GEOMETRIC CONDITIONS OF RIGIDITY IN NONGENERIC SETTINGS

OLEG KARPENKOV

Contents

1. Introduction

This is the first draft of the chapter. The following things are yet to be done.

- To write introduction (I will do this at the very end).
- Should I add multidimensional examples of geometric conditions (aren't they already in the book)?

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Key words and phrases. Tensegrity, self-tensional equilibrium frameworks.

- Should I write regarding tie-downs (or it is already written somewhere)?
- To make the notation more friendly with the other chapters.
- Other minor corrections.

2. Stratification of the space of tensegrities

In this section we give a general definition of the space of tensegrities and its stratification and consider the example of stratification for graphs on 4 vertices.

2.1. **General definitions.** Let us consider the configuration space of all ordered n-tuples of points in $\mathbb{R}^d$. Note that it is equivalent to $(\mathbb{R}^d)^n$. Denote by $\mathrm{Tens}_G(P)$ the space of all tensegrities on $G(P)$. For every graph G on n vertices we define the function

$$f_G : (\mathbb{R}^d)^n \to \mathbb{Z}_{\geq 0}, \quad \text{where} \quad f_G(P) = \dim(\mathrm{Tens}_G(P)).$$

Here the n-tuple P is taken to the dimension of the space of all tensegrities for a given graph G and the n-tuple P.

The function f_G give rise to a natural stratification of the space of all ordered n-tuples of points in $\mathbb{R}^d$, where the strata are the level sets of f_G.

Definition 2.1. The **general stratification of the space of all tensegrities on** n **points** is the intersection of all the stratifications defined by functions f_G where G runs over all graphs on n-elements.

It is known that all the strata are semialgebraic sets [1].

2.2. **Tensegrities on 4 points in the plane.** In this subsection we consider the general stratification of the space of all tensegrities on 4 points in the plane. for a generic 4-tuple of points P we have: *no three points in a line, no two points coincide.* A generic P admits the unique up to a scalar multiplication tensegrity for a complete graph K_4 and no tensegrities for all other graphs on 4 points. There are exactly 14 connected components of generic points.

The strata of codimension 1 correspond to the configurations P where *three out of four vertices of the graph lie in a line.* For such configurations there are some $K_3 \subset K_4$ graphs that admits a nonzero tensegrity. The number of such strata is 24.

Combinatorial adjacency structure of the full dimension and codimension 1 strata is shown on Figure 1. Each oval corresponds to the union of 6 strata of codimension 1 having the same triples of points in a line. The ovals divide the plane into 14 connected component representing strata of full dimension. Finally the large dots represent the strata of higher codimension (which is rather non-trivial to show on the picture).

All the strata of codimension greater than 1 are the intersections of the closures of the codimension 1 strata. Below is the table of all these strata.

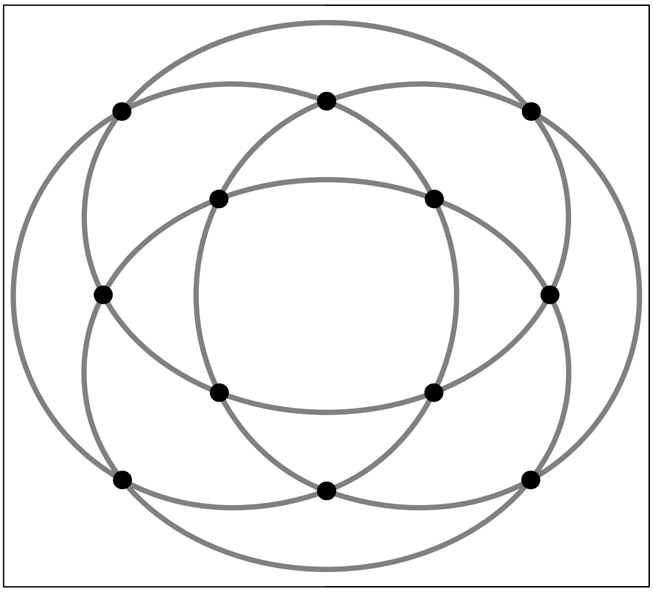


FIGURE 1. Adgacency of codimension 1 and full dimension strara of tensegrities on 4 points in the plane.

Stratum description	**codim**	**quantity**
four points in a line	2	12
two points coincide	2	12
four points in a line, two of which coincide	3	18
three points coincide	4	4
two pairs of points coincide	4	3
all points coincide	6	1

For a more detailed description of the configuration spaces we refer to papers [1]. The case of planar tensegrities on 5 points is exhaustively studied in [5]. It has the following amount of strata. The stratum of codimension 8 is the stratum corresponding to the case when all the points coincide, it is of dimension 2.

Codimension of a stratum	0	1	2	3	4	5	6	7	8
Number of strata	264	600	810	300	170	75	15	0	1

3. Geometric conditions

If the number of points is greater than 6 then a more interesting strata appear. They are described via geometric conditions of the extended Cayley geometry which we study in this section.

3.1. **Extended Cayley geometry.** First of all we define three elementary operations on points and line in the plane which have much in common with *join* and *meet* operations of

Cayley algebra (i.e., $\vee$ and $\wedge$). For more information on Cayley algebras we refer to [2], [7], and [3].

Operation I (2-point operation). Denote the first operation by $(*, *)$. This operation is a binary operation defined on the set of all points in the plane and the additional element *true*, as follows.

$(*, *)$	$\mathbf{p_1}$	$\mathbf{p_2}(\neq \mathbf{p_1})$	**true**
$\mathbf{p_1}$	*true*	$p_1 \vee p_2$	*true*
$\mathbf{p_2}(\neq \mathbf{p_1})$	$p_1 \vee p_2$	*true*	*true*
true	*true*	*true*	*true*

Here p_1 and p_2 are arbitrary distinct points and $p_1 \vee p_2$ denotes the line through them.

In case if there is no confusion we write $p_1 p_2$ instead of (p_1, p_2).

Operation II (2-line operation). Similarly we define the binary operation $* \cap *$ on the the set of all lines in the plane and the additional element *true*.

$* \cap *$	$\boldsymbol{\ell_1}$	$\boldsymbol{\ell_2}(\neq \boldsymbol{\ell_1})$	**true**
$\boldsymbol{\ell_1}$	*true*	$\ell_1 \wedge \ell_2$	*true*
$\boldsymbol{\ell_2}(\neq \boldsymbol{\ell_1})$	$\ell_1 \wedge \ell_2$	*true*	*true*
true	*true*	*true*	*true*

Here ℓ_1 and ℓ_2 are arbitrary distinct lines and $\ell_1 \vee \ell_2$ denotes intersection point of these lines.

Choice operations. Further we need two choice operations:

Operation III **Point choice**	**Operation IV** **Line choice**
Pick a point on a given line avoiding a given discrete subset of points	Pick a line through a given point avoiding a given discrete subset of lines

Let us study the following curious example.

Remark 3.1. Consider the following 4-tiple of points of $\mathbb{R}P^2$:

$$p_1 = [0 : 0 : 1], \quad p_2 = [1 : 0 : 1], \quad p_3 = [0 : 1 : 1], \quad p_4 = [1 : 1 : 1]$$

and let $p = [a : b : c]$. Then the existence of a sequence of Operations I and II expressing p in terms of the points p_1, p_2, p_3, and p_4 is equivalent to the fact that $p \in \mathbb{Q}P^2$ (i.e., (a, b, c) is proportional to a triple of rational numbers).

Relations. Finally we define the following relations:

Relation	**Notation**	**It is fulfilled in the following cases**
3-point	$(p_1, p_2, p_3) = true$	1) One of the entries is *true*; 2) Two points coincide; 3) p_1, p_2, p_3 are in a line.
point-line	$p \in \ell = true$	1) One of the entries is *true*; 2) p is contained in ℓ.
3-line	$\ell_1 \cap \ell_2 \cap \ell_3 = true$	1) One of the entries is *true*; 2) Two lines coincide; 3) ℓ_1, ℓ_2, ℓ_3 are concurrent.

3.1.1. *Geometric relations on configuration spaces of points and lines.* Let us consider geometric relations for special configuration spaces of lines. For a fix n-point configuration P consider the configuration space of (non-fixed) lines passing through prescribed points, namely

$$\big\{(\ell_1, \dots, \ell_m)\big|\ell_i \text{ passes through } p_{j(i)},\ i = 1, \dots, m\big\}$$

We denote it by $\Xi_P(R)$, where R is the list of inclusion conditions defining the configuration space (i.e., the conditions $p_{j(i)} \in \ell_i$, $i = 1, \dots, m$).

Remark 3.2. One may think of lines $\ell_1, \dots, \ell_m$ to be variables of equations while points $p_1, \dots, p_n$ to be parameters on which equations depend.

Let us give the following general definitions.

Definition 3.3. Consider $\Xi_P(R)$ as above.

- A *geometric condition* on $\Xi_P(R)$ is a composition of several geometric operations and one geometric relation on points $(p_1, \dots, p_n)$ and lines $(\ell_1, \dots, \ell_m)$ of $\Xi_P(R)$.
- We say that a system of geometric conditions on $\Xi_P(R)$ is *fulfilled* at P if there exists a choice of m lines satisfying all the conditions R such that every geometric condition is "*true*" for this choice of lines.
- Two systems of geometric conditions on n-tuples of points and m-tuples of lines satisfying conditions R are *equivalent* if for every configuration P these systems are simultaneously either fulfilled or not fulfilled for $\Xi_P(R)$ at P.

Let us consider the following simple example.

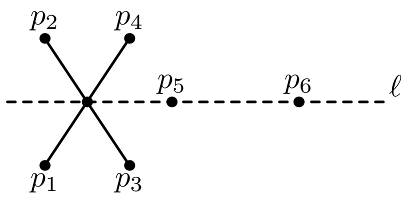
Example 3.4. For the configuration space

$$\Xi_{(p_1,\dots,p_6)}(p_5 \in \ell_1)$$

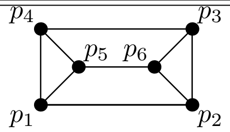
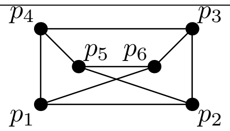
we consider the following system of geometric conditions:

$$\begin{cases} p_1p_4 \cap p_2p_3 \cap \ell = true \\ p_6 \in \ell = true \end{cases}.$$

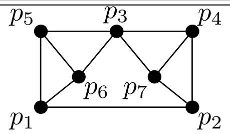
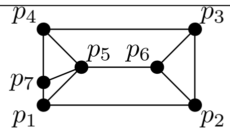
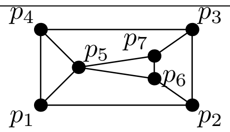
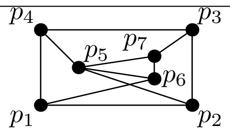
Then a generic configuration of 6 points does not fulfill this system, while the following configuration does.



3.2. **Examples in the plane.** In all the examples of this subsection the list of non-fixed lines is empty. So we have simply $\Xi_P()$. There are two interesting examples on 6 vertices, they are listed below.

Graph (6 vert.)	**Sufficient geometric conditions**
	$p_1p_2 \cap p_3p_4 \cap p_5p_6 = true$
	$\big(p_1p_2 \cap p_4p_5, p_2p_3 \cap p_5p_6, p_3p_4 \cap p_6p_1\big) = true$ (Equivalently: the six points p_1, p_2, p_3, p_4, p_5, and v_6 are on a conic)

We have the following collection of graphs on 7 vertices together with geometric conditions for them.

Graph (7 vert.)	**Sufficient geometric conditions**
	$(p_1, p_2, p_3) = true$
	$p_1p_2 \cap p_3p_4 \cap p_5p_6 = true$
	$p_1p_2 \cap p_3p_4 \cap (p_5, p_2p_6 \cap p_3p_7) = true$
	$\big(p_1p_2 \cap p_4p_5, p_2p_3 \cap (p_5, p_1p_6 \cap p_3p_7), p_3p_4 \cap (p_1, p_1p_6 \cap p_3p_7)\big) = true$

4. Geometric conditions in terms of Extended Cayley algebra

4.1. **Frameworks in general position.** Recall that a *cycle* is a graph homeomorphic to the circle. A *simple cycle* in a graph G is a subgraph of G homeomorphic to the circle.

Definition 4.1. Let us consider the following notions of general position.

- An n-tuple of lines in the projective plane are said to be *in general position* if no three lines meet in a point.
- Let $C(P)$ be a realization of a cycle C in the projective plane, where $P = (p_1, \ldots, p_n)$. We say that $C(P)$ is in general position if the lines passing through the edges of $C(P)$ are in general position.
- A framework $G(P)$ is said to be *in general position* if every simple cycle of at most $n-1$ vertex is in general position (recall that G has n vertices).

In particular if a cycle is in general position, then all its edges are of nonzero length.

4.2. **Geometric conditions for non-parallelizable tensegrities.** In this section we briefly describe the objects that are involved in the geometric conditions for non-parallelizable tensegrities. The actual algorithm to write them would be given in further subsections.

Consider an arbitrary graph G on n vertices, and let $G(P)$ be a framework with distinct points P. Consider the following arrangement of lines: *At each point $p_i \in P$ we choose $\deg p_i - 3$ ordered lines passing through p_i.* Denote by $\Xi_G(P)$ the configuration space of all such arrangements.

Remark 4.2. We have
$$\dim(\Xi_G(P)) = \sum_{i=1}^{n} \big(\deg(p_i) - 3\big),$$
In particular, if all vertices are of degree 3, then the configuration space $\Xi_G(P)$ is empty.

We use the configuration space $\Xi_G(P)$ for the detection non-parallelizable tensegrities.

Theorem 4.3. *A framework $G(P)$ in general position admits a non-parallelizable tensegrity if and only if this framework satisfies a certain system of geometric conditions on points of P and lines of $\Xi_G(P)$ for all simple cycles of G.*

Remark 4.4. The system of geometric conditions is explicitly described by the algorithm of Subsection 6.6, see also Theorem 6.19.

The proof of this theorem is rather technical, we refer an interested reader to [4].

4.3. **Conjecture on strong geometric conditions for tensegrities.** As we have seen in the examples of the tables above, all the geometric conditions are written entirely in terms of the vertices of the framework P, none of the lines of $\Xi_G(P)$ for them are involved. One might expect that this situation is general for planar tensegrities. Let us briefly discuss this here.

Definition 4.5. Let G be a graph and let $G(P)$ be one of the frameworks for G. We say that a geometric condition on points P is a *strong geometric condition* for $G(P)$ if it does not involve choice operations (i.e., operations III and IV).

In many cases geometric conditions on points and lines passing through them are equivalent to certain strong geometric conditions on points only. However this is not always the case, we illustrate this with the following example.

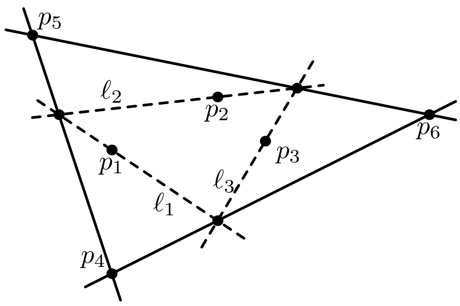
Example 4.6. In this example we deal with the configuration space

$$\Xi_{(p_1,p_2,p_3,p_4,p_5,p_6)}\big(p_1 \in \ell_1, p_2 \in \ell_2, p_3 \in \ell_3\big)$$

and the following system of geometric conditions for it:

$$\begin{cases} \ell_1 \cap \ell_2 \cap p_4p_5 = true \\ \ell_2 \cap \ell_3 \cap p_5p_6 = true \\ \ell_3 \cap \ell_1 \cap p_6p_4 = true \end{cases}$$

Below is the example of a 9-point and 3-line configuration satisfying the above system of condition.



From the one hand, this system is not equivalent to any system of strong geometric conditions on P. From the other hand this system does not come from any graph G. So it would be interesting to check if the following statement holds.

Conjecture ([4]). For every graph G there exists a system of strong geometric conditions such that a framework $G(P)$ in general position admits a non-parallelizable tensegrity if and only if P satisfies this system of strong geometric conditions.

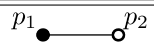
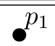
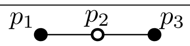
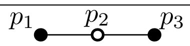
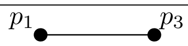
In other words, this conjecture implies that all the non-parallelizable tensegrities are described in terms of Cayley algebra operations on the *vertices* of frameworks.

5. Surgeries on graphs

Surgeries on graphs is a techniques to obtain geometric conditions on graphs while knowing geometric conditions on some another graphs. Here we change the structure of the graph locally leaving most of the vertices and vertices of the original graph unchanged. The more surgeries one knows, the smaller the set of initial graphs one needs to investigate in order to study geometric conditions for all graphs. This techniques is rather successful in the planar case and rather unstudied in higher dimensional cases. Below we describe most of the currently known graph surgeries.

In the diagrams we show a part of a graph. Black dots indicate vertices that might have some edges that we do not see on the corresponding diagram (i.e., the edges connecting such vertices with the vertices that are not in this part of the graph). The edges of white dots are precisely the edges shown in the diagram. Finally we would like to mention that surgeries usually works only if certain conditions of genericity are fulfilled. Such conditions are indicated in the last column.

Basic surgeries. We start with the simplest possible type of surgeries. These surgeries remove the points of degree 1 and 2. Here is the complete list of them.

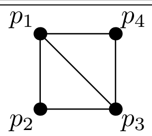
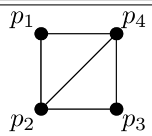
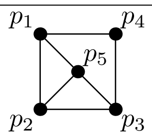
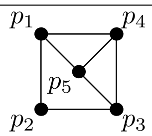
Source	**Target**	**Conditions**
		$p_1 \neq p_2$.
		The points p_1, p_2 and p_3 are not in a line.
		The points p_1, p_2 and p_3 are in a line and distinct to each other.

Dimension 1 subgraph surgeries. The second class of surgeries is rather wide. First of all we give the following definition.

Definition 5.1. We say that a triple (G, e_1, e_2) where G is a graph and e_1 and e_2 are its edges *fulfills the condition* $\langle G, e_1, e_2 \rangle$ at a framework (G, P) if the following two conditions hold.

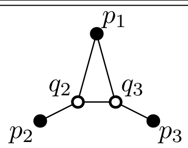
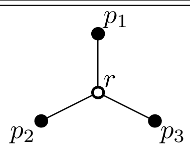
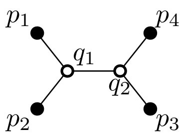
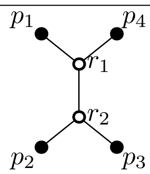
- The graph G has a unique (up to a scalar) non-zero tensegrity.
- The stresses at edges e_1 and e_2 for non-zero tensegrities on (G, P) are non-zero.

If a (G, e_1, e_2) fulfills the condition $\langle G, e_1, e_2 \rangle$ then the tensegrities for the graph $G \cup e_1$ admits non-zero tensegrities at P if and only if the graph $G \cup e_2$ admits non-zero tensegrities at P. Here we have the following list of examples.

Source	**Target**	**Conditions**
		All the triples of points are not in a line.
		The triples of points are not in a line: (p_4, p_1, p_5) and (p_i, p_{i+1}, p_5) where $i = 1, 2, 3$, and the points (p_1, p_2, p_3, p_4) are not in one line.
...	...	...
$H \setminus e_1$	$H \setminus e_2$	There exists $G \subset H$ with $e_1, e_2 \in G$ satisfying Consdition $\langle G, e_1, e_2 \rangle$.

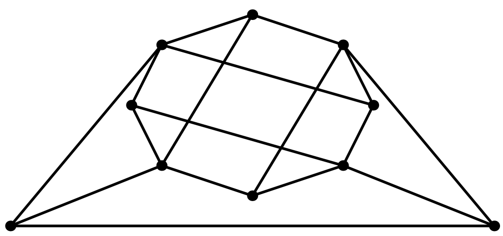
One should be careful while using these surgeries, since the condition $\langle G, e_1, e_2 \rangle$ might already contain nontrivial configuration with non-zero tensegrities. So while removing such cases one might remove possible realizations.

Degree 3 vertex surgeries. The last class of surgeries plays an important role in the planar case.

Source	**Target**	**Relations**	**Conditions**
		$r = p_2q_2 \cap p_3q_3$	the triples of points are not in a line: (p_1, q_2, q_3), (p_2, p_3, q_i), and (p_1, p_i, q_i), where $i = 2, 3$.
		$r_1 = p_1q_1 \cap p_4q_2$ $r_2 = p_2q_1 \cap p_3q_2$	the triples of points are not in a line: (p_1, p_2, q_1), (p_3, p_4, q_2), and (p_i, q_1, q_2) for $i = 1, \ldots, 4$.

The last surgery is called an HΦ-surgery. It is essentially used in the study of planar tensegrities, see in [4].

Remark on strong geometric conditions for graphs on small number of vertices. Using these surgeries one might find strong geometric conditions of realizability to all graphs having 9 vertices and less (see in [4]). The complete list of codimension 1 graphs for 8 points and less can be found in [1]. The first example of a graph which we unable to reduce to 9-point graphs using the above surgeries is as follows:



This graph is a good candidate as a counterexample to the above conjecture.

6. Algorithm to write geometric conditions of realizability of generic tensegrities

In this section we show how to construct geometric conditions for cycles mentioned in Theorem 4.3.

We start this section with the study of framed cycles, here for each framed cycle in general position we introduce a certain geometric condition related to it. This already allows us to define geometric conditions for trivalent graphs (see Subsection 6.1). For the graphs having vertices of degree greater than 3 we introduce resolution schemes for such vertices (see Subsection 6.2). Further in Subsections 6.3 and 6.4 using resolution schemes we construct the framings for simple cycles of a general graph G, which provides us with desired geometric conditions for G. Finally we summarize the construction techniques of geometric conditions defining tensegrities in Subsection 6.5 (see also Theorem 6.19).

6.1. **Framed cycles in general position.**

6.1.1. *Basic definitions.* Let us start with the following general definition.

Definition 6.1. Let $P = (p_1, \dots, p_k)$.

- A realization of the cycle $C(P)$ in the projective plane has a *framing* if every vertex p_i is equipped with a line ℓ_i passing through it.
- The realization $C(P)$ together with its framing is called the *framed cycle*. Denote it by

$$C(P, L) = \big((p_1, \dots, p_k), (\ell_1, \dots, \ell_k)\big).$$

We use the following notion of genericity for framed cycles.

Definition 6.2. A framed cycle $C(P, L)$ is in *general position* if
— the cycle $C(P)$ is in general position;
— for every admissible i we have: the line ℓ_i does not contain the points p_{i-1} and p_{i+1}.

Let $C(P, L)$ be a cycle with $k \geq 4$ vertices. for an arbitrary $i \in \{1, 2, \dots, k\}$ we set

$$\begin{aligned} p'_i &= p_{i-1}p_i \cap p_{i+1}p_{i+2}; \\ \ell'_i &= p'_i(\ell_i \cap \ell_{i+1}). \end{aligned}$$

(Here we set $p_0 = p_k$, $p_{k+1} = p_1$, and $p_{k+2} = p_2$.)

The following surgery on the framed cycle $C(P, L)$ is called a *projection operation* of a cycle:

$$\omega_i(C(P, L)) = C(P', L'),$$

where

$$\begin{aligned} P' &= (p_1, p_2, \dots, p_{i-1}, p'_i, p_{i+2}, \dots, p_k), \quad \text{and} \\ L' &= (\ell_1, \ell_2, \dots, \ell_{i-1}, \ell'_i, \ell_{i+2}, \dots, \ell_k). \end{aligned} \tag{1}$$

In other words the projection operation $\omega_i(C(P, L))$ removes the face p_ip_{i+1} by prolonging the edges $p_{i-1}p_i$ and $p_{i+1}p_{i+2}$ towards the intersection point of their lines (i.e., p'_i) and introduce a framing ℓ'_i to p'_i. See an example of $\omega_2(C(P, L))$ on Figure 2.

Remark 6.3. It is clear that the projection operation is entirely expressed by the elementary operations (Operations II and I).

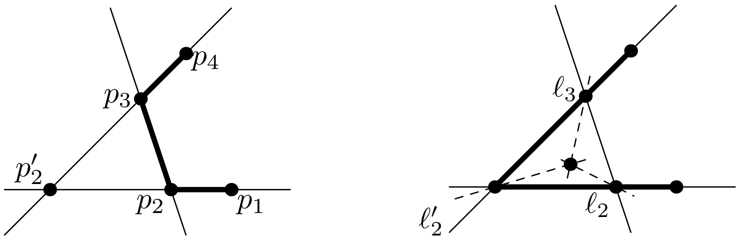


FIGURE 2. A projection operation ω_2. On the left we have: $p_2' = p_1p_2 \cap p_3p_4$; on the right: $\ell_2' = (p_2', \ell_2 \cap \ell_3)$.

6.1.2. *Geometric conditions for framed cycles.* Consider a framed cycle

$$C(P, L) = C\big((p_1, \ldots, p_k), (\ell_1, \ldots, \ell_k)\big)$$

in general position. Then the composition of $k-3$ projection operations applied to $C(P, L)$ will result in a framed triangular cycle in general position. Denote the resulting cycle by

$$C(\hat{P}, \hat{L}) = C((\hat{p}_1, \hat{p}_2, \hat{p}_3), (\hat{\ell}_1, \hat{\ell}_2, \hat{\ell}_3)).$$

Definition 6.4. Let $C(P, L)$ and $C(\hat{P}, \hat{L})$ be as above. Then the condition

$$\hat{\ell}_1 \cap \hat{\ell}_2 \cap \hat{\ell}_3 = true \tag{2}$$

is the *geometric condition* defined by C.

- The geometric condition for $C(P, L)$ does not depend on the choice of projection operations. The resulting geometric relations are equivalent.
- The geometric condition for $C(P, L)$ is a combination of $9k-9$ Operations I; $6k-6$ Operations II; and one 3-line relation on points P and lines L. (Hence it is a true geometric condition).

For simplicity one might always fix the following composition of projection operators:

$$\underbrace{\omega_1 \circ \ldots \circ \omega_1}_{k-3 \text{ times}}(C(P, L)).$$

The expression in terns of Operations I, II and one 3-line relation ir written directly from (1).

Example 6.5. For the cycles on 3, 4, and 5 vertices we have the following geometric relations:

k=3: $\ell_1 \cap \ell_2 \cap \ell_3 = true$;
k=4: $(\ell_1 \cap \ell_4, \ell_2 \cap \ell_3, p_1p_2 \cap p_3p_4) = true$;
k=5: $(\ell_2 \cap \ell_3, p_1p_2 \cap p_3p_4) \cap \ell_1 \cap (\ell_4 \cap \ell_5, p_1p_3 \cap p_3p_4) = true$.

6.1.3. *Geometric conditions for trivalent graphs.* In the case of trivalent graphs we can already write down all the geometric conditions for cycles mentioned in Theorem 4.3.

Let G be a trivalent graph and $G(P)$ be a framework in general position. Consider a cycle C in G and and set the natural framing for $C(P(C))$ (here $P(C) \subset P$) as follows. let the vertex $p_i \in C$ by adjacent to the edges $p_ip_{i,1}$, $p_ip_{i,2}$, and $p_ip_{i,3}$ of the framework $G(P)$. Without loss of generality we assume that the edges $p_ip_{i,1}$, $p_ip_{i,2}$ are edges of $C(P)$ at vertex p_i. Then we set $\ell_i(C) = p_i, p_{i,3}$. Set

$$L(C) = (\ell_1(C), \ldots, \ell_k(C)).$$

The geometric condition for the cycle $C \in G$ in Theorem 4.3 are precisely the geometric condition (2) of Definition 6.4 for $C(P(C), L(C))$. So Theorem 4.3 can be reformulated as follows.

Theorem 6.6. *Let G be a trivalent graph. A framework $G(P)$ in general position admits a non-parallelizable tensegrity if and only if every simple (framed) cycle $C(P(C), L(C))$ of G satisfies the geometric condition (2) for $C(P(C), L(C))$.*

6.2. **Resolution schemes.** Suppose now G has vertices of degree greater than 3, so we cannot write geometric conditions for cycles using Theorem 6.19. The main idea here is to consider *resolutions* at vertices of degree $k > 3$ replacing them by unrooted full binary trees with k leaves. Then one can define a similar geometric conditions for the resulting graph. We show the main steps to do that below.

6.2.1. *Definition of resolution schemes.* Let us study how to replace a vertex of the framework by a unrooted full binary tree. Recall that an edge of a tree is a *leaf* if one of its vertices is of degree 1. All other edges are *interior* edges of a tree. Recall also that an unrooted full binary tree is a tree without the root where the degree of every vertex of T is either 1 or 3.

Denote by $\mathrm{Gr}(1, \mathbb{R}P^2)$ the Grassmannian of 2-dimensional planes in $\mathbb{R}^3$ (i.e., $\mathrm{Gr}(1, \mathbb{R}P^2)$ is the set of all lines in the projective plane).

Definition 6.7. Consider an unrooted full binary tree T and let

$$\mathcal{L} : E(T) \to \mathrm{Gr}(1, \mathbb{R}P^2).$$

We say that a pair $(T, \mathcal{L})$ is a *resolution scheme* at point $p \in \mathbb{R}P^2$ if for every edge $e \in T$ it holds $p \in \mathcal{L}(e)$. Denote it by $(T, \mathcal{L})_p$.

6.2.2. *Resolution of a framework.* In what follows we restrict ourselves to graphs whose vertices are all of degree 3 or greater.

Definition 6.8. Let G be a graph on n vertices and let $G(P)$ be its framework on $P = (p_1, \ldots, p_n)$. We say that the collection

$$(G(P), ((T_1, \mathcal{L}_1)_{p_1}, \ldots, (T_n, \mathcal{L}_n)_{p_n}))$$

is a *resolution of* $G(P)$ if for every i we have:

- the resolution scheme $(T_i, \mathcal{L}_i)_{p_i}$ has $\deg p_i$ leaves.

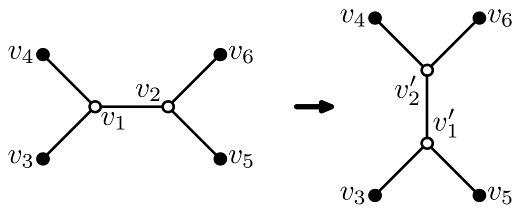
- the edges of G adjacent to p_i are enumerated by the leaves $(T_i, \mathcal{L}_i)_{p_i}$ (i.e., the one-to-one correspondence between the adjacent edges and the leaves is fixed).
- let v be a leaf at of T_i corresponding to an edge p_ip_j then

$$\mathcal{L}_i(v) = (p_i, p_j).$$

We denote it by $G(P)^{\mathcal{T}}_{\mathcal{L}}$.

6.2.3. HΦ-*surgeries on completely generic resolution schemes.* In this section we describe a surgery for a certain class of generic resolution schemes. This surgery is similar to HΦ-surgery (the second degree 3 vertex surgery shown on page 9) for graphs.

Let $(T, \mathcal{L})_p$ be a resolution scheme. Consider the tree T' obtained from T by the following flip operation:



Here white vertices are interior while black vertices can be both interior or leaves.

Let us construct the line ℓ geometrically by the following sequence of Operations I–IV.

- Operation IV: Pick a point $p_\infty \neq p \in \mathcal{L}(v_1v_2)$;
- Operation III: Pick a line $\ell_\infty \neq \mathcal{L}(v_1v_2)$ through p_∞;
- Operation III: Pick a point $p' \notin \{p, p_\infty\}$;
- Operations I and II: Define $\hat{\ell} = p(\mathcal{L}(v_1v_2) \cap \ell_\infty)$;
- Operations I: $p'' = \hat{\ell} \cap \mathcal{L}(v_2v_5)$;
- Operations I and II: $\ell' = p'(\mathcal{L}(v_1v_4) \cap \ell_\infty)$;
- Operations I and II: $\ell'' = p''(\mathcal{L}(v_2v_6) \cap \ell_\infty)$;
- Operations I: $p''' = \ell' \cap \ell''$;
- Operations II: $\ell = pp'''$.

See Figure 3.

Definition 6.9. An HΦ-*surgery on* $(T, \mathcal{L})_p$ *at the interior edge* v_1v_2 is the operation that replaces $(T, \mathcal{L})$ with $(T', \mathcal{L}')$ where $\mathcal{L}'$ is defined as follows:

$$\begin{array}{ll} \mathcal{L}'(v_1'v_2') = \ell; & \mathcal{L}'(v_1'v_3) = \mathcal{L}(v_1, v_3); \\ \mathcal{L}'(v_1'v_5) = \mathcal{L}(v_2, v_5); & \mathcal{L}'(v_2'v_4) = \mathcal{L}(v_1, v_4); \\ \mathcal{L}'(v_2'v_6) = \mathcal{L}(v_2, v_6); & \mathcal{L}'(e) = \mathcal{L}(e) \text{ for any other edge } e \end{array}$$

In fact, the resulting resolution scheme is not always well-defined due to some non-genericity phenomena (e.g, when the lines v_1v_3 and v_2v_5 coincide). So the following definition is actual here.

Definition 6.10. We say that a resolution scheme $(T, \mathcal{L})$ is *completely generic* if every composition of HΦ-surgeries is well-defined.

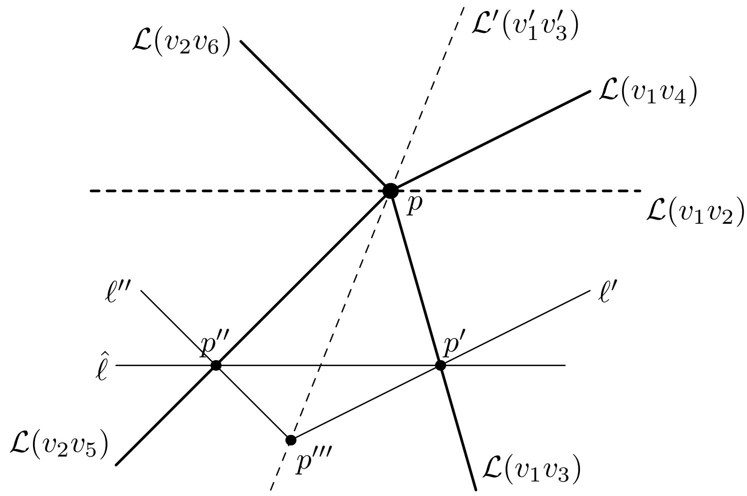


FIGURE 3. Geometric construction of $\ell = \mathcal{L}'(v_1'v_2')$.

For a geometric description of completely generic resolution trees we refer to [4].

Remark 6.11. Let $(T, \mathcal{L})_p$ be a completely generic resolution scheme (where T has n leaves). Then applying all possible different compositions of HΦ-surgeries one gets precisely $(2n-1)!!$ distinct resolutions scheme, which is the number of the all unrooted binary full trees with n marked leaves (e.g., see ex. 5.2.6 in [6]). In fact, these schemes are in a natural one-to-one correspondence with the set of all unrooted binary full trees with n marked leaves ($(T, \mathcal{L}) \to T$).

Definition 6.12. We say that a resolution of a graph is *generic* if every its resolution scheme is completely generic.

6.3. **Construction of framing for pairs of leaves in completely generic resolution schemes.** Assume that we are given by a completely generic resolution scheme $(T, \mathcal{L})_p$. Let us define a special line for every pair of leaves u, v.

Definition 6.13. Let $(T, \mathcal{L})$ be a completely generic resolution scheme. Consider a composition of HΦ-surgeries ϕ and a resolution scheme $(T', \mathcal{L}')_p$ such that the following two conditions hold

- $\phi\big((T, \mathcal{L})_p\big) = (T', \mathcal{L}')_p$;
- the leaves u and v are adjacent at the tree T' (here we use the fact that any HΦ-surgery does not affect leaves, so u and v are leaves at T').

Assume that w is the third edge of T' adjacent to the common point of the leaves u and v. Set

$$\ell_p(u, v) = \mathcal{L}'(w).$$

The following two statements clarify the correctness of Definition 6.13.

Proposition 6.14. *For a completely generic resolution scheme* $(T, \mathcal{L})$ *we have:*

- *there exists a pair* $\big(\phi, (T', \mathcal{L}')_p\big)$ *satisfying both conditions of Definition 6.13.*
- *the line* $\ell_p(v, w)$ *does not depend on the choice of* ϕ *and* $(T', \mathcal{L}')_p$.

We will skip the proof of the second statement and provide the algorithm to construct a certain pair $\big(\phi, (T', \mathcal{L}')_p\big)$ below.

Remark 6.15. **(Construction of $\ell_p(v, w)$.)** Let $v_1 \ldots v_s$ be a simple path connecting the leaf $u = v_1v_2$ where $\mathcal{L}(v_1v_2) = p_ip_j$ and the leaf $v = v_{s-1}v_s$ with framing to $\mathcal{L}(v_{s-1}v_s) = p_ip_k$. Then we consequently apply $s-3$ HΦ-surgeries along the edges v_2v_3, $v_3v_4, \ldots, v_{s-2}v_{s-1}$. As a result we have a resolution scheme $(T_i', \mathcal{L}')_{p_i}$ whose leaves $\mathcal{L}'^{-1}(e_{ij})$ and $\mathcal{L}'^{-1}(e_{ik})$ share a common vertex.

6.4. **Framed cycles associated to generic resolutions of a graph.** First, we define the framing for two adjacent edges of frameworks.

Definition 6.16. Consider a generic resolution $G(P)^{\mathcal{T}}_{\mathcal{L}}$ of a framework $G(P)$. Let p_ip_j and p_ip_k be two edges in $G(P)$ with a common vertex p_i and let v and w be the associated leaves in the resolution scheme $(T_i, \mathcal{L}_i)_{p_i}$. Then the line $\ell_{p_i}(v, w)$ introduced in Definition 6.13 is the *associated framing for the pair of edges* (e_{ij}, e_{ik}) *at* p_i. We denote it by $\ell_{j,i,k}$.

The above definition leads to the natural notion of framed cycles associated to generic resolutions of a graph.

Definition 6.17. Consider a generic resolution $G(P)^{\mathcal{T}}_{\mathcal{L}}$ of a framework $G(P)$. Let $C = p_1 \ldots p_s$ be a cycle in $G(P)$. Denote by $C(G, \mathcal{T}, P, \mathcal{L})$ the framed cycle with vertices $p_1 \ldots p_s$ such that

$$\ell_{i-1,i,i+1} \text{ is a framing at } q_i \text{ (for } i = 1, \ldots, s).$$

We say that this cycle is a *framed cycle associated to* $G(P)^{\mathcal{T}}_{\mathcal{L}}$.

Definition 6.18. Any line $\ell_{j,i,k}$ in the framing associated to a $G(P)^{\mathcal{T}}_{\mathcal{L}}$ is explicitly expressed in terms of Operations I–IV on the lines of $\mathcal{L}(E(G_{\mathcal{T}}))$ (by Definition 6.9 and Remark 6.15). Let us fix one of the possible composition of Operations I–IV defining the framing $\ell_{j,i,k}$ and call it the *sequence of geometric operations* defining $\ell_{j,i,k}$.

6.5. **Natural correspondences between $\Xi_G(P)$ and the set of all resolutions for $G(P)$.** Given a framework $G(P)$ and the corresponding configuration space $\Xi_G(P)$. Let us fix the following data and notation:

- Fix a resolution tree T_i at each vertex p_i and denote $\mathcal{T} = (T_1, \ldots, T_n)$.
- Denote by $G(P)^{\mathcal{T}}_{*}$ the configuration space of all resolutions for the framework $G(P)$ whose resolution schemes at p_i has a tree T_i for $i = 1 \ldots, n$.
- Enumerate all interior edges every tree T_i.
- Enumerate all the lines of $\Xi_G(P)$ passing through every point p_i.

Once the above is done, we have a natural isomorphism between $G(P)^{\mathcal{T}}_{*}$ and $\Xi_G(P)$. Here the line l_j at point p_i of the configuration in $\Xi_G(P)$ corresponds to the line $\mathcal{L}_i(v_j)$ of the j-th interior edge of the resolution scheme $(T_i, \mathcal{L}_i)_{p_i}$.

6.6. **Techniques to construct geometric conditions defining tensegrities.** Finally let us show step by step how to write down the system of geometric conditions for the existence of non-parallelizable tensegrities.

Input Data. We start with a framework $G(P)$ in general position.

Step 1. Fix $\mathcal{T} = (T_1, \ldots, T_n)$ in resolution schemes at all vertices and associate the configuration space $\Xi_G(P)$ with $G(P)_*^{\mathcal{T}}$ (see Subsection 6.5).

Step 2. Pick all simple cycles $C_1, \ldots, C_N$ in G that does not pass through all the points of G.

Step 3. Write all lines $\ell_{j,i,k}$ in terms of compositions of Operations I–IV on the points of P and the lines corresponding to the interior edges of $G(P)_*^{\mathcal{T}}$. (See Definition 6.18.)

Step 4. Define framed cycles $C_i(G, \mathcal{T}, P, \mathcal{L})$ related to C_i for $i = 1, \ldots, N$. Here we use the lines obtained in Step 3 for framings.

Step 5. Write down geometric conditions for $C_i(G, \mathcal{T}, P, \mathcal{L})$ for $i = 1, \ldots, N$ in terms of lines $\ell_{i,j,k}$. (See the construction of Subsubsection 6.1.2.)

Step 6. Combining together Step 3 and Step 5 we write down geometric conditions for framed cycles $C_i(G, \mathcal{T}, P, \mathcal{L})$ for $i = 1, \ldots, N$ in terms of P and the lines of $\Xi_G(P)$ (which is isomorphic to $G(P)_*^{\mathcal{T}}$, see Subsubsection 6.5).

Output data. As an output we get the system of geometric conditions on the space Ξ_G. By Theorem 4.3 this system is fulfilled if and only if there exists a non-parallelizable tensegrity at Ξ_G.

Theorem 6.19. *The above algorithm produces geometric conditions for Theorem 4.3.*

Remark 6.20. In fact, at Step 2 it is sufficient to pick only the simple cycles generating $H_1(G)$. In practice it is sufficient to choose even less cycles to get the corresponding geometric existence condition of a non-parallelizable tensegrity.

For further details and justification of the above algorithm we refer to [4].

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E-mail address, Oleg Karpenkov: karpenk@liv.ac.uk