A CPHD approximation based on a discrete-Gamma cardinality model – Supplementary Material –

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APPENDIX
PROOF OF PROPOSITION 1

Proof of Proposition 1: Consider the p.g.fl. of the prior state RFS,

$$\begin{aligned} G_{k|k-1}[h] &= \int h^{\mathrm{X}} p_{\Xi_{k|k-1}}(\mathrm{X}|\mathrm{Z}_{1:k-1})\,\delta\mathrm{X} \\ &= \int h^{\mathrm{X}} \left\{ \int p_{\Xi|\Xi_{k-1}}(\mathrm{X}|\mathrm{X}')\, p_{\Xi_{k-1}}(\mathrm{X}'|\mathrm{Z}_{1:k-1})\,\delta\mathrm{X}' \right\} \delta\mathrm{X} \\ &= \int G_t[h|\mathrm{X}'] p_{\Xi_{k-1}}(\mathrm{X}'|\mathrm{Z}_{1:k-1})\,\delta\mathrm{X}', \end{aligned}$$

where $G_t[h|\mathrm{X}']$ is the p.g.fl. of the multi-target transition kernel. Because targets are assumed to move independently (*Assumption 1*) and new targets are born independently (*Assumption 3*), we know that $G_t[h|\mathrm{X}'] = G_b[h] \cdot \prod_{i=1}^{n} G_t[h|\mathrm{x}'_i]$ where $G_t[h|\mathrm{x}'] = q_s(\mathrm{x}') + p_s(\mathrm{x}')p_{t,h}(\mathrm{x}')$ (using *Assumption 2*), with $p_{t,h}(\mathrm{x}') \triangleq \int_{\mathcal{X}} h(\mathrm{x}) p_t(\mathrm{x}|\mathrm{x}') d\mathrm{x}$, and $p_t(\cdot|\mathrm{x}')$ is the single-target transition kernel. Since the birth RFS is assumed to be a Poisson point process (*Assumption 3*) with mean $\mu_b = \hat{N}_b$ and intensity $D_b(\mathrm{x}) = \mu_b b(\mathrm{x})$:

$$\begin{aligned} G_{k|k-1}[h] &= G_b[h] \int (q_s + p_s p_{t,h})^{\mathrm{X}'} p_{\Xi_{k-1}}(\mathrm{X}'|\mathrm{Z}_{1:k-1})\,\delta\mathrm{X}' \\ &= e^{\mu_b b_h - \mu_b} G_{k-1}[q_s + p_s p_{t,h}], \end{aligned}$$

where $b_h \triangleq \int_{\mathcal{X}} h(\mathrm{x}) b(\mathrm{x}) d\mathrm{x}$. By *Assumption 8*, both the posterior and prior state random finite sets follow multi-object discrete-Gamma i.i.d. cluster processes, i.e., their p.g.fl. take the form of (30). Obtaining the first functional derivative of the prior p.g.fl.:

$$\begin{aligned} \frac{\delta}{\delta\mathrm{x}} G_{k|k-1}[h] &= \frac{\delta}{\delta\mathrm{x}} \left(e^{\mu_b b_h - \mu_b} G_{k-1}[q_s + p_s p_{t,h}] \right) \\ &= e^{\mu_b b_h - \mu_b} G_{k-1}[q_s + p_s p_{t,h}] \mu_b \frac{\delta b_h}{\delta\mathrm{x}} \\ &+ e^{\mu_b b_h - \mu_b} G^{(1)}_{k-1}[q_s + p_s p_{t,h}] \frac{\delta \langle q_s + p_s p_{t,h}, \varsigma_{k-1} \rangle}{\delta\mathrm{x}} \\ &= e^{\mu_b b_h - \mu_b} G_{k-1}[q_s + p_s p_{t,h}] \mu_b b(\mathrm{x}) \\ &+ e^{\mu_b b_h - \mu_b} G^{(1)}_{k-1}[q_s + p_s p_{t,h}] \langle p_s p_t(\mathrm{x}|\cdot), \varsigma_{k-1} \rangle. \end{aligned} \tag{A.1}$$

where $\langle f, \varsigma_{k-1} \rangle = \int_{\mathcal{X}'} f(\mathrm{x}') \varsigma(\mathrm{x}') d\mathrm{x}'$ and $D_{k-1}(\mathrm{x}) = G^{(1)}_{k-1}[1]\varsigma_{k-1}(\mathrm{x})$. Set $f(\mathrm{x}') \coloneqq q_s + p_s p_{t,h}$ and recall that

$$\begin{aligned} G_{k-1}[f] &= \frac{\mathrm{Li}_{-\alpha_{k-1}+1}(e^{-\beta_{k-1}} \langle f, \varsigma_{k-1} \rangle)}{\mathrm{Li}_{-\alpha_{k-1}+1}(e^{-\beta_{k-1}})}, \\ G^{(1)}_{k-1}[f] &= \frac{\langle f, \varsigma_{k-1} \rangle^{-1} \mathrm{Li}_{-\alpha_{k-1}}(e^{-\beta_{k-1}} \langle f, \varsigma_{k-1} \rangle)}{\mathrm{Li}_{-\alpha_{k-1}+1}(e^{-\beta_{k-1}})}. \end{aligned}$$

For $f = 1$, $\langle 1, \varsigma_{k-1} \rangle = 1$, hence

$$\begin{aligned} G_{k-1}[1] &= \frac{\mathrm{Li}_{-\alpha_{k-1}+1}(e^{-\beta_{k-1}})}{\mathrm{Li}_{-\alpha_{k-1}+1}(e^{-\beta_{k-1}})} = 1, \\ G^{(1)}_{k-1}[1] &= \frac{\mathrm{Li}_{-\alpha_{k-1}}(e^{-\beta_{k-1}})}{\mathrm{Li}_{-\alpha_{k-1}+1}(e^{-\beta_{k-1}})} \approx \alpha_{k-1}\beta^{-1}_{k-1}. \end{aligned}$$

By noticing that $b_h|_{h=1} = \int_{\mathcal{X}} b(\mathrm{x})d\mathrm{x} = 1$ and $p_{t,h}|_{h=1} = \int_{\mathcal{X}} p_t(\mathrm{x}|\mathrm{x}')d\mathrm{x} = 1$, then the prior intensity function, $D_{k|k-1}(\mathrm{x}) = \frac{\delta}{\delta \mathrm{x}} G_{k|k-1}[1]$, can be obtained from (A.1) as

$$\begin{aligned} D_{k|k-1}(\mathrm{x}) &= e^{\mu_b 1 - \mu_b} G_{k-1}[1]\mu_b b(\mathrm{x}) \\ &\quad + e^{\mu_b 1 - \mu_b} G_{k-1}^{(1)}[1]\langle p_s p_t(\mathrm{x}|\cdot), \varsigma_{k-1}\rangle \\ &\approx 1 \cdot \mu_b b(\mathrm{x}) + 1 \cdot \alpha_{k-1}\beta_{k-1}^{-1}\langle p_s p_t(\mathrm{x}|\cdot), \varsigma_{k-1}\rangle, \\ D_{k|k-1}(\mathrm{x}) &= D_b(\mathrm{x}) + \int_{\mathcal{X}'} p_s(\mathrm{x}')p_t(\mathrm{x}|\mathrm{x}')D_{k-1}(\mathrm{x}')d\mathrm{x}'. \end{aligned}$$

Since the previous posterior and prior state random finite sets are i.i.d. cluster processes, then the probability generating functions of their cardinality distributions are given by $G_{k-1}(y) = G_{k-1}[y]$ and $G_{k|k-1}(y) = G_{k|k-1}[y]$, for $y \in \mathbb{R}$, $|y| \le 1$. Take the first two derivatives of $G_{k|k-1}(y)$,

$$\begin{aligned} G_{k|k-1}^{(1)}(y) &= e^{\mu_b y - \mu_b} G_{k-1}[q_s + p_s y]\mu_b \\ &\quad + e^{\mu_b y - \mu_b} G_{k-1}^{(1)}[q_s + p_s y]\langle p_s, \varsigma_{k-1}\rangle, \end{aligned} \tag{A.2}$$

$$\begin{aligned} G_{k|k-1}^{(2)}(y) &= e^{\mu_b y - \mu_b} G_{k-1}[q_s + p_s y]\mu_b^2 \\ &\quad + 2e^{\mu_b y - \mu_b} G_{k-1}^{(1)}[q_s + p_s y]\mu_b\langle p_s, \varsigma_{k-1}\rangle \\ &\quad + e^{\mu_b y - \mu_b} G_{k-1}^{(2)}[q_s + p_s y]\langle p_s, \varsigma_{k-1}\rangle^2, \end{aligned} \tag{A.3}$$

where

$$\begin{aligned} G_{k-1}^{(2)}[q_s + p_s y] &= \frac{\langle q_s + p_s y, \varsigma_{k-1}\rangle^{-2}\mathrm{Li}_{-\alpha_{k-1}-1}(e^{-\beta_{k-1}}\langle q_s + p_s y, \varsigma_{k-1}\rangle)}{\mathrm{Li}_{1-\alpha_{k-1}}(e^{-\beta_{k-1}})} \\ &\quad - \frac{\langle q_s + p_s y, \varsigma_{k-1}\rangle^{-1}\mathrm{Li}_{-\alpha_{k-1}}(e^{-\beta_{k-1}}\langle q_s + p_s y, \varsigma_{k-1}\rangle)}{\mathrm{Li}_{1-\alpha_{k-1}}(e^{-\beta_{k-1}})}. \end{aligned} \tag{A.4}$$

Evaluate (A.2) and (A.3) at $y = 1$, and notice that $G_{k-1}^{(2)}[1] \approx \alpha_{k-1}(\alpha_{k-1} + 1)\beta_{k-1}^{-2} - \alpha_{k-1}\beta_{k-1}^{-1}$ to compose the first two cardinality moments:

$$\mu_{N,k|k-1} = G_{k|k-1}^{(1)}(1) \approx \underbrace{\mu_b}_{N_b} + \underbrace{\alpha_{k-1}\beta_{k-1}^{-1}\langle p_s, \varsigma_{k-1}\rangle}_{N_s},$$

$$\begin{aligned} \sigma_{N,k|k-1}^2 &= G_{k|k-1}^{(2)}(1) - \mu_{N,k|k-1}^2 + \mu_{N,k|k-1} \\ &\approx \mu_b^2 + 2\mu_b\alpha_{k-1}\beta_{k-1}^{-1}\langle p_s, \varsigma_{k-1}\rangle \\ &\quad + \left[\alpha_{k-1}(\alpha_{k-1} + 1)\beta_{k-1}^{-2} - \alpha_{k-1}\beta_{k-1}^{-1}\right]\langle p_s, \varsigma_{k-1}\rangle^2 \\ &\quad - \left[\mu_b + \alpha_{k-1}\beta_{k-1}^{-1}\langle p_s, \varsigma_{k-1}\rangle\right]^2 + \mu_{N,k|k-1} \\ &= \mu_{N,k|k-1} + \alpha_{k-1}\beta_{k-1}^{-1}\left(\beta_{k-1}^{-1} - 1\right)\langle p_s, \varsigma_{k-1}\rangle^2. \end{aligned}$$

The prior state RFS is assumed to follow a multi-object discrete-Gamma process with $\mu_{N,k|k-1} \approx \alpha_{k|k-1}\beta_{k|k-1}^{-1}$ and $\sigma_{N,k|k-1}^2 \approx \alpha_{k|k-1}\beta_{k|k-1}^{-2}$, for $\alpha_{k|k-1}, \beta_{k|k-1} \in \mathbb{R}$, therefore

$$\alpha_{k|k-1} \approx \frac{\mu_{N,k|k-1}^2}{\sigma_{N,k|k-1}^2},\ \beta_{k|k-1} \approx \frac{\mu_{N,k|k-1}}{\sigma_{N,k|k-1}^2}.$$

■

Proof of Proposition 2

Proof of Proposition 2: From [1] we invoke the expression for the p.g.fl. of a CPHD posterior process, which is generally valid for *Assumptions 4–7* and i.i.d. cluster processes, and reads

$$\begin{aligned} G_k[h] &= \frac{\frac{\delta F[0,h]}{\delta \mathrm{Z}_k}}{\frac{\delta F[0,1]}{\delta \mathrm{Z}_k}} \\ &= \frac{\sum_{j=0}^{m} G_c^{(m-j)}(0) \cdot G_{k|k-1}^{(j)}(\langle hq_d, \varsigma_{k|k-1}\rangle)\sigma_j(\mathrm{Z}_k, h)}{\sum_{i=0}^{m} G_c^{(m-i)}(0) \cdot G_{k|k-1}^{(i)}(\langle\ q_d, \varsigma_{k|k-1}\rangle)\sigma_i(\mathrm{Z}_k, 1)}, \end{aligned}$$

where $m = |\mathrm{Z}_k|$, $G_{k|k-1}(\cdot)$ is the p.g.f. of the prior cardinality distribution,

$$\sigma_i(\mathrm{Z}_k, h) = \sigma_{m,i}\left(\frac{\langle hp_d\ell_{\mathrm{z}_1}, \varsigma_{k|k-1}\rangle}{c(\mathrm{z}_1)}, \ldots, \frac{\langle hp_d\ell_{\mathrm{z}_m}, \varsigma_{k|k-1}\rangle}{c(\mathrm{z}_m)}\right)$$

is the elementary symmetric function of degree i in $\frac{\langle hp_d\ell_{z_1}, \varsigma_{k|k-1}\rangle}{c(z_1)}, \ldots, \frac{\langle hp_d\ell_{z_m}, \varsigma_{k|k-1}\rangle}{c(z_m)}$, and

$$F[g,h] = \int\int h^{X} g^{Z} p_{\Psi}\left(Z_k|X\right) p_{\Xi,k|k-1}\left(X|Z_{1:k-1}\right)\delta X \delta Z$$

is the joint p.g.fl. on the state and observation random finite sets. First, we invoke another result from [1], the posterior intensity function estimated by the CPHD filter according to:

$$\begin{aligned} D_k(x) &= \frac{q_d(x)}{G^{(1)}_{k|k-1}(1)} \Upsilon_k\left[Z_k\right] D_{k|k-1}(x) \\ &\quad + \frac{p_d(x)}{G^{(1)}_{k|k-1}(1)} \sum_{z\in Z_k} \frac{\ell_z(x)}{c(z)} \Upsilon_k\left[Z_k \setminus \{z\}\right] D_{k|k-1}(x), \end{aligned}$$

with

$$\Upsilon_k[Z] = \frac{\sum_{j=0}^{|Z|} G_c^{(|Z|-j)}(0) G^{(j+1)}_{k|k-1}(\langle q_d, \varsigma_{k|k-1}\rangle)\sigma_j(Z\ \ ,1)}{\sum_{i=0}^{m} G_c^{(m-i)}(0) G^{(i)}_{k|k-1}\left(\langle q_d, \varsigma_{k|k-1}\rangle\right)\sigma_i(Z_k,1)}, \tag{A.5}$$

whose proof follows from evaluating $D_k(x) = \frac{\delta}{\delta x} G_k[1]$ (see [1]). In the DG-CPHD context, we use *Assumptions 5* (clutter RFS follows a Poisson point process) and *8* to further simplify (A.5). The p.g.fl. of the clutter process is given by $G_c[g] = \exp\left(\lambda \int_{\mathcal{Z}} g(z)c(z)dz - \lambda\right)$. Set $G_c^{(i)}(y) := e^{\lambda y - \lambda}$ and notice that $G_c^{(\ell)}(0) = \lambda^{\ell} e^{-\lambda}$ to obtain

$$\begin{aligned} \Upsilon_k[Z] &= \frac{\lambda^{|Z|}\sum_{j=0}^{|Z|} \lambda^{-j} G^{(j+1)}_{k|k-1}(\langle q_d, \varsigma_{k|k-1}\rangle)\sigma_j(Z\ \ ,1)}{\lambda^m \sum_{i=0}^{m} \lambda^{-i} G^{(i)}_{k|k-1}\left(\langle q_d, \varsigma_{k|k-1}\rangle\right)\sigma_i(Z_k,1)} \\ &= \frac{\lambda^{|Z|}\sum_{j=0}^{|Z|} G^{(j+1)}_{k|k-1}(\langle q_d, \varsigma_{k|k-1}\rangle)\bar{\sigma}_j(Z\ \ )}{\lambda^m \sum_{i=0}^{m} G^{(i)}_{k|k-1}\left(\langle q_d, \varsigma_{k|k-1}\rangle\right)\bar{\sigma}_i(Z_k)} \\ &\triangleq \lambda^{|Z|-m}\Theta_k[Z], \end{aligned} \tag{A.6}$$

where

$$\bar{\sigma}_i(Z) \triangleq \sigma_{m=|Z|,i}\left(\frac{\langle p_d\ell_{z_1}, \varsigma\rangle}{\lambda c(z_1)}, \ldots, \frac{\langle p_d\ell_{z_m}, \varsigma\rangle}{\lambda c(z_m)}\right). \tag{A.7}$$

From (A.6), (A.7), and observing that $\Upsilon_k[Z_k] = \Theta_k[Z_k]$ and $\Upsilon_k\left[Z_k \setminus \{z\}\right] = \lambda^{-1}\Theta_k[Z_k \setminus \{z\}]$, the posterior intensity for the DG-CPHD, according to (35), follows straightforwardly.

As follows, we use *Assumption 8* to derive the posterior cardinality parameters. Under this assumption, the posterior state RFS follows an i.i.d. cluster process and so the posterior cardinality p.g.f. is given by $G_k(y) = G_k[y]$, where $y \in \mathbb{R}$, $|y| \leq 1$, which results in

$$\begin{aligned} G_k(y) &= \frac{\sum_{j=0}^{m} \lambda^{-j} G^{(j)}_{k|k-1}(y\langle q_d, \varsigma_{k|k-1}\rangle)\sigma_j(Z_k, y)}{\sum_{i=0}^{m} \lambda^{-i} G^{(i)}_{k|k-1}(\ \ \langle q_d, \varsigma_{k|k-1}\rangle)\sigma_i(Z_k,1)} \\ &= \frac{\sum_{j=0}^{m} y^j G^{(j)}_{k|k-1}(y\langle q_d, \varsigma_{k|k-1}\rangle)\bar{\sigma}_i(Z_k)}{\sum_{i=0}^{m} \quad G^{(i)}_{k|k-1}(\ \ \langle q_d, \varsigma_{k|k-1}\rangle)\bar{\sigma}_j(Z_k)}. \end{aligned} \tag{A.8}$$

We compute the first two derivatives of $G_k(y)$ as

$$\begin{aligned} G_k^{(1)}(y) &= \frac{\sum_{j=0}^{m} j \cdot y^{j-1} G^{(j)}_{k|k-1}(y\langle q_d, \varsigma_{k|k-1}\rangle)\bar{\sigma}_j(Z_k)}{\sum_{i=0}^{m} G^{(i)}_{k|k-1}(\langle q_d, \varsigma_{k|k-1}\rangle)\bar{\sigma}_i(Z_k)} \\ &\quad + \frac{\sum_{j=0}^{m} y^j G^{(j+1)}_{k|k-1}(y\langle q_d, \varsigma_{k|k-1}\rangle)\bar{\sigma}_j(Z_k)}{\sum_{i=0}^{m} G^{(i)}_{k|k-1}(\langle q_d, \varsigma_{k|k-1}\rangle)\bar{\sigma}_i(Z_k)}\langle q_d, \varsigma_{k|k-1}\rangle, \end{aligned} \tag{A.9}$$

$$\begin{aligned} G_k^{(2)}(y) &= \frac{\sum_{j=0}^{m} j(j-1) \cdot y^{j-2} G^{(j)}_{k|k-1}(y\langle q_d, \varsigma_{k|k-1}\rangle)\bar{\sigma}_j(Z_k)}{\sum_{i=0}^{m} G^{(i)}_{k|k-1}(\langle q_d, \varsigma_{k|k-1}\rangle)\bar{\sigma}_i(Z_k)} \\ &\quad + 2\frac{\sum_{j=0}^{m} j \cdot y^{j-1} G^{(j+1)}_{k|k-1}(y\langle q_d, \varsigma_{k|k-1}\rangle)\bar{\sigma}_j(Z_k)}{\sum_{i=0}^{m} G^{(i)}_{k|k-1}(\langle q_d, \varsigma_{k|k-1}\rangle)\bar{\sigma}_i(Z_k)}\langle q_d, \varsigma_{k|k-1}\rangle \\ &\quad + \frac{\sum_{j=0}^{m} y^j G^{(j+2)}_{k|k-1}(y\langle q_d, \varsigma_{k|k-1}\rangle)\bar{\sigma}_j(Z_k)}{\sum_{i=0}^{m} G^{(i)}_{k|k-1}(\langle q_d, \varsigma_{k|k-1}\rangle)\bar{\sigma}_i(Z_k)}\langle q_d, \varsigma_{k|k-1}\rangle^2. \end{aligned} \tag{A.10}$$

Evaluate (A.9) and (A.10) at $y = 1$ to obtain

$$G_k^{(1)}(1) = \frac{\sum_{j=0}^{m} j \cdot G_{k|k-1}^{(j)}(\langle q_d, \varsigma_{k|k-1}\rangle)\bar{\sigma}_j(\mathrm{Z}_k)}{\sum_{i=0}^{m} G_{k|k-1}^{(i)}(\langle q_d, \varsigma_{k|k-1}\rangle)\bar{\sigma}_i(\mathrm{Z}_k)} + \frac{\sum_{j=0}^{m} G_{k|k-1}^{(j+1)}(\langle q_d, \varsigma_{k|k-1}\rangle)\bar{\sigma}_j(\mathrm{Z}_k)}{\sum_{i=0}^{m} G_{k|k-1}^{(i)}(\langle q_d, \varsigma_{k|k-1}\rangle)\bar{\sigma}_i(\mathrm{Z}_k)}\langle q_d, \varsigma_{k|k-1}\rangle,$$

$$\begin{aligned} G_k^{(2)}(1) = &\frac{\sum_{j=0}^{m} j^2 G_{k|k-1}^{(j)}(\langle q_d, \varsigma_{k|k-1}\rangle)\bar{\sigma}_j(\mathrm{Z}_k)}{\sum_{i=0}^{m} G_{k|k-1}^{(i)}(\langle q_d, \varsigma_{k|k-1}\rangle)\bar{\sigma}_i(\mathrm{Z}_k)} \\ &- \frac{\sum_{j=0}^{m} j\, G_{k|k-1}^{(j)}(\langle q_d, \varsigma_{k|k-1}\rangle)\bar{\sigma}_j(\mathrm{Z}_k)}{\sum_{i=0}^{m} G_{k|k-1}^{(i)}(\langle q_d, \varsigma_{k|k-1}\rangle)\bar{\sigma}_i(\mathrm{Z}_k)} \\ &+ 2\frac{\sum_{j=0}^{m} j\, G_{k|k-1}^{(j+1)}(\langle q_d, \varsigma_{k|k-1}\rangle)\bar{\sigma}_j(\mathrm{Z}_k)}{\sum_{i=0}^{m} G_{k|k-1}^{(i)}(\langle q_d, \varsigma_{k|k-1}\rangle)\bar{\sigma}_i(\mathrm{Z}_k)}\langle q_d, \varsigma_{k|k-1}\rangle \\ &+ \frac{\sum_{j=0}^{m} G_{k|k-1}^{(j+2)}(\langle q_d, \varsigma_{k|k-1}\rangle)\bar{\sigma}_j(\mathrm{Z}_k)}{\sum_{i=0}^{m} G_{k|k-1}^{(i)}(\langle q_d, \varsigma_{k|k-1}\rangle)\bar{\sigma}_i(\mathrm{Z}_k)}\langle q_d, \varsigma_{k|k-1}\rangle^2. \end{aligned}$$

By defining

$$\theta_{u,v} \triangleq \frac{\sum_{j=0}^{m_k} j^u G_{k|k-1}^{(j+v)}(\langle q_d, \varsigma_{k|k-1}\rangle)\bar{\sigma}_j(\mathrm{Z}_k)}{\sum_{i=0}^{m_k} \quad G_{k|k-1}^{(i)}(\langle q_d, \varsigma_{k|k-1}\rangle)\bar{\sigma}_i(\mathrm{Z}_k)},$$

one has

$$\begin{aligned} G_k^{(1)}(1) &= \theta_{1,0} + \theta_{0,1} \cdot \langle q_d, \varsigma_{k|k-1}\rangle, \\ G_k^{(2)}(1) &= \theta_{2,0} - \theta_{1,0} + 2\theta_{1,1} \cdot \langle q_d, \varsigma_{k|k-1}\rangle + \theta_{0,2} \cdot \langle q_d, \varsigma_{k|k-1}\rangle^2, \end{aligned}$$

from which the posterior cardinality moments are obtained as given in (37) and (38). The posterior state RFS is assumed to follow a multi-object discrete-Gamma process with cardinality characterized by $\mu_{N,k} \approx \alpha_k \beta_k^{-1}$ and $\sigma_{N,k}^2 \approx \alpha_k \beta_k^{-2}$, therefore (36) holds. Since the prior cardinality distribution is also assumed to be a discrete-Gamma distribution, we write

$$\begin{aligned} G_{k|k-1}^{(\ell)}(y) &= \frac{\mathrm{d}^\ell}{\mathrm{d}y^\ell}\left(\frac{\mathrm{Li}_{-\alpha_{k|k-1}+1}(e^{-\beta_{k|k-1}}y)}{\mathrm{Li}_{-\alpha_{k|k-1}+1}(e^{-\beta_{k|k-1}}\quad)}\right) \\ G_{k|k-1}^{(\ell)}(y) &= \frac{\frac{\mathrm{d}^\ell}{\mathrm{d}y^\ell}\left(\mathrm{Li}_{-\alpha_{k|k-1}+1}(e^{-\beta_{k|k-1}}y)\right)}{\mathrm{Li}_{-\alpha_{k|k-1}+1}(e^{-\beta_{k|k-1}})}, \\ &\triangleq \frac{\hat{G}_{k|k-1}^{(\ell)}(y)}{\hat{G}_{k|k-1}^{(0)}(1)}. \end{aligned}$$

The proof is complete by noting that, in all terms of (A.9) and (A.10), $1/\hat{G}_{k|k-1}^{(0)}(1)$ appears both in the numerator and denominator and can be cancelled out to leave only terms depending on $\hat{G}_{k|k-1}^{(\ell)}(y)$. ■

Proof of Proposition 3

Lemma 1. *Let $f : [0, \infty) \to [0, \infty)$ be a nonnegative increasing function, such that $0 < \sum_{\ell=0}^{L} f(\ell) < \infty$ for $L \in \mathbb{N}_0$. If there is a nonnegative function $g : [0, \infty) \to [0, \infty)$ satisfying $g(\ell) = \mathcal{O}(f(\ell))$ such that $\frac{f(\ell)}{\sum_{\ell=0}^{L} f(\ell)} \geq \frac{g(\ell)}{\sum_{\ell=0}^{L} g(\ell)}$ for $\ell \geq L_0$, $L_0 \in \mathbb{R}_+$, then*

$$\frac{\sum_{\ell=0}^{L} \ell \cdot f(\ell)}{\sum_{\ell=0}^{L} f(\ell)} \geq \frac{\sum_{\ell=0}^{L} \ell \cdot g(\ell)}{\sum_{\ell=0}^{L} g(\ell)}, \tag{A.11}$$

for $L \geq L_0 + \frac{L_0 - \lfloor L_0 \rfloor}{2}$.

Proof: Write $w_{f,\ell} := f(\ell)/\sum_{\ell=0}^{L} f(\ell)$ and $w_{g,\ell} := g(\ell)/\sum_{\ell=0}^{L} g(\ell)$. Note that $\sum_{\ell=0}^{L} w_{f,\ell} = \sum_{\ell=0}^{L} w_{g,\ell} = 1$ and so

$$\begin{aligned} 0 \leq \sum_{\ell=\lceil L_0 \rceil}^{L} (w_{f,\ell} - w_{g,\ell}) &= \sum_{\ell=0}^{\lfloor L_0 \rfloor} (w_{g,\ell} - w_{f,\ell}) < 1, \\ \sum_{\ell=\lceil L_0 \rceil}^{L} L_0 (w_{f,\ell} - w_{g,\ell}) &= \sum_{\ell=0}^{\lfloor L_0 \rfloor} L_0 (w_{g,\ell} - w_{f,\ell}), \\ \sum_{\ell=\lceil L_0 \rceil}^{L} \ell (w_{f,\ell} - w_{g,\ell}) &\geq \sum_{\ell=0}^{\lfloor L_0 \rfloor} \ell (w_{g,\ell} - w_{f,\ell}). \end{aligned} \tag{A.12}$$

Therefore $\sum_{\ell=0}^{L} \ell \cdot w_{f,\ell} \geq \sum_{\ell=0}^{L} \ell \cdot w_{g,\ell}$, which corresponds to (A.11). The condition $L \geq L_0 + (L_0 - \lfloor L_0 \rfloor)/2$ must be met to ensure $2(L - L_0)M \geq (L_0 - \lfloor L_0 \rfloor)M$, where

$$M = \frac{(w_{f,L} - w_{g,L}) - (w_{f,\lfloor L_0 \rfloor} - w_{g,\lfloor L_0 \rfloor})}{L - \lfloor L_0 \rfloor},$$

so that $2(w_{f,2} - w_{g,2}) \geq 1(w_{g,1} - w_{f,1})$ in the particular case when $\lfloor L_0 \rfloor = 1$ and $L = 2$. ■

Proof of Proposition 3: We resort to a special type of Chebyshev's inequality that is appropriate for bounding the probabilities of non-symmetric intervals, [2], viz.

$$\varepsilon = \Pr\{0 < n < n_{\max}\} \geq 4\frac{\mu_{N,k} n_{\max} - (\mu_{N,k}^2 + \sigma_{N,k}^2)}{n_{\max}^2}. \tag{A.13}$$

where $0 < \sigma_{N,k}^2 \leq \mu_{N,k} n_{\max} - \mu_{N,k}^2$. Rearranging terms and solving the inequality for $n_{\max}$ corresponding to the upper root:

$$n_{\max} \geq \frac{2\mu_{N,k}}{\varepsilon}\left(1 + \sqrt{(1-\varepsilon) - \varepsilon\left(\frac{\sigma_{N,k}}{\mu_{N,k}}\right)^2}\right). \tag{A.14}$$

As follows, we bound $\mu_{N,k}$ as per the moment update equation [3]:

$$\begin{aligned}
\mu_{N,k} &= \frac{\sum_{n\geq 0} n \cdot \sum_{j=0}^{\min(m,n)} G_c^{(m-j)}(0) n^{\underline{j}} q_d^{n-j} \sigma_j(\mathrm{Z}) \cdot p_{k|k-1}(n)}{\sum_{n\geq 0} \sum_{j=0}^{\min(m,n)} G_c^{(m-j)}(0) n^{\underline{j}} q_d^{n-j} \sigma_j(\mathrm{Z}) \cdot p_{k|k-1}(n)} \\
&= \frac{\sum_{n\geq 0} n \cdot \sum_{j=0}^{m} \mathbb{1}_{m\leq n} \lambda^{m-j} e^{-\lambda} n^{\underline{j}} q_d^{n-j} \sigma_j(\mathrm{Z}) \cdot p_{k|k-1}(n)}{\sum_{n\geq 0} \sum_{j=0}^{m} \mathbb{1}_{m\leq n} \lambda^{m-j} e^{-\lambda} n^{\underline{j}} q_d^{n-j} \sigma_j(\mathrm{Z}) \cdot p_{k|k-1}(n)} \\
&\geq \frac{\sum_{n\geq 0} \sum_{j=0}^{m} j \cdot \mathbb{1}_{m\leq n} \lambda^{m-j} e^{-\lambda} n^{\underline{j}} q_d^{n-j} \sigma_j(\mathrm{Z}) \cdot p_{k|k-1}(n)}{\sum_{n\geq 0} \sum_{j=0}^{m} \mathbb{1}_{m\leq n} \lambda^{m-j} e^{-\lambda} n^{\underline{j}} q_d^{n-j} \sigma_j(\mathrm{Z}) \cdot p_{k|k-1}(n)} \\
&= \frac{\sum_{j=0}^{m} j \cdot \mathbb{E}_{N,k|k-1}\left[\mathbb{1}_{m\leq N} \lambda^{-j} N^{\underline{j}} q_d^{N-j}\right] \sigma_j(\mathrm{Z})}{\sum_{j=0}^{m} \quad \mathbb{E}_{N,k|k-1}\left[\mathbb{1}_{m\leq N} \lambda^{-j} N^{\underline{j}} q_d^{N-j}\right] \sigma_j(\mathrm{Z})} \\
&\geq \frac{\sum_{j=0}^{m} j \cdot \mathbb{1}_{m\leq n^\star} \lambda^{-j} \mu_{N,k|k-1}^{\underline{j}} q_d^{\mu_{N,k|k-1}-j} \sigma_j(\mathrm{Z})}{\sum_{j=0}^{m} \quad \mathbb{1}_{m\leq n^\star} \lambda^{-j} \mu_{N,k|k-1}^{\underline{j}} q_d^{\mu_{N,k|k-1}-j} \sigma_j(\mathrm{Z})},
\end{aligned} \tag{A.15}$$

where $n^{\underline{j}} \triangleq n(n-1)\ldots(n-j+1)$ is the falling factorial (Pochhamer's symbol), $n^\star \triangleq \min\left(\mu_{N,k|k-1}, \arg\max_j \sigma_j(\mathrm{Z})\right)$, and the first inequality holds because $\sum_{j=0}^{\min(m,n)} j \cdot f(j) \leq \sum_{j=0}^{\min(m,n)} n \cdot f(j)$ for any function $f : \mathbb{N}_0 \to \mathbb{R}$, $f(j) \geq 0$. Per Jensen's inequality

$$\begin{aligned}
\mathbb{E}_{N,k|k-1}\left[\mathbb{1}_{m\leq N} \varphi_j(N)\right] &\geq \mathbb{E}_{N,k|k-1}\left[\mathbb{1}_{m\leq n^\star} \varphi_j(N)\right] \\
&\geq \mathbb{1}_{m\leq n^\star} \varphi_j(\mu_{N,k|k-1}),
\end{aligned}$$

where $\varphi_j(n) := \lambda^{-j} n^{\underline{j}} q_d^{n-j}$ is convex in n, and *Lemma 1* is applied in (A.15) with $f(j) = \mathbb{E}_{N,k|k-1}\left[\mathbb{1}_{m\leq n^\star} \varphi_j(N)\right] \sigma_j(\mathrm{Z})$ and $g(j) = \mathbb{1}_{m\leq n^\star} \varphi_j(\mu_{N,k|k-1}) \sigma_j(\mathrm{Z})$ for $j \in [0..n^\star]$. This bound can be simplified by applying *Lemma 1* once more as $\varphi_j(\mu_{N,k|k-1}) \sigma_j(\mathrm{Z})$ is a nonnegative increasing function in the interval $j \in [0..n^\star]$ to give

$$\begin{aligned}
\frac{\sum_{j=0}^{n^\star} j \cdot \varphi_j(\mu_{N,k|k-1}) \sigma_j(\mathrm{Z})}{\sum_{j=0}^{n^\star} \quad \varphi_j(\mu_{N,k|k-1}) \sigma_j(\mathrm{Z})} &\gtrapprox \frac{\sum_{j=0}^{\mu_{N,k|k-1}} j \cdot \inf_j \left[\varphi_j(\mu_{N,k|k-1}) \sigma_j(\mathrm{Z})\right]}{\sum_{j=0}^{\mu_{N,k|k-1}} \quad \inf_j \left[\varphi_j(\mu_{N,k|k-1}) \sigma_j(\mathrm{Z})\right]} \\
&= \frac{\sum_{j=0}^{\mu_{N,k|k-1}} j}{\sum_{j=0}^{\mu_{N,k|k-1}} 1} = \frac{\mu_{N,k|k-1}}{2}.
\end{aligned} \tag{A.16}$$

Substituting this bound in (A.14) we conclude that there exists a $K \geq 1$ such that $n_{\max} \geq K \cdot \mu_{N,k|k-1}$. ■

Remark 2. Note that for a certain $x \in \mathbb{R}_+$ such that

$$\prod_{j=1}^{m}\left(1 - \frac{\langle p_d \ell_{z_j}, \varsigma_{k|k-1}\rangle}{c(z_j)}\right) = (1-x)^m = \sum_{j=0}^{m}\binom{m}{j} x^j,$$

$j_{\text{max}} \approx \arg\max_j \sigma_j(\mathrm{Z})$ can be computed as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}j}\left[\binom{m}{j}x^j\right] &= 0,\\
\left[\frac{\mathrm{d}}{\mathrm{d}j}\binom{m}{j}\right]x^j + \binom{m}{j}x^j \log x &= 0,\\
-\binom{m}{j}\left(\sum_{\ell=1}^{j}\ell^{-1} - \sum_{\ell=1}^{m-j}\ell^{-1}\right) + \binom{m}{j}\log x &= 0,\\
\sum_{\ell=2}^{j}\ell^{-1} - \sum_{\ell=2}^{m-j}\ell^{-1} &= \log x,\\
\log j - \log(m-j) &\approx \log x,\\
j_{\text{max}} &\approx m\frac{x}{1+x},
\end{aligned}$$

where

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}j}\binom{m}{j} &= m!\frac{-D_j(j!)\cdot(m-j)! - (j!)D_j((m-j)!)}{[j!(m-j)!]^2}\\
&= -\binom{m}{j}\frac{j!\left(-\gamma + \sum_{\ell=1}^{j}\ell^{-1}\right)(m-j)!}{j!(m-j)!}\\
&\quad -\binom{m}{j}\frac{-j!(m-j)!\left(-\gamma + \sum_{\ell=1}^{m-j}\ell^{-1}\right)}{j!(m-j)!}\\
&= -\binom{m}{j}\left(\sum_{\ell=1}^{j}\ell^{-1} - \sum_{\ell=1}^{m-j}\ell^{-1}\right),
\end{aligned}$$

and γ is the Euler–Mascheroni constant.

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