection of codimension l, is equal to

$$\frac{(M+l-1)(M+l-2)}{2}.$$

Therefore, estimating the codimension of $\mathcal{X}$, we may consider only l-uples such that their zero set Q_Π is an irreducible reduced complete intersection; in particular, the l quadratic forms are linearly independent. For

$$\lambda = (\lambda_1 : \dots : \lambda_l) \in \mathbb{P}^{l-1}$$

set

$$W(\lambda) = \{\lambda_1 q^*_{1,2} + \cdots + \lambda_l q^*_{l,2} = 0\} \subset \mathbb{P}^{M+l-1}$$

to be the corresponding quadric hypersurface in the linear system generated by $(q^*_{j,2})$. We will use the following simple observation, which for $k = 2$ was used in the proof of Proposition 3.3.9.

Lemma 4.2.6. *For any point $p \in \operatorname{Sing} Q_\Pi$ there is $\lambda \in \mathbb{P}^{l-1}$ such that $p \in \operatorname{Sing} W(\lambda)$.*

Proof. Obvious computations. Q.E.D.

Corollary 4.2.7. *The following inclusion holds:*

$$\operatorname{Sing} Q_\Pi \subset \bigcup_{\lambda \in \mathbb{P}^{l-1}} \operatorname{Sing} W(\lambda).$$

Set $\mathcal{R}_{\leqslant a} \subset \mathcal{P}_{2,M+l}$ to be the closed subset of quadratic forms of rank $\leqslant a$. It is well known that

$$\operatorname{codim} \mathcal{R}_{\leqslant a} \subset \mathcal{P}_{2,M+l} = \frac{(M+l+1-a)(M+l+2-a)}{2}.$$

Now for every $e = 1, \ldots, l$ consider the closed subset $\mathcal{X}_{e,a} \subset \mathcal{P}^{\times e}_{2,M+l}$, consisting of e-uples $(g_1, \ldots, g_e)$ such that the linear span $\langle g_1, \ldots, g_e \rangle$ has a positive-dimensional intersection with $\mathcal{R}_{\leqslant a}$.

Lemma 4.2.8. *The following estimate holds:*

$$\operatorname{codim}(\mathcal{X}_{e,a} \subset \mathcal{P}^{\times e}_{2,M+l}) \geqslant \operatorname{codim}(\mathcal{R}_{\leqslant a} \subset \mathcal{P}_{2,M+l}) - (e-1).$$

Proof. Consider the natural projections

$$\begin{tikzcd}[column sep=small] & \mathcal{P}^{\times e-1}_{2,M+l} \times \mathcal{P}_{2,M+l} \arrow[dl, "\Phi"'] \arrow[dr, "\varphi"] & \\ \mathcal{P}^{\times e-1}_{2,M+l} & & \mathcal{P}_{2,M+l} \end{tikzcd}$$

Take the minimal e such that $\langle g_1, \ldots, g_e \rangle$ intersects $\mathcal{R}_{\leqslant a}$ but $\langle g_1, \ldots, g_{e-1} \rangle$ does not intersect $\mathcal{R}_{\leqslant a}$. This implies g_e belongs to the cone with vertex $\langle g_1, \ldots, g_{e-1} \rangle$ and base $\mathcal{R}_{\leqslant a}$, by definition $\langle g_1, \ldots, g_e \rangle \in \mathcal{X}_{e,a}$ but $\langle g_1, \ldots, g_{e-1} \rangle \notin \mathcal{X}_{e-1,a}$. The cone C has dimension $\dim \mathcal{R}_{\leqslant a} + (e-1)$, by taking a fibre of the projection φ and by a dimension count the following inequality holds

$$\dim(\mathcal{X}_{e,a} \subset \mathcal{P}^{\times e}_{2,M+l}) \leqslant \dim C + \dim \mathcal{P}^{\times (e-1)}_{2,M+l}.$$

The required inequality now follows, Q.E.D. for the lemma.

Now we can complete the proof of Proposition 4.2.5. By Definition 4.2.3, the l-uple $(q^*_{j,2}\,|\,j=1,\dots,l)$ is not correct when

$$\operatorname{codim}(\operatorname{Sing} Q_\Pi \subset Q_\Pi) \leqslant b-1$$

or, equivalently, if

$$\dim(\operatorname{Sing} Q_\Pi) \geqslant M+l-b.$$

By Corollary 4.2.7 we conclude that for an incorrect tuple $(q^*_{1,2},\dots,q^*_{1,l})$ the inequality

$$\max_{\lambda\in\mathbb{P}^{l-1}}\{\dim \operatorname{Sing} W(\lambda)\} \geqslant M+1-b$$

is satisfied. In its turn, this implies that

$$(g_1,\dots,g_e)\notin \mathcal{X}_{l,a}$$

for $a=l+b-2$. Now lemma 4.2.8 completes the proof of Proposition 4.2.5 Q.E.D. for the proposition.

This completes the proof of Theorem 4.0.2 as well, as the minimum of the estimate obtained in Proposition 4.2.5 occurs for $l=k$.

4.3 Regular complete intersections

In this section we prove Theorem 4.0.5. In Subsection 4.3.1 we produce the estimates for the set of non-regular tuples of polynomials given by the projection method. After that, the proof of Theorem 4.0.5 is reduced to showing a purely analytical fact: estimating the minimum of an integral sequence, consisting of certain binomial coefficients, depending on several integral parameters. The required computations are non-trivial. We perform them in several steps. In Subsection 4.3.2 a number of reductions simplify the task. In Subsection 4.3.3 we employ the classical Stirling formula to approximate, with good precision the expressions to be minimized by a smooth function and study that function using the standard tools of calculus. In Subsections 4.3.4 and 4.3.5 we complete the proof showing the required estimates.

4.3.1 The projection method.

We use the notations of Section 2.8. Since an elementary dimension count relates the codimension of the set of globally non-regular tuples $\underline{f}$ (which is what Theorem

4.0.5 estimates) to the codimension of the set of tuples $\underline{f}$ that are non-regular at a fixed point $o \in V(\underline{f})$ (see Theorem 4.3.1 and the comments below), we concentrate on the local problem: fix a point $o \in \mathbb{P}$, a system of affine coordinates $z_1, \ldots, z_{M+k}$ with the origin at o and consider (non-homogeneous) tuples $\underline{f}$ such that $o \in V(\underline{f})$.

Next, we fix $l \in \{0, 1, \ldots, k\}$ and assume that the rank of the set of linear forms $q_{i,1}$, $i = 1, \ldots, k$, is equal to $k - l$, so that in the sequence (2.15) exactly the first $k - l$ polynomials are linear forms. We fix them, so that the linear subspace

$$\Pi = \{h_1 = \ldots = h_{k-l} = 0\} \cong \mathbb{C}^{M+l}$$

of the space $\mathbb{C}^{M+k}_{z_*}$ is also fixed. Recall the notation

$$N_l = M - \max\{[2 \log k], l\},$$

introduced in Subsection 4.0.2. Set

$$g_i = h_{k-l+i}|_{\mathbb{P}(\Pi)},$$

$i = 1, \ldots, N_l$. This is a sequence of N_l homogeneous polynomials of non-decreasing degrees $m_i = \deg g_i$ on the projective space $\mathbb{P}(\Pi) \cong \mathbb{P}^{M+l-1}$. Define the space of such sequences:

$$\mathcal{G}(\underline{d}, l) = \prod_{i=1}^{N_l} \mathcal{P}_{m_i, M+l}.$$

It is obvious that the point $o \in V$ is regular (as a multi-quadratic point of type 2^l in the sense of Definition 2.8.1) if and only if the sequence

$$g_1, \ldots, g_{N_l}$$

is regular, that is to say, if the closed algebraic set

$$\{g_1 = \cdots = g_{N_l} = 0\} \subset \mathbb{P}(\Pi)$$

has codimension N_l. Set $\mathcal{Y} = \mathcal{Y}(\underline{d}, l) \subset \mathcal{G}(\underline{d}, l)$ to be the closed set of *non*-regular tuples.

Theorem 4.3.1. *Assume that* $M \geqslant 8k \log k$ *and* $k \geqslant 20$. *Then*

$$\operatorname{codim}(\mathcal{Y} \subset \mathcal{G}(\underline{d}, l)) \geqslant \frac{(M - 5k)(M - 6k)}{2} + M + k$$

Taking into account that the point o varies in $\mathbb{P}$ and the original tuple $\underline{f}$ satisfies the conditions $f_1(o) = \cdots = f_k(o) = 0$ and $\dim\langle q_{i,1} \mid 1 \leqslant i \leqslant k\rangle = k - l$, an elementary dimension count gives Theorem 4.0.5 as an immediate corollary of Theorem 4.3.1.

The rest of this section is the **proof of Theorem 4.3.1**. Our main tool is *the projection method* used in Subsection 3.3.2, the idea is to represent

$$\mathcal{Y} = \coprod_{e=1}^{N_l} \mathcal{Y}_e$$

as a disjoint union of subsets $\mathcal{Y}_e$, consisting of tuples $(g_1, \dots, g_{N_l})$ such that the closed set

$$\{g_1 = \dots = g_{e-1} = 0\} \subset \mathbb{P}(\Pi)$$

is of codimension $e-1$ and g_e vanishes on some irreducible component of that set (if $e = 1$, this means simply that the quadratic form g_1 is identically zero). The projection method estimates the codimension of $\mathcal{Y}_e$ in $\mathcal{G}(\underline{d}, l)$ as follows:

$$\operatorname{codim}(\mathcal{Y}_e \subset \mathcal{G}(\underline{d}, l) \geqslant \gamma(e, \underline{d}, l) = h^0(\mathbb{P}^{M+l-e}, \mathcal{O}_{\mathbb{P}^{M+l-e}}(m_e)) = \binom{M+l-e+m_e}{M+l-e},$$

where $m_e = \deg g_e$, see Subsection 3.3.3 or [53, Chapter 3]. Therefore, in order to prove Theorem 4.3.1, we must show that the numbers $\gamma(e, \underline{d}, l)$ for $e = 1, \dots, N_l$ are not smaller than the right hand side of the inequality of Theorem 4.3.1. This is what we are going to do. The task is non-trivial. First, we do some preparatory work in order to simplify the inequalities to be shown and reduce the number of integral parameters on which the numbers $\gamma(e, \underline{d}, l)$ depend.

4.3.2 Reductions

If the original tuple $\underline{f}$ of defining polynomials consists of k_2 quadrics, k_3 cubics, ..., k_m polynomials of degree $m = d_k \geqslant 8 \log k + 1$, then

$$k_2 + k_3 + \dots + k_m = k$$

and

$$2k_2 + 3k_3 + \dots + mk_m = |\underline{d}| = d_1 + \dots + d_k.$$

It is easy to see that

$$m_e = \deg g_e = \min \left\{ j \;\middle|\; \sum_{\alpha=2}^{j} \left(\sum_{\beta=\alpha}^{m} k_\beta \right) \geqslant e \right\}.$$

This explicit presentation gives us the first reduction.

Proposition 4.3.2. *The following estimate holds:*

$$\gamma(e, \underline{d}, l) \geqslant \gamma(e, \underline{d}^*, l),$$

where the k*-uple* $\underline{d}^* = (d_1^*, \ldots, d_k^*)$ *is defined by the equalities*

$$d_1^* = \ldots = d_r^* = a + 1, \qquad d_{r+1}^* = \ldots = d_k^* = a + 2 \tag{4.7}$$

and $M = ka + (k - r)$*, where* $0 \leqslant r \leqslant k - 1$.

Proof. Explicitly, the proposition states that

$$\binom{M + l - e + m_e}{M + l - e} \geqslant \binom{M + l - e + m_e^*}{M + l - e},$$

where m_e^* is calculated for the tuple $\underline{d}^*$. It is easy to see that $m_e \geqslant m_e^*$, which proves the proposition. Q.E.D.

The second reduction simplifies the situation further, allowing us to consider only the case when all degrees d_i are equal.

Proposition 4.3.3. *For the tuple* $\underline{d}^+ = (d_1^+, \ldots, d_k^+)$ *such that* $d_1^+ = \cdots = d_k^+$*, with* $M^+ + k = |\underline{d}^+|$ *and* $M^+ \geqslant 8k \log k - k$ *the estimate*

$$\gamma(e, \underline{d}^+, l) \geqslant \frac{(M^+ - 4k)(M^+ - 5k)}{2} + M^+ + 2k$$

holds for all $e = 1, \ldots, N_l^+ = M^+ - \max\{[2 \log k], l\}$.

Let us show that Theorem 4.3.1 follows from Propositions 4.3.2 and 4.3.3.

Indeed, by Proposition 4.3.2 it is sufficient to prove the inequality

$$\gamma(e, \underline{d}^*, l) \geqslant \frac{(M - 5k)(M - 6k)}{2} + M + k$$

for $e = 1, \ldots, N_l$. Let us consider the tuple $\underline{d}^+$ with

$$d_1^+ = \cdots = d_k^+ = a + 1$$

for the constant a defined in Proposition 4.3.2. Set $M^+ = ka$. Obviously, $\gamma(e, \underline{d}^*, l) \geqslant \gamma(e, \underline{d}^+, l)$ for $e = 1, \ldots, N_l^+$ as $M \geqslant M^+$. If $N_l > N_l^+$, then for $i = 0, \ldots, N_l - N_l^+ - 1$ we have a similar estimate "from the other end":

$$\gamma(N_l - i, \underline{d}^*, l) \geqslant \gamma(N_l^+ - i, \underline{d}^+, l)$$

(note that $N_l - N_l^+ = M - M^+ \leqslant k$). Therefore,

$$\gamma(\underline{d}^*, l) = \min_{1 \leqslant e \leqslant N_l} \{\gamma(e, \underline{d}^*, l)\} \geqslant \gamma(\underline{d}^+, l) = \min_{1 \leqslant e \leqslant N_l^+} \{\gamma(e, \underline{d}^+, l)\},$$

and applying Proposition 4.3.3 and taking into account that $M^+ \geqslant M - k$, we get the claim of Theorem 4.3.1.

The third reduction allows us to remove the integral parameter $l \in \{0, 1, \dots, k\}$. In order to simplify our notations, we write d_i for d_i^+, thus assuming that $d_1 = \dots = d_k = a + 1$, so that $M = ka$. We use the notation $\gamma(\underline{d}, l)$ for the minimum of the numbers $\gamma(e, \underline{d}, l)$, $e = 1, \dots, N_l$, introduced above.

Proposition 4.3.4. *The following inequality holds:*

$$\gamma(\underline{d}, l) \geqslant \gamma(\underline{d}, 0)$$

for all $l = 0, 1, \dots, k$.

Proof. Since for $l \geqslant 1$ we have $N_0 > N_l$. It is sufficient to compare the integers $\gamma(e, \underline{d}, l)$ and $\gamma(e, \underline{d}, 0)$ for the same values of $e = 1, \dots, N_l$. They are

$$\binom{M + l - e + m_e}{M + l - e} \quad \text{and} \quad \binom{M - e + m_e}{M - e},$$

so the claim becomes obvious. Q.E.D.

Remark 4.3.5. We could as well do the third reduction as the first one: show that the minimum of the integers $\gamma(e, \underline{d}, l)$ is attained for $l = 0$ (which corresponds to regular non-singular points of V), and after that prove that the worst estimates correspond to the case (4.7).

The last (fourth) reduction makes the computations more compact. Recall that now all degrees d_i are equal to $a + 1$. Introduce the integer-valued function $\beta\colon \{2, \dots, a\} \to \mathbb{Z}_+$ by the formula

$$\beta(t) = \binom{k(a - t + 1) + t}{t} = \binom{kb(t) + t}{t},$$

where $b(t) = (a - t + 1)$. Set also

$$\alpha = \alpha(M, k) = \binom{a + 1 + [2 \log k]}{a + 1}.$$

Proposition 4.3.6. *The following estimate holds:*

$$\gamma(\underline{d}, 0) \geqslant \min\left\{ \min_{t \in \{2, \dots, a\}} \{\beta(t)\}, \alpha \right\}.$$

Proof. This follows immediately from the fact that for the special tuple $\underline{d}$ of equal degrees

$$m_{ki+1} = m_{ki+2} = \dots = m_{ki+k} = i + 2$$

for $i = 0, \ldots, a-1$. Q.E.D.

Therefore, the statement of Theorem 4.3.1 is implied by the following facts. In both propositions below we assume that $M \geqslant 8k\log k - k$ and $k \geqslant 20$.

Proposition 4.3.7. *The minimum of the function $\beta(t)$ on the set $\{2, 3, \ldots, a\}$ is attained at $t = 2$.*

Proposition 4.3.8. *The following inequality holds:*

$$\alpha(M, k) \geqslant A(M, k) = \frac{(M-4k)(M-5k)}{2} + (M+2k).$$

Remark 4.3.9. The proof of Proposition 4.3.7 only requires $k \geqslant 10$, it is Proposition 4.3.8 that requires $k \geqslant 20$. To obtain a lower bound of k in Proposition 4.3.8 one can change the function $A(M,k)$, which would give a weaker estimate in Theorem 4.3.1 and Theorem 4.0.1.

The rest of this section is a proof of the last two propositions, which requires analytic arguments.

4.3.3 The Stirling formula

The strategy of the proof of Proposition 4.3.7 is as follows. Using the Stirling formula, we construct a smooth function $\varepsilon \colon \mathbb{R}_+ \to \mathbb{R}$ such that $\varepsilon(t) \leqslant \beta(t)$ for $t = 2, \ldots, a$ and ε approximates β with a good precision. Then we show that the minimum of the function $\varepsilon(t)$ on the interval $[2, a]$ occurs at one of the end points $t = 2$ and $t = a$. From this we deduce the claim of Proposition 4.3.7.

Recall that by the Stirling formula

$$n! = \sqrt{2\pi n} n^n \exp(-n) \exp\left(\frac{\theta_n}{12n}\right)$$

for some θ_n between 0 and 1. The integral parameter e, enumerating the polynomials g_e, will not be used again in this paper, so we use the symbol e for the number $\exp(1)$. Set

$$\varepsilon(t) = \frac{\sqrt{2\pi}}{e^2}(kb(t)+t)^{(kb(t)+t(t)+\frac{1}{2})}(kb(t))^{-(kb(t)+\frac{1}{2})}t^{-(t+\frac{1}{2})},$$

by the Stirling formula $\beta(t) \geqslant \varepsilon(t)$.

Lemma 4.3.10. *The smooth function $\varepsilon(t)$ for $k \geqslant 3$ has only one critical point on the interval $[2, a]$, which is a maximum, so that the minimum of that function is attained at one of the end points.*

Proof. This is shown by demonstrating that

(1) for $2 \leqslant t \leqslant \frac{M+k}{2k}$ the function $\log \varepsilon(t)$ is strictly increasing,

(2) for $\frac{M+1}{k+1} \leqslant t \leqslant \frac{M}{k}$ it is strictly decreasing,

(3) for $\frac{M+k}{2k} \leqslant t \leqslant \frac{M+1}{k+1}$ the second derivative of $\log \varepsilon(t)$ is strictly negative (this is where the maximum lies).

The first derivative $\frac{d}{dt}\log\varepsilon(t)$ is equal to

$$\frac{t^2 - kb(t)^2}{2b(t)t(kb(t)+t)} - k\log\left(1 + \frac{t}{kb(t)}\right) + \log\left(1 + \frac{kb(t)}{t}\right), \tag{4.8}$$

the second derivative $\frac{d^2}{dt^2}\log\varepsilon(t)$ is given by the formula

$$\frac{1}{b(t)t} + \frac{(t^2 - kb(t))^2)^2}{2b(t)^2t^2(kb(t)+t)^2} + \frac{(k-1)(t^2 - kb(t)^2)}{b(t)t(kb(t)+t)^2} - \frac{k(t+b(t))^2}{tb(t)(kb(t)+t)}. \tag{4.9}$$

We present the derivatives in these forms in order to use the inequality

$$\left|\frac{t^2 - kb(t)^2}{2b(t)t(kb(t)+t)}\right| \leqslant \frac{1}{2b(t)}. \tag{4.10}$$

Now let us consider the domains (1)-(3) separately.

(1) Assume that $2 \leqslant t \leqslant \frac{M+k}{2k}$. Note that on this interval $b(t) \geqslant 2$ so that

$$\left|\frac{t^2 - kb(t)^2}{2b(t)t(kb(t)+t)}\right| \leqslant \frac{1}{4}.$$

The last term in the expression (4.8) can be estimated as

$$\log\left(1 + \frac{kb(t)}{t}\right) \geqslant \log(1+k) \geqslant \log 4,$$

since on the interval $[2, \frac{M+k}{2k}]$ we have $t \leqslant b(t)$. Finally, for the second term in (4.8) we get

$$-k\log\left(1 + \frac{t}{kb(t)}\right) \geqslant -\frac{t}{b(t)} \geqslant -1.$$

Combining these estimates, we obtain the inequality

$$\frac{d}{dt}\log(\varepsilon(t))\bigg|_{2\leqslant t\leqslant \frac{M+k}{2k}} \geqslant -\frac{5}{4} + \log 4 > 0,$$

so that indeed $\varepsilon(t)$ is increasing on the interval under consideration.

(2) Assume now that $\frac{M+1}{k+1} \leqslant t \leqslant \frac{M}{k}$. Here $t \geqslant kb(t)$, which gives the inequality

$$\left|\frac{t^2 - kb(t)^2}{2b(t)t(kb(t)+t)}\right| \leqslant \frac{1}{2}.$$

For the other two terms in the expression (4.8) we get the estimates

$$-k\text{log}\left(1+\frac{t}{kb(t)}\right) \leqslant -k\log 2$$

and

$$\log\left(1+\frac{kb(t)}{t}\right) \leqslant \text{log}2.$$

Combining these inequalities, we see that

$$\frac{d}{dt}\text{log}(\varepsilon(t)) \leqslant \frac{1}{2} - 2\log 2 < 0$$

for $t \in [\frac{M+1}{k+1}, \frac{M}{k}]$ as we claimed above.

(3) Finally, assume that $\frac{M+k}{2k} \leqslant t \leqslant \frac{M+1}{k+1}$. On this interval $b(t) \leqslant t \leqslant kb(t)$. Let us show that the second derivative (4.9) is negative. Using again the inequality (4.10), we get that $\frac{d^2}{dt^2}\text{log}\varepsilon(t)$ on the interval under consideration is not higher than

$$\frac{1}{b(t)t} + \frac{1}{2(b(t))^2} + \frac{(k-1)}{b(t)(kb(t)+t)} - \frac{k(t+b(t))^2}{tb(t)(kb(t)+t)} =$$

$$= \frac{t^2 + kb(t)(-2t^2 - 4b(t)t + 3t - 2b(t)^2 + 2b(t))}{2tb(t)^2(kb(t)+t)}.$$

Elementary computations, together with the inequality $t \leqslant a$, show that the expression in brackets in the numerator is not higher than $-2a^2 - a$. Therefore, the whole numerator is not higher than

$$t^2 - kb(t)(2a^2 + a) \leqslant kb(t)(t - 2a^2 - a) < 0.$$

We have shown that $\frac{d^2}{dt^2}\text{log}\varepsilon(t) < 0$ for $t \in [\frac{M+k}{2k}, \frac{M+1}{k+1}]$. This completes the proof of Lemma 4.3.10 Q.E.D.

4.3.4 Proof of Proposition 4.3.7.

In view of the inequality $\varepsilon(t) \leqslant \beta(t)$ and Lemma 4.3.10, Proposition 4.3.7 follows from the two lemmas stated below.

Lemma 4.3.11. *The inequality* $\beta(2) \leqslant \varepsilon(3)$ *holds.*

Lemma 4.3.12. *The inequality* $\beta(2) \leqslant \varepsilon(a)$ *holds.*

Proof of Lemma 4.3.11. We need to estimate the error in Stirling's approximation in order to be able to use $\beta(3)$ instead of $\varepsilon(3)$. The number $\beta(3)$ is a polynomial in M, k, which makes the task easier. From the Stirling formula we get:

$$1.126 \cdot \varepsilon(3) \leqslant \beta(3) \leqslant 1.132 \cdot \varepsilon(3).$$

The inequality of the lemma will follow if it is shown that $1.14 \cdot \beta(2) < \beta(3)$ and this is equivalent to the inequality $G_1(M,k) = 6(\beta(3) - 1.14 \cdot \beta(2)) > 0$. Here $G_1(M,k)$ is given explicitly by the expression

$$M^3 + M^2(2.58 - 6k) + M(12k^2 - 17.16k + 0.74) - 8k^3 + 20.58k^2 - 11.74k - 0.84.$$

It is easy to check that $G_1(8k\log k - k, k)$ and the partial derivative $\frac{\partial}{\partial M}G_1(M,k)$ are both positive for $k \geqslant 20$ and $M \geqslant 8k\log k - k$. This completes the proof. Q.E.D.

Proof of Lemma 4.3.12. The claim of the lemma is equivalent to the inequality

$$G_2(M,k) = \log\varepsilon(a) - \log\beta(2) \geqslant 0.$$

A direct calculation gives $G_2(160\log(20) - 20, 20) > 0$. Set

$$G_3(t) = \frac{d}{dt}G_2(8t\log t - t, t).$$

Lemma 4.3.13. $G_3(t) > 0$ *for* $t \geqslant 20$.

Proof. Explicitly,

$$G_3(t) = \log\left(1 + \frac{8\log(t) - 1}{t}\right) + \frac{8}{t}\log\left(1 + \frac{t}{8\log(t) - 1}\right) - \frac{1}{2t} + H_1(t) + H_2(t),$$

where

$$H_1(t) = \left(\frac{8}{t} + 1\right)\frac{8\log t + t - 0.5}{8\log t + t - 1} - \left(\frac{8}{t}\right)\frac{8\log t - 0.5}{8\log t - 1} - 1,$$

$$H_2(t) = -(8\log t + 6)\left(\frac{1}{8t\log t - 2t + 2} + \frac{1}{8t\log t - 2t + 1}\right).$$

Using the power series expansion of $\log(1+x)$, we obtain the inequality

$$G_3(t) > -\frac{1}{2t} + \frac{8\log(t) - 1}{t} - \frac{(8\log(t) - 1)^2}{2t^2} + \frac{8}{t}\log\left(1 + \frac{t}{8\log(t) - 1}\right) + H_1(t) + H_2(t).$$

For $t \geqslant 20$ then $H_1(t) \geqslant 0$ and $H_2(t) \geqslant -\frac{4}{t}$, which can be checked directly, giving the inequality

$$G_3(t) > \frac{16\log t - 11}{2t} - \frac{(8\log(t) - 1)^2}{2t^2} + \frac{8}{t}\log\left(1 + \frac{t}{8\log(t) - 1}\right).$$

The right hand side of the last inequality is higher than

$$\frac{1}{2t^2}(16t\log t - 11t - 64\,(\log t)^2 + \log t - 1) > 0$$

which is positive for $t \geqslant 20$. Q.E.D. for Lemma 4.3.13.

We conclude that $G_2(8t\log(t)-t,t)>0$ for $t\geqslant 20$. The claim of Lemma 4.3.12 will be proven if we show that for $k\geqslant 20$ and $M\geqslant 8k\log(k)-k$ the function $G_2(M,k)$ is an increasing function of M. Set

$$G_4(s,t)=\frac{\partial}{\partial s}G_2(s,t).$$

Lemma 4.3.14. $G_4(s,t)>0$ *for* $t\geqslant 20, s\geqslant 8t\log(t)-t$.

Proof. Explicitly,

$$G_4(s,t)=\frac{1}{t}\log\left(1+\frac{t^2}{s}\right)-\frac{t^2}{2s(t^2+s)}-\frac{2s+3-2t}{s^2+(3-2t)s+t^2-3t+2}.$$

First we consider the case when $s\leqslant t^2$ and get

$$G_4|_{s\leqslant t^2}\geqslant\frac{1}{t}\log(2)-\frac{t^2}{2s(t^2+s)}-\frac{2s+3-2t}{s^2+(3-2t)s+t^2-3t+2}.$$

It is easy to see that the minimum of the right hand side occurs when $s=8t\log(t)-t$ is the smallest possible so that for $s\leqslant t^2$ the function $G_4(s,t)$ is bounded from below by the expression

$$\frac{1}{t}\log(2)-\frac{1}{(16\log(t)-2)(t+8\log(t)-1)}$$
$$-\frac{16t\log t+3-4t}{t^2(8\log t-1)^2+(3-2t)(8t\log(t)-t)+t^2-3t+2},$$

which is positive for $t\geqslant 20$.

Now let us consider the region $s\geqslant t^2$. Here we get

$$G_4(s,t)\geqslant\frac{t}{s}-\frac{t^3}{2s^2}-\frac{t^2}{2s(t^2+s)}-\frac{2s+3-2t}{s^2+(3-2t)s+t^2-3t+2}.$$

A direct check shows that for $t\geqslant 20$ the expression in the right hand side is positive. Q.E.D. for Lemmas 4.3.14, 4.3.12 and Proposition 4.3.7.

4.3.5 Proof of Proposition 4.3.8

This proof is obtained in the same way as that of Proposition 4.3.7. In order to prove the inequality $\alpha(M,k)\geqslant A(M,k)$, we use the Stirling approximation of $\alpha(M,k)$. Namely, we introduce the function $G_5(s,t,r)$ of three real variables by the formula

$$G_5(s,t,r)=\left(\frac{s}{t}+r+\frac{3}{2}\right)\log\left(\frac{s}{t}+r+1\right)-\left(r+\frac{1}{2}\right)\log r-$$

$$-\left(\frac{s}{t}+\frac{3}{2}\right)\log\left(\frac{s}{t}+1\right)+\log\left(\frac{\sqrt{2\pi}}{e^2}\right)-\log A(s,t).$$

By the Stirling approximation, Proposition 4.3.8 follows from the inequality

$$G_5(M,k,[2\log k])\geqslant 0.$$

It is easy to see that

$$G_5(M,k,[2\log k])\geqslant G_5(M,k,2\log k-1),$$

so we set $G_6(s,t)=G_5(s,t,2\log t-1)$ and prove the inequality

$$G_6(s,t)\geqslant 0$$

for $s\geqslant 8t\log(t)-t$, $t\geqslant 20$. First of all, explicit computations show that

$$G_6(8t\log(t)-t,t)\geqslant 0$$

for $t=20$. Set

$$G_7(t)=\frac{d}{dt}G_6(8t\log(t)-t,t),$$

which is given by the expression

$$\frac{2}{t}\log\left(1+\frac{8\log t}{2\log t-1}\right)+\frac{8}{t}\log\left(1+\frac{2\log t-1}{8\log t}\right)+H_3(t)+H_4(t),$$

where

$$H_3(t)=\frac{1}{t}\left(\frac{5}{10\log t}-\frac{1}{2\log t-1}-\frac{1}{2\log t}\right)\geqslant-\frac{1}{t}\left(\frac{1}{18(\log t)^3}-\frac{10}{9\log t}\right),$$

for $t\geqslant 20$ and

$$H_4(t)=-\frac{80t\log t+16\log t-28t+18}{64t^2(\log t)^2-88t^2\log t+30t^2-16t\log t+2t}\geqslant-\frac{2}{t}$$

for $t\geqslant 20$. The expression has a factor of $\frac{1}{t}$ removed as this does not effect the positivity of the function, then the logarithm terms of $G_7(t)$ are now approximated to get $G_7(t)$ upto a factor of $\frac{1}{t}$

$$\geqslant 2\log(5)+2-\frac{1}{\log t}\left(1+\frac{(2\log t-1)^2}{16\log t}\right)-\left(\frac{1}{18(\log t)^3}-\frac{10}{9\log t}\right)-2.$$

This can easily be shown to be

$$\geqslant 2\log(5)-\frac{1}{\log t}-\frac{1}{4}-\left(\frac{1}{18(\log t)^3}-\frac{10}{9\log t}\right)>0,$$

for $t \geqslant 20$. Set
$$G_8(s,t) = \frac{\partial}{\partial s} G_6(s,t).$$
It remains to show that for $s \geqslant 8t\log(t) - t$, $t \geqslant 20$ $G_8(s,t) \geqslant 0$. Explicitly, $G_7(s,t)$ is given by the expression

$$\frac{1}{t}\log\left(1 + \frac{2\log t - 1}{\frac{s}{t}+1}\right) - \frac{2\log t - 1}{2t(\frac{s}{t}+1)(\frac{s}{t}+2\log t)} - \frac{2s - 9t + 2}{s^2 - 9ts + 2s + 20t^2 + 4t}.$$

Now the inequality $G_8(s,t) \geqslant 0$ is obtained by tedious but straightforward computations, using the estimate

$$\frac{1}{t}\log\left(1 + \frac{2\log t - 1}{\frac{s}{t}+1}\right) > \frac{2\log t - 1}{t(\frac{s}{t}+1)} - \frac{(2\log t - 1)^2}{2t(\frac{s}{t}+1)^2}.$$

Q.E.D. for Proposition 4.3.8 and Theorem 4.3.1. This completes the proof of Theorem 4.0.5.

Conclusion

Chapter 4 contains lengthy calculations to obtain bounds for the codimension of the locus of non-superrigid complete intersections of index 1 and high codimension. Similar bounds were obtained for complete intersections of codimension $k = 2$ in Chapter 3 and in [17] a bound for the codimension of the locus of non-superrigid hypersurfaces of index 1 was given. Also similar bounds were found for double quadrics and cubics (which could be understood as complete intersections of codimension 2 in a weighted projective space) in [35]. Such bounds are important for investigations of birational geometry of Fano fibre spaces with a higher-dimensional base. Let us describe a possible use for these bounds in the following example.

Consider the fibre space $X = \mathbb{P}^{M+k} \times \mathbb{P}^m \to S = \mathbb{P}^m$ with

$$m < \frac{(M-5k)(M-6k)}{2},$$

and the following inequalitities $M \geqslant 8k \log k$, $k \geqslant 20$ hold. Let $W \subset X$ be a sufficiently general subvariety given by k polynomials $f_1, \ldots, f_k$ of bidegree (d_i, l) such that $d_1 + d_2 + \ldots + d_k = M + k$, and l satisfies some inequality, which depends on m, M and k. The fibre space $W \to S$ satisfies all the assumptions of Theorem 1 in [68]. This states that every birational map $\chi : W \dashrightarrow W'$ onto the total space of a rationally connected fibre space W/S (with $\dim W = \dim W'$) is fibre-wise, that is, there exists a rational dominant map $\beta : S \dashrightarrow S'$, such that the following diagram commutes

$$\begin{tikzcd} W \arrow[r, dashed, "\chi"] \arrow[d] & W' \arrow[d] \\ S \arrow[r, dashed, "\beta"] & S'. \end{tikzcd}$$

In the direction of proving birational superrigidity for complete intersections there are certain open families which have yet to be studied. The following generic complete intersections $V \subset \mathbb{P}^{M+k}$ of type $\underline{d}$ with $|\underline{d}| = M + k$ and $M \geqslant 2k + 1$ were proved to be birationally superrigid in [59]. In [67] superrigidity was extended to the families with $M \geqslant k + 3$, $M \geqslant 7$ and $d_k = \max\{d_i\} \geqslant 4$, and in [66] to complete intersections of k_2 quadrics and k_3 cubics such that $M \geqslant 12$ and $k_3 \geqslant 2$. Today birational superrigidity of complete intersections in $\mathbb{P}^N$ remains an open problem only for three infinite series: complete intersections of type $\underline{d}$, where $\underline{d}$ is

$$(2, \ldots, 2) \quad \text{or} \quad (2, \ldots, 2, 3) \quad \text{or} \quad (2, \ldots, 2, 4)$$

and finitely many families with $M \leqslant 11$.

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