ON THE CONTINUOUS PART OF CODIMENSION TWO ALGEBRAIC CYCLES ON THREE-DIMENSIONAL VARIETIES

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Abstract. Let X be a non-singular projective threefold over an algebraically closed field, and let $A^2(X)$ be the group of algebraically trivial codimension 2 algebraic cycles on X modulo rational equivalence with coefficients in $\mathbb{Q}$. Assume X is, up to birational equivalence, fibered over an integral curve C with the generic fiber $X_{\bar{\eta}}$ satisfying the following three conditions: (i) the motive $M(X_{\bar{\eta}})$ is finite-dimensional, (ii) $H^1_{et}(X_{\bar{\eta}}, \mathbb{Q}_l) = 0$ and (iii) $H^2_{et}(X_{\bar{\eta}}, \mathbb{Q}_l(1))$ is spanned by divisors on $X_{\bar{\eta}}$. We prove that, provided these three assumptions, there exists a curve Y and a correspondence z on $Y \times X$, such that z induces an epimorphism $A^1(Y) \to A^2(X)$, where $A^1(Y)$ is isomorphic to $\mathrm{Pic}^0(Y)$ tensored with $\mathbb{Q}$.

1. Introduction

Let X be a non-singular projective variety of dimension d over an algebraically closed field k. For any integer $0 \leq i \leq d$ let $CH^i(X)$ be a Chow group of codimension i algebraic cycles on X with coefficients in $\mathbb{Q}$ modulo rational equivalence on X. In this paper we are studying a subgroup $A^i(X)$ in $CH^i(X)$ generated by cycles algebraically equivalent to zero on X. Recall that an algebraic cycle is called algebraically equivalent to zero if it can be deformed to 0 in a trivial family over a non-singular projective curve. In that sense $A^i(X)$ can be viewed as a continuous part of the group $CH^i(X)$, see [2]. The group $A^1(X)$ can be identified with the Picard group $\mathrm{Pic}^0(X)$ tensored with $\mathbb{Q}$. In the highest codimension, $A^d(X)$ is a group of zero-cycles of degree zero modulo rational equivalence relation on X.

The group $A^i(X)$ is said to be (weakly) representable if there exists a curve Y and a cycle class z in $CH^i(Y \times X)$, such that the induced homomorphism $z_* : A^1(Y) \to A^i(X)$ is surjective. Working with integer coefficients, it is required moreover that the kernel of the homomorphism z_* is a closed algebraic subgroup in the jacobian of the curve Y. In the

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present paper we will systematically use rational coefficients in Chow groups, so that the matter of closeness of the kernel is omitted and representability of $A^i(X)$ means the existence of a surjective homomorphism z_*.

The group $A^1(X)$ is representable if X is a curve. For a surface X over $\mathbb{C}$, if the kernel of the Albanese homomorphism is trivial the group $A^2(X)$ is representable too. Representability of 0-cycles on a surface without non-trivial everywhere holomorphic 2-forms was conjectured by S.Bloch, and it was a part of the intuition leading to the whole Bloch-Beilinson motivic vision of algebraic cycles, [8]. On the other hand, if X is a surface with $p_g > 0$, the group $A^2(X)$ is too far from to be representable, [10]. Recently it was discovered that $A^2(X)$ is representable for a surface X with $p_g = 0$ if and only if the motive $M(X)$ of X is finite-dimensional in the sense of Kimura, [7], [9].

The aim of the present paper is to show that motivic finite-dimensionality can be also useful for the study of $A^2(X)$ in three-dimensional case. We will show that for a certain type of threefolds X representability of $A^2(X)$ follows from finite-dimensionality of the motive of the generic fiber of an appropriate fibration of X over a curve. In full generality this statement should not be true, of course.

To state our result precisely we need to fix some more notation. Let $f : X \to C$ be a fibering of a threefold over an integral curve, $\eta = \mathrm{Spec}(k(C))$ be the generic point on C, $\bar{\eta} = \mathrm{Spec}(\overline{k(C)})$ the spectrum of the closure of $k(C)$, and $X_{\bar{\eta}}$ the generic fiber of the morphism f over $\bar{\eta}$. Let also $H^1_{et}(X_{\bar{\eta}}, \mathbb{Q}_l)$ and $H^2_{et}(X_{\bar{\eta}}, \mathbb{Q}_l(1))$ be the etale l-adic cohomology groups, where l is a prime different from the characteristic of the ground field k.

Theorem 1. *Let X be a non-singular projective threefold over an algebraically closed field k. Assume that, up to a birational equivalence (via blow ups and blow downs with smooth centers), can be fibered over a curve, and the generic fiber $X_{\bar{\eta}}$ of this fibering satisfies the following three assumptions:*

(i) the motive $M(X'_{\bar{\eta}})$ is finite-dimensional;
(ii) $H^1_{et}(X'_{\bar{\eta}}, \mathbb{Q}_l) = 0$ and
(iii) $H^2_{et}(X'_{\bar{\eta}}, \mathbb{Q}_l(1))$ is spanned by divisors on $X'_{\bar{\eta}}$.

Then the group $A^2(X)$ is representable in the above sense.

Let us see how much that theorem is applicable. If $k = \mathbb{C}$ then, according to the Minimal Model Program over $\mathbb{C}$, all non-singular projective threefolds can be divided into two parts: those which are birational to a $\mathbb{Q}$-factorial X' with at most terminal singularities and $K_{X'}$ nef, and those which are birational to a $\mathbb{Q}$-factorial X' with at most terminal singularities with an extremal contraction $X' \to C$ being a Mori fibration. The second case can be divided into 3 subcases: if $\dim(C) = 0$ then X'

is Fano, if $\dim(C) = 2$ then X' is a conic bundle, and if $\dim(C) = 1$ then the generic fiber X'_η is a non-singular projective Del Pezzo surface over η. In the last case X satisfies the assumptions of Theorem 1, whence representability of $A^2(X)$. A typical example here is a general non-singular cubic X in $\mathbb{P}^4$. The group $A^2(X)$ for X is known to be representable (H.Clemens, P.Griffiths) and the motive of the generic hyperplane section is finite-dimensional. If we take a general quartic $X \subset \mathbb{P}^4$, then $A^2(X)$ is representable as well (S.Bloch, J.Murre), but its non-singular hyperplane sections are K3's - those surfaces whose motivic finite-dimensionality is not known yet.

The paper is divided into two parts. In Section 2 we recall some known facts and also prove preliminary results on representability of the continuous part of codimension two algebraic cycles on threefolds. Section 3 is the main part of the paper where we make main computations and prove Theorem 1.

2. *Preliminary results*

Let S be a non-singular connected quasi-projective variety over k, and let X and Y be two non-singular and projective schemes over S. Let $X = \cup_j X_j$ be the connected components of X. For any non-negative m let

$$Corr_S^m(X, Y) = \oplus_j CH^{e_j+m}(X_j \times_S Y)$$

be the group of relative correspondences of degree m from X to Y over S, where e_j is the relative dimension of X_j over S. For example, given a morphism $f : X \to Y$ over S, the transpose Γ_f^t of its graph Γ_f is in $Corr_S^0(X, Y)$. For any two correspondences $f : X \to Y$ and $g : Y \to Z$ their composition $g \circ f$ is defined, as usual, by the formula

$$g \circ f = p_{13*}(p_{12}^*(f) \cdot p_{23}^*(g)) ,$$

where the central dot denotes the intersection of cycle classes in the sense of [4], and the projections are projections of a fibered product over S. The category $\mathscr{M}(S)$ of Chow-motives over S with coefficients in $\mathbb{Q}$ can be defined then as a pseudoabelian envelope of the category of correspondences over S with certain "Tate twists" indexed by integers. Objects in $\mathscr{M}(S)$ are triples

$$(X/S, p, n) ,$$

where X is non-singular projective over S, $p \in Corr_S^0(X, X)$ and n is an integer. For any X/S its motive $M(X/S)$ is defined by the relative diagonal $\Delta_{X/S}$, $M(X/S) = (X/S, \Delta_{X/S}, 0)$, and for any morphism $f : X \to Y$ over S the correspondence Γ_f^t defines a morphism $M(f) : M(Y/S) \to M(X/S)$. The category $\mathscr{M}(S)$ is rigid with a tensor product satisfying the formula

$$M(X/S) \otimes M(Y/S) = M(X \times_S Y) ,$$

so that the functor M from the category of non-singular projective schemes over S to $\mathscr{M}(S)$ is tensor. The scheme S/S indexed by 0 gives the unite $\mathbb{1}_S$ in $\mathscr{M}(S)$, and when it is indexed by -1, it gives the Lefschetz motive $\mathbb{L}_S$. If E is a multisection of degree $w > 0$ of X/S, we set

$$\pi_0 = \frac{1}{w}\,[E \times_S X] \quad \text{and} \quad \pi_{2e} = \frac{1}{w}\,[X \times_S E]\ ,$$

where e is the relative dimension of X/S. Then one has the standard isomorphisms $\mathbb{1}_S \cong (X, \pi_0, 0)$ and $\mathbb{L}_S^{\otimes e} \cong (X, \pi_{2e}, 0)$. Finally, if $f : T \to S$ is a morphism of base schemes over k, then f gives a base change tensor functor $f^* : \mathscr{M}(S) \to \mathscr{M}(T)$. All the details about Chow motives over a non-singular base can be found, for instance, in [5].

Below we will use basic facts from the theory of finite dimensional motives, or, more generally, finite dimensional objects, see [9] or [1]. Roughly speaking, once we have a tensor $\mathbb{Q}$-linear pseudo-abelian category $\mathscr{C}$, one can define wedge and symmetric powers of any object in $\mathscr{C}$. Then we say that $X \in \mathrm{Ob}(\mathscr{C})$ is finite-dimensional, [9], if it can be decomposed into a direct sum, $X = Y \oplus Z$, such that $\wedge^m Y = 0$ and $\mathrm{Sym}^n Z = 0$ for some non-negative integers m and n. The property to be finite dimensional is closed under direct sums, tensor products, etc.

Let now X and Y be objects in $\mathscr{C}$. As $\mathscr{C}$ is rigid its "internal Hom" is connected with the duality in $\mathscr{C}$ through the formula

$$\underline{Hom}(X, Y) \cong \check{X} \otimes Y\ .$$

In particular, we have an isomorphism $\underline{Hom}(X, X) \cong \check{X} \otimes X$ whose composition with the evaluation morphism $\check{X} \otimes X \to \mathbb{1}$ gives a morphism

$$\underline{Hom}(X, X) \longrightarrow \mathbb{1}\ .$$

Applying the functor $\mathrm{Hom}(\mathbb{1}, -)$ to the both parts we get so-called trace morphism

$$\mathrm{tr}_X : \mathrm{End}(X) \longrightarrow \mathrm{End}(\mathbb{1})\ .$$

In all rigid categories used below $\mathrm{End}(\mathbb{1}) = \mathbb{Q}$, so that for any $f : X \to X$ its trace $\mathrm{tr}_X(f)$ is just a number in $\mathbb{Q}$. A morphism $f : X \to Y$ in $\mathscr{C}$ is said to be numerically trivial if for any morphism $g : Y \to X$ the trace $\mathrm{tr}(g \circ f)$ is equal to zero, [1, 7.1.1-7.1.2].

Proposition 2. *Let $\mathscr{C}$ be a tensor $\mathbb{Q}$-linear pseudo-abelian category, let X be a finite-dimensional object in $\mathscr{C}$, and let f be a numerically trivial endomorphism of X. Then f is nilpotent in the ring* $\mathrm{End}(X)$.

Proof. See [9, 7.5] for Chow motives and [1, 9.1.14] in the abstract setting. □

Lemma 3. *Let $F : \mathscr{C}_1 \to \mathscr{C}_2$ be a tensor functor between two rigid tensor $\mathbb{Q}$-linear pseudo-abelian categories. Assume F induces an injection*

$\mathrm{End}(\mathbb{1}_{\mathscr{C}_1}) \hookrightarrow \mathrm{End}(\mathbb{1}_{\mathscr{C}_2})$. *Then, if* X *is a finite-dimensional object in* $\mathscr{C}_1$ *and* $F(X) = 0$, *it follows that* $X = 0$ *as well.*

Proof. Let g be an endomorphism of X. Then $F(\mathrm{tr}(g \circ 1_X)) = \mathrm{tr}(F(g \circ 1_X))$, see [3], page 116. Since $F(X) = 0$, we have $F(g \circ 1_X) = 0$. Then $\mathrm{tr}(g \circ 1_X) = 0$ because F is an injection on the rings of endomorphisms of units. Hence, the identity morphism 1_X is numerically trivial. Since X is finite dimensional, it is trivial by Proposition 2. □

The following lemma was pointed out to me by J. Ayoub:

Lemma 4. *Let* S *be a non-singular integral variety over* k, *let* η *be the generic point of* S, *and let* W *be the set of Zariski open subsets in* S. *Then*

$$\mathscr{M}(\eta) = \mathrm{colim}_{\,U \in W} \mathscr{M}(U) \ ,$$

where the colimit is taken in the 2*-category of pseudo-abelian tensor categories with coefficients in* $\mathbb{Q}$.

Proof. Assume we are given with a tensor $\mathbb{Q}$-linear pseudo-abelian category $\mathscr{T}$ and a set of tensor functors $G_U : \mathscr{M}(U) \to \mathscr{T}$, compatible with restriction functors $i^* : \mathscr{M}(U) \to \mathscr{M}(V)$ for each inclusion of Zariski open subsets $i : V \subset U$. We need to show that there exists a unique functor $F : \mathscr{M}(\eta) \to \mathscr{T}$, such that the composition of a pull-back $\mathscr{M}(U) \to \mathscr{M}(\eta)$ with F coincides with G_U for each U from W. The proof is, actually, just a systematic use of spreads of algebraic cycles and the localization sequence for Chow groups. Indeed, let $M = (X, p, n)$ be a motive over η, and let X' and p' be spreads of X and p respectively over some Zariski open subset U in S. Shrinking U if necessary and applying the localization for Chow groups we may assume that $p'^2 = p'$, i.e. p' is a relative projector over U. The triple $M' = (X', p', n)$ is an object in $\mathscr{M}(U)$, so that we define $F(M)$ to be $G_U(M')$. Systematically shrinking Zariski open subsets one can easily show that such defined $F(M)$ does not depend on the choice of spreads. On morphisms F can be defined in a similar way, because morphisms in Chow motives are just algebraic cycles. The uniqueness is evident when taking into account the localization for Chow groups again. □

Also we need to recall an equivalent reformulation of representability of $A^2(X)$. Let X be a non-singular projective threefold X over an algebraically closed field k. As we pointed out in Introduction, the group $A^2(X)$ is representable if there exists a non-singular projective curve Y and a correspondence $a \in CH^2(Y \times X)$, such the homomorphism $a_* : A^1(Y) \to A^2(X)$ induced by a is surjective, where $A^1(Y)$ is $\mathrm{Pic}^0(Y) \cong \mathrm{Alb}(Y)$ tensored with $\mathbb{Q}$. By technical reasons it is convenient to introduce another one definition: $A^2(X)$ is representable if there exist a finite collection of non-singular projective curves $Y_1, \dots, Y_m$ and correspondences $a_i \in CH^2(Y_i \times X)$, such that the homomorphism

$$\sum_{i=1}^{m} (a_i)_* : \oplus_{i=1}^{m} A^1(Y_i) \longrightarrow A^2(X)$$

is surjective. Evidently, the first definition implies the second one. The following argument for the inverse implication has been taken from [2]. By Bertini's theorem, we take a non-singular one-dimensional linear section Y of $Y_1 \times \dots \times Y_m$. The inclusion $Y \hookrightarrow Y_1 \times \dots \times Y_m$ gives rise to a surjection on Albanese varieties $\mathrm{Alb}(Y) \to \mathrm{Alb}(Y_1 \times \dots \times Y_m)$. Fix a closed point y_i on each Y_i. For any $1 \leq i \leq m$ the points $y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_m$ give the embedding $Y_i \hookrightarrow Y_1 \times \dots \times Y_m$ sending any point $y \in Y_i$ into the point $(y_1, \dots, y_{i-1}, y, y_{i+1}, \dots, y_m)$, where y stays on i-th place. Composing such an embedding with the corresponding projection we get the identity map on Y_i. Applying the functor Alb we get the homomorphisms $\mathrm{Alb}(Y_i) \to \mathrm{Alb}(Y_1 \times \dots \times Y_m)$ induced by the above embeddings and the homomorphisms $\mathrm{Alb}(Y_1 \times \dots \times Y_m) \to \mathrm{Alb}(Y_i)$ induced by the projections. Since a finite product of abelian groups is also a coproduct, we get two homomorphisms $\mathrm{Alb}(Y_1) \times \dots \times \mathrm{Alb}(Y_m) \to \mathrm{Alb}(Y_1 \times \dots \times Y_m)$ and $\mathrm{Alb}(Y_1 \times \dots \times Y_m) \to \mathrm{Alb}(Y_1) \times \dots \times \mathrm{Alb}(Y_m)$ whose composition is identity on $\mathrm{Alb}(Y_1) \times \dots \times \mathrm{Alb}(Y_1)$. It follows that the second homomorphism is surjective. Composing it with the above surjective homomorphism $\mathrm{Alb}(Y) \to \mathrm{Alb}(Y_1 \times \dots \times Y_m)$ we get a surjective homomorphism $\mathrm{Alb}(Y) \to \times_{i=1}^{m} \mathrm{Alb}(Y_i) = \oplus_{i=1}^{m} \mathrm{Alb}(Y_i)$. Since Y and all $Y_1, \dots, Y_m$ are non-singular projective curves, we can replace Alb by A^1 getting a surjective homomorphism $A^1(Y) \to \oplus_{i=1}^{m} A^1(Y_i)$. Composing it with the above homomorphism $\sum_{i=1}^{m} (a_i)_* : \oplus_{i=1}^{m} A^1(Y_i) \to A^2(X)$ we get a surjective homomorphism $A^1(Y) \to A^2(X)$, as required. Thus, both definitions are equivalent.

3. *The proof of Theorem 1*

Let X be a non-singular projective threeefold over an algebraically closed field k satisfying the assumptions of Theorem 1. The below proof can be divided into two parts - chain of reductions and computations with relative correspondences. The reductions consist, roughly speaking, in finite extensions of the base curve C, cutting finite collections of points out of C and, respectively, removing their fibers out of X. Then, of course, we will work with a new family dealing with A^2 in the non-compact case. The point is that representability of $A^2(X)$ can be defined also for a non-compact X, and if X is a threefold fibered over a curve, representability of $A^2(X)$ is local on the base.

To be more precise, we start with the following two lemmas both proved in [2]. Since we do not require the closeness of kernel in the definition of representability of $A^2(X)$ (see Introduction) both lemmas follow from the general properties of Chow groups considered in [4].

Lemma 5. *Let $Y \to X$ be a morphism of non-singular projective varieties of finite degree over k. If $A^2(Y)$ is representable, then $A^2(X)$ is representable as well.*

Lemma 6. *Let X and X' be two non-singular projective varieties over a field. Assume that X is birationally equivalent to X' by means of a chain of blowing ups and blowing downs with non-singular centers. Then representability of $A^2(X)$ is equivalent to representability of $A^2(X')$.*

Reduction 1

By Lemma 6 we can think X as a threefold fibered over a curve,

$$f : X \longrightarrow C ,$$

with a generic fiber satisfying the three assumptions in Theorem 1.

Reduction 2

Let $\eta = \mathrm{Spec}(k(C))$ be the generic point of the curve C and let $\bar{\eta}$ be the spectrum of an algebraic closure $\overline{k(C)}$ of the function field $k(C)$. We assume that $H^1_{et}(X_{\bar{\eta}}, \mathbb{Q}_l) = 0$ and that $H^2_{et}(X_{\bar{\eta}}, \mathbb{Q}_l(1))$ is algebraic. Let b_2 be the second Betti number for the generic fiber $X_{\bar{\eta}}$ and let

$$D_1, \ldots, D_{b_2}$$

be divisors over $\bar{\eta}$ generating $H^2_{et}(X_{\bar{\eta}}, \mathbb{Q}_l)$. We can extend C by a finite map $C' \to C$ so that, if $X' = X \times_C C'$ is a base change, the divisors D_i are defined over the generic point η' of the curve C'. Since $X' \to X$ is a morphism of finite degree, we may assume that the divisors $D_1, \ldots, D_{b_2}$ are defined over η by Lemma 5. And, of course, to remain within the category of non-singular varieties we need to resolve singularities on X' and apply Lemma 6 once again.

Now we need the following easy lemma:

Lemma 7. *Let X be a non-singular projective threefold over k and let $A^2(X) = V \oplus W$ be a splitting of the $\mathbb{Q}$-vector space $A^2(X)$ into two subspaces. Assume, furthermore, that V is finite-dimensional. Then $A^2(X)$ is representable if and only if there exists a finite collection of non-singular projective curves $Y_1, \ldots, Y_m$ and correspondences $a_i \in CH^2(Y_i \times X)$, such that the homomorphism $\sum_{i=1}^m (a_i)_* : \oplus_{i=1}^m A^1(Y_i) \to A^2(X)$ is onto the subspace W. In other words, representability of $A^2(X)$ is up to a finite-dimensional subspace in $A^2(X)$.*

Proof. Let $\{v_1, \dots, v_n\}$ be a basis for V. For each index j let B_j be an algebraically equivalent to zero algebraic cycle representing the cycle class v_j. By the definition of algebraic equivalence there exists a non-singular projective curve T_j, two closed points p_j and q_j on T_j, and an algebraic cycle Z_j on $T_j \times X$, such that Z_j intersects the divisors $p_j \times X$ and $q_j \times X$ properly, and these intersections are B_j and 0 respectively. In that case, if z_j is a class of the cycle A_j, the value of the homomorphism $(z_j)_*$ on the class of the algebraically trivial zero-cycle $p_j - q_j$ is exactly the cycle class v_j. Thus, the homomorphism $\sum (z_j)_* : \oplus A^1(T_j) \to A^2(X)$ covers the space V. To complete the proof we need only to add the curves Y_i from the assumptions and use the second equivalent definition of representability of $A^2(X)$. □

Reduction 3

Let $\{p_1, \dots, p_m\}$ be a finite collection of closed points on the curve C including all the singular points of the morphism f, let

$$U = C - \{p_1, \dots, p_m\}$$

and let

$$Y = X - f^{-1}(\{p_1, \dots, p_m\}) \ .$$

We will denote the map

$$f : Y \longrightarrow U$$

by the same letter f. By the localization exact sequence

$$\oplus_{j=1}^m CH^1(X_{p_j}) \longrightarrow CH^2(X) \longrightarrow CH^2(Y) \to 0$$

the $\mathbb{Q}$-vector space $CH^2(X)$ splits as

$$CH^2(X) = CH^2(Y) \oplus I \ ,$$

where

$$I = \mathrm{im}(\oplus_j CH^1(X_{p_j}) \longrightarrow CH^2(X)) \ ,$$

or, equivalently, the localization homomorphism $CH^2(X) \to CH^2(Y)$ has a section $CH^2(Y) \to CH^2(X)$. It is not hard to see that for any cycle class $\alpha \in A^2(Y)$ one can find a cycle class β in the preimage of α with respect to the surjective homomorphism $CH^2(X) \longrightarrow CH^2(Y)$, such that β is algebraically trivial as well. Then we have a surjective localization homomorphism

$$A^2(X) \longrightarrow A^2(Y)$$

and the splitting

$$A^2(X) = A^2(Y) \oplus J \ ,$$

where $J = I \cap A^2(X)$ inside $CH^2(X)$, if we fix a section $A^2(Y) \to A^2(X)$ and identify $A^2(Y)$ with its image in $A^2(X)$.

For each index j let $\tilde{X}_{p_j}$ be a resolution of singularities of the surface X_{p_j} ($\tilde{X}_{p_j} = X_{p_j}$ if X_{p_j} is non-singular). Note that for surfaces we have resolution of singularities in any characteristic, so we need not to make any restrictions on the ground field k here. By Bertini's theorem, for any irreducible component Z of $\tilde{X}_{p_j}$ there exists a smooth linear section T of the Picard variety $P = \mathrm{Pic}^0(Z)$ of Z. Applying Albanese functor to the embedding $T \hookrightarrow P$ we get a surjective homomorphism $J = \mathrm{Alb}(C) \to \mathrm{Alb}(P) \cong P$ where J is the Jacobian of the curve C. But any such homomorphism can be induced by a divisor D on the product $C \times Z$ after tensoring with $\mathbb{Q}$, see [12]. In other words, $D_* : A^1(T) \to A^1(Z)$ is surjective. Since the push-forward of a blowing up is surjective on Chow groups, [4, 6.7(b)], one can easily construct a finite collection of non-singular curves $T_1, \ldots, T_m$, and divisors D_{ij} on $T_i \times X_{p_j}$, such that the corresponding homomorphism $\sum_{i,j}(D_{ij})_*$ from $\sum_i A^1(T_i)$ to $\sum_j A^1(X_{p_j})$ is surjective. Since Neron-Severi group $NS_{\mathbb{Q}}(Z) = CH^1(Z)/A^1(Z)$ is finite-dimensional for each component Z in $\tilde{X}_{p_j}$, the complement of $\sum_j A^1(X_{p_j})$ in $\sum_j CH^1(X_{p_j})$ is a finite-dimensional $\mathbb{Q}$-vector space. It follows, that the kernel J of the localization homomorphism $A^2(X) \to A^2(Y)$ splits into two $\mathbb{Q}$-vector subspaces V and W, where W is covered by A^1 of the above curves T_i via the divisors D_{ij}, and the second subspace V is finite-dimensional. Then, by Lemma 7, and also taking into account the second definition of representability of $A^2(X)$, we see that in order to prove that $A^2(X)$ is representable we need only to show that there exists another one finite collection of non-singular projective curves and correspondences from them to X, such that the corresponding homomorphism covers the complement $A^2(Y)$ of $J = V \oplus W$ inside $A^2(X)$. In other words, in proving representability of $A^2(X)$ we can cut out any finite number of fibers of the map $X \to C$ taking into account the group $A^2(Y)$ only.

Reduction 4

Extending C more, if necessary, we may assume that X_η has a rational point over η. This rational point induces a section of the map $f : Y \to U$. If E is an image of that section, its self-intersection $E \cdot E$ is trivial on Y because the codimension of E in X is equal to two (or, for instance, because of another one cutting of additional fibers of the map f). Then it follows that the relative projectors

$$\pi_0 = E \times_U Y \ , \quad \pi_4 = Y \times_U E$$

are pairwise orthogonal, and hence

$$\pi_2 = \Delta_{Y/U} - \pi_0 - \pi_4$$

is a relative second Murre projector for the whole family $f : Y \to U$.

Reduction 5

Let $M^2(Y/U)$ be the relative motive defined by the projector π_2. Then we get the following standard decomposition:

$$M(Y/U) = \mathbb{1}_U \oplus M^2(Y/U) \oplus \mathbb{L}_U^{\otimes 2} .$$

By the assumption of Theorem 1, the motive $M(X_{\bar\eta})$ is finite-dimensional. If we look through the definition of motivic finite-dimensionality from the geometrical viewpoint, it means an existence of some algebraic cycles on $X_{\bar\eta}^{\times 2N} \times \mathbb{P}^1$ providing rational triviality of the wedge and symmetric powers of even and odd components of the diagonal for $X_{\bar\eta}$. These algebraic cycles have their common minimal field of definition, which is a finite extension of $k(C)$. Extending C by a finite extension again, we may assume, without loss of generality, that the motive $M(Y_\eta)$ is finite dimensional over η itself.

On the other hand, $\mathscr{M}(\eta)$ is a colimit of the categories of Chow motives over Zariski open subsets in S by Lemma 4. It means that there exists U, such that the relative Chow motive $M(Y/U)$ is finite-dimensional in the category $\mathscr{M}(U)$.

Main computations

Finite-dimensionality of $M(Y/U)$ implies, of course, that $M^2(Y/U)$ is finite dimensional. But, actually, it is evenly finite dimensional of dimension b_2. One can show that either using Lemma 4 or by the following argument. Apriori one has

$$M^2(Y/U) = A \oplus B ,$$

where $\wedge^m A = 0$ and $\mathrm{Sym}^n B = 0$. The base change functor

$$\Xi : \mathscr{M}(U) \longrightarrow \mathscr{M}(\eta) .$$

is tensor, so it respects finite-dimensionality. In addition, Ξ induces an isomorphism between $\mathrm{End}(\mathbb{1}_U) = \mathbb{Q}$ and $\mathrm{End}(\mathbb{1}_\eta) = \mathbb{Q}$. Since

$$\Xi(M^2(Y/U)) = M^2(Y_\eta)$$

it follows that

$$\Xi(B) = 0$$

because $M^2(Y_\eta)$ is evenly finite dimensional. Then

$$B = 0$$

by Lemma 3. Thus, $M^2(Y/U)$ is evenly finite dimensional. Using the the same arguments one can also show that it can be annihilated by $\wedge^{b_2+1}$.

The divisors $D_1, \ldots, D_{b_2}$ are defined over η by Reduction 2, and they generate the second cohomology group $H^2(Y_{\bar\eta})$ via the cycle class map

$$CH^1(X_\eta) \longrightarrow H^2(Y_{\bar\eta}) .$$

Since the motive $M(Y_\eta)$ is finite dimensional, $H^1(Y_{\bar\eta}) = 0$ and $H^2_{\mathrm{tr}}(Y_{\bar\eta}) = 0$, the second piece $M^2(Y_\eta)$ in the Murre decomposition of $M(Y_\eta)$ can be computed as follows:

$$M^2(Y_\eta) = \mathbb{L}_\eta^{\oplus b_2} \ ,$$

see [6, Theorem 2.14]. Actually, these b_2 copies of $\mathbb{L}_\eta$ arise from the collection of divisor classes

$$[D_1], \dots, [D_{b_2}]$$

and their Poincaré dual classes

$$[D'_1], \dots, [D'_{b_2}]$$

on Y_η, loc.cit. In other words, if

$$(\pi_2)_\eta = \Xi(\pi_2)$$

is a projector determining the middle motive $M^2(Y_\eta)$, the difference

$$\varrho_\eta = (\pi_2)_\eta - \sum_{i=1}^{b_2} [D_i \times_\eta D'_i]$$

is homologically trivial. Then

$$\varrho_\eta^n = 0$$

in the associative ring

$$\mathrm{End}(M^2(Y_\eta))$$

for some n by Kimura's nilpotency theorem (see Proposition 7.2 in [9]).

Now let

$$W_i \qquad \text{and} \qquad W'_i$$

be spreads of the above divisors D_i and D'_i over U (shrink U some more, if necessary). By definition, W_i and W'_i are algebraic cycles of codimension one on Y, such that their pull-backs to the generic fiber are the divisors D_i and D'_i respectively. In other terms:

$$\Xi[W_i] = D_i \qquad \text{and} \qquad \Xi[W'_i] = D'_i \ .$$

These W_i and W'_i are defined not uniquely, of course. But for us the only important thing is that they give D_i and D'_i while passing to the generic point. The cycles $W_i \times_U W'_i$ are in $Corr^0_U(Y \times_U Y)$, and we set

$$\varrho = \pi_2 - \sum_{i=1}^{b_2} [W_i \times_U W'_i] \ .$$

Then we have:

$$\Xi(\varrho) = \varrho_\eta \ .$$

Let ω be any endomorphism of the motive $M^2(Y/U)$ and let $\omega_\eta = \Xi(\omega)$. Then:

$$\Xi(\mathrm{tr}(\omega \circ \varrho)) = \mathrm{tr}(\Xi(\omega \circ \varrho)) = \mathrm{tr}(\omega_\eta \circ \varrho_\eta) = 0$$

because ϱ_η is homologically trivial. Here we use the formula on page 116 in [3], i.e. the compatibility of tensor functors with traces, once again. Since the functor Ξ induces an isomorphism $\mathrm{End}(\mathbb{1}_U) \cong \mathrm{End}(\mathbb{1}_\eta)$,

$$\mathrm{tr}(\omega \circ \varrho) = 0$$

for any ω, i.e. ϱ is numerically trivial. Therefore,

$$\varrho^n = 0$$

in $\mathrm{End}(M^2(Y/U))$ by Proposition 2.

Let now $\bar{W}'_i$ be a divisor on X whose restriction on Y coincides with W'_i. Then

$$\theta_i = \Gamma^t_f \cdot [C \times \bar{W}'_i]$$

is a cycle class of codimension 2 in $C \times X$. Here the fibered product $\times$ is assumed to be taken over k, and the cycle class Γ^t_f is a transpose of the graph of the map f, i.e. the push-forward of the class $[X]$ with respect to the proper morphism $\tau : X \to C \times X$ defined by the map $f : X \to C$ and the identity $\mathrm{id} : X \to X$. Let

$$(\theta_i)_* : CH^1(C) \longrightarrow CH^2(X)$$

be a homomorphism induced by the correspondence θ_i. This homomorphism can be computed also by another formula. Indeed, for any cycle class a from $CH^1(C)$ we have:

$$f^*(a) = \tau^* p^*_C(a) = \tau^* p^*_C(a) \cdot [X] \ .$$

By the projection formula we get:

$$\tau_* f^*(a) = p^*_C(a) \cdot \tau_*[X] = p^*_C(a) \cdot \Gamma^t_f \ .$$

Then we can compute:

$$\begin{aligned} (\theta_i)_*(a) &= (p_X)_*(p^*_C(a) \cdot \theta_i) \\ &= (p_X)_*(p^*_C(a) \cdot \Gamma^t_f \cdot [C \times \bar{W}'_i]) \\ &= (p_X)_*(\tau_* f^*(a) \cdot [C \times \bar{W}'_i]) \\ &= (p_X)_*(\tau_* f^*(a) \cdot p^*_X[\bar{W}'_i]) \\ &= (p_X)_* \tau_* f^*(a) \cdot [\bar{W}'_i] \\ &= f^*(a) \cdot [\bar{W}'_i] \end{aligned}$$

This gives us a possibility to define a homomorphism

$$(\theta_i)_* : CH^1(U) \longrightarrow CH^2(Y)$$

in the non-compact case by the analogous formula:

$$(\theta_i)_*(a) = f^*(a) \cdot [W'_i]$$

for any cycle class a in $CH^1(U)$, where f is the map $f : Y \to U$. Then, for each index i, we have a commutative square

$$\begin{array}{ccc} CH^1(C) & \xrightarrow{(\theta_i)_*} & CH^2(X) \\ \downarrow & & \downarrow \\ CH^1(U) & \xrightarrow{(\theta_i)_*} & CH^2(Y) \end{array}$$

In other words, the homomorphisms $(\theta_i)_*$ are compatible in compact and non-compact cases.

Evidently, the homomorphisms θ_i respect algebraic equivalence, so that we also have the corresponding commutative square for groups $A^*(-)$:

$$\begin{array}{ccc} A^1(C) & \xrightarrow{(\theta_i)_*} & A^2(X) \\ \downarrow & & \downarrow \\ A^1(U) & \xrightarrow{(\theta_i)_*} & A^2(Y) \end{array}$$

A resulting commutative diagram we need is then as follows:

$$\begin{array}{ccc} \oplus_{i=1}^{b_2} A^1(C) & \xrightarrow{\theta_*} & A^2(X) \\ \downarrow & & \downarrow \\ \oplus_{i=1}^{b_2} A^1(U) & \xrightarrow{\theta_*} & A^2(Y) \end{array}$$

Both homomorphisms θ_* here are defined by summing of homomorphisms $(\theta_i)_*$.

An elementary diagram chasing shows now that, if the bottom homomorphism θ_* is surjective, the image of the top homomorphism θ_* coincides with $A^2(X)$ modulo a finite-dimensional subspace J in $A^2(X)$ appeared in Reduction 3. Let us recall that, as soon as we fix a section $A^2(Y) \to A^2(X)$ we have that $A^2(X) = A^2(Y) \oplus J$, $J = W \oplus V$, W is covered by A^1 of curves and V is finite-dimensional. Then, according to what was shown in Reduction 3, in order to prove representability of $A^2(X)$ we need only to show that the bottom θ_* is onto.

Let $y \in CH^i(Y)$. For any correspondence $c \in Corr_U^j(Y, Y)$ let, as usual,

$$c_*(y) = p_{2*}(p_1^*(y) \cdot c) \in CH^{i+j}(Y)$$

be the action of the correspondence c on y in the relative sense, i.e. p_1 and p_2 are two projections from $Y \times_U Y$ onto Y. In particular, one has a decomposition

$$\begin{aligned} y &= \Delta_{Y/U}{}_*(y) \\ &= (\pi_0)_*(y) + (\pi_2)_*(y) + (\pi_4)_*(y) \end{aligned}$$

in $CH^2(Y)$. Now we have:

$$\begin{aligned} (\pi_0)_*(y) &= p_{2*}(p_1^*(y) \cdot \pi_0) \\ &= p_{2*}(p_1^*(y) \cdot p_1^*[E]) \\ &= p_{2*}p_1^*(y \cdot [E]) \\ &= f^* f_*(y \cdot [E]) \end{aligned}$$

From now on we assume that y is of codimension two. Then, as y and $[E]$ are both of codimension two in a three-dimensional variety, we have that $y \cdot [E] = 0$, whence

$$(\pi_0)_*(y) = 0 \ .$$

Assume, furthermore, that y is algebraically trivial, i.e. $y \in A^2(Y)$. In that case $f_*(y) = 0$. Then we compute:

$$\begin{aligned} (\pi_4)_*(y) &= p_{2*}(p_1^*(y) \cdot \pi_4) \\ &= p_{2*}((y \times_U [Y]) \cdot ([Y] \times_U [E])) \\ &= p_{2*}(y \times_U [E]) \\ &= f_*(y) \times_U [E] \\ &= 0 \end{aligned}$$

(here and below we use nice properties of intersections of relative cycles over a non-singular one-dimensional base, see Chapter 2.2 in [4]). As a result we have:

$$y = (\pi_2)_*(y) \ .$$

On the other hand,

$$\pi_2 = \sum_{i=1}^{b_2} [W_i \times_U W_i'] + \varrho \ ,$$

whence we get:

$$y = (\pi_2)_*(y) = \sum_{i=1}^{b_2} [W_i \times_U W_i']_*(y) + \varrho_*(y) \ .$$

Let us emphasize once more that the correspondences here act as relative correspondences.

Now let

$$v_1 = -\sum_{i=1}^{b_2} [W_i \times_U W_i']_*(y) \ ,$$

so that

$$\varrho_*(y) = y + v_1 \ .$$

Write y as a class of a linear combination

$$\sum_j n_j Z_j \ ,$$

where Z_j are integral curves on Y. For any i and j one has:

$$\begin{array}{rcl} [W_i \times_U W_i']_*[Z_j] & = & p_{2*}([Z_j \times_U Y] \cdot [W_i \times_U W_i']) \\ & = & p_{2*}(([Z_j] \cdot [W_i]) \times_U ([Y] \cdot [W_i'])) \\ & = & p_{2*}(([Z_j] \cdot [W_i]) \times_U [W_i']) \end{array}$$

By linearity:

$$[W_i \times_U W_i']_*(y) = p_{2*}((y \cdot [W_i]) \times_U [W_i'])$$

Since y is of codimension two and W_i is of codimension one in the non-singular threefold Y, the intersection $y \cdot [W_i]$ is zero-dimensional cycle class on Y. Let

$$a_i = f_*(y \cdot [W_i])$$

be its push-forward to U with respect to the proper map $f : Y \to U$. Using proper-flat base change and the projection formula we compute:

$$\begin{array}{rcl} [W_i \times_U W_i']_*(y) & = & p_{2*}((y \cdot [W_i]) \times_U [W_i']) \\ & = & p_{2*}(p_1^*(y \cdot [W_i]) \cdot p_2^*([W_i'])) \\ & = & p_{2*}p_1^*(y \cdot [W_i]) \cdot [W_i'] \\ & = & f^* f_*(y \cdot [W_i]) \cdot [W_i'] \\ & = & f^*(a_i) \cdot [W_i'] \\ & = & (\theta_i)_*(a_i) \ . \end{array}$$

Moreover, as y is in $A^2(Y)$, it follows that each a_i is in $A^1(U)$. Then we get:

$$v_1 = -\sum_{i=1}^{b_2} [W_i \times_U W_i']_*(y) = -\sum_{i=1}^{b_2} (\theta_i)_*(a_i) = \theta_*(c_1) \ ,$$

where

$$c_1 = (-a_1, \dots, -a_{b_2})$$

is in $\oplus_{i=1}^{b_2} A^1(U)$.

Applying ϱ_* to the both sides of the equality

$$\varrho_*(y) = \theta_*(c_1) + y$$

we get:

$$\varrho_*^2(y) = \theta_*(c_2) + y$$

for some c_2 in $\oplus_{i=1}^{b_2} A^1(U)$, and so forth. After n steps we will get:

$$\varrho_*^n(y) = \theta_*(c_n) + y \ ,$$

where e_n is in $\oplus_{i=1}^{b_2} A^1(U)$. But we know that ϱ is a nilpotent correspondence in $\mathrm{End}(M(Y/U))$, so that

$$\varrho_*^n(y) = 0$$

for big enough n. It follows that

$$y = -\theta_*(c_n)$$

is in the image of the homomorphism θ_*, and we are done.

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