A CPHD approximation based on a discrete-Gamma cardinality model
– Supplementary Material –

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APPENDIX

PROOF OF PROPOSITION 1

**Proof of Proposition 1**: Consider the p.g.f.l. of the prior state RFS,

\[
G_{k|k-1}[h] = \int h^X p_{B|Z_{k-1}}(X)\,dX
\]

\[
= \int h^X \left\{ \int p_{B|Z_{k-1}}(X) p_{B|Z_{k-1}}(X'|Z_{1:k-1})\,dX' \right\} \,dX
\]

\[
= \int G_t[X'] p_{Z_{k-1}}(X'|Z_{1:k-1})\,dX',
\]

where \(G_t[X']\) is the p.g.f.l. of the multi-target transition kernel. Because targets are assumed to move independently (Assumption 1) and new targets are born independently (Assumption 3), we know that \(G_t[X'] = G_0[h] \cdot \prod_{i=1}^n G_{i}[h|x'_i]\) where \(G_i[h|x'_i] = q_s(x'_i) + p_s(x'_i)p_{t,h}(x'_i)\) (using Assumption 2), with \(p_{t,h}(x'_i) = \int_X h(x)p_{t,h}(x|x')\,dx\), and \(p_{t,h}(x|x')\) is the single-target transition kernel. Since the birth RFS is assumed to be a Poisson point process (Assumption 3) with mean \(\mu_b = N_b\) and intensity \(D_b(x) = \mu_b b(x)\):

\[
G_{k|k-1}[h] = G_0[h] \int (q_s + p_s p_{t,h})^{X'} p_{Z_{k-1}}(X'|Z_{1:k-1})\,dX'
\]

\[
= e^{\mu_b h} \cdot \mu_s G_{k-1}[q_s + p_s p_{t,h}],
\]

where \(b_h = \int_X h(x)b(x)\,dx\). By Assumption 8, both the posterior and prior state random finite sets follow multi-object discrete-Gamma i.i.d. cluster processes, i.e., their p.g.f.l. take the form of (30). Obtaining the first functional derivative of the prior p.g.f.l.:

\[
\frac{\delta}{\delta X} G_{k|k-1}[h] = \frac{\delta}{\delta X} \left( e^{\mu_b h} \cdot \mu_s G_{k-1}[q_s + p_s p_{t,h}] \right)
\]

\[
= e^{\mu_b h} \cdot \mu_s G_{k-1}[q_s + p_s p_{t,h}] \frac{\delta h}{\delta X} + e^{\mu_b h} \cdot \mu_s G_{k-1}[q_s + p_s p_{t,h}] \frac{\delta}{\delta X} (q_s + p_s p_{t,h} b(x))
\]

\[
+ e^{\mu_b h} \cdot \mu_s G_{k-1}[q_s + p_s p_{t,h}] \frac{\delta}{\delta X} \left( \int^X h(x)p_{t,h}(x|x')\,dx \right),
\]

\[
(\alpha_k - 1)(\frac{\delta}{\delta X} G_{k-1}[f, \langle x \rangle]) + \frac{\delta}{\delta X} \left( \int^X h(x)p_{t,h}(x|x')\,dx \right),
\]

where \(\langle f, \langle x \rangle \rangle = \int^X f(x')(x')\,dx'\) and \(D_{k-1}(x) = \int^X h(x)p_{t,h}(x)\,dx\). Set \(f(x') := q_s + p_s p_{t,h}\) and recall that

\[
G_{k-1}[f] = \frac{\text{Li}_{-\alpha_k - 1} + 1(e^{-\beta_k - 1}(f, \langle x \rangle))}{\text{Li}_{-\alpha_k - 1} + 1(e^{-\beta_k - 1})},
\]

\[
G^{(1)}_{k-1}[f] = \frac{\langle f, \langle x \rangle \rangle \text{Li}_{-\alpha_k - 1} + 1(e^{-\beta_k - 1}(f, \langle x \rangle))}{\text{Li}_{-\alpha_k - 1} + 1(e^{-\beta_k - 1})}.
\]

For \(f = 1, \langle 1, \langle x \rangle \rangle = 1\), hence

\[
G_{k-1}[1] = \frac{\text{Li}_{-\alpha_k - 1} + 1(e^{-\beta_k - 1})}{\text{Li}_{-\alpha_k - 1} + 1(e^{-\beta_k - 1})} = 1,
\]

\[
G^{(1)}_{k-1}[1] = \frac{\text{Li}_{-\alpha_k - 1} + 1(e^{-\beta_k - 1})}{\text{Li}_{-\alpha_k - 1} + 1(e^{-\beta_k - 1})} \approx \alpha_k - 1\beta_k^{-1}.
\]
By noticing that \( b_h|_{h=1} = \int_X b(x)dx = 1 \) and \( p_{t,h}|_{h=1} = \int_X p_t(x|x')dx = 1 \), then the prior intensity function, \( D_{k|k-1}(x) = \frac{\mu}{\sigma^2} G_{k|k-1}[1] \), can be obtained from (A.1) as

\[
D_{k|k-1}(x) = e^{\mu_1 - \mu} G_{k-1}[1] \mu b(x) + e^{\mu_1 - \mu} G_{k-1}^{(1)}(p_s p_t(x|\cdot), \kappa_{k-1}) + e^{\mu_1 - \mu} G_{k-1}^{(2)}(p_s p_t(x|\cdot), \kappa_{k-1}) \\
\approx 1 \cdot \mu b(x) + 1 \cdot \alpha_{k-1}^2 \beta_{k-1}^{-1} (p_s p_t(x|\cdot), \kappa_{k-1}) \\
D_{k|k-1}(x) = D_b(x) + \int_{X'} p_s(x') p_t(x'|x) D_{k-1}(x')dx'.
\]

Since the previous posterior and prior state random finite sets are i.i.d. cluster processes, then the probability generating functions of their cardinality distributions are given by \( G_{k-1}(y) = G_{k-1}[y] \) and \( G_{k|k-1}(y) = G_{k|k-1}[y] \), for \( y \in \mathbb{R} \), \( |y| \leq 1 \). Take the first two derivatives of \( G_{k|k-1}(y) \),

\[
G_{k|k-1}^{(1)}(y) = e^{\mu_1 - \mu} G_{k-1}[q_s + p_s y] \mu b \\
+ e^{\mu_1 - \mu} G_{k-1}^{(1)}(q_s + p_s y) \mu b, \quad (A.2)
\]

\[
G_{k|k-1}^{(2)}(y) = e^{\mu_1 - \mu} G_{k-1}[q_s + p_s y] \mu b^2 \\
+ 2e^{\mu_1 - \mu} G_{k-1}^{(1)}(q_s + p_s y) \mu b \mu b. \quad (A.3)
\]

where

\[
G_{k|k-1}^{(2)}(q_s + p_s y) = \begin{pmatrix} \frac{(q_s + p_s y, \kappa_{k-1}) - 2L_{1-\alpha_{k-1}}(e^{-\beta_{k-1}}(q_s + p_s y, \kappa_{k-1}))}{L_{1-\alpha_{k-1}}(e^{-\beta_{k-1}})} \\
- \frac{(q_s + p_s y, \kappa_{k-1}) - 4L_{1-\alpha_{k-1}}(e^{-\beta_{k-1}}(q_s + p_s y, \kappa_{k-1}))}{L_{1-\alpha_{k-1}}(e^{-\beta_{k-1}})} \end{pmatrix}. \quad (A.4)
\]

Evaluate (A.2) and (A.3) at \( y = 1 \), and notice that \( G_{k|k-1}^{(2)}[1] \approx \alpha_{k-1}(\alpha_{k-1} + 1) \beta_{k-1}^{-1} - \alpha_{k-1}^2 \beta_{k-1}^{-2} \) to compose the first two cardinality moments:

\[
\mu_{N,k|k-1} = G_{k|k-1}^{(1)}(1) \approx \frac{\mu_2}{N_b} + \frac{\alpha_{k-1}^2 \beta_{k-1}^{-1}}{N_b},
\]

\[
\sigma^2_{N,k|k-1} = G_{k|k-1}^{(2)}(1) - \mu^2_{N,k|k-1} + \mu N_{N,k|k-1} \approx \frac{\mu_2^2}{N_b} + 2 \mu_2 \alpha_{k-1} \beta_{k-1}^{-1} (p_s, \kappa_{k-1}) \\
+ \left[ \alpha_{k-1}(\alpha_{k-1} + 1) \beta_{k-1}^{-2} - \alpha_{k-1}^2 \beta_{k-1}^{-1} \right] (p_s, \kappa_{k-1})^2 \\
- \left[ \mu_2 + \alpha_{k-1} \beta_{k-1}^{-1} (p_s, \kappa_{k-1}) \right]^2 + \mu N_{N,k|k-1} \\
= \mu N_{N,k|k-1} + \alpha_{k-1} \beta_{k-1}^{-1} (p_s, \kappa_{k-1})^2.
\]

The prior state RFS is assumed to follow a multi-object discrete-Gamma process with \( \mu_{N,k|k-1} \approx \alpha_{k|k-1} \beta_{k|k-1}^{-1} \) and \( \sigma^2_{N,k|k-1} \approx \alpha_{k|k-1}^2 \beta_{k|k-1}^{-2} \), for \( \alpha_{k|k-1}, \beta_{k|k-1} \in \mathbb{R} \), therefore

\[
\alpha_{k|k-1} \approx \frac{\mu^2_{N,k|k-1}}{\sigma^2_{N,k|k-1}}, \quad \beta_{k|k-1} \approx \frac{\mu_{N,k|k-1}}{\sigma^2_{N,k|k-1}}.
\]

**Proof of Proposition 2:** From [1] we invoke the expression for the p.g.f. of a CPHD posterior process, which is generally valid for Assumptions 4–7 and i.i.d. cluster processes, and reads

\[
G_k[h] = \frac{dF^{(0)}[h]}{dZ_k} \frac{dF^{[0]}[0]}{dZ_k} = \sum_{j=0}^{m} G^{(m-j)}(0) \cdot G^{(j)}_{k|k-1}(h q_{k|k-1}) \sigma_j(Z_k, h) \\
+ \sum_{i=0}^{m} G^{(m-i)}(0) \cdot G^{(i)}_{k|k-1}(q_{k|k-1}) \sigma_i(Z_k, h),
\]

where \( m = |Z_k| \), \( G^{(i)}_{k|k-1}(\cdot) \) is the p.g.f. of the prior cardinality distribution,

\[
\sigma_i(Z_k, h) = \sigma_{m,i} \left( \frac{(h p_d f_{z_1}, s_{k|k-1})}{c_{z_1}}, \ldots, \frac{(h p_d f_{z_m}, s_{k|k-1})}{c_{z_m}} \right)
\]

PROOF OF PROPOSITION 2
is the elementary symmetric function of degree $i$ in $(y_{p,q_1}, \cdots, y_{p,q_{m-1}})$, and
\[
F[g, h] = \int \int h^{X_g} g^{X_h} p_{q, (Z_k | X)} p_{\Xi, k| k-1} (X | Z_{k| k-1}) \delta X \delta Z
\]
is the joint p.g.f.l. on the state and observation random finite sets. First, we invoke another result from [1], the posterior intensity function estimated by the CPHD filter according to:
\[
D_k(x) = \frac{q_d(x)}{G_{k|k-1}^{(1)}} \Upsilon_k [Z_k] D_{k|k-1}(x) + \frac{p_a(x)}{G_{k|k-1}^{(1)}} \sum_{z \in Z_k} \Upsilon_k [Z_k \setminus \{x\}] D_{k|k-1}(x),
\]
with
\[
\Upsilon_k[Z] = \frac{\sum_{j=0}^{\lvert Z \rvert} G_c^{(\lvert Z \rvert - j)}(0) G_{k|k-1}^{(j+1)} (\langle q_d, s_k|k-1 \rangle) \sigma_j \langle Z_k, 1 \rangle}{\sum_{m=0}^{\lvert Z \rvert} G_c^{(m-1)}(0) G_{k|k-1}^{(m)} (\langle q_d, s_k|k-1 \rangle) \sigma_m \langle Z_k, 1 \rangle}, \tag{A.5}
\]
whose proof follows from evaluating $D_k(x) = \hat{f}_d G_k[1]$ (see [1]). In the DG-CPHD context, we use Assumptions 5 (clutter RFS follows a Poisson point process) and 8 to further simplify (A.5). The p.g.f.l. of the clutter process is given by $G_c[g] = \exp(\lambda \int_{\mathbb{R}} g(z) c(z) dz - \lambda)$. Set $G_c^{(i)}(y) := e^{\lambda y - \lambda}$ and notice that $G_c^{(i)}(0) = \lambda^i e^{-\lambda}$ to obtain
\[
\Upsilon_k[Z] = \frac{\lambda^{|Z|} \sum_{j=0}^{|Z|} \lambda^{-j} G_c^{(j+1)}(\langle q_d, s_k|k-1 \rangle) \sigma_j \langle Z_k, 1 \rangle}{\lambda^m \sum_{m=0}^{|Z|} \lambda^{-m} G_c^{(m)}(\langle q_d, s_k|k-1 \rangle) \sigma_m \langle Z_k, 1 \rangle}.
\]
where
\[
\sigma_i(Z) := \begin{pmatrix} \langle p_d f_{z_1}, 1 \rangle & \cdots & \langle p_d f_{z_m}, 1 \rangle \end{pmatrix}.
\]
From (A.6), (A.7), and observing that $\Upsilon_k[Z_k] = \Theta_k[Z_k]$ and $\Upsilon_k[Z_k \setminus \{z\}] = \lambda^{-1} \Theta_k[Z_k \setminus \{z\}]$, the posterior intensity for the DG-CPHD, according to (35), follows straightforwardly.

As follows, we use Assumption 8 to derive the posterior cardinality parameters. Under this assumption, the posterior state RFS follows an i.i.d. cluster process and so the posterior cardinality p.g.f.l. is given by $G_k(y) = G_k[y]$, where $y \in \mathbb{R}$, $|y| \leq 1$, which results in
\[
G_k(y) = \frac{\sum_{j=0}^{m} \lambda^{-j} G_{k|k-1}^{(j)} (\langle q_d, s_k|k-1 \rangle) \sigma_j \langle Z_k, y \rangle}{\sum_{m=0}^{|Z|} \lambda^{-m} G_c^{(m)}(\langle q_d, s_k|k-1 \rangle) \sigma_m \langle Z_k, 1 \rangle}.
\]
We compute the first two derivatives of $G_k(y)$ as
\[
G_k^{(1)}(y) = \frac{\sum_{j=0}^{m} \lambda^{-j} G_c^{(j+1)}(\langle q_d, s_k|k-1 \rangle) \sigma_j \langle Z_k \rangle}{\sum_{j=0}^{m} \lambda^{-j} G_c^{(j)}(\langle q_d, s_k|k-1 \rangle) \sigma_j \langle Z_k \rangle},
\]
and
\[
G_k^{(2)}(y) = \frac{\sum_{j=0}^{m} \lambda^{-j} G_c^{(j+2)}(\langle q_d, s_k|k-1 \rangle) \sigma_j \langle Z_k \rangle}{\sum_{j=0}^{m} \lambda^{-j} G_c^{(j+1)}(\langle q_d, s_k|k-1 \rangle) \sigma_j \langle Z_k \rangle}.
\]
Evaluate (A.9) and (A.10) at \( y = 1 \) to obtain

\[
G_k^{(1)}(1) = \sum_{i=0}^{m} \beta_i \cdot G_{k-1}^{(i)}(\langle q_d, s_{k|k-1} \rangle) \sigma_i(Z_k)
\]

\[
G_k^{(2)}(1) = \sum_{i=0}^{m} \beta_i \cdot G_{k-1}^{(i)}(\langle q_d, s_{k|k-1} \rangle) \sigma_i(Z_k)
\]

By defining

\[
\theta_{u,v} = \sum_{i=0}^{m_k} \frac{\beta_i \cdot G_{k-1}^{(i)}(\langle q_d, s_{k|k-1} \rangle) \sigma_i(Z_k)}{G_{k-1}^{(i)}(\langle q_d, s_{k|k-1} \rangle) \sigma_i(Z_k)}
\]

one has

\[
G_k^{(1)}(1) = \theta_{1,0} + \theta_{0,1} \cdot \langle q_d, s_{k|k-1} \rangle,
\]

\[
G_k^{(2)}(1) = \theta_{2,0} - \theta_{1,1} \cdot \langle q_d, s_{k|k-1} \rangle + \theta_{0,2} \cdot \langle q_d, s_{k|k-1} \rangle^2,
\]

from which the posterior cardinality moments are obtained as given in (37) and (38). The posterior state RFS is assumed to follow a multi-object discrete-Gamma process with cardinality characterized by \( \mu_{N,k} \approx \alpha_k \beta_k^{-1} \) and \( \sigma_{N,k}^2 \approx \alpha_k \beta_k^{-2} \), therefore (36) holds. Since the prior cardinality distribution is also assumed to be a discrete-Gamma distribution, we write

\[
G_{k-1}^{(H)}(y) = \frac{d}{dy} \left( \frac{\alpha_{k-1} \cdot \beta_{k-1} (e^{-\beta_{k-1} y})}{\alpha_{k-1} + \beta_{k-1}} \right)
\]

The proof is complete by noting that, in all terms of (A.9) and (A.10), \( 1/G_{k-1}^{(0)}(y) \) appears both in the numerator and denominator and can be cancelled out to leave only terms depending on \( G_{k-1}^{(H)}(y) \).
Therefore $\sum_{\ell=0}^{L} \ell \cdot w_{f,\ell} \geq \sum_{\ell=0}^{L} \ell \cdot w_{g,\ell}$, which corresponds to (A.11). The condition $L \geq L_0 + (L_0 - \lfloor L_0 \rfloor)/2$ must be met to ensure $2(L - L_0)M \geq (L_0 - \lfloor L_0 \rfloor)M$, where

$$M = \frac{(w_{f,L} - w_{g,L}) - (w_{f,\lfloor L_0 \rfloor} - w_{g,\lfloor L_0 \rfloor})}{L - \lfloor L_0 \rfloor},$$

so that $2(w_{f,2} - w_{g,2}) \geq 1(w_{g,1} - w_{f,1})$ in the particular case when $\lfloor L_0 \rfloor = 1$ and $L = 2$.

**Proof of Proposition 3:** We resort to a special type of Chebyshev’s inequality that is appropriate for bounding the probabilities of non-symmetric intervals, [2], viz.

$$\varepsilon = \Pr \{ 0 < n < n_{max} \} \geq 4 \frac{\mu_{N,k} n_{max} - (\mu_{N,k}^2 + \sigma_{N,k}^2)}{n_{max}^2}.$$  

(A.13)

where $0 < \sigma_{N,k}^2 \leq \mu_{N,k} n_{max} - \mu_{N,k}^2$. Rearranging terms and solving the inequality for $n_{max}$ corresponding to the upper bound:

$$n_{max} \geq \frac{2 \mu_{N,k} \varepsilon}{\varepsilon} \left(1 + \sqrt{(1 - \varepsilon) - \varepsilon \left(\frac{\sigma_{N,k}^2}{\mu_{N,k}^2}\right)^2}\right).$$  

(A.14)

As follows, we bound $\mu_{N,k}$ as per the moment update equation [3]:

$$\mu_{N,k} = \frac{\sum_{j=0}^{m} \sum_{n=0}^{\min(m,n)} \sigma_j(Z) \cdot p_{k-1}(n)}{\sum_{n=0}^{\min(m,n)} \sigma_j(Z) \cdot p_{k-1}(n)} = \frac{\sum_{j=0}^{m} \sum_{n=0}^{\min(m,n)} \mu_{N,k}^{\mu_{N,k}} \sigma_j(Z) \cdot p_{k-1}(n)}{\sum_{n=0}^{\min(m,n)} \sigma_j(Z) \cdot p_{k-1}(n)} \geq \frac{\sum_{j=0}^{m} \sum_{n=0}^{\min(m,n)} \mu_{N,k}^{\mu_{N,k}} \sigma_j(Z) \cdot p_{k-1}(n)}{\sum_{n=0}^{\min(m,n)} \sigma_j(Z) \cdot p_{k-1}(n)},$$

(A.15)

where $n^\mu \triangleq n(n-1) \cdots (n-j+1)$ is the falling factorial (Pochhammer’s symbol), $n^\ast \triangleq \min (\mu_{N,k} \ast 1, \arg \max_j \sigma_j(Z))$, and the first inequality holds because $\sum_{j=0}^{\min(m,n)} \mu_{N,k}^{\mu_{N,k}} \sigma_j(Z) \cdot p_{k-1}(n) \leq \sum_{j=0}^{\min(m,n)} n \cdot f(j)$ for any function $f : N_0 \rightarrow \mathbb{R}$, $f(j) \geq 0$. Per Jensen’s inequality

$$\mathbb{E}_{N,k-1} \left[ \sum_{n=0}^{n^\ast} \varphi_j(N) \right] \geq \mathbb{E}_{N,k-1} \left[ \sum_{n=0}^{\min(n^\mu, n^\ast)} \varphi_j(N) \right] \geq \mathbb{E}_{m \leq n^\mu} \varphi_j(N),$$

where $\varphi_j(n) := \lambda \mu_{N,k}^{\mu_{N,k}} \sigma_j(Z)$ is convex in $n$, and Lemma 1 is applied in (A.15) with $f(j) = \mathbb{E}_{N,k-1} \left[ \sum_{m=0}^{n^\ast} \varphi_j(N) \right] \sigma_j(Z)$ and $g(j) = \mathbb{E}_{m \leq n^\mu} \varphi_j(N)$. This bound can be simplified by applying Lemma 1 once more as $\varphi_j(\mu_{N,k} \ast 1) \sigma_j(Z)$ is a nonnegative increasing function in the interval $j \in [0..n^\ast]$. For $j \in [\mu_{N,k} \ast 1]$, the bound in (A.14) can be concluded that there exists a $K \geq 1$ such that $n_{max} \geq K \cdot \mu_{N,k} \ast 1$.

**Remark 2.** Note that for a certain $x \in \mathbb{R}_+$ such that

$$1 - \frac{\mu_{N,k} \ast 1}{c_{\ast}} = (1 - x)^m = \sum_{j=0}^{m} \binom{m}{j} x^j,$$
$j_{\text{max}} \approx \arg \max_j \sigma_j(Z)$ can be computed as

\[
\frac{d}{dj} \left[ \binom{m}{j} x^j \right] = 0,
\]

\[
\left[ \frac{d}{dj} \binom{m}{j} \right] x^j + \binom{m}{j} x^j \log x = 0,
\]

\[- \binom{m}{j} \left( \sum_{\ell=1}^{j} \ell^{-1} - \sum_{\ell=1}^{m-j} \ell^{-1} \right) + \binom{m}{j} \log x = 0,
\]

\[
\sum_{\ell=2}^{j} \ell^{-1} - \sum_{\ell=2}^{m-j} \ell^{-1} = \log x,
\]

\[
\log j - \log(m - j) \approx \log x,
\]

\[
\frac{d}{dj} \binom{m}{j} = m! \frac{-D_j(j!) \cdot (m - j)! - (j!)D_j((m - j)!) \cdot [j!(m - j)!]^2}{j!(m - j)!}
\]

\[
= - \binom{m}{j} \left( -\gamma + \sum_{\ell=1}^{m-j} \ell^{-1} \right) \frac{(m - j)!}{j!(m - j)!}
\]

\[
= - \binom{m}{j} \left( \sum_{\ell=1}^{j} \ell^{-1} - \sum_{\ell=1}^{m-j} \ell^{-1} \right),
\]

and $\gamma$ is the Euler–Mascheroni constant.

REFERENCES

