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A Heuristic Algorithm of Possibilistic Clustering with Partial Supervision for Classification of the Intuitionistic Fuzzy Data

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The paper deals with the problem of clustering of intuitionistic fuzzy data. A modification of a heuristic algorithm of possibilistic clustering for intuitionistic fuzzy data that account for the information coming from the labeled objects is proposed. The paper describes the basic ideas of the method and gives the plan of the partially supervised version of a direct possibilistic clustering algorithm. Illustrative examples of application of the method to two intuitionistic fuzzy data sets are provided. Preliminary conclusions are formulated and some perspectives outlined, notably for the analysis of agricultural value chain.

Keywords: Intuitionistic fuzzy set, intuitionistic fuzzy tolerance, clustering, allotment among intuitionistic fuzzy clusters, membership degree, non-membership degree, labeled object, partial supervision, agricultural value chan

1 INTRODUCTION

1.1 Partially supervised fuzzy clustering

Clustering is the unsupervised classification of objects into groups. It can be considered as a useful approach for finding similarities in data and putting similar objects into dissimilar sets known as clusters. Needless to say that possible applications of clustering span an extremely wide spectrum of areas.

Cluster analysis aims at identifying groups of related objects and so that it helps to discover assignment of objects and correlations in large data sets. The idea of data grouping is simple to use and in its nature is very near to human thinking; whenever people are presented with a large amount of data, humans tend to summarize these the data in a small number of classes or categories so as to further facilitate the analysis.

Fuzzy clustering is used, when the boundaries among the clusters are uncertain and confusing. Fuzzy clustering is a well established area, and fuzzy clustering algorithms are standard tools in unsupervised machine learning and applied statistics. Different methods have been developed in fuzzy clustering, based on different assumptions on the data and on different properties that the resulting clusters should satisfy. Heuristic methods, hierarchical methods and objective function-based methods are the main approaches in fuzzy clustering.

Partially supervised fuzzy clustering is used in discovering structure in data when labeled patterns are present. This setting applies to many systems in many areas. To just name a few, on the one hand, speech recognition systems or robot vision systems can be mentioned, and on the other hand, problems related to the analysis of data in many socio-economic systems, exemplified by broadly perceived agricultural systems.

A priori knowledge of labels of some objects is very useful for classification which has implied the development of fuzzy clustering with partial supervision. Such algorithms were proposed, for instance, by Pedrycz [1]. Numerical experiments show that knowing the membership of a small portion of patterns can significantly improvey clustering results in the sense that the partition matrix can better reflects a real structure existing in the data set. Moreover, the speed of convergence of the scheme can be considerably improved. These facts were demonstrated by Pedrycz [2].

The idea of partial supervision in fuzzy clustering was developed by different researchers. For example, an original semi-supervised modification of the FCM-algorithm was proposed by Bensaid, Hall, Bezdek and Clarke in [3]. The method is well suited to problems like image segmentation. In particular, the procedure was effectively applied to magnetic resonance image segmentation [3]. Interesting and important results in the fuzzy clustering with partial supervision were presented by Bouchachia and Pedrycz [4].

The objective function based approach is the most commonly used in fuzzy clustering. Yet, the heuristic algorithms of fuzzy clustering display a low level of complexity and a high level of clarity and comprehensibility. Some heuristic clustering algorithms are based on a very definition of the cluster concept and they aim at detecting clusters fulfilling a given definition.

Such algorithms are called algorithms of direct classification or direct clustering algorithms [5].

A possibilistic approach to clustering was proposed by Krishnapuram and Keller [6] and this approach can be considered as a special case of the fuzzy approach to clustering because all methods of possibilistic clustering are the objective function based methods. On the other hand, constraints in the possibilistic approach to clustering are weaker than constraints in the fuzzy objective function based approach to clustering and values of the membership function of a possibilistic partition can be considered as typicality degrees. So, the possibilistic approach to clustering is more general and flexible than the fuzzy approach. Many fuzzy and possibilistic clustering algorithms can be found in the books [7, 8, 9, 10], in particular cf. the recent Wierzchoń and Kłopotek's [36] book.

A heuristic approach to possibilistic clustering is proposed in [11]. Basically, the essence of the heuristic approach to possibilistic clustering is that the sought clustering structure of the set of observations is formed directly on the basis of a formal definition of a fuzzy cluster, and possibilistic memberships are determined, also directly, from the values of the pairwise similarity of observations. The notion of the allotment among fuzzy clusters is the basic concept of the approach considered and the allotment among fuzzy clusters is a special case of the possibilistic partition [6]. It should be noted that partially supervised algorithms of possibilistic clustering are absent in the framework of the objective function based approach to clustering. On the other hand, a heuristic algorithm of possibilistic clustering with partial supervision is presented in [11].

1.2 Brief introduction to intuitionistic fuzzy clustering approaches

Since the original Atanassov's [12] paper, the theory of intuitionistic fuzzy sets has been applied to many areas and new concepts were introduced, cf. [13, 14] for a comptehensive account. In particular, intuitionistic fuzzy clustering procedures were elaborated by different researchers. There are relational and prototype based intuitionistic fuzzy clustering procedures. The matrix of an intuitionistic fuzzy tolerance or intolerance relation is the input for the relational procedures. Let us consider in brief some of the methods proposed.

A fuzzy clustering method, based on the intuitionistic fuzzy tolerance relations, was proposed by Hung, Lee and Fuh [15]. An intuitionistic fuzzy similarity relation matrix is obtained starting from an intuitionistic fuzzy tolerance relation matrix by using an extended n-step procedure involving the application of the composition of intuitionistic fuzzy relations. A hard partition obtained for some thresholds α and β is the result of classification. Several types of the max-T and min-S compositions can be used in the Hung, Lee and Fuh's approach, where T is some T-norm (trangular norm) and S is a corresponding S-norm (triangular co-norm).

Then, the concepts of the association matrix and of the equivalent association matrix were defined by Xu, Chen and Wu [16] who introduced some methods for calculating the association coefficients of intuitionistic fuzzy sets. The proposed clustering algorithm uses the association coefficients of intuitionistic fuzzy sets to construct an association matrix, and utilizes a procedure to transform it into an equivalent association matrix. The α-cutting matrix of the equivalent association matrix is used to cluster the given intuitionistic fuzzy sets. So, a hard partition for some value of α is the result of classification.

In turn, Cai, Lei and Zhao [17] presented a clustering technique based on the intuitionistic fuzzy dissimilarity matrix and (α,β)-cutting matrices. The method is based on the transitive closure technique.

Then, a method of constructing an intuitionistic fuzzy tolerance matrix from a set of intuitionistic fuzzy sets and a method to cluster intuitionistic fuzzy sets via the corresponding intuitionistic fuzzy tolerance matrix were presented by Wang, Xu, Liu and Tang [18]. A hard partition is here the result of classification and the clustering depends on the chosen value of the confidence level $\alpha \in [0,1]$.

Let us consider some prototype-based intuitionistic fuzzy clustering methods. These methods are based on the representation of the initial data by a matrix of attributes. Some of these methods are objective function-based clustering procedures. So, Pelekis, Iakovidis, Kotsifakos and Kopanakis [19] proposed a variant of the well-known FCM algorithm that copes with uncertainty and involves a similarity measure between intuitionistic fuzzy sets which is then appropriately integrated in the clustering algorithm. The ordinary fuzzy c-partition is the clustering result. An application of the proposed clustering technique to image segmentation was described in [20].

Then, Torra, Miyamoto, Endo and Domingo-Ferrer [21] proposed a clustering method, based on the FCM-algorithm, for constructing an intuitionistic fuzzy partition. In the clustering method, the intuitionistic fuzzy partition deals with the uncertainty present in different executions of the same clustering procedure. The authors considered intuitionistic fuzzy partitions for the traditional fuzzy c-means, intuitionistic fuzzy partitions for the entropy-based fuzzy c-means, and intuitionistic fuzzy partitions for the fuzzy c-means with tolerance.

Following the above line of reasoning, an intuitionistic fuzzy approach to distributed fuzzy clustering was considered by Visalakshi, Thangavel and Parvathi in [22]. The corresponding IFDFC algorithm works at two different levels: local and global. At the local level, ordinary numerical data are converted into intuitionistic fuzzy data and the data are clustered independently from each other using the modified FCM algorithm. At the global level, the global centroid is calculated by clustering all local cluster centroids and the global centroid is again transmitted to local sites to update the local cluster models.

A simple clustering technique, based on calculating cluster etalons, was proposed by Todorova and Vassilev [23]. Their technique assumes that the number of clusters is equal two. The algorithm stops when all objects are assigned to crisp clusters according to the similarity measure adopted.

Agglomerative hierarchical clustering algorithms for classification of ordinary intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets were proposed by Xu [24]. On each stage of both binary clustering procedures, the centers of clusters are recalculated using the average of intuitionistic fuzzy sets assigned to the cluster, and the distance between two clusters is to be determined as the distance between the centers of each cluster.

In turn, an intuitionistic fuzzy c-means method to cluster intuitionistic fuzzy sets was presented by Xu and Wu [25]. The corresponding IFCM algorithm assumes that the initial data are in the form of a set of intuitionistic fuzzy sets, defined on the universe of attributes. The method is extended for clustering interval-valued intuitionistic fuzzy sets and the corresponding IVIFCM algorithm is also described in [25]. The fuzzy c-partition is obtained from both algorithms.

Following this idea, the WIFCM-algorithm, based on weighted intuitionistic fuzzy sets was proposed in [26]. The concepts of an equivalent classification object and a weighted intuitionistic fuzzy set were introduced, and the objective function for the WIFCM algorithm was derived from these concepts.

Then, an intuitionistic fuzzy possibilistic c-means algorithm to cluster intuitionistic fuzzy sets was proposed in [27]. The corresponding IFPCM algorithm is based on hybridizing the concepts of the FPCM clustering method [28], intuitionistic fuzzy sets and distance measures. The IFPCM-algorithm resolves the inherent problems with the availability of information on the membership values of objects in each cluster by generalizing the membership and non-membership with a hesitancy degree. Moreover, the IFPCM algorithm is extended in [27] for clustering interval-valued intuitionistic fuzzy sets, leading to interval-valued intuitionistic fuzzy possibilistic c-means algorithm. This IVIFPCM-algorithm involves the membership and non-membership degrees as intervals. Different intuitionistic fuzzy-set clustering methods are also described by Xu [29].

A heuristic approach to possibilistic clustering is generalized for the case of an intuitionistic fuzzy tolerance and the corresponding D-PAIFC algorithm is proposed in [11]. Moreover, the intuitionistic fuzzy prototype based heuristic D-PAIFC-TC algorithm of possibilistic clustering is given in [30] and the relational D-AIFC algorithm is presented in [31].

The aim of the present paper is to propose a new intuitionistic fuzzy relational heuristic algorithm of possibilistic clustering with a partial supervision. The structure of the paper is as follows: in the second section basic definitions of the intuitionistic fuzzy sets theory are given, basic concepts of the intuitionistic fuzzy generalization of the heuristic approach to possibilistic clustering are considered and the semi-supervised D-AIFC-PS(c) algorithm is proposed, in the third section two illustrative examples are provided, in fourth section some conclusions are formulated, both related to the new algorithm itself and its potential further applications, and perspectives of future investigations are outlined.

2 OUTLINE OF THE NEW HEURISTIC APPROACH TO INTUITIONISTIC FUZZY CLUSTERING

2.1 Basic definitions of the intuitionistic fuzzy sets theory

Let $X = \{x_1, \ldots, x_n\}$ be a set of elements from some universe of discours X. An intuitionistic fuzzy set IA in X is given by the ordered triple $IA = \{\langle x_i, \mu_{IA}(x_i), \nu_{IA}(x_i)\rangle \, | x_i \in X\}$, where $\mu_{IA}, \nu_{IA} : \quad X \to [0, 1]$ should satisfy

$$0 \leq \mu_{IA}(x_i) + \nu_{IA}(x_i) \leq 1 \tag{1}$$

for all $x_i \in X$. The values $\mu_{IA}(x_i)$ and $\nu_{IA}(x_i)$ denote the degree of membership and the degree of non-membership of element $x_i \in X$ to IA, respectively.

For each intuitionistic fuzzy set IA in X an intuitionistic fuzzy index hesitation margin) of an element $x_i \in X$ in IA can be defined as follows [12]

$$\rho_{IA}(x_i) = 1 - (\mu_{IA}(x_i) + \nu_{IA}(x_i)) \,. \tag{2}$$

The intuitionistic fuzzy index $\rho_{IA}(x_i)$ can be considered as a hesitation degree of x_i to IA. Naturally, $0 \leq \rho_{IA}(x_i) \leq 1$ for all $x_i \in X$. Obviously, when $\nu_{IA}(x_i) = 1 - \mu_{IA}(x_i)$, for each $x_i \in X$, the intuitionistic fuzzy set IA is an ordinary fuzzy set in X. For each ordinary fuzzy set A in X, we have $\rho_A(x_i) = 0$, for all $x_i \in X$.

Let IFS(X) denote the family of all intuitionistic fuzzy sets in X. The basic operations on the intuitionistic fuzzy sets were defined by Atanassov [12, 13, 14]. In particular, if $IA, IB \in \text{IFS}(X)$, then

$$IA \cap IB = \left\{ \begin{array}{l} \langle x_i, \mu_{IA}(x_i) \wedge \mu_{IB}(x_i), \nu_{IA}(x_i) \vee \nu_{IB}(x_i) \rangle \\ | x_i \in X \end{array} \right\}, \tag{3}$$

and

$$IA \cup IB = \left\{ \begin{array}{l} \langle x_i, \mu_{IA}(x_i) \vee \mu_{IB}(x_i), \nu_{IA}(x_i) \wedge \nu_{IB}(x_i) \rangle \\ | x_i \in X \end{array} \right\}, \tag{4}$$

Moreover, some properties of the intuitionistic fuzzy sets were also given in [32]. For example, if $IA, IB \in \text{IFS}(X)$, then

$$\begin{array}{l} IA \subseteq IB \Leftrightarrow \mu_{IA}(x_i) \leq \mu_{IB}(x_i) \text{ and} \\ \nu_{IA}(x_i) \geq \nu_{IB}(x_i), \ \forall x_i \in X \end{array}, \tag{5}$$

$$\begin{array}{l} IA \subset IB \Leftrightarrow \mu_{IA}(x_i) \leq \mu_{IB}(x_i) \text{ and} \\ \nu_{IA}(x_i) \leq \nu_{IB}(x_i), \ \forall x_i \in X \end{array}, \tag{6}$$

$$IA = IB \Leftrightarrow IA \subseteq IB \text{ and } IA \supseteq IB, \ \forall x_i \in X, \tag{7}$$

$$\bar{IA} = \{ \langle x_i, \nu_{IA}(x_i), \mu_{IA}(x_i) \rangle \, | x_i \in X \} . \tag{8}$$

Some definitions will be useful for further considerations. In particular, an α, β-level of an intuitionistic fuzzy set IA in X can be defined as

$$IA_{\alpha,\beta} = \{ x_i \in X | \mu_{IA}(x_i) \geq \alpha, \ \nu_{IA}(x_i) \leq \beta \} , \tag{9}$$

where the condition

$$0 \leq \alpha + \beta \leq 1, \tag{10}$$

is met for any values α and β, $\alpha, \beta \in [0, 1]$.

The concept of the (α, β)-level intuitionistic fuzzy set was defined in [11] as follows. The (α, β)-level intuitionistic fuzzy set $IA_{(\alpha,\beta)}$ in X is given as:

$$IA_{(\alpha,\beta)} = \left\{ \left\langle \begin{array}{l} x_i \in IA_{\alpha,\beta}, \mu_{IA_{(\alpha,\beta)}}(x_i) = \mu_{IA}(x_i), \\ \nu_{IA_{(\alpha,\beta)}}(x_i) = \nu_{IA}(x_i) \end{array} \right\rangle \right\}, \tag{11}$$

where $\alpha, \beta \in [0, 1]$ should satisfy the condition (10) and $IA_{\alpha,\beta}$ is the α, β-level of an intuitionistic fuzzy set IA satisfying (9).

If IA is an intuitionistic fuzzy set in X, where X is the set of elements, then the (α, β)-level intuitionistic fuzzy set $IA_{(\alpha,\beta)}$ in X, for which

$$\mu_{IA_{(\alpha,\beta)}}(x_i) = \begin{cases} \mu_{IA}(x_i), & \text{if } \mu_{IA}(x_i) \geq \alpha \\ 0, & \text{otherwise} \end{cases}, \tag{12}$$

and

$$\nu_{IA_{(\alpha,\beta)}}(x_i) = \begin{cases} \nu_{IA}(x_i), & \text{if } \nu_{IA}(x_i) \leq \beta \\ 0, & \text{otherwise} \end{cases}, \tag{13}$$

is called an (α, β)-level intuitionistic fuzzy subset $IA_{(\alpha,\beta)}$ of the intuitionistic fuzzy set IA in X, for some $\alpha, \beta \in [0, 1]$, $0 \leq \alpha + \beta \leq 1$.

Obviously, the condition $IA_{(\alpha,\beta)} \preceq IA$ is met for any intuitionistic fuzzy set IA and by its (α, β)-level intuitionistic fuzzy subset $IA_{(\alpha,\beta)}$, for any $\alpha, \beta \in [0, 1]$, $0 \leq \alpha + \beta \leq 1$. This important property will be very useful in further considerations.

Let us recall some basic definitions, which were considered by Burillo and Bustince [32, 33]. In cluster analysis, one is only interested in relations in a set X of classified objects. So, let $X = \{x_1, \ldots, x_n\}$ be an ordinary non-empty set. The binary intuitionistic fuzzy relation IR on X is an intuitionistic fuzzy subset IR of $X \times X$, which is given by the expression

$$IR = \left\{ \left\langle (x_i, x_j), \mu_A(x_i, x_j), \nu_A(x_i, x_j) \right\rangle | x_i, x_j \in X \right\}, \tag{14}$$

where $\mu_{IR} : X \times X \to [0, 1]$ and $\nu_{IR} : X \times X \to [0, 1]$ satisfy $0 \leq \mu_{IR}(x_i, x_j) + \nu_{IR}(x_i, x_j) \leq 1$, for each $(x_i, x_j) \in X \times X$.

Let IFR(X) denote the set of all intuitionistic fuzzy relations on X, and let us consider some basic properties of the intuitionistic fuzzy relations. An intuitionistic fuzzy relation $IR \in$ IFR(X) is reflexive, if for each $x_i \in X$, $\mu_{IR}(x_i, x_i) = 1$ and $\nu_{IR}(x_i, x_i) = 0$. An intuitionistic fuzzy relation $IR \in$ IFR(X) is symmetric, if for all $(x_i, x_j) \in X \times X$, the conditions $\mu_{IR}(x_i, x_j) = \mu_{IR}(x_j, x_i)$ and $\nu_{IR}(x_i, x_j) = \nu_{IR}(x_j, x_i)$ are met.

An intuitionistic fuzzy relation IT in X is called an intuitionistic fuzzy tolerance, if it is reflexive and symmetric. An intuitionistic fuzzy relation IS in X is called an intuitionistic fuzzy similarity relation, if it is reflexive, symmetric and transitive.

An n-step procedure, using the composition of the intuitionistic fuzzy relations, beginning with an intuitionistic fuzzy tolerance, can be used for the construction of the transitive closure of an intuitionistic fuzzy tolerance IT and the transitive closure is an intuitionistic fuzzy similarity relation IS. The procedure is the basis of the clustering algorithm, proposed by Hung, Lee and Fuh [15].

An α, β-level of an intuitionistic fuzzy relation IR in X was defined in [15] as

$$IR_{\alpha,\beta} = \left\{(x_i, x_j)|\mu_R(x_i, x_j) \geq \alpha,\ \nu_R(x_i, x_j) \leq \beta\right\}, \tag{15}$$

where condition (10) is met for any values α and β, $\alpha, \beta \in [0, 1]$. So, if $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ and $0 \leq \beta_2 \leq \beta_1 \leq 1$ with $0 \leq \alpha_1 + \beta_1 \leq 1$ and $0 \leq \alpha_2 + \beta_2 \leq 1$, then $IR_{\alpha_2,\beta_2} \subseteq IR_{\alpha_1,\beta_1}$. The respective proposition was formulated in [15].

The (α, β)-level intuitionistic fuzzy relation $IR_{(\alpha,\beta)}$ in X was defined in [11] as follows:

$$IR_{(\alpha,\beta)} = \left\{ \left\langle \begin{array}{l} (x_i, x_j) \in IR_{\alpha,\beta}, \\ \mu_{IR_{(\alpha,\beta)}}(x_i, x_j) = \mu_{IR}(x_i, x_j), \\ \nu_{IR_{(\alpha,\beta)}}(x_i, x_j) = \nu_{IR}(x_i, x_j) \end{array} \right\rangle \right\}, \tag{16}$$

where $\alpha, \beta \in [0, 1]$ should satisfy condition (10) and $IR_{\alpha,\beta}$ is the α, β-level of an intuitionistic fuzzy relation IR, which satisfies condition (15). The concept of the (α, β)-level intuitionistic fuzzy relation will be very useful in further considerations.

2.2 An intuitionistic fuzzy generalization of the heuristic approach to possibilistic clustering

Let us now consider intuitionistic extensions of the basic concepts of the D-PAFC algorithm which was proposed in [11]. Let $X = \{x_1, ..., x_n\}$ be the initial set of elements from $X$and IT be some binary intuitionistic fuzzy tolerance on $X = \{x_1, ..., x_n\}$, $\mu_{IT}(x_i, x_j) \in [0, 1]$ being its membership function and $\nu_{IT}(x_i, x_j) \in [0, 1]$ its non-membership function. Let α and β be the α, β-level values ofIT, $\alpha \in (0, 1]$, $\beta \in [0, 1)$, $0 \leq \alpha + \beta \leq 1$. The columns or lines of the intuitionistic fuzzy tolerance matrix are intuitionistic fuzzy sets $\{IA^1, ..., IA^n\}$.

Let $\{IA^1, ..., IA^n\}$ be intuitionistic fuzzy sets on X, which are generated by an intuitionistic fuzzy tolerance IT. The (α, β)-level intuitionistic fuzzy set, defined as

$$IA^l_{(\alpha,\beta)} = \left\{ \begin{array}{l} (x_i, \mu_{IA^l}(x_i), \nu_{IA^l}(x_i))| \\ \mu_{IA^l}(x_i) \geq \alpha, \nu_{IA^l}(x_i) \leq \beta, x_i \in X \end{array} \right\}$$

is an intuitionistic fuzzy (α, β)-cluster or, simply, an intuitionistic fuzzy cluster.

So $IA^l_{(\alpha,\beta)} \subseteq IA^l$, $\alpha \in (0, 1]$, $\beta \in [0, 1)$, $IA^l \in \{IA^1, \ldots, IA^n\}$, and μ_{li} is the membership degree of $x_i \in X$, for some intuitionistic fuzzy cluster $IA^l_{(\alpha,\beta)}$, $\alpha \in (0, 1]$, $\beta \in [0, 1)$, $l \in \{1, \ldots, n\}$. On the other hand, ν_{li} is the non-membership degree of $x_i \in X$ for the cluster $IA^l_{(\alpha,\beta)}$. The value of α is the tolerance threshold of the elements of the intuitionistic fuzzy cluster and value of β is the difference threshold of the elements of the intuitionistic fuzzy.

The membership degree of element $x_i \in X$ for some intuitionistic fuzzy cluster $IA^l_{(\alpha,\beta)}$, $\alpha \in (0, 1]$, $\beta \in [0, 1)$, $0 \leq \alpha + \beta \leq 1$, $l \in \{1, \ldots, n\}$ can be defined as

$$\mu_{li} = \begin{cases} \mu_{IA^l}(x_i), & x_i \in IA^l_{\alpha,\beta} \\ 0, & \text{otherwise} \end{cases}, \tag{17}$$

where an α, β-level $IA^l_{\alpha,\beta}$ of an intuitionistic fuzzy set IA^l is the support of the intuitionistic fuzzy cluster $IA^l_{(\alpha,\beta)}$. So, the condition $IA^l_{\alpha,\beta} = Supp(IA^l_{(\alpha,\beta)})$ is met for each intuitionistic fuzzy cluster $IA^l_{(\alpha,\beta)}$. The membership degree μ_{li} can be interpreted as a degree of typicality of an element to an intuitionistic fuzzy cluster.

The non-membership degree of the element $x_i \in X$ for an intuitionistic fuzzy cluster $IA^l_{(\alpha,\beta)}$, $\alpha \in (0, 1]$, $\beta \in [0, 1)$, $0 \leq \alpha + \beta \leq 1$, $l \in \{1, \ldots, n\}$ can be defined as

$$\nu_{li} = \begin{cases} \nu_{IA^l}(x_i), & x_i \in IA^l_{\alpha,\beta} \\ 0, & \text{otherwise} \end{cases}. \tag{18}$$

and can be interpreted as a degree of non-typicality of an element to an intuitionistic fuzzy cluster.

In other words, if columns or lines of the intuitionistic fuzzy tolerance matrix IT are intuitionistic fuzzy sets $\{IA^1, ..., IA^n\}$ on X, then intuitionistic fuzzy clusters $\{IA^1_{(\alpha,\beta)}, ..., IA^n_{(\alpha,\beta)}\}$ are intuitionistic fuzzy subsets of fuzzy sets $\{IA^1, ..., IA^n\}$, for some values $\alpha \in (0, 1]$ and $\beta \in [0, 1)$, $0 \leq \alpha + \beta \leq 1$. So, the condition $0 \leq \mu_{li} + \nu_{li} \leq 1$ is met for some intuitionistic fuzzy cluster $IA^l_{(\alpha,\beta)}$.

If the conditions $\mu_{li} = 0$ and $\nu_{li} = 0$ are met for some element $x_i \in X$ and for an intuitionistic fuzzy cluster $IA^l_{(\alpha,\beta)}$, then the element will be called the residual element of the intuitionistic fuzzy cluster $IA^l_{(\alpha,\beta)}$. The value zero for a fuzzy set membership function is equivalent to the non-belongingness of an element to a fuzzy set. That is why the values of tolerance threshold α are considered in the interval $(0, 1]$. So, the value of the membership function of each element of the intuitionistic fuzzy cluster is the degree of similarity of the object to some typical object of the fuzzy cluster. On the other hand, the value one for an intuitionistic fuzzy set non-membership function is equivalent to the non-belongingness of an element to an intuitionistic fuzzy set. That is why the values of difference threshold β are considered in the interval $[0, 1)$.

Let IT be an intuitionistic fuzzy tolerance on X, where X is the set of elements, and $\{IA^1_{(\alpha,\beta)}, ..., IA^n_{(\alpha,\beta)}\}$ be the family of intuitionistic fuzzy clusters for some $\alpha \in (0, 1]$ and $\beta \in [0, 1)$. The point $\tau^l_e \in IA^l_{\alpha,\beta}$, for which

$$\tau^l_e = \arg\max_{x_i} \mu_{li}, \quad \forall x_i \in IA^l_{\alpha,\beta} \tag{19}$$

is called a typical point of the intuitionistic fuzzy cluster $IA^l_{(\alpha,\beta)}$. Obviously, the membership degree of a typical point of an intuitionistic fuzzy cluster is equal one because an intuitionistic fuzzy tolerance IT is a reflexive intuitionistic fuzzy relation. So, the non-membership degree of a typical point of an intuitionistic fuzzy cluster is equal zero. Moreover, a typical point of an intuitionistic fuzzy cluster does not depend on the value of the tolerance threshold and an intuitionistic fuzzy cluster can have several typical points. That is why we use symbol e to denote the index of the typical point.

Let $IR^{\alpha,\beta}_{c(z)}(X) = \left\{ \begin{array}{l} IA^l_{(\alpha,\beta)} | l = \overline{1,c},\ c \leq n, \\ \alpha \in (0,1],\ \beta \in [0,1) \end{array} \right\}$ be a family of intuitionistic fuzzy clusters for some value of the tolerance threshold $\alpha \in (0, 1]$ and some value of the difference threshold $\beta \in [0, 1)$, $0 \leq \alpha + \beta \leq 1$. These intuitionistic fuzzy clusters are generated by some intuitionistic fuzzy tolerance IT on the initial set of elements $X = \{x_1, ..., x_n\}$. If the conditions

$$\sum_{l=1}^{c} \mu_{li} > 0, \quad \forall x_i \in X \tag{20}$$

and

$$\sum_{l=1}^{c} \nu_{li} \geq 0, \quad \forall x_i \in X \tag{21}$$

are met for all $IA^l_{(\alpha,\beta)}$, $l = \overline{1,c}$, $c \leq n$, then the family is the allotment of elements of the set $X = \{x_1, \ldots, x_n\}$ among the intuitionistic fuzzy clusters $\{IA^l_{(\alpha,\beta)},\ l = \overline{1,c},\ 2 \leq c \leq n\}$, for some value of the tolerance threshold $\alpha \in (0, 1]$ and some value of the difference threshold $\beta \in [0, 1)$. It should be noted that several allotments $IR^{\alpha,\beta}_{c(z)}(X)$ can exist for a pair of thresholds α and β. That is why we introduce the symbol z as the index of an allotment.

Condition (20) requires each object x_i, $i = \overline{1, n}$ to be assigned to at least one intuitionistic fuzzy cluster $IA^l_{(\alpha)}$, $l = \overline{1, c}$, $c \leq n$ with the membership degree higher than zero and this condition is similar to the definition of the possibilistic partition [6, 11]. The condition $2 \leq c \leq n$ requires that the number c of intuitionistic fuzzy clusters in $IR^{\alpha,\beta}_{c(z)}(X)$ be higher than two. Otherwise, the unique intuitionistic fuzzy cluster will contain all objects, possibly with different membership and non-membership degrees.

The number c of fuzzy clusters can be equal the number of objects, n. This is taken into account in our further considerations.

The allotment $IR^{\alpha,\beta}_I(X) = \left\{ \begin{array}{l} IA^l_{(\alpha,\beta)} | l = \overline{1, n}, \\ \alpha \in (0, 1],\ \beta \in [0, 1) \end{array} \right\}$ of the set of objects among n intuitionistic fuzzy clusters, for some pair of thresholds α and β, $0 \leq \alpha + \beta \leq 1$, is the initial allotment of the set $X = \{x_1, ..., x_n\}$. In other words, if the initial data are represented by a matrix of an intuitionistic fuzzy tolerance relation IT, then lines or columns of this matrix are the intuitionistic fuzzy sets IA^l, $l = \overline{1, n}$ and (α, β)-level fuzzy sets $IA^l_{(\alpha,\beta)}$, $l = \overline{1, n}$, $\alpha \in (0, 1]$, $\beta \in [0, 1)$ are the intuitionistic fuzzy clusters. These intuitionistic fuzzy clusters constitute an initial allotment for a pair of thresholds α and β, and they can be considered as clustering components.

If the conditions

$$\bigcup_{l=1}^{c} IA^l_{\alpha,\beta} = X, \tag{22}$$

and

$$\begin{array}{l} card(IA^l_{\alpha,\beta} \cap IA^m_{\alpha,\beta}) = 0, \\ \forall IA^l_{(\alpha,\beta)}, IA^m_{(\alpha,\beta)}, l \neq m,\ \alpha, \beta \in (0, 1] \end{array}, \tag{23}$$

are met for all clusters $IA^l_{(\alpha,\beta)}$, $l = \overline{1, c}$ of the allotment $IR^{\alpha,\beta}_{c(z)}(X) = \left\{ \begin{array}{l} IA^l_{(\alpha,\beta)} | l = \overline{1, c},\ c \leq n, \\ \alpha \in (0, 1],\ \beta \in [0, 1) \end{array} \right\}$, then the allotment is the allotment among fully separate intuitionistic fuzzy clusters.

The intuitionistic fuzzy clusters in the sense of definitions (17), (18) can have intersections. If the intersection area of any pair of different intuitionistic fuzzy clusters is empty, then the conditions (22) and (23) are met and the intuitionistic fuzzy clusters are called fully separate intuitionistic fuzzy clusters. Otherwise, the intuitionistic fuzzy clusters are called particularly separate intuitionistic fuzzy clusters and $w \in \{0, \ldots, n\}$ is the maximum number of elements in the intersection of different intuitionistic fuzzy clusters. For $w = 0$, the intuitionistic fuzzy clusters are fully separate. Thus, the conditions (22) and (23) can be generalized for the case of particularly separate intuitionistic fuzzy clusters. Therefore, the conditions

$$\begin{array}{l}\sum_{l=1}^{c} card(A^{l}_{\alpha,\beta}) \geq card(X), \\ \forall A^{l}_{(\alpha,\beta)} \in IR^{\alpha,\beta}_{?(z)}(X),\ \alpha \in (0,1],\ \beta \in [0,1)\end{array}, \tag{24}$$

and

$$\begin{array}{l}card(A^{l}_{\alpha,\beta} \cap A^{m}_{\alpha,\beta}) \leq w, \\ \forall A^{l}_{(\alpha,\beta)}, A^{m}_{(\alpha,\beta)},\ l \neq m,\ \alpha \in (0,1], \beta \in [0,1)\end{array}, \tag{25}$$

are generalizations of the conditions (22) and (23). The conditions (24) and (25) were formulated in [33]. Obviously, if $w = 0$ in (24) and (25), then the conditions (22) and (23) are met. The adequate allotment $IR^{\alpha,\beta}_{?(z)}(X)$ for some values of tolerance threshold $\alpha \in (0,1]$ and difference threshold $\beta \in [0,1)$ is a family of fuzzy clusters which are elements of the initial allotment $IR^{\alpha,\beta}_{I}(X)$ for values of α and β, and the family of fuzzy clusters should satisfy the conditions (24) and (25). So, the construction of the adequate allotments $IR^{\alpha,\beta}_{?(z)}(X) = \{A^{l}_{(\alpha,\beta)} | l = \overline{1,c},\ c \leq n\}$ for values α and β is a trivial combinatorial problem.

Thus, the problem of cluster analysis boils downs, in general, to establishing a unique allotment $IR^{*}_{c}(X)$, corresponding to either the most natural allocation of objects among intuitionistic fuzzy clusters or to the researcher's opinion about their classification. In the first case, the number of intuitionistic fuzzy clusters ? is not fixed. In the second case, the researcher's opinion determines the kind of the allotment sought and the number of clusters ? can be fixed. Finding the number c of the partially separated intuitionistic fuzzy clusters can be considered as the aim of classification.

Several allotments among intuitionistic fuzzy clusters can exist for a pair of thresholds α and β. Thus, the problem consists in selecting the unique principal allotment $IR^{*}_{c}(X)$ among c intuitionistic fuzzy clusters from the set $B(c)$ of allotments, $B(c) = \{IR^{\alpha,\beta}_{c(z)}(X)\}$, which is the class of possible solutions of the concrete classification problem. Here, z is the index of allotments. The selection of the unique allotment $IR^{*}_{c}(X)$ from the set $B(c) = \{IR^{\alpha,\beta}_{c(z)}(X)\}$ of allotments is made on the basis of the criterion

$$\begin{array}{l}F(IR^{\alpha,\beta}_{c(z)}(X), \alpha, \beta) = \left(\sum_{l=1}^{c} \frac{1}{n_l} \sum_{i=1}^{n_l} \mu_{li} - \alpha \cdot c \right) - \\ - \left(\sum_{l=1}^{c} \frac{1}{n_l} \sum_{i=1}^{n_l} \nu_{li} - \beta \cdot c \right)\end{array}, \tag{26}$$

where c is the number of intuitionistic fuzzy clusters in the allotment $IR^{\alpha,\beta}_{c(z)}(X)$ and $n_l = card(IA^l_{\alpha,\beta})$, $IA^l_{(\alpha,\beta)} \in IR^{\alpha,\beta}_{c(z)}(X)$ is the number of elements in the support of the intuitionistic fuzzy cluster $IA^l_{(\alpha,\beta)}$, and they can be used for the evaluation of allotments. The criterion (26) is the intuitionistic extension of the one proposed in [11] for the D-AFC(c) algorithm.

The maximum value of criterion (26) corresponds to the best allotment of objects among an a priori given number c of intuitionistic fuzzy clusters. So, the classification problem consists in finding $IR^*_c(X)$ satisfying

$$IR^*_c(X) = \arg \max_{IR^{\alpha,\beta}_{c(z)}(X) \in B(c)} F(IR^{\alpha,\beta}_c(X), \alpha, \beta), \tag{27}$$

where $B(c) = \{IR^{\alpha,\beta}_{c(z)}(X)\}$ is the set of allotments of objects among an a priori given number c of intuitionistic fuzzy clusters corresponding to the pair of thresholds α and β.

A clustering procedure is then based on decomposition of the initial intuitionistic fuzzy tolerance IT [11]. The basic concepts of the method of decomposition will now be presented. Let IT be an intuitionistic fuzzy tolerance in X, $IT_{(\alpha,\beta)}$ an (α,β)-level intuitionistic fuzzy relation and the condition (10) be met for values α and β,$\alpha \in (0,1]$, $\beta \in [0,1)$. Let $IT_{\alpha,\beta}$ be the α,β-level of an intuitionistic fuzzy tolerance IT in X and $IT_{\alpha,\beta}$ be the support of $IT_{(\alpha,\beta)}$. The membership function $\mu_{IT_{(\alpha,\beta)}}(x_i, x_j)$ can be defined as

$$\mu_{IT_{(\alpha,\beta)}}(x_i, x_j) = \begin{cases} \mu_{IT}(x_i, x_j), & if \ \mu_{IT}(x_i, x_j) \geq \alpha \\ 0, & otherwise \end{cases}, \tag{28}$$

and the non-membership function $\nu_{IT_{(\alpha,\beta)}}(x_i, x_j)$ as

$$\nu_{IT_{(\alpha,\beta)}}(x_i, x_j) = \begin{cases} \nu_{IT}(x_i, x_j), & if \ \nu_{IT}(x_i, x_j) \leq \beta \\ 0, & otherwise \end{cases}. \tag{29}$$

Obviously, the condition $IT_{(\alpha,\beta)} \subset IT$ is met for any intuitionistic fuzzy tolerance IT and (α,β)-level intuitionistic fuzzy relation $IT_{(\alpha,\beta)}$, for any $\alpha \in (0,1]$, $\beta \in [0,1)$, $0 \leq \alpha + \beta \leq 1$. So, we have the proposition that if $\alpha_{\ell(\alpha)} \leq \alpha_{\ell+1(\alpha)}$ and $\beta_{\ell+1(\beta)} \leq \beta_{\ell(\beta)}$ with $0 \leq \alpha_{\ell(\alpha)} + \beta_{\ell(\beta)} \leq 1$, $0 \leq \alpha_{\ell+1(\alpha)} + \beta_{\ell+1(\beta)} \leq 1$, then the condition $IT_{(\alpha_{\ell+1(\alpha)},\beta_{\ell+1(\beta)})} \subset IT_{(\alpha_{\ell(\alpha)},\beta_{\ell(\beta)})}$ is met. Hence, the ordered sequences $0 < \alpha_0 \leq \ldots \leq \alpha_{\ell(\alpha)} \leq \ldots \leq \alpha_{Z(\alpha)} \leq 1$ and $0 \leq \beta_{Z(\beta)} \leq \ldots \leq \beta_{\ell(\beta)} \leq \ldots \leq \beta_0 < 1$ must be constructed for the decomposition of an intuitionistic fuzzy tolerance IT. A method of construction of such sequences was developed in [11].

2.3 A note on constructing the intuitionistic fuzzy tolerance relation

The method for constructing the intuitionistic fuzzy tolerance relation was proposed by Wang, Xu, Liu and Tang in [18]. The corresponding similarity measure is based on the normalized Hamming distance and the similarity measure can be expressed by the formula

$$r(IA, IB) = \begin{cases} (1,\ 0), & IA = IB \\ \left(\begin{array}{l} 1 - \frac{1}{n}\sum_{i=1}^{n} |\nu_{IA}(x_i) - \nu_{IB}(x_i)| - \\ -\frac{1}{n}\sum_{i=1}^{n} |\rho_{IA}(x_i) - \rho_{IB}(x_i)|, \\ \frac{1}{n}\sum_{i=1}^{n} |\nu_{IA}(x_i) - \nu_{IB}(x_i)| \end{array} \right), & IA \neq IB \end{cases}, \quad (30)$$

for all $i, j = 1, \ldots, n$. That is why the closeness degree $r(IA, IB) = (\mu_{IT}(IA, IB),\ \nu_{IT}(IA, IB))$ of the intuitionistic fuzzy sets IA and IB can be constructed according to (29). Obviously, if all the differences of values of the non-membership degree and the differences of values of the intuitionistic fuzzy index of two objects IA and IB with respect to attributes x_i, $i = 1, \ldots, n$ are smaller, then the two objects are more similar to each other.

The corresponding intuitionistic fuzzy relation has the properties of symmetry and reflexivity. Moreover, the condition $0 \le \mu_{IT}(IA, IB) + \nu_{IT}(IA, IB) \le 1$ is met for any intuitionistic fuzzy sets IA and IB cf. [18]. It should be noted that other similarity measures were also proposed in [34] and [35].

2.4 Partial supervision and the D-AIFC-PS(c) algorithm

Let us consider a subset of labeled objects $X_L = \{x_{L(1)}, \ldots, x_{L(c)}\}$ and $X_L \subset X$.The condition $card(X_L) = c$ must hold for the subset. Let the membership grades $y_{l(j)}, l = 1, \ldots, c,\ j = 1, \ldots, c$ and non-membership grades $f_{l(j)}, l = 1, \ldots, c,\ j = 1, \ldots, c,$ correspond to each labeled object $x_{L(j)} \in X_L,\ j = 1, \ldots, c,$ as follows: if $x_i \in X_L$ and $x_i = x_{L(j)}$, the values of $y_{l(j)}$ and $f_{l(j)}$ are given by the analyst. So, the construction of the allotment $IR_c^*(X)$ among an a priori given number c of partially separate intuitionistic fuzzy clusters can be considered as the aim of classification and each labeled object must be assigned to a unique fuzzy cluster. Moreover, for each labeled object $x_i = x_{L(j)}$ its membership value $\mu_{li},\ l = 1, \ldots, c,\ i = 1, \ldots, n,$ in the allotment $IR_?^*(X)$ sought must be greater than an a priori determined membership grade $y_{l(j)} \in (0, 1]$ and its non-membership value $\nu_{li}, l = 1, \ldots, c, i = 1, \ldots, n$ in the allotment $IR_?^*(X)$ should be lower than an a priori given non-membership grade $f_{l(j)} \in [0, 1)$.

The corresponding D-AIFC-PS(c) algorithm for the determination of the allotment $IR_c^*(X)$ is a twelve-step procedure of classification as given below:

1. The following condition is checked:

if for any labeled object $x_{L(j)} = x_i,\ j = 1, \dots, c,\ i \in \{1, \dots, n\}$
the condition $0 \leq y_{li} + f_{li} \leq 1$ is met
then go to step 2
else print 'Conditions of classification are incorrect' and stop;

2. Let $w := 0$;
3. Construct ordered sequences $0 < \alpha_0 \leq \dots \leq \alpha_{\ell(\alpha)} \leq \dots \leq \alpha_{Z(\alpha)} \leq 1$ and $0 \leq \beta_{Z(\beta)} \leq \dots \leq \beta_{\ell(\beta)} \leq \dots \leq \beta_0 < 1$ of threshold values; let $\ell(\alpha) := 0$ and $\ell(\beta) := 0$;
4. The following condition is checked:

if the condition $0 \leq \alpha_{\ell(\alpha)} + \beta_{\ell(\beta)} \leq 1$ is met
then construct the (α, β)-level intuitionistic fuzzy relation $IT_{(\alpha,\beta)}$ in the sense of definition (16) and go to step 5
else the following condition is checked:
if the condition $\ell(\beta) < Z(\beta)$ is met
then let $\ell(\beta) := \ell(\beta) + 1$ and go to step 4;

5. Construct the initial allotment $IR_I^{\alpha,\beta}(X) = \{IA^l_{(\alpha,\beta)} | l = \overline{1, n},\ \alpha \in (0, 1],\ \beta \in [0, 1)\}$ for calculated values $\alpha_{\ell(\alpha)}$ and $\beta_{\ell(\beta)}$;
6. The following condition is checked:

if for some intuitionistic fuzzy cluster $IA^l_{(\alpha,\beta)} \in IR_I^{\alpha,\beta}(X)$ the condition $card(IA^l_{\alpha,\beta}) = n$ is met
then let $\ell(\beta) := \ell(\beta) + 1$ and go to step 4
else go to step 7;

7. Construct allotments among an a priori given number c of intuitionistic fuzzy clusters $IR_{c(z)}^{\alpha,\beta}(X) = \{IA^l_{(\alpha,\beta)} | l = \overline{1, c}, c \leq n\}$, $\alpha = \alpha_{\ell(\alpha)}$, $\beta = \beta_{\ell(\beta)}$, which satisfy the conditions (24) and (25) for the pair of values $\alpha_{\ell(\alpha)}$ and $\beta_{\ell(\beta)}$ from the sequences $0 < \alpha_0 \leq \dots \leq \alpha_{\ell(\alpha)} \leq \dots \leq \alpha_{Z(\alpha)} \leq 1$ and $0 \leq \beta_{Z(\beta)} \leq \dots \leq \beta_{\ell(\beta)} \leq \dots \leq \beta_0 < 1$;
8. The following condition is checked:

if allotments among a priori given number c of intuitionistic fuzzy clusters $IR_{c(z)}^{\alpha,\beta}(X) = \{IA^l_{(\alpha,\beta)} | l = \overline{1, c}, c \leq n\}, \alpha = \alpha_{\ell(\alpha)}, \beta = \beta_{\ell(\beta)}$ which satisfy conditions (24) and (25) are not constructed
then the following condition is checked:
if condition $\ell(\beta) = Z(\beta)$ is met, let$\ell(\alpha) := \ell(\alpha) + 1$ and $\ell(\beta) := 0$ and go to step 4
else let $\ell(\beta) := \ell(\beta) + 1$ and go to step 4
else go to step 9;

9. The following condition is checked:

if allotments among a priori given number c of intuitionistic fuzzy clusters $IR^{\alpha,\beta}_{c(z)}(X) = \{IA^l_{(\alpha,\beta)} | l = \overline{1,c}, c \leq n\}, \alpha = \alpha_{\ell(\alpha)}, \beta = \beta_{\ell(\beta)}$ which satisfy conditions (24) and (25) are not constructed
then let $w := w + 1$ and go to step 3
else go to step 10;

10. Construct the class of possible solutions of the classification problem $B(c) = \{IR^{\alpha,\beta}_{c(z)}(X)\}$, which satisfy conditions (24) and (25) for the calculated pair of values $\alpha_{\ell(\alpha)}$ and $\beta_{\ell(\beta)}$ and for the given number of fuzzy clusters c as follows:

if for some allotment $IR^{\alpha,\beta}_{c(z)}(X)$ the condition $card(IR^{\alpha,\beta}_{c(z)}(X)) = c$ is met
and for each labeled object $x_{L(j)} = x_i,\ j = 1, \ldots, c,\ i \in \{1, \ldots, n\}$ conditions $\mu_{li} \geq y_{li}, \nu_{li} \leq f_{li}, IA^l_{(\alpha,\beta)} \in IR^{\alpha,\beta}_{c(z)}(X),\ j = 1, \ldots, c$ are met
then $IR^{\alpha,\beta}_{c(z)}(X) \in B(c)$
else the following condition is checked:
if the condition $w < n - c$ is met
then let $w := w + 1$ and go to step 4
else print 'There is no solution of the classification problem for given conditions' and stop;

11. Calculate the value of the criterion (26) for each allotment $IR^{\alpha,\beta}_{c(z)}(X) \in B(c)$;
12. The result $IR^*_c(X)$ of classification is formed as follows:

if for some unique allotment $IR^{\alpha,\beta}_{c(z)}(X) \in B(c)$ the condition (27) is met
then the allotment is the result of classification $IR^*_c(X)$
else print 'The number c of classes is suboptimal' and stop.

The unique principal allotment $IR^*_c(X)$ among the a priori given number c of partially separate intuitionistic fuzzy clusters and the corresponding values of the tolerance threshold α and the difference threshold β, $0 \leq \alpha + \beta \leq 1$ are the results of classification.

The functioning of the proposed D-AIFC-PS(c) algorithm for the intuitionistic fuzzy data will now be explained by some illustrative examples.

3 EXAMPLES

3.1 Example 1

Let us consider the application of the proposed D-AIFC-PS(c) algorithm to the intuitionistic fuzzy data which originally appeared in [16]. A car dealer wishes to classify five different cars x_i, $i = 1, \ldots, 5$. There are 6 car evaluation criteria: x^1: fuel consumption, x^2: friction coefficient, x^3: price, x^4: comfort, x^5: design, x^6: safety. Evaluations of the cars regarding the criteria x^{t_1}, $t_1 = 1, \ldots, 6$ are represented by the intuitionistic fuzzy sets, shown in Table 1.

Cars	Criteria					
	x^1	x^2	x^3	x^4	x^5	x^6
x_1	(0.3, 0.5)	(0.6, 0.1)	(0.4, 0.3)	(0.8, 0.1)	(0.1, 0.6)	(0.5, 0.4)
x_2	(0.6, 0.3)	(0.5, 0.2)	(0.6, 0.1)	(0.7, 0.1)	(0.3, 0.6)	(0.4, 0.3)
x_3	(0.4, 0.4)	(0.8, 0.1)	(0.5, 0.1)	(0.6, 0.2)	(0.4, 0.5)	(0.3, 0.2)
x_4	(0.2, 0.4)	(0.4, 0.1)	(0.9, 0.0)	(0.8, 0.1)	(0.2, 0.5)	(0.7, 0.1)
x_5	(0.5, 0.2)	(0.3, 0.6)	(0.6, 0.3)	(0.7, 0.1)	(0.6, 0.2)	(0.5, 0.3)

TABLE 1
Car characteristics

IT	x_1	x_2	x_3	x_4	x_5
x_1	(1.00, 0.00)				
x_2	(0.80, 0.10)	(1.00, 0.00)			
x_3	(0.72, 0.12)	(0.82, 0.08)	(1.00, 0.00)		
x_4	(0.75, 0.13)	(0.72, 0.10)	(0.70, 0.05)	(1.00, 0.00)	
x_5	(0.65, 0.22)	(0.68, 0.18)	(0.63, 0.23)	(0.63, 0.25)	(1.00, 0.00)

TABLE 2
Intuitionistic fuzzy tolerance relation matrix

So, the data constitute of the family $X = \{x_1, \ldots, x_5\}$ of the intuitionistic fuzzy sets, where $x_i = \left\{\left\langle x^{t_1}, \mu_{x_i}(x^{t_1}), \nu_{x_i}(x^{t_1})\right\rangle | x^{t_1} \in X^{t_1}\right\}$, $X^{t_1} = \{x^1, \ldots, x^6\}$. By applying the method of constructing the intuitionistic fuzzy tolerance relation (30) to the set X, the matrix of the intuitionistic fuzzy tolerance relation IT was obtained, as shown in Table 2.

The proposed D-AIFC-PS(c)valgorithm was applied to the matrix of the intuitionistic fuzzy tolerance relation IT. The first experiment was made for the set of labeled objects $X_L = \{x_1 = x_{L(1)}, x_5 = x_{L(2)}\}$ with their membership functions $y_{1(1)} = 1.0$, $y_{2(5)} = 1.0$ and non-membership functions $f_{1(1)} = 0.0$, $f_{2(5)} = 0.0$. By executing the D-AIFC-PS(c) algorithm, the allotment among two fully separated intuitionistic fuzzy clusters was obtained for $\alpha = 0.70$ and $\beta = 0.25$. The membership and non-membership values are presented in Figure 1.

The second experiment was performed for the set of labeled objects $X_L = \{x_4 = x_{L(1)}, x_5 = x_{L(2)}\}$ with their membership functions $y_{1(4)} = 1.0$, $y_{2(5)} = 0.9$ and non-membership functions $f_{1(4)} = 0.0$, $f_{2(5)} = 0.1$. The resulting allotment among two fully separated intuitionistic fuzzy clusters was obtained for $\alpha = 0.70$ and $\beta = 0.25$. The membership and non-membership values are presented in Figure 2.

In order to compare the proposed D-AIFC-PS(c)balgorithm with the D-PAIFC-algorithm, the principal allotment among two fully separated intuitionistic fuzzy clusters was obtained from the D-PAIFC algorithm for $\alpha = 0.63$ and $\beta = 0.13$. The membership and non-membership values are presented in Figure 3.

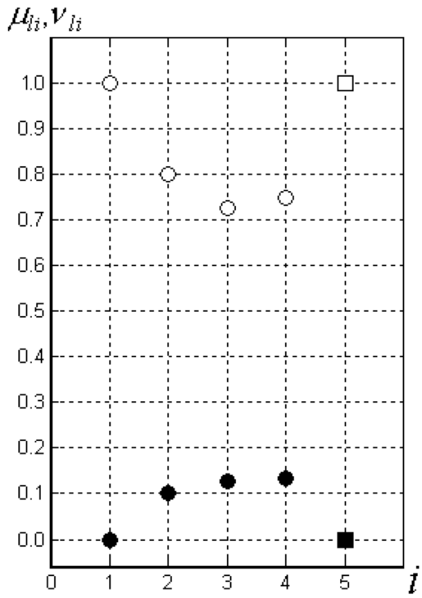

FIGURE 1
Membership and non-membership values of two intuitionistic fuzzy clusters obtained by using the D-AIFC-PS(c) algorithm for $y_{1(1)} = 1.0$, $f_{1(1)} = 0.0$, and $y_{2(5)} = 1.0$, $f_{2(5)} = 0.0$

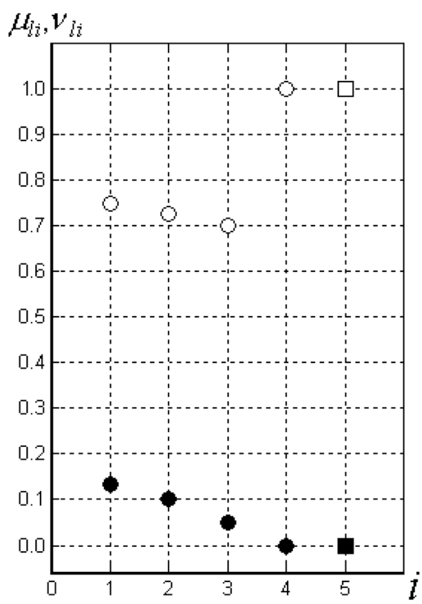

FIGURE 2
Membership and non-membership values of two intuitionistic fuzzy clusters obtained by using the D-AIFC-PS(c) algorithm for $y_{1(4)} = 1.0$, $f_{1(4)} = 0.0$, and $y_{2(5)} = 0.9$, $f_{2(5)} = 0.1$

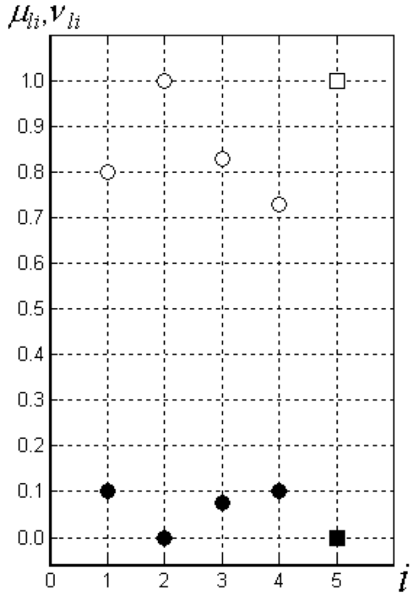

FIGURE 3
Membership and non-membership values of two intuitionistic fuzzy clusters obtained by using the D-PAIFC-algorithm

The membership values of the first class are represented by ○, non-membership values of the first class are represented by •, membership values of the second class are represented by □ and non-membership values of the second class are represented by ■ in all three figures.

The third experiment was made for the set of labeled objects $X_L = \{x_3 = x_{L(1)}, x_4 = x_{L(2)}\}$ with their membership functions $y_{1(3)} = 0.8$, $y_{2(4)} = 0.7$ and non-membership functions $f_{1(3)} = 0.1$, $f_{2(4)} = 0.2$. The allotment among two intuitionistic fuzzy clusters does not exist for these conditions.

3.2 Example 2

Let us consider an application of the proposed D-AIFC-PS(c) algorithm to Hung's [15] relational intuitionistic fuzzy data. The respective matrix of intuitionistic fuzzy tolerance relation is given in Table 3.

The proposed D-AIFC-PS(c) algorithm was applied to the matrix of the intuitionistic fuzzy tolerance relation IT for $c = 4$. The first experiment was made for the set of labeled objects $X_L = \{x_2 = x_{L(1)}, x_3 = x_{L(2)}, x_5 = x_{L(3)}, x_8 = x_{L(4)}\}$ with their membership functions $y_{1(2)} = 0.7$, $y_{2(3)} = 0.5$, $y_{3(5)} = 0.6$, $y_{4(8)} = 0.5$, and non-membership functions $f_{1(2)} = 0.2$, $f_{2(3)} = 0.3$, $f_{3(5)} = 0.3$, $f_{4(8)} = 0.4$. By executing the D-AIFC-PS(c) -algorithm, the allotment among

IT	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_1	(1.0, 0.0)									
x_2	(0.2, 0.7)	(1.0, 0.0)								
x_3	(0.5, 0.5)	(0.3, 0.6)	(1.0, 0.0)							
x_4	(0.8, 0.1)	(0.6, 0.4)	(0.5, 0.4)	(1.0, 0.0)						
x_5	(0.6, 0.3)	(0.7, 0.2)	(0.3, 0.6)	(0.7, 0.2)	(1.0, 0.0)					
x_6	(0.2, 0.7)	(0.9, 0.1)	(0.4, 0.5)	(0.3, 0.6)	(0.2, 0.7)	(1.0, 0.0)				
x_7	(0.3, 0.7)	(0.2, 0.7)	(0.1, 0.9)	(0.5, 0.4)	(0.4, 0.5)	(0.1, 0.7)	(1.0, 0.0)			
x_8	(0.9, 0.1)	(0.8, 0.2)	(0.3, 0.6)	(0.4, 0.6)	(0.5, 0.5)	(0.3, 0.7)	(0.6, 0.3)	(1.0, 0.0)		
x_9	(0.4, 0.5)	(0.3, 0.7)	(0.7, 0.2)	(0.1, 0.8)	(0.8, 0.1)	(0.7, 0.2)	(0.1, 0.8)	(0.0, 0.9)	(1.0, 0.0)	
x_{10}	(0.3, 0.7)	(0.2, 0.7)	(0.6, 0.3)	(0.3, 0.7)	(0.9, 0.1)	(0.2, 0.7)	(0.3, 0.7)	(0.2, 0.8)	(0.1, 0.8)	(1.0, 0.0)

TABLE 3
Matrix of intuitionistic fuzzy tolerance relation

four partially separated intuitionistic fuzzy clusters was obtained for α = 0.1 and β = 0.4. The membership and non-membership values are presented in Figure 4.

The membership values of the first class are represented by • in Figure 4, non-membership values of the first class are represented by □, membership values of the second class are represented by ∇, non-membership values of the second class are represented by ▼, membership values of the third class are represented by Δ, non-membership values of the third class are

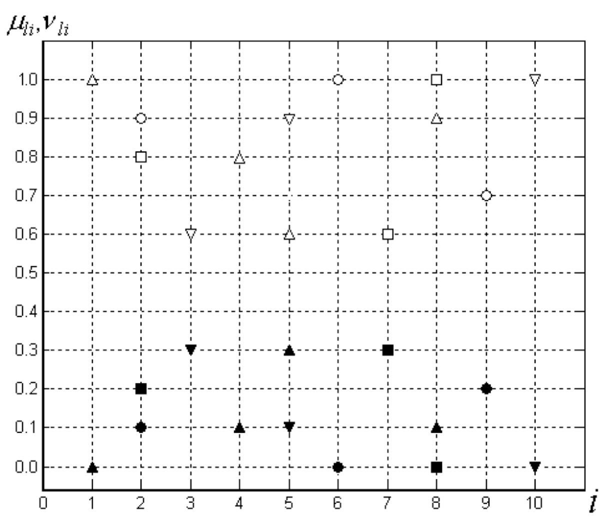

FIGURE 4
Membership and non-membership values of four intuitionistic fuzzy clusters obtained by using the D-AIFC-PS(c)-algorithm for $y_{1(2)} = 0.7$, $f_{1(2)} = 0.2$, $y_{2(3)} = 0.5$, $f_{2(3)} = 0.3$, $y_{3(5)} = 0.6$, $f_{3(5)} = 0.3$ and $y_{4(8)} = 0.5$, $f_{4(8)} = 0.4$

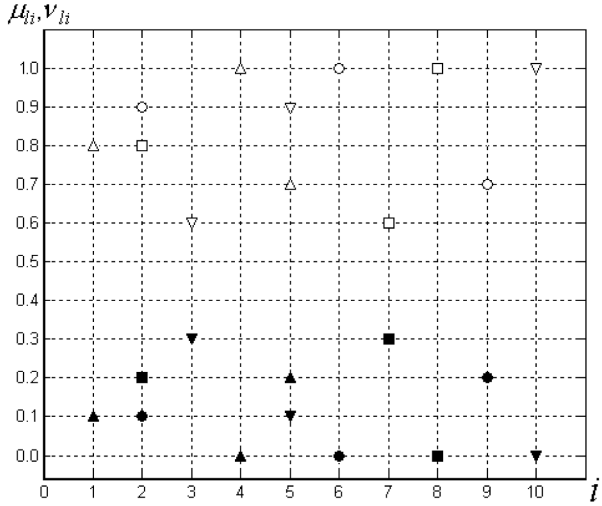

FIGURE 5
Membership values and non-membership values of four intuitionistic fuzzy clusters obtained by using the D-AIFC(c)-algorithm for $c = 4$

represented by ▲, membership values of the fourth class are represented by □ and non-membership values of the fourth class are represented by ■.

In order to compare the proposed D-AIFC-PS(c) algorithm with the D-AIFC(c) algorithm proposed in [35], the allotment among four partially separated intuitionistic fuzzy clusters was obtained from the D-AIFC(c)-algorithm for $\alpha = 0.1$ and $\beta = 0.3$. The membership and non-membership values are presented in Figure 5, where the designations of memberships and non-memberships are the same as in Figure 4.

4 CONCLUDING REMARKS

The heuristic D-AIFC-PS(c) algorithm of possibilistic clustering of the intuitionistic fuzzy data is proposed in the paper. The algorithm is based on the mechanism of partial supervision. Numerical experiments show that the results obtained by using the D-AIFC-PS(c) algorithm to the intuitionistic fuzzy data set depends on the choice of the labeled objects and on their a priori membership and non-membership functions.

The D-AIFC-PS(c)-algorithm can be applied directly to the data given as the matrix of intuitionistic fuzzy tolerance coefficients. Thus, it can be used with the object by attribute data, by choosing a similarity measure (29), or it can be used in situations where object by object proximity data is available.

The results of application of the D-AIFC-PS(c) algorithm to two intuitionistic fuzzy data sets show that the D-AIFC-PS(c) algorithm is a precise, and effective and effective numerical procedure for solving the classification problem in the presence of labeled objects.

The given membership and non-membership values can be different for different labeled objects. A problem of choice of the membership and non-membership function values for the labeled objects must be investigated. Moreover, the method can be extended to the case of presence of a few labeled objects for each class in the sought allotment among the intuitionistic fuzzy clusters. These issues will be investigated in the future.

It should also be noted that the algorithm can be very useful for the broadly perceived problems related to the analysis of value chains in agriculature considered within the RUC-APS: Enhancing and implementing Knowledge based ICT solutions within high Risk and Uncertain Conditions for Agriculture Production Systems. Basically, due to their inherent complexity the agricultural data are often not only imprecisely known but also described by human testimonies and judgments in which a "pro" – "con" structure, i.e. involving arguments in favor and against a particular statement, can be very convenient and human consistent. This may be quite effectively and efficiently modeled by the intuitionistic fuzzy sets. Moreover, due to the above mentioned complexity of the agricultural value chains, it may be expedient to determine some groups (clusters) of elements, issues, etc. to be considered consecutively, which should reduce the numerical complexity. Therefore, the intuitionistic fuzzy clustering can come to the rescue, and the algorithm presented here can be applied. Its application to the agriculatural value chain analyses will be presented in next papers.

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