Idempotents of Hecke algebras of type A.

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Abstract

We use a skein-theoretic version of the Hecke algebras of type A to present three-dimensional diagrammatic views of Gyoja's idempotent elements, based closely on the corresponding Young diagram λ. In this context we give straightforward calculations for the eigenvalues f_λ and m_λ of two natural central elements in the Hecke algebras, namely the full curl and the sum of the Murphy operators. We discuss their calculation also in terms of the framing factor associated to the appropriate irreducible representation of the quantum group, $SU(N)_q$.

1 Introduction

The Hecke algebras of type A have played a central role in the original discovery of the Homfly polynomial for links, as described by Ocneanu [17]. They also relate to the representation theory of the quantum groups $SU(N)_q$ in a very direct counterpart to the classical relations between the symmetric group algebras and the representation theory of the linear groups $GL(N)$, [21, 24]. As a result they form a natural means of transition between the invariants of a knot determined from quantum-group representations and those determined by using the Homfly polynomial of suitable decorations of the knot.

Because of the close connection with the Homfly polynomial there are well-established skein-theoretic ways of viewing the type A Hecke algebras in terms of linear combinations of braids or tangles, modulo some simple linear relations, [16].

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Irreducible representations of the Hecke algebras are known to be associated to Young diagrams via quantum analogues of the symmetriser and anti-symmetriser. In this paper we construct three-dimensional skein-theoretic versions of the idempotent elements in these algebras, based on Gyoja's algebraic versions in [8], which can be directly visualised in terms of the corresponding Young diagram.

We give direct skein-theoretic proofs of some of their properties. These lead to a simple calculation of the eigenvalues f_λ and m_λ for the full curl and for the sum of the Murphy operators when applied to the idempotent for the Young diagram λ in the Hecke algebra H_n with $n = |\lambda|$. While these eigenvalues are already known, [5, 12], we believe that our approach brings out certain unexpected features.

We exploit the relationships between the Hecke algebras and Homfly skein theory to express the idempotents as linear combinations of the positive permutation braids.

The three-dimensional idempotents described here use a natural set of parameters x, v and s. The closure of the idempotent for the Young diagram λ in the skein of the annulus gives a linear combination of patterns which determines a knot invariant with parameters x, v and s, depending on λ. The substitution $s = e^{h/2}$, along with $v = e^{-Nh/2} = s^{-N}$ and $x = e^{-h/2N} = s^{-1/N}$ then gives for each N an invariant of the knot which is the $SU(N)_q$–invariant of the knot when coloured by the irreducible representation of $SU(N)_q$ with the same Young diagram λ, where $q = e^h$. This gives an explicit way to calculate the $SU(N)_q$–invariants of the knot in terms of the Homfly polynomials of its satellites.

Yokota [26] has described a rather similar skein-theoretic version of the idempotents, although he does this individually for each N in terms of a single parameter, while here we make use of the extra variables x and v from the Homfly skein to handle all values of N at once.

Akutsu, Deguchi and Wadati [3, 4] have described idempotents in the cases of individual $SU(N)_q$ which can be used to give knot invariants based on a chosen Young diagram, as well as a means for extending these to 2-variable invariants, which are essentially these Homfly based satellite invariants.

There are accounts by Wenzl [23, 22] both in this case and in the setting of the quantum groups of types B, C and D, of the general theory connecting quantum invariants to 2-variable knot invariants using idempotents of suitable algebras.

Our motivation is to show how the skein theory approach, based on the Homfly polynomial, can be used very simply and explicitly in dealing with certain aspects of the quantum $SU(N)$ invariants. In many cases the quantum and skein techniques are complementary, and it is useful to have a better understanding of the transitions between them, so as to translate readily from one context to the other.

For example, the eigenvalues f_λ for the full curl give the eigenvalues for the Casimir operators of $SU(N)_q$, and thus the framing factors, in a way which focuses attention on the nature of their dependence on N. Equally either version of knot invariants gives a convenient starting place for the Witten-Reshetikhin-

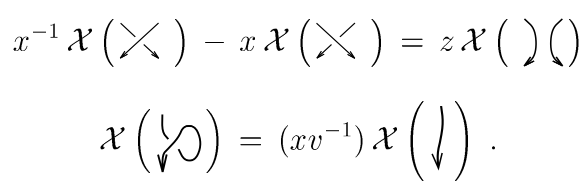


Figure 1: The skein relations for the framed Homfly polynomial

Turaev $SU(N)$ manifold invariants, which can be constructed by suitable substitution of a root of unity in a combination of link invariants, [15]. The facility to change readily between the quantum and skein invariants then allows for flexibility in interpretation and a consequent anticipation of potentially interesting features.

2 The Hecke algebra and Homfly skein theory

We give a brief description of skein theory based on planar pieces of knot diagrams and a framed version of the Homfly polynomial. The ideas go back to Conway and have been substantially developed by Lickorish and others. A fuller version of this account can be found in [13]. At a later stage we shall expand our view from diagrams to actual pieces of knot lying in controlled regions of 3-dimensional space, under suitable equivalence.

We shall work with the framed Homfly polynomial $\mathcal{X}$. This is an invariant of framed oriented links, constructed from the Homfly polynomial by setting $\mathcal{X}(L) = (xv^{-1})^{\omega(D)}P(L)$. Here $\omega(D)$ is the writhe (the sum of the signs of the crossings) of any diagram D of the framed link L which realises the chosen framing by means of the 'blackboard parallel'. It is determined, up to a scalar, by the skein relations in Fig. 1 for conventionally framed link diagrams which differ only as shown. We will normalise $\mathcal{X}$ to take the value 1 on the empty knot, taking $\mathcal{X} = \dfrac{v^{-1}-v}{z}$ for the unknot with zero framing.

Let F be a planar surface and fix a (possibly empty) set of distinguished points on the boundary. We consider diagrams in F, consisting of oriented arcs joining any distinguished boundary points and oriented closed curves, up to Reidemeister moves II and III. They carry the implicit framing defined by the parallel curves in the diagram. Define the *framed Homfly skein* of F, denoted by $\mathcal{S}(F)$, to consist of linear combinations of diagrams in F modulo the skein relations in Fig. 2 As a consequence, the relation in Fig. 3 holds in $\mathcal{S}(F)$, for any diagram D, where the oriented circle denotes a null homotopic loop.

We are interested in three specific cases, namely when F is the whole plane $\mathbf{R}^2$, the annulus $S^1 \times I$ or the rectangle $R_n^n \cong I \times I$ with n distinguished points on its top and bottom edge. In the last case we insist that any arcs in R_n^n enter at

Figure 2: The skein relations for the Homfly skein of a surface

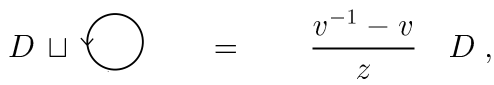


Figure 3: The skein relation for a null homotopic loop

the top and leave at the bottom. Diagrams in R_n^n are termed *oriented n-tangles*, and include the case of n-string braids.

A *positive permutation braid* is defined for each permutation $\pi \in S_n$. It is the n-string braid, ω_π, uniquely determined by the properties

i) all strings are oriented from top to bottom

ii) for $i = 1, \ldots, n$ the ith string joins the point numbered i at the top of the braid to the point numbered $\pi(i)$ at the bottom of the braid,

iii) all the crossings occur with positive sign and each pair of strings cross at most once.

These were first defined by Elrifai and Morton [6]. We can think of the braid strings as sitting in layers, with the first string at the back and the nth string at the front.

We define the *negative permutation braid* for π in exactly the same manner as the positive permutation braid except that we demand that all the crossing be negative. We shall denote this braid by $\overline{\omega}_\pi$. The inverse of ω_π is the negative permutation braid with permutation π^{-1}, thus $\omega_\pi^{-1} = \overline{\omega}_{\pi^{-1}}$.

The skein $\mathcal{S}(\mathbf{R}^2)$ is just the set of linear combinations of framed link diagrams, modulo the skein relations. Every diagram D represents a scalar multiple of the empty diagram, with the multiple being simply $\mathcal{X}(D)$.

It is shown in [16] that the skein $\mathcal{S}(R_n^n)$ is spanned by the $n!$ positive permutation braids, and that these are linearly independent. The elementary braid σ_i, which is the positive permutation braid for the transposition $(i\, i{+}1)$, satisfies the relation $x^{-1}\sigma_i - x\sigma_i^{-1} = z$ in the skein. The skein forms an algebra over Λ with multiplication derived from the concatenation of diagrams. As is conventional for braids, we write ST for the diagram given by placing diagram S above diagram T. The resulting algebra is a quotient of the braid-group algebra. It is shown in [16] that the algebra $\mathcal{S}(R_n^n)$ is isomorphic to the Hecke algebra H_n of type A,

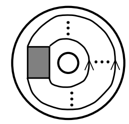


Figure 4: The wiring of $\mathcal{S}(R_n^n)$ into $\mathcal{S}(S^1 \times I)$

with the explicit presentation

$$H_n \quad = \quad \left\langle \quad \sigma_i \quad : \quad i=1,\ldots,n-1 \qquad \middle| \quad \begin{array}{l} \sigma_i\sigma_j = \sigma_j\sigma_i \ : \ |i-j|>1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \\ x^{-1}\sigma_i - x\sigma_i^{-1} = z \ , \end{array} \quad \right\rangle .$$

There are various presentations of the Hecke algebra in the literature. Here we have used a coefficient ring Λ with 3 variables x, v and z. The variable v is needed in the skein when we want to write a general tangle in terms of the basis of permutation braids, but it does not appear in the relations. The variable x keeps track of the writhe of a diagram, and can be dropped without affecting the algebraic properties. Define an algebra $H_n(1,z)$, obtained from H_n by setting $x=1$, in terms of generators ρ_i where the quadratic relation is given by $\rho_i - \rho_i^{-1} = z$.

In algebra texts such as [12, 5] the Hecke algebra is usually presented with a quadratic relation whose roots are q and -1. We will denote this presentation by $H_q(n)$. With generators τ_i, the quadratic relation is $\tau_i^2 = (q-1)\tau_i + q$. Note that these presentations are all isomorphic.

$$\begin{array}{ccccc} H_n & \cong & H_n(1,z) & \cong & H_q(n) \\ \sigma_i & \mapsto & x\rho_i & & \\ & & \rho_i & \mapsto & s^{-1}\tau_i \end{array}$$

where we relate q and z by setting $z = s - s^{-1}$ and $q = s^2$.

In what follows we shall generally use the skein form $\mathcal{S}(R_n^n)$ of H_n, but the results can be transferred immediately to the other versions by these isomorphisms.

A *wiring* W of a surface F into another surface F' is a choice of inclusion of F into F' and a choice of a fixed diagram of curves and arcs in $F' - F$ whose boundary is the union of the distinguished sets of F and F'. A wiring W determines naturally a Λ-linear map $\mathcal{S}(W) : \mathcal{S}(F) \to \mathcal{S}(F')$.

We can wire the rectangle R_n^n into the annulus as indicated in Fig. 4. The resulting diagram in the annulus is called the *closure* of the oriented tangle. We shall also use the term ‘closure’ for the Λ-linear map from each Hecke algebra H_n to the skein of the annulus induced by this wiring.

The skein $\mathcal{S}(S^1 \times I)$ of the annulus itself forms an algebra, whose product is given by stacking the annuli one inside the other. This product is obviously commutative (lift the inner annulus up and stretch it so that the outer one will fit on the inside of it). Write $\mathcal{C}$ for $\mathcal{S}(S^1 \times I)$ regarded as a Λ-algebra in this way. Let $\mathcal{C}^+$ be the sub-algebra spanned by the closures of oriented tangles. Turaev [19] showed that $\mathcal{C}^+$ is freely generated as an algebra by $\{A_m,\ m \in \mathbf{N}\}$, where A_m is the closure of the positive permutation braid for the cycle $(1\,2\,\ldots\,m)$.

Write $\mathcal{C}^{(n)}$ for the linear subspace spanned by the closures of oriented n-tangles. This is spanned by all terms of the form $(A_{i_1})^{j_1}(A_{i_2})^{j_2}\cdots(A_{i_p})^{j_p}$ where $i_k, j_k \in \mathbf{N}$ and $\sum_{k=1}^{p} i_k j_k = n$. Hence the algebra $\mathcal{C}^+$ is graded as

$$\mathcal{C}^+ = \bigoplus_{n=0}^{\infty} \mathcal{C}^{(n)}\ ,$$

and each subspace $\mathcal{C}^{(n)}$ is the image of the Hecke algebra H_n under the closure map.

3 Young diagrams

There is a wealth of detail about the features of Young tableaux in many texts such as [25, 7, 10]. Here we emphasize certain properties which will be to the fore in this article.

A partition of n can be represented by a *Young diagram*; a collection of n cells arranged in rows, with λ_1 cells in the first row, λ_2 cells in the second row up to λ_k cells in the kth row where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ and $\sum_{i=1}^{k} \lambda_i = n$. We shall denote both the partition and its Young diagram by λ. The Young diagram for (0) is the empty diagram. We denote the number of cells in λ by $|\lambda|$. The *conjugate* $\lambda^\vee$ of λ is the Young diagram whose rows form the columns of λ. Any cell for which a legitimate Young diagram remains after it has been removed will be called an *extreme cell*. To each extreme cell we associate an *extreme rectangle*, namely those cells above and to the left of it in the Young diagram. When there is a cell in the ith row and jth column of λ we write $(i,j) \in \lambda$, and refer to (i,j) as the *coordinates* of the cell.

We will work with the example $\nu = (4,2,1)$ throughout this paper. The conjugate of ν is $\nu^\vee = (3,2,1,1)$ and $|\nu| = |\nu^\vee| = 7$. The Young diagrams for ν and $\nu^\vee$ are shown in Fig. 5. There are three extreme cells in ν, marked by a cross in Fig. 6 with their associated extreme rectangles shaded in. The coordinates of the extreme cells are $(1,4)$, $(2,2)$ and $(3,1)$ respectively.

Let $T'(\lambda)$ be an assignment of the numbers 1 to n to the cells of λ such that the numbers increase from left to right along the rows and top to bottom down the columns. We call $T'(\lambda)$ a *standard tableau*. In particular $T(\lambda)$ will denote the tableau where the cells of the Young diagram are numbered from 1 to n along the rows.

Figure 5: The Young diagram $\nu = (4, 2, 1)$ and its conjugate

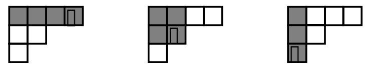


Figure 6: The extreme cells and rectangles of ν

Note that the transposition of rows and columns doesn't take $T(\lambda)$ to $T(\lambda^\vee)$. We define the permutation π_λ by $\pi_\lambda(i) = j$ where the transposition of λ carries the cell i in $T(\lambda)$ to the cell j in $T(\lambda^\vee)$. The tableaux $T(\nu)$ and $T(\nu^\vee)$ are shown in Fig. 7 and $\pi_\nu = (2\,4\,7\,3\,6\,5)$.

We now give some standard combinatorial results about permutations and Young diagrams, in a context which extends readily to positive permutation braids and the Hecke algebras.

Let λ and μ be Young diagrams with $|\lambda| = |\mu| = n$. We say that $\pi \in S_n$ *separates* λ from μ if no pair of numbers in the same row of $T(\lambda)$ are mapped by π to the same row of $T(\mu)$. The permutation π_λ, for example, separates λ from its conjugate $\lambda^\vee$.

Write $R(\lambda) \subset S_n$ for the subgroup of permutations which preserve the rows of $T(\lambda)$. Each $R(\lambda)$ is generated by some subset of the elementary transpositions $(i\,i+1)$. For example $R(\nu)$ is generated by $\{(12), (23), (34), (56)\}$.

It is easy to see that if π separates λ from μ then so does $\rho\pi\sigma$ for any $\rho \in R(\lambda), \sigma \in R(\mu)$. Conversely, it can be shown that if π separates λ from $\lambda^\vee$ then $\pi = \rho\pi_\lambda\sigma$ with $\rho \in R(\lambda)$ and $\sigma \in R(\lambda^\vee)$. We say that λ is *just separable* from $\lambda^\vee$. If no permutation $\pi \in S_n$ separates λ from μ then we call λ and μ *inseparable.*

Order the Young diagrams by lexicographical ordering of their rows. Thus $\lambda > \mu$ when there exists t with $\lambda_t > \mu_t$ and $\lambda_i = \mu_i$ for $i < t$.

Lemma 3.1 *If $\lambda > \mu$ then λ and $\mu^\vee$ are inseparable.*

Figure 7: The tableaux $T(\nu)$ and $T(\nu^\vee)$

Proof. By induction on t. Suppose first that $t = 1$, and let $\pi \in S_n$. Since $\mu^\vee$ has $\mu_1 < \lambda_1$ rows, π must map at least two numbers from the first row of λ to the same row of $\mu^\vee$. Hence π does not separate λ from $\mu^\vee$.

When $t > 1$, consider the Young diagrams $r(\lambda)$ and $r(\mu)$, where $r(\alpha)$ is the Young diagram obtained from α by removing the first row. We have $r(\lambda) > r(\mu)$, and the diagrams first differ in the $(t-1)$st row. Hence, by the induction hypothesis, $r(\lambda)$ and $r(\mu)^\vee$ are inseparable. It remains to prove that if λ were separable from $\mu^\vee$ then $r(\lambda)$ would be separable from $r(\mu)^\vee$, giving us a contradiction.

Suppose that π separates λ from $\mu^\vee$. Since $\lambda_1 = \mu_1$, the number of cells in the first row of λ is equal to the number of rows of $\mu^\vee$. Then π must send exactly one cell from the first row of λ to each row of $\mu^\vee$. We can suppose, without loss of generality, that it is the first cell in each row of $\mu^\vee$, for if not, there is a transposition $\tau \in R(\mu^\vee)$ which will switch the first cell of the row with the image of the cell in the first row of λ, and $\pi\tau$ will also separate λ from $\mu^\vee$. Restrict π to all but the first row of λ. Its image will be exactly the cells of $r(\mu)^\vee$, and the restricted permutation separates $r(\lambda)$ from $r(\mu)^\vee$. □

Corollary 3.2 *If $\lambda \neq \mu$ then either λ and $\mu^\vee$ are inseparable or $\lambda^\vee$ and μ are inseparable.*

Lemma 3.3 *Let π be a permutation which separates λ from $\lambda^\vee$. Then the positive permutation braid ω_π can be written as $\omega_\pi = \omega_\rho \omega_{\pi_\lambda} \omega_\sigma$ for some $\rho \in R(\lambda)$ and $\sigma \in R(\lambda^\vee)$.*

Proof. We already know that we can write $\pi = \rho\pi_\lambda\sigma$. It is enough to show that $\omega_\rho \omega_{\pi_\lambda} \omega_\sigma$ is a positive permutation braid, for it is then determined as ω_π by its permutation π.

The key feature of ω_{π_λ} is that pairs of strings which start in the same row of $T(\lambda)$ or finish in the same row of $T(\lambda^\vee)$ don't cross. The only pairs which cross in ω_ρ or in ω_σ are in the same row of $T(\lambda)$ or $T(\lambda^\vee)$ respectively. Consequently each pair of strings in the product braid $\omega_\rho \omega_{\pi_\lambda} \omega_\sigma$ crosses at most once. All crossings are positive, so this is a positive permutation braid. □

This is a special case of a property of positive permutation braids which will be useful later.

Lemma 3.4 *In each right coset $R(\lambda)\pi'$ of the subgroup $R(\lambda)$ there is a unique $\overline{\pi}$ such that $\omega_\pi = \omega_\rho \omega_{\overline{\pi}}$, where $\rho \in R(\lambda)$ and $\pi = \rho\overline{\pi}$ runs through the coset.*

Proof. Choose $\overline{\pi}$ so that no two strings which start in the same row of λ cross. Then $\omega_\rho \omega_{\overline{\pi}}$ is a positive permutation braid. □

Corollary 3.5 *If $\pi = \rho\pi'$ with $\rho \in R(\lambda)$ then $\omega_\pi = \omega_{\rho'} \overline{\omega}_{\rho''} \omega_{\pi'}$, for some $\rho', \rho'' \in R(\lambda)$. A similar result holds for left cosets.*

4 Idempotents

The Hecke algebra H_n is closely related to the group algebra of S_n, whose idempotents are described by the classical Young symmetrisers. For a Young diagram λ its Young symmetriser is the product of the sum of permutations which preserve the rows of $T(\lambda)$ and the alternating sum of permutations which preserve columns. With care it is possible to make a similar construction of idempotents in H_n, replacing permutations by suitably weighted positive permutation braids. Jones [11] gives a good description of the two idempotents corresponding to single row and column Young diagrams. Other authors, for example Wenzl and Cherednik, have given descriptions for general λ, but we shall here adapt the construction of Gyoja [8] to construct idempotents in H_n regarded as the skein $\mathcal{S}(R_n^n)$. We shall follow the account in [13] for the basic row and column idempotents, and use these to construct an idempotent for each Young diagram λ.

We start from the visually appealing 3-dimensional picture for the idempotent as a linear combination of braids in a 3-ball based very closely on the diagram λ rather than as a linear combinations of diagram in a rectangle.

We consider a 3-ball $B \cong B^3$, with a chosen subset P of $2n$ points on its boundary sphere, designated as n inputs P_I and n outputs P_O. An oriented tangle T in (B, P) is made up of n oriented arcs in B joining the points P_I to the points P_O, together with any number of oriented closed curves. The arcs and curves of T are assumed to carry a framing defined by a specific choice of parallel for each component.

The skein $\mathcal{S}(B, P)$ is defined as linear combinations of such tangles, modulo the framed Homfly skein relations applied to tangles which differ only as in Fig. 2 inside some ball. The case when $B = D^2 \times I$ and the points P_I and P_O are lined up along the top and bottom respectively, gives a skein which can readily be identified with $\mathcal{S}(R_n^n) = H_n$. There is a homeomorphism mapping any other pair (B', P') to this pair, when $|P'| = 2n$. This induces a linear isomorphism from each $\mathcal{S}(B', P')$ to the Hecke algebra H_n.

As in the case of diagrams, a *wiring* W of (B, P) into (B', P') is an inclusion of the ball B into the interior of B' and a choice of framed oriented arcs in $B - B'$ ending with compatible orientation at the boundary points $P \cup P'$. Given a tangle T in (B, P) and a wiring W their union determines a tangle $W(T)$ in (B', P'), and induces a linear map $\mathcal{S}(W) : \mathcal{S}(B, P) \to \mathcal{S}(B', P')$.

The region between two balls is homeomorphic to $S^2 \times I$. A simple example of wiring W consists of n arcs with each lying monotonically in the I coordinate, sometimes called an n-braid in S^2. In such a case the map $\mathcal{S}(W)$ is a linear isomorphism whose inverse is induced by the inverse braid. Such a wiring can always be chosen to determine an explicit isomorphism from any $\mathcal{S}(B, P)$ to $H_n = \mathcal{S}(R_n^n)$ when $|P| = 2n$.

In terms of calculation the 3-dimensional viewpoint has advantages and dis-

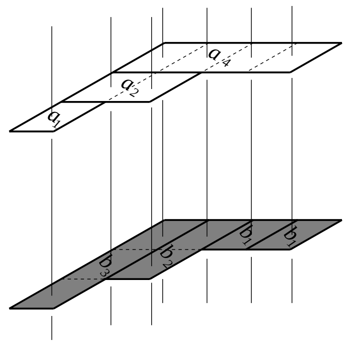


Figure 8: The 3-dimensional quasi-idempotent associated to ν

advantages. There may not be a natural way to compose tangles in the skein $\mathcal{S}(B, P)$, so the isomorphism with H_n does not immediately carry any algebra information. On the other hand a suitable choice of wiring may show when certain elements are zero in the skein, and hence exhibit relations in H_n.

The following picture gives the heart of the construction of the idempotent for the Young diagram λ. It lies in the skein of $B^3 \cong D^2 \times I$ where the points P_I and P_O are the centres of cells of templates in the shape of λ at the top and bottom respectively. The strings in each rows are first grouped together using a linear combination a_j of braids for a row with j cells. Below this the strings in the columns are grouped with linear combinations b_j of braids, to define an element E_λ of the skein. The elements a_j and b_j are Jones' basic row and column quasi-idempotents, described shortly in more detail.

We give the 3-dimensional picture for E_ν in Fig. 8. Note how the strings are first grouped in rows and then in columns. Although the braids run vertically the boxes have been drawn in the horizontal plane to emphasize how the idempotent is related to the shape of the Young diagram.

This 3-dimensional view will be discussed further in the next section, along with some closely related choices of wiring. For the moment we will work with a 2-dimensional picture given by using the standard tableau $T(\lambda)$ to produce a wiring which straightens out the strings to lie in order on a line. The corresponding figure, more immediately related to Gyoja's idempotents, is shown in Fig. 10.

We first define the row and column elements a_j and b_j, following the account in Morton [13].

Write $E_n(\sigma_1, \sigma_2, \cdots, \sigma_{n-1}) = \sum_{\pi \in S_n} \omega_\pi$ for the sum of the positive permutation braids.

Theorem 4.1 ([13]) *For each i, we can factorise E_n in H_n as*

$$E_n = E_n^{(i)}(\sigma_i + 1) = (\sigma_i + 1)E_n^{(i)}$$

where $E_n^{(i)} = \sum_{\pi(i)<\pi(i+1)} \omega_\pi$.

The elementary braids σ_i satisfy the quadratic relation $(\sigma_i - a)(\sigma_i - b) = 0$ in H_n, where $a = -xs^{-1}$ and $b = xs$, with the variants $-s^{-1}, s$ or $-1, q$ for the roots a and b in the other versions of the Hecke algebra. Define a_n and b_n by substituting $-a^{-1}\sigma_i$ or $-b^{-1}\sigma_i$ respectively for σ_i in E_n. Thus

$$a_n = \sum_\pi (-a)^{-l(\pi)}\omega_\pi \ , \quad b_n = \sum_\pi (-b)^{-l(\pi)}\omega_\pi$$

where $l(\pi)$ is the writhe of ω_π, known in algebraic terms as the length of the permutation π.

Corollary 4.2 *We can factorise a_n and b_n in H_n as*

$$a_n = (\sigma_i - a)a_n^{(i)} = a_n^{(i)}(\sigma_i - a) \ , \quad b_n = (\sigma_i - b)b_n^{(i)} = b_n^{(i)}(\sigma_i - b) \ .$$

Proof. Substitute $-a^{-1}\sigma_i$ or $-b^{-1}\sigma_i$ for σ_i in Theorem 4.1. □

In our diagrams the elements a_n and b_n are drawn as rectangles. Those denoting b_n are shaded to distinguish them from those carrying a_n. In the classical case, with $x = s = 1$, a_n reduces to a sum of permutations and b_n to an alternating sum.

Applying Cor. 4.2 and the factorisation of the quadratic relation we obtain the following result.

Lemma 4.3 ([13]) *Let ϕ_a and ϕ_b be the linear homomorphisms from the Hecke algebra, H_n, to the ring of scalars Λ defined by $\phi_a(\sigma_i) = a$ and $\phi_b(\sigma_i) = b$ for $i = 1, \ldots, n-1$. Then for all $h \in H_n$,*

$$a_n h = h a_n = \phi_b(h) a_n \quad \textit{and} \quad b_n h = h b_n = \phi_a(h) b_n \ .$$

In particular, we note the following consequence of Lemma 4.3. A copy of an a_i can be swallowed (from above or below) by an a_k, if $i \le k$, at the expense of multiplying the resulting diagram by a scalar. This scalar is $\alpha_{(i)}$ (see page 13), which is non-zero. Thus an a_k can also throw out extra copies of a_i, multiplying the resulting diagram by $\alpha_{(i)}^{-1}$. Further, there is no net effect if we introduce and then later remove an a_i. This works equally well with b in place of a. We make further use of this property, with variants, in the next section.

We now define the quasi-idempotent elements $e_\lambda \in H_n$ for each Young diagram $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. To each cell of λ we assign a braid string, ordered according to $T(\lambda)$. Define $E_\lambda(a) \in H_n$ as a linear combination of braids by placing a_{λ_i} on the strings corresponding to the ith row of λ for each i, and similarly $E_\lambda(b)$ using b_{λ_i}. The element $E_\nu(a)$ is depicted in Fig. 9.

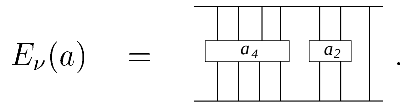


Figure 9: The element $E_\nu(a)$

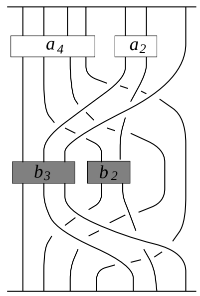


Figure 10: The quasi-idempotent e_ν

Now define $e_\lambda = E_\lambda(a)\omega_{\pi_\lambda}E_{\lambda^\vee}(b)\omega_{\pi_\lambda}^{-1} \in H_n$. The element e_ν is shown in Fig. 10. This picture can be obtained from the 3-dimensional picture of Fig. 8 by sliding the rows apart at the top of the diagram and sliding the columns apart at the bottom, using the standard tableau $T(\nu)$ to determine how to order the strings.

For each λ write $H(\lambda) \subset H_n$ for the subspace spanned by $\{\omega_\rho, \rho \in R(\lambda)\}$. Then $E_\lambda(a), E_\lambda(b) \in H(\lambda)$ and $E_\lambda(a)h = hE_\lambda(a) = \phi_b(h)E_\lambda(a)$ for $h \in H(\lambda)$; a similar result holds with a and b interchanged.

Lemma 4.4 *Let λ and μ be Young diagrams with n cells, and let $\pi \in S_n$ be a permutation which does not separate λ from μ. Then*

$$E_\lambda(a)\omega_\pi E_\mu(b) = 0 = E_\lambda(b)\omega_\pi E_\mu(a).$$

Proof. Since π does not separate λ from μ, there are two cells in some row of λ, the lth say, which are sent to two cells in the same row of μ, the pth say, by π. We can find $\rho \in R(\lambda)$ and $\sigma \in R(\mu)$ such that $\pi' = \rho\pi\sigma$ sends the two adjacent cells i and $i+1$ in the lth row of λ to cells j and $j+1$ in the pth row of μ. By Cor. 3.5 we can write $\omega_\pi = \omega_{\rho'}\overline{\omega}_{\rho''}\omega_{\pi'}\overline{\omega}_{\sigma'}\omega_{\sigma''}$, with $\rho', \rho'' \in R(\lambda)$ and $\sigma', \sigma'' \in R(\mu)$. Hence $\omega_\pi = h\omega_{\pi'}h'$, with $h \in H(\lambda)$ and $h' \in H(\mu)$. Then $E_\lambda(a)\omega_\pi E_\mu(b) = \phi_b(h)\phi_a(h')E_\lambda(a)\omega_{\pi'}E_\mu(b)$.

We can thus replace π by π', and assume that there are two adjacent cells, i and $i+1$, with $\pi(i+1) = \pi(i)+1 = j+1$, where $(i\ i+1) \in R(\lambda)$ and $(j\ j+1) \in R(\mu)$. Since strings i and $i+1$ finish at adjacent points, any other string in ω_π either crosses above or below both strings. Then $\sigma_i\,\omega_\pi = \omega_\pi\,\sigma_j$, and so $(\sigma_i - a)\omega_\pi = \omega_\pi(\sigma_j - a)$. Knowing that $(i\ i+1) \in R(\lambda)$ we can use Cor. 4.2 to write $E_\lambda(a) = h(\sigma_i - a)$, and similarly $E_\mu(b) = (\sigma_j - b)h'$. Then

$$\begin{aligned} E_\lambda(a)\omega_\pi E_\mu(b) &= h(\sigma_i - a)\omega_\pi(\sigma_j - b)h' \\ &= h\omega_\pi(\sigma_j - a)(\sigma_j - b)h' = 0. \end{aligned}$$

Interchange a and b to get the other equation. □

Corollary 4.5 *If λ and μ are inseparable then*

$$E_\lambda(a)H_nE_\mu(b) = 0 = E_\lambda(b)H_nE_\mu(a)\ .$$

Proof. H_n is spanned by $\{\omega_\pi; \pi \in S_n\}$, and no π separates λ from μ. □

Lemma 4.6 *Let λ have n cells. Then $E_\lambda(a)\omega_\pi E_{\lambda^\vee}(b)$ is a scalar multiple of $E_\lambda(a)\omega_{\pi_\lambda}E_{\lambda^\vee}(b)$ for each $\pi \in S_n$.*

Proof. By Lemma 4.4, we need only consider the case when π separates λ from $\lambda^\vee$.

In this case $\omega_\pi = \omega_\rho\omega_{\pi_\lambda}\omega_\sigma$ with $\rho \in R(\lambda), \sigma \in R(\lambda^\vee)$. Then

$$E_\lambda(a)\omega_\pi E_{\lambda^\vee}(b) = E_\lambda(a)\omega_\rho\omega_{\pi_\lambda}\omega_\sigma E_{\lambda^\vee}(b) = \phi_a(\omega_\rho)\phi_b(\omega_\sigma)E_\lambda(a)\omega_{\pi_\lambda}E_{\lambda^\vee}(b).$$

□

Theorem 4.7 *Let λ and μ be Young diagrams with n cells. Then*

$$\begin{aligned} e_\lambda e_\mu &= 0 \quad \textit{for } \lambda \neq \mu, \\ e_\lambda^2 &= \alpha_\lambda e_\lambda \quad \textit{for some scalar } \alpha_\lambda. \end{aligned}$$

Thus distinct Young diagrams determine orthogonal elements, while each e_λ is a quasi-idempotent element of H_n.

Proof. By definition,

$$e_\lambda e_\mu = E_\lambda(a)\omega_{\pi_\lambda}E_{\lambda^\vee}(b)\omega_{\pi_\lambda}^{-1}E_\mu(a)\omega_{\pi_\mu}E_{\mu^\vee}(b)\omega_{\pi_\mu}^{-1}\ .$$

If $\lambda \neq \mu$ then either $\lambda^\vee$ and μ are inseparable, and so $E_{\lambda^\vee}(b)\left(\omega_{\pi_\lambda}^{-1}\right)E_\mu(a) = 0$, or λ and $\mu^\vee$ are inseparable and then $E_\lambda(a)\left(\omega_{\pi_\lambda}E_{\lambda^\vee}(b)\omega_{\pi_\lambda}^{-1}E_\mu(a)\omega_{\pi_\mu}\right)E_{\mu^\vee}(b) = 0$. Thus $e_\lambda e_\mu = 0$ when $\lambda \neq \mu$.

When $\lambda = \mu$ we can write

$$E_\lambda(a)\left(\omega_{\pi_\lambda}E_{\lambda^\vee}(b)\omega_{\pi_\lambda}^{-1}E_\lambda(a)\omega_{\pi_\lambda}\right)E_{\lambda^\vee}(b) = \alpha_\lambda E_\lambda(a)\omega_{\pi_\lambda}E_{\lambda^\vee}(b),$$

by Lemma 4.6. Then $e_\lambda^2 = \alpha_\lambda E_\lambda(a)\omega_{\pi_\lambda}E_{\lambda^\vee}(b)\omega_{\pi_\lambda}^{-1} = \alpha_\lambda e_\lambda$. □

The scalar α_λ is calculated explicitly in [1] as

$$\alpha_\lambda = \prod_{(i,j)\in\lambda} s^{j-i}[\lambda_i + \lambda_j^\vee - i - j + 1]\ ,$$

where $[k] = \dfrac{s^k - s^{-k}}{s - s^{-1}}$ denotes the 'quantum integer k'.

Note that $\lambda_i + \lambda_j^\vee - i - j + 1$ is the *hook length* of the cell in the ith row and jth column.

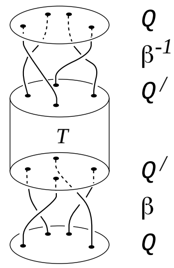


Figure 11: The wiring of $(D^2 \times I, P')$ into $(D^2 \times I, P)$

5 Framing and the Murphy operators

In this section we return to the 3-dimensional views of the skein $\mathcal{S}(B, P)$ for a ball $B \cong B^3$ with a set P of $2n$ distinguished points on its boundary, introduced in Sect. 4.

We noted earlier that this skein is linearly isomorphic to H_n. In order to translate the results about H_n from the previous section into a 3-dimensional setting we shall use some explicit choices of isomorphism in a couple of standard cases.

We start with the case when $B = D^2 \times I$ and the set $P = P_I \cup P_O$ of input and output points consists of $P_I = Q \times \{1\}$ and $P_O = Q \times \{0\}$ for some $Q \subset D^2$ with $|Q| = n$. Write $\mathcal{S}(B, P) = H_Q$ in this case, which will be identified with H_n when the points of Q lie in a straight line across D^2. We can compare $H_{Q'}$ and H_Q by choosing an n-string braid β in $D^2 \times I$ with end points $Q \times \{0\}$ and $Q' \times \{1\}$, or equivalently a homeomorphism from (D^2, Q) to (D^2, Q'). Now construct a wiring as shown in Fig. 11, where the strings meet the level disks in Q or Q' as indicated and are connected by β or β^{-1}. This induces a linear map $\mathcal{S}(\beta) : H_{Q'} \to H_Q$, where the image of a tangle T in $H_{Q'}$ is the extended tangle in H_Q shown in Fig. 11.

Note that β is a braid in $D^2 \times I$ in the sense that its strings are monotone in the last coordinate. It should properly be regarded as an element of the 'braid groupoid' of D^2.

The skein H_Q is clearly an algebra under the obvious stacking operation, and $\mathcal{S}(\beta)$ is an algebra homomorphism. Indeed it is an algebra isomorphism with inverse $\mathcal{S}(\beta^{-1})$. Thus $H_Q \cong H_n$ for each Q with $|Q| = n$.

In the previous section we constructed an element $E_\lambda \in H_Q$ where the points Q lie in the cells of the Young diagram λ. The element $e_\lambda \in H_n$ has the form $e_\lambda = \mathcal{S}(\beta)(E_\lambda)$ where the braid β lines up the cells of λ according to the tableau $T(\lambda)$. Then E_λ is itself a quasi-idempotent in H_Q. Now E_λ is constructed as the composite $E_\lambda = E^I_\lambda E^O_\lambda$ of two elements of H_Q, with E^I_λ made from the row boxes with elements a_j and E^O_λ made from the column boxes with b_j. In the notation of the last chapter we have $\mathcal{S}(\beta)(E^I_\lambda) = E_\lambda(a)$ and $\mathcal{S}(\beta)(E^O_\lambda) = \omega_{\pi_\lambda} E_{\lambda^\vee}(b)\omega^{-1}_{\pi_\lambda}$. To understand the behaviour of e_λ further, by considering E_λ and other elements of the skeins H_Q, we shall show the following.

Lemma 5.1 *Let $Q \subset D^2$ with $|Q| = n$. There is a unique element a_Q in the algebra H_Q which corresponds to $a_n \in H_n$ under* every *choice of isomorphism $\mathcal{S}(\beta)$.*

Proof. The element $a_n \in H_n$ is central, by Lemma 4.3. Given Q select any braid β to connect Q to the straight line of n points, and define $a_Q = \mathcal{S}(\beta)(a_n)$. Let γ be another such braid. Then $\gamma^{-1}\beta$ represents an element of H_n which commutes with a_n. Hence $\mathcal{S}(\gamma^{-1}\beta)(a_n) = a_n$, and $\mathcal{S}(\gamma)(a_n) = \mathcal{S}(\beta)(a_n)$.

A similar element b_Q can be constructed from b_n. □

For every subdisk $D' \subset D^2$ with $R = Q \cap D'$ there is an induced inclusion of algebras $H_R \subset H_Q$, where a tangle in $D' \times I$ is extended by the trivial strings on $Q - R$. The discussion following Lemma 4.3 shows that $a_Q a_R$ is a non-zero multiple of a_Q, and similarly $b_Q b_R$ is a non-zero multiple of b_Q.

We now look at some consequences of the work in Sect. 4 within the context of the general skein $\mathcal{S}(B, P)$. Take $B = B^3$ and $P = P_O \cup P_I \subset S^2$. Define a *geometric partition* ω of P to be a family of disjoint discs $\{D_\alpha\}$ in S^2 containing the points of P, such that no disc D_α contains both output and input points.

The geometric partition ω determines two partitions $\lambda(\omega)$ and $\mu(\omega)$ of n, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ are the numbers of points of P_I in the individual disks D_α of the partitioning family, and μ is determined similarly by the output points P_O.

Given a geometric partition ω construct a wiring in $S^2 \times I$ as follows. For each disk D_α containing a subset $P_\alpha \subset P$ insert the skein element a_{P_α} or b_{P_α} into $D_\alpha \times I \subset S^2 \times I$, choosing a_{P_α} if $P_\alpha \subset P_O$ and b_{P_α} if $P_\alpha \subset P_I$. The union of these gives a skein element in $S^2 \times I$, which induces a linear map

$$\mathcal{S}(\omega) : \mathcal{S}(B, P) \to \mathcal{S}(B, P)$$

by attaching $S^2 \times I$ as a 'shell' around B^3.

In many cases the nature of the map $\mathcal{S}(\omega)$ depends only on the partitions $\lambda(\omega)$ and $\mu(\omega)$, as described in the following lemma, which is an immediate consequence of Lemmas 4.4 and 4.6.

Lemma 5.2 *Let $\mathcal{S}(\omega) : \mathcal{S}(B, P) \to \mathcal{S}(B, P)$ be the linear map induced from a geometric partition ω of P.*

(a) If $\lambda(\omega)$ and $\mu(\omega)$ are inseparable then $\mathcal{S}(\omega) = 0$.

(b) If $\lambda(\omega)$ is just separable from $\mu(\omega)$ (when $\mu(\omega) = \lambda(\omega)^\vee$) then $\mathcal{S}(\omega)$ has rank 1. Its image is spanned by $\mathcal{S}(\omega)(T)$, where T is any tangle whose arcs separate $\lambda(\omega)$ from $\mu(\omega)$.

Proof. Choose a homeomorphism from B to $D^2 \times I$ which carries the disks D_α containing P_O to $D^2 \times \{0\}$ and those containing P_I to $D^2 \times \{1\}$, arranged in partition order along a straight line with the input or output points lined up inside each disk. This induces a linear isomorphism, ϕ say, from $\mathcal{S}(B,P)$ to H_n. The homeomorphism also extends to a map of $S^2 \times I$ which carries the wiring determined by ω to the element $E_{\lambda(\omega)}(a)$ at the top of $D^2 \times I$ along with $E_{\mu(\omega)}(b)$ at the bottom. Then $\phi \circ \mathcal{S}(\omega) \circ \phi^{-1} : H_n \to H_n$ is the linear map given by $T \mapsto E_{\lambda(\omega)}(a)\; T\; E_{\mu(\omega)}(b)$.

In case (a) this map is zero by Lemma 4.4, and so $\mathcal{S}(\omega) = 0$.

In case (b) its image has dimension 1, spanned by the image of any T which separates $\lambda(\omega)$ from $\mu(\omega)$, by Lemmas 3.3 and 4.6. The isomorphism ϕ gives the corresponding result for $\mathcal{S}(\omega)$ using tangles in $\mathcal{S}(B,P)$ which correspond to separating tangles T under the homeomorphism. □

Even where ω is separable we can deduce easily that $\mathcal{S}(\omega)(T) = 0$ for certain tangles T.

Lemma 5.3 *Let ω' be a subpartition of ω, in the sense that every disk of ω' is contained in a disk of ω. Then $\mathcal{S}(\omega)\,(\mathcal{S}(\omega')(T))$ is a non-zero multiple of $\mathcal{S}(\omega)(T)$ for every T.*

Proof. This follows at once from the construction of $\mathcal{S}(\omega)$ and the properties of the elements a_Q and b_Q, when the shell for ω' is viewed as lying inside the shell for ω. □

Hence if $\mathcal{S}(\omega')(T) = 0$ then $\mathcal{S}(\omega)(T) = 0$ also.

Lemma 5.4 *Suppose that some of the strings of a tangle T can be enclosed in a ball $B' \subset B$ which does not meet the remaining strings of T. A given geometric partition ω of $P = T \cap \partial B$ determines a partition ω' of $P' = P \cap B'$, using disks $D_\alpha \cap B'$. If $\lambda(\omega')$ and $\mu(\omega')$ are inseparable then $\mathcal{S}(\omega)(T) = 0$.*

Proof. Extend ω' to a partition of P by choosing the trivial partition on the points of $P - P'$. Then $\mathcal{S}(\omega')(T) = 0$, by Lemma 5.2 applied to B'. Now ω' is a subpartition of ω, so $\mathcal{S}(\omega)(T) = 0$, by Lemma 5.3. □

The simplest case occurs when B' contains just two strings whose inputs and outputs each lie in a disk in B' contained in one of the disks of ω. Then $\lambda(\omega') = \mu(\omega') = (2)$ are inseparable partitions.

We return now to the study of $e_\lambda \in H_n$ by means of $E_\lambda = E^I_\lambda E^O_\lambda$ in the skein H_Q based on the Young diagram λ. Write ω for the geometric partition grouping the input points by the rows of λ and the output points by its columns. Then $\lambda(\omega) = \lambda$ and $\mu(\omega) = \lambda^\vee$ are just separable, and $\mathcal{S}(\omega)(T) = E^I_\lambda T E^O_\lambda$ for any $T \in H_Q$. By Lemma 5.2(b) we can then write $E^I_\lambda T E^O_\lambda = \varphi_T E^I_\lambda E^O_\lambda = \varphi_T E_\lambda$ for

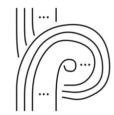
$F_n \quad = $ 

Figure 12: The full curl F_n on n strings

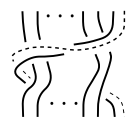


Figure 13: The braid T_i

some scalar φ_T. In particular, if T is central then $TE_\lambda = \varphi_T E_\lambda$, and consequently $\mathcal{S}(\beta)(T)e_\lambda = \varphi_T e_\lambda$ in H_n, for the appropriate β.

Thus any central element $c \in H_n$ satisfies $ce_\lambda = c_\lambda e_\lambda$ for some scalar c_λ. The eigenvalue c_λ may be found as φ_T when we choose $T = \mathcal{S}(\beta^{-1})(c)$ in the appropriate skein H_Q. We will now calculate these eigenvalues for two important central elements of H_n.

The first of these is the full curl $F_n \in H_n$, which is closely involved with changes of framing. In the skein H_Q we can describe the full curl F_Q as follows. Take the trivial framed braid $Q \times I \subset D^2 \times I$ and rotate $D^2 \times \{1\}$ through 2π anticlockwise while keeping $D^2 \times \{0\}$ fixed. The resulting framed braid represents F_Q. Note that the framing of each string in F_Q is 1. It is clear that $\mathcal{S}(\beta)F_{Q'} = F_Q$ for any β. The full curl F_n can be drawn, with blackboard framing, as in Fig. 12. We can describe F_Q recursively by selecting a subdisk containing $n-1$ of the points, R say. Then F_Q is the composite of F_R with the braid which takes the remaining string once around all the others in the subdisk.

Using tensor product to denote the juxtaposition of oriented tangles in H_n, the recursive definition gives

$$F_n = T_n(F_{n-1} \otimes 1) = \prod_{i=1}^{n} T_i \otimes 1^{n-i} \ ,$$

where T_i is the i-string tangle in Fig. 13 (all strings have blackboard framing except the ith which has its framing indicated by the dotted line in the figure).

Theorem 5.5 *Let λ be a Young diagram with n cells. Then $F_n e_\lambda = f_\lambda e_\lambda$, where $f_\lambda = x^{|\lambda|^2} v^{-|\lambda|} s^{n_\lambda}$ and $n_\lambda = \sum_{(i,j)\in\lambda} 2(j-i)$.*

Proof. By induction on the number of strings. When $n = 1$ we have $\lambda = \square$. Then e_λ is the single string, and $f_\lambda = xv^{-1}$ by the skein relations.

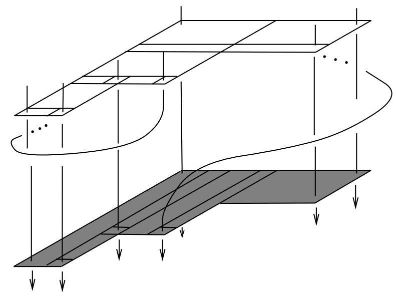


Figure 14: The element $E^I_\lambda T E^O_\lambda = \mathcal{S}(\omega)(T)$

Assume now that the result holds for all Young diagrams with fewer than n cells. Let λ be a Young diagram with n cells and let μ denote the Young diagram obtained from λ by deleting a fixed choice of extreme cell, with coordinates (p, r).

We shall work in the skein H_Q based on λ, and show that $F_Q E_\lambda = f_\lambda E_\lambda$ by calculating $\mathcal{S}(\omega)(F_Q)$. Take $R \subset Q$ to be defined by the subdiagram μ. We can include the algebra H_R in H_Q by adjoining an extra trivial string to the skein. Then $F_Q = F_R T$, where T is a version of Fig. 13 in which the extreme string winds once around the strings from R. It is geometrically clear that T commutes with the subalgebra H_R.

Now since μ is a subdiagram of λ we have $E^I_\lambda E^I_\mu = k E^I_\lambda$ and $E^O_\mu E^O_\lambda = l E^O_\lambda$ for some non-zero k, l. Then $F_Q E_\lambda = E^I_\lambda F_Q E^O_\lambda$ since F_Q is central and

$$\begin{aligned}
\mathcal{S}(\omega)(F_Q) = E^I_\lambda F_Q E^O_\lambda &= (lk)^{-1} E^I_\lambda E^I_\mu F_R T E^O_\mu E^O_\lambda \\
&= (lk)^{-1} E^I_\lambda (E^I_\mu F_R E^O_\mu) T E^O_\lambda \\
&= f_\mu (lk)^{-1} E^I_\lambda (E^I_\mu E^O_\mu) T E^O_\lambda \\
&= f_\mu E^I_\lambda T E^O_\lambda = f_\mu \mathcal{S}(\omega)(T).
\end{aligned}$$

It is enough to prove that

$$E^I_\lambda T E^O_\lambda = x^{2|\mu|+1} v^{-1} s^{2(r-p)}\, E_\lambda\ , \tag{1}$$

since then $F_Q E_\lambda = f_\mu x^{2|\mu|+1} v^{-1} s^{2(r-p)}\, E_\lambda$. Rewriting f_μ by the induction hypothesis we obtain

$$f_\lambda = x^{|\mu|^2+2|\mu|+1} v^{-1-|\mu|} s^{2(r-p)+n_\mu} = x^{|\lambda|^2} v^{-|\lambda|} s^{n_\lambda}\ .$$

The picture of $E^I_\lambda T E^O_\lambda = \mathcal{S}(\omega)(T)$ is given in Fig. 14.

We now represent some simple braids B in the skein H_Q by schematic pictures looking down on B from above. The points of Q are points on the grid. Any string not explicitly shown is assumed to pass straight down, while a path with an

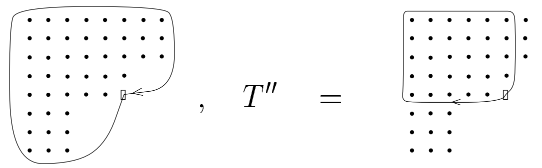


Figure 15: Reduction to the extreme rectangle

Figure 16: Smoothed braids arising from crossings outside the extreme rectangle

arrow indicates the projection of a string descending from the top to the bottom in $D^2 \times I$. When all strings pass straight down we have the identity braid Id, with $\mathcal{S}(\omega)(\mathrm{Id}) = E_\lambda$.

The schematic picture for T is shown in Fig. 15. There is one string in this braid starting and finishing from the chosen extreme position, marked $\times$, which encircles the vertical strings based on μ. We now alter T systematically through a sequence of braids which retain the vertical strings while taking the encircling string round successively fewer verticals. When we pass from one such braid T' to another T'' by crossing through one vertical string, noted on the accompanying diagrams by $\bullet$, the skein relation gives $T' = x^2 T'' + x(s - s^{-1})L$, where the braid L arises by smoothing at the crossing of the vertical and encircling strings. Thus the diagram for L has two non-vertical arcs, one from $\times$ to $\bullet$ and one from $\bullet$ to $\times$. The braids L which arise in this way will be shown, using Lemma 5.4, to satisfy $\mathcal{S}(\omega)(L) = 0$, giving $\mathcal{S}(\omega)(T') = x^2 \mathcal{S}(\omega)(T'')$, when the string $\bullet$ does not lie on the edge of the extreme rectangle determined by $\times$, in the sense of Sect. 3.

Starting then with T, pull the encircling string successively through the vertical strings outside the extreme rectangle for $\times$, working leftward through the columns and upwards through the rows until it encircles the extreme rectangle. The diagram of a typical smoothed braid L is shown in Fig. 16.

Each L has a subbraid made up of an L-shaped array of vertical strings taken from the row and column containing $\times$ and $\bullet$, along with a single string joining $\times$ and $\bullet$. This subbraid can be enclosed in a ball B' for which the induced geometric partition ω' determines an inseparable pair of partitions $\lambda(\omega')$ and $\mu(\omega')$. Thus $\mathcal{S}(\omega)(L) = 0$ in each case, by Lemma 5.4. A typical example is shown in Fig. 17.

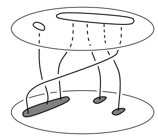


Figure 17:

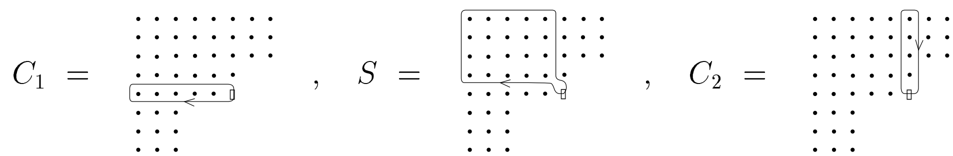


Figure 18:

The encircling string in T can then be pulled through $|\lambda| - rp$ strings to give the braid T'' in which it encircles the extreme rectangle, as in Fig. 15, with $\mathcal{S}(\omega)(T) = x^{2(|\lambda|-rp)}\mathcal{S}(\omega)(T'')$.

Now write $T'' = C_1SC_2$, where C_1, S and C_2 are shown in Fig. 18. A similar systematic move of the encircling string in S through the vertical strings inside the extreme rectangle, working through the columns from the left and from the top down, will replace S by a braid with all strings vertical, at the expense of a factor $x^{2(r-1)(p-1)}$. A typical smoothed braid L arising from the skein relation at an intermediate crossing change is shown in Fig. 19, again with an L-shaped subdiagram which ensures that $\mathcal{S}(\omega)(L) = 0$.

The framing of the eventual vertical string through $\times$ is still $+1$; when this is returned to 0 we get $\mathcal{S}(\omega)(S) = x^{2(r-1)(p-1)}xv^{-1}E_\lambda$. The calculations for Eq. (1) are completed by noting that $E^I_\lambda C_1 = (xs)^{2(r-1)}E^I_\lambda$ and $C_2E^O_\lambda = (-xs^{-1})^{2(p-1)}E^O_\lambda$, by Lemma 4.3.

Then $\mathcal{S}(\omega)(T'') = x^{2(r+p-2)}s^{2(r-p)}\mathcal{S}(\omega)(S)$, giving

$$\mathcal{S}(\omega)(T) = x^{2|\lambda|-1}s^{2(r-p)}v^{-1}E_\lambda$$

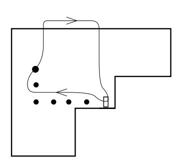


Figure 19: A smoothed braid arising from a crossing inside the extreme rectangle

as claimed. $\square$

Note that $\displaystyle\sum_{(i,j)\in\lambda} 2j = \sum_{i=1}^{k}\sum_{j=1}^{\lambda_i} 2j = \sum_{i=1}^{k}\lambda_i(\lambda_i+1) = \sum_{i=1}^{k}\lambda_i^2 + |\lambda|$. Similarly, $\displaystyle\sum_{(i,j)\in\lambda} 2i = \sum_{j=1}^{m}(\lambda_j^\vee)^2 + |\lambda|$. Thus, by Theorem 5.5, $\displaystyle n_\lambda = \sum_i \lambda_i^2 - \sum_j (\lambda_j^\vee)^2$.

As we discuss in Sect. 6, Theorem 5.5 is a skein theoretic calculation of the framing factors for the irreducible representations of $SU(N)_q$. There are various other well-known formulae, in terms of the weights of the representation or the length of either the columns or rows of the Young diagram. In [14] Morton gives the 3-variable formula for f_λ, with n_λ defined recursively in terms of smaller Young diagrams. Alternatively, the fact that a Young diagram with exactly N cells in the first column indexes the same representation as the Young diagram obtained by removing this column gives an inductive proof of a formula in terms of the column length. For a given N, the framing factor becomes a function of s alone. Reshetikhin [] proved that $f_\lambda(N) = s^{\Delta_\lambda}$ where Δ_λ is the value of the Casimir operator on the irreducible representation associated to λ. An expression for Δ_λ in terms of the weights of the representation, was described to the authors by Kohno, from which a form of n_λ in terms of row length alone can be derived, namely, $n_\lambda = \sum \lambda_i^2 + |\lambda| - \sum 2i\lambda_i$.

However, this lacks the simplistic charm of stating that n_λ is twice the sum of the contents.

The other central element of H_n which we consider is the sum M of the Murphy operators. These are discussed in [5] and are most easily described in the version $H_n(1,z)$ of the Hecke algebras, converting by the appropriate isomorphism as needed. In $H_n(1,z)$ the jth *Murphy operator* $\displaystyle M(j) = \sum_{i=1}^{j-1}\omega_{(i\,j)}$ is the sum of all the positive *transposition* braids involving the string j and strings of smaller label. Then write $M = \sum_{j=1}^{n} M(j) = \sum \omega_\tau$, where the sum is over all transpositions $\tau \in S_n$, to give an element which can readily be shown by skein theory to be central in H_n. We know from our work above that eigenvalues m_λ can be found such that $Me_\lambda = m_\lambda e_\lambda$. As above we shall calculate m_λ from the equation $E^I_\lambda \mathcal{S}(\beta)(M)E^O_\lambda = m_\lambda E_\lambda$ in H_Q, with Q defined by the diagram λ. We again use schematic diagrams to consider the constituent braids $\mathcal{S}(\beta)(\omega_{(i\,j)}) = \beta^{-1}\omega_{(i\,j)}\beta$ inside H_Q. This allows an easy calculation of the scalar $m_\lambda^{(j)}$ which satisfies $E^I_\lambda \mathcal{S}(\beta)(M(j))E^O_\lambda = m_\lambda^{(j)} E_\lambda$ and hence gives $m_\lambda = \sum_j m_\lambda^{(j)}$. It provides an alternative to the original algebraic proof of Theorem 5.7 by Dipper and James [5].

One of the advantages of the central element M over, say, the full curl is demonstrated in [12]. There are many examples of distinct Young diagrams with the same framing factor, which is the eigenvalue of the full curl. The smallest example occurs with six cells, where the partitions $(4,1,1)$ and $(3,3)$ both have

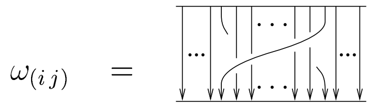


Figure 20:

framing factor $x^{36}v^{-6}s^{6}$. Any pair of Young diagrams with the same number of cells, which are self conjugate will have the same framing factor, for example $(4,2,1,1)$ and $(3,3,2)$. In contrast, the scalar m_λ uniquely determines λ, given the number of cells.

Recall that the Young tableau $T(\lambda)$ was defined by labelling the cells of λ from 1 to $|\lambda|$ along the rows. For the three-dimensional picture, we use this to indicate the ordering of the strings within the Young diagram, and to determine the conversion braid β used in the isomorphism $\mathcal{S}(\beta)$ with the Hecke algebra.

As we saw in the case of the framing coefficient, the power of x really only keeps track of the number of strings. We shall set $x=1$ in what follows. This gives one of the presentations of the Hecke algebra, $H_n(1,z)$, described in Sect. 2. The result for H_n follows directly from an application of the isomorphism with $H_n(1,z)$.

We can write $M(j)$ as the sum of positive transposition braids

$$M(j)=\sum_{i=1}^{j}\omega_{(i\,j)}\ ,$$

where $\omega_{(i\,j)}$ is the positive permutation braid in Fig. 20.

Lemma 5.6 *Suppose that the jth cell of λ has coordinates (k,l). Then*

$$E^I_\lambda\mathcal{S}(\beta)(M(j))E^O_\lambda=s^{l-k}[l-k]\ E_\lambda\ .$$

Proof. Work as above in the algebra H_Q based on the diagram λ. Write $j@i$ for the braid drawn schematically in Fig. 21. If we pull the string j through

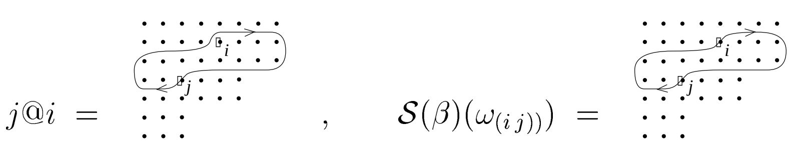


Figure 21:

the vertical string i we get the braid $j@(i+1)$, while smoothing at the crossing of these strings gives the braid $\mathcal{S}(\beta)(\omega_{(i\,j)})$, shown also in Fig. 21. Then

$$j@i = (s - s^{-1})\mathcal{S}(\beta)(\omega_{(i\,j)}) + j@(i+1),$$

by the skein relation (with $x = 1$). Summing over all $i < j$ gives

$$j@1 - j@j = (s - s^{-1})\mathcal{S}(\beta)(M(j)).$$

From the calculation in Theorem 5.5 we have

$$\mathcal{S}(\omega)(j@1) = E^I_\lambda(j@i)E^O_\lambda = s^{2(l-k)}E_\lambda$$

while $\mathcal{S}(\omega)(j@j) = E_\lambda$, where ω is the geometric partition determined, as before, by the diagram λ. Therefore

$$E^I_\lambda \mathcal{S}(\beta)(M(j))E^O_\lambda = \mathcal{S}(\omega)(j@1 - j@j) = \frac{(s^{2(l-k)} - 1)}{(s - s^{-1})}\; E_\lambda\ .$$

□

We can thus prove the result of Dipper and James [5] directly using skein theory. For completeness, we also include the proof that the eigenvalues of the sum of Murphy operators uniquely determines the Young diagrams, which is due to Katriel, Abdesselam and Chakrabarti [12]

Theorem 5.7 *Let λ be a Young diagram with n cells. Set $M = \sum_{j=1}^n M(j)$. Then M is central in H_n and $Me_\lambda = m_\lambda e_\lambda$ where*

$$m_\lambda = \sum_{(k,l)\in\lambda} s^{(l-k)}[l-k] = \sum_{(k,l)\in\lambda} \frac{s^{2(l-k)} - 1}{s - s^{-1}}\ .$$

The Young diagram λ is determined by m_λ and n.

Proof. First we show that M is central. It is enough to show that σ_i commutes with M, for $i = 1, \ldots, n-1$. We know that $M = \sum_{k<l} \omega_{(k\,l)}$ is the sum of all the positive transposition braids, working as above in $H_n(1, z)$. If neither k nor l is equal to i or $i+1$ then σ_i commutes with $\omega_{(k\,l)}$. It also commutes with $\omega_{(i\; i+1)} = \sigma_i$.

The remaining elements in M can be written as

$$A_i = \sum_{k<i} \left(\omega_{(k\ i)} + \omega_{(k\ i+1)}\right) + \sum_{i+1<l} \left(\omega_{(i\ l)} + \omega_{(i+1\ l)}\right)\ .$$

Now $\sigma_i\,\omega_{(k\ i)}\,\sigma_i = \omega_{(k\ i+1)}$, for $k < i$. Similarly $\sigma_i\,\omega_{(i+1\ l)}\,\sigma_i = \omega_{(i\ l)}$, for $i+1 < l$.

Hence $A_i = X + \sigma_i\, X\, \sigma_i$, where $X = \sum_{k<i} \omega_{(k\ i)} + \sum_{i+1<l} \omega_{(i+1\ l)}$. Then

$$\begin{aligned} \sigma_i\, A_i &= \sigma_i\, X + \sigma_i^2\, X\, \sigma_i \\ &= \sigma_i\, X + z\, \sigma_i\, X\, \sigma_i + X\, \sigma_i \\ &= A_i\, \sigma_i, \end{aligned}$$

since $\sigma_i^2 = z\,\sigma_i + 1$ in $H_n(1,z)$. This completes the proof that M is central.

Then $\mathcal{S}(\beta)(M)$ is central in H_Q and so $\mathcal{S}(\beta)(M)E_\lambda = E_\lambda^I \mathcal{S}(\beta)(M)E_\lambda^O = m_\lambda E_\lambda$. The value of m_λ as claimed follows directly from Lemma 5.6 by writing $M = \sum_j M(j)$.

To work in the version of the Hecke algebra used by Dipper and James we need simply apply the isomorphism from $H_n(1,z)$ to $H_n(q)$. A further small change is needed to give their exact result, as the Murphy operators in their paper correspond to $s\,M(j)$ here, and so their value of m_λ is s times our value.

To recover λ from m_λ calculate the Laurent polynomial

$$(s - s^{-1})m_\lambda + n = \sum_{(k,l)\in\lambda} s^{2(l-k)} = \sum c_j s^{2j} \text{ say.}$$

Then the integer c_j gives the number of cells in λ on the diagonal at distance j to the right of the leading diagonal. □

6 Connections with the quantum group invariants

Here we discuss briefly the relationship between the Homfly polynomial and the $SU(N)_q$–invariants and the bearing the idempotent elements of Sect. 4 have on this relationship. For more details, the reader is referred to [2] and [15].

Let Q_λ denotes the closure of $(1/\alpha_\lambda)\, e_\lambda$ in the Homfly skein of the annulus, or equally of $(1/\alpha_\lambda)\, E_\lambda$ in the skein of $D^2 \times S^1$.

Theorem 6.1 ([20]) *The quantum invariant $J(L; V_\square, \cdots, V_\square)$ of the link L with each component decorated by the fundamental representation is given as a function of s by the framed Homfly polynomial $\mathcal{X}(L)$, evaluated at $x = s^{-1/N}$ and $v = s^{-N}$. We will denote this evaluation of $\mathcal{X}$ by $\mathcal{X}_N$.*

Every element of $\mathcal{S}(R_n^n)$ determines an endomorphism of the n-fold tensor product of the fundamental representation, $V_\square^{\otimes n}$. Let Φ denote the map which takes an (n,n) tangle to this endomorphism. Jimbo [9] showed that Φ is a surjective ring homomorphism.

Proposition 6.2 ([2]) *For every Young diagram λ with n cells, the endomorphism of $V_\square^{\otimes n}$ determined by $\Phi(e_\lambda)$ is a scalar multiple of the projection map onto a single copy of the irreducible $SU(N)_q$–module V_λ.*

Theorem 6.3 ([2]) *Let C be a framed knot coloured by the irreducible representation V_λ. Let S be the satellite knot $C * Q_\lambda$ with companion C and pattern Q_λ. Then*

$$J(C; V_\lambda) = \mathcal{X}_N(S) \ .$$

The result also holds for links where each component coloured by V_λ is replaced by its satellite, with pattern Q_λ.

It is not difficult to show that the elements Q_λ form a linear basis for the skein of the annulus, by expressing them in terms of Turaev's basis, as discussed in Sect. 2. There is no nice skein theory proof that the expression for $Q_\lambda Q_\mu$ in terms of this basis is identical to that for the decomposition of the tensor product of the irreducible representations $V_\lambda \otimes V_\mu$. It can, however, be proved more circuitously that

$$Q_\lambda Q_\mu = \sum_{\nu \in \lambda\mu} a_{\lambda\mu\nu} Q_\nu \ ,$$

where the $a_{\lambda\mu\nu}$ are the Littlewood-Richardson coefficients.

Theorem 6.3 has been used to calculate the quantum dimensions of representations of $SU(N)_q$. The details are discussed in [1], where the following formula is proved skein theoretically;

$$\mathcal{X}(\hat{e}_\lambda) = \prod_{(i,j)\in\lambda} s^{j-i} \frac{(v^{-1}s^{j-i} - vs^{i-j})}{(s - s^{-1})} \ .$$

Making the substitutions prescribed by Theorem 6.1, we obtain the formula for the quantum dimension $\mathcal{X}_N(Q_\lambda)$, which was originally established by Reshetikhin [18],

$$\mathcal{X}_N(Q_\lambda) = \prod_{(i,j)\in\lambda} \frac{[N+j-i]}{[\lambda_i + \lambda_j^\vee - i - j + 1]}.$$

The classical dimension is given by the same formula but with integers in the place of quantum integers.

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