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The minimum norm multi-input multi-output receptance method for partial pole placement

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ABSTRACT

A closed-form analytical solution is developed for the first time that fully addresses the problem of choosing feedback gains that minimize the control effort required for partial pole placement in multi-input, multi-output systems. The norm of the feedback gain matrix is shown to take the form of an inverse Rayleigh quotient, such that the optimal closed-loop system eigenvectors are given as a function of the dominant (highest) eigenvectors of the matrix in the quotient. The feedback gains that deliver the required pole placement with minimum effort may then be determined using standard procedures. The original formulation by the receptance method proposed an arbitrary choice of the closed loop eigenvectors that assigned the poles exactly but was generally wasteful of control effort that might otherwise be conserved or put to good use in satisfying additional control objectives. The analytical solution is validated against a set of numerical examples.

1. Introduction

Partial quadratic eigenvalue placement for flexible structures may be carried out numerically [1–3], as can minimum norm partial pole assignment [4–6]. In the latter case, the norm of the feedback gains is minimized, thereby minimising the control effort, while guaranteeing that the poles of the system are assigned. In general, these techniques require knowledge of the mass, stiffness and damping matrices $\mathbf{M}, \mathbf{K}, \mathbf{C}$.

Alternatively, the receptance method, proposed by Ram and Mottershead [7,8], offers a straightforward methodology for partial pole placement using only measured receptances. For controlling structures with a single control input and several outputs, the method considers a proportional and derivative output feedback and uses the measured receptances to determine the controller gains. Its efficiency has been demonstrated on several occasions [9–14].

Recently, the method has been generalized for Multi-Input Multi-Output systems by the same authors [13] and has seen several numerical and experimental validations [12–18]. However, in the generalized formulation, in addition to pole placement, the method requires the choice a priori of the closed loop eigenvectors of the system. Though, it does not indicate how this choice should be made, the closed loop eigenvector affects significantly the norm of feedback gains, and thus the control effort. This paper is concerned with the choice of the closed loop eigenvectors in order to minimize the norm of the gain terms. For relatively small sized systems, the optimal eigenvector which minimizes the norm of the feedback gains may be determined using numerical optimization, as in [19], where the system matrices $\mathbf{M}, \mathbf{K}, \mathbf{C}$ are required in addition to the

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measured receptances. In this paper, the optimal solution is derived analytically, and relies only on the measured receptances. The norm of the feedback gain matrix is written as a function of the assigned eigenvectors, and appears as an inverse Rayleigh quotient, $\mathbf{x}^T\mathbf{x}/\mathbf{x}^T\mathbf{A}\mathbf{x}$; the optimal system eigenvector is then defined by the dominant eigenvector of the matrix $\mathbf{A}$.

The paper is organised in four main sections: Section 2 summarises the receptance method for MIMO systems; Section 3 demonstrates the optimal eigenvectors which minimizes the norm of the feedback gains; while Section 5 is dedicated to numerical illustration. Finally, the outcomes are reviewed in a Conclusions section.

2. Pole placement with the receptance method

Consider the partial pole placement of a dynamic structure, described with its mass, damping and stiffness matrices, $\mathbf{M}_{n\times n}, \mathbf{C}_{n\times n}$ and $\mathbf{K}_{n\times n}$,

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{B}\mathbf{u}, \tag{1}$$

where $\mathbf{u}_{m\times 1}$ is a vector of m control input forces, and $\mathbf{B} = [\mathbf{b}_1 \ \mathbf{b}_2 \cdots \mathbf{b}_\mathbf{m}]$ is a $n \times m$ topology matrix, describing the control force distribution over the structure.

Using the receptance method, and assuming a multi-input multi-output position feedback, i.e. $\mathbf{u} = \left(\mathbf{F}^T s + \mathbf{G}^T\right)\mathbf{x}$, we aim to determine the feedback gains $\mathbf{F}_{n\times m} = [\mathbf{f}_1 \ \mathbf{f}_2 \cdots \mathbf{f}_m]$ and $\mathbf{G}_{n\times m} = [\mathbf{g}_1 \ \mathbf{g}_2 \cdots \mathbf{g}_m]$ which places the poles of the system at the desired location.

2.1. The method

The quadratic eigenvalue problems associated with the open loop and the closed loop system, respectively, are given by,

$$(\lambda_k^2\mathbf{M} + \lambda_k\mathbf{C} + \mathbf{K})\mathbf{v}_k = \mathbf{0}, \ k = 1, \cdots, 2n \tag{2}$$

$$(\mu_k^2\mathbf{M} + \mu_k\mathbf{C} + \mathbf{K})\mathbf{w}_k = \mathbf{B}\left(\mu_k\mathbf{F}^T + \mathbf{G}^T\right)\mathbf{w}_k, \ k = 1, \cdots, 2n \tag{3}$$

Each assigned eigenvalue $\{\mu_k\}_{k=1}^{p}$ is assumed distinct from the uncontrolled system eigenvalues $\{\lambda_k\}_{k=1}^{2n}$, while for $k = p+1, \cdots, 2n$, the eigenvalues are kept invariant $\mu_k = \lambda_k$. A non-trivial solution of Eq. (3) may then be written as,

$$\mathbf{w}_k = \mathbf{v}_k, \ \text{for } k = p+1, \cdots, 2n \tag{4}$$

and

$$\mathbf{B}\left(\lambda_k\mathbf{F}^T + \mathbf{G}^T\right)\mathbf{v}_k = \mathbf{0} \tag{5}$$

Then, by virtue of (2), and since $\mathbf{B}$ is arbitrary,

$$\left(\lambda_k\mathbf{F}^T + \mathbf{G}^T\right)\mathbf{v}_k = \mathbf{0}, \ \text{for } k = p+1, \cdots, 2n \tag{6}$$

The first p equations of Eq. (3) may be re-cast as,

$$\mathbf{w}_k = (\mu_k^2\mathbf{M} + \mu_k\mathbf{C} + \mathbf{K})^{-1}\mathbf{B}\left(\mu_k\mathbf{F}^T + \mathbf{G}^T\right)\mathbf{w}_k \tag{7}$$

The receptance matrix is now defined as,

$$\mathbf{H}(s) = (s^2\mathbf{M} + s\mathbf{C} + \mathbf{K})^{-1} \tag{8}$$

and the transfer matrix as,

$$\mathbf{r}_s = \mathbf{H}(s)\mathbf{B} \tag{9}$$

so that, from Eq. (7),

$$\mathbf{w}_k = \mathbf{r}_{\mu_k}\left(\mu_k\mathbf{F}^T + \mathbf{G}^T\right)\mathbf{w}_k \tag{10}$$

where $\mathbf{r}_{\mu_k} = \mathbf{H}(\mu_k)\mathbf{B}$. The matrix $\mathbf{H}(s)$ is invertible at $s = \{\mu_k\}_{k=1}^{p}$, since the assigned eigenvalues μ_k are distinct from the open loop eigenvalues λ_k. In most practical problems, only the transfer matrix $\mathbf{r}_s$ is measurable, with a finite number of sensors.

By introducing

$$\boldsymbol{\alpha}_k = \left(\mu_k\mathbf{F}^T + \mathbf{G}^T\right)\mathbf{w}_k \tag{11}$$

It becomes apparent from Eq. (10) that

$$\mathbf{w}_k = \mathbf{r}_{\mu_k}\boldsymbol{\alpha}_k \tag{12}$$

Eq. (11) may be now be cast in matrix terms

$$\begin{bmatrix} \mu_k \mathbf{w}_k^T & 0 & \dots & 0 & \mathbf{w}_k^T & 0 & \dots & 0 \\ 0 & \mu_k \mathbf{w}_k^T & \dots & 0 & 0 & \mathbf{w}_k^T & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_k \mathbf{w}_k^T & 0 & 0 & \dots & \mathbf{w}_k^T \end{bmatrix} \begin{pmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_m \\ \mathbf{g}_1 \\ \vdots \\ \mathbf{g}_m \end{pmatrix} = \boldsymbol{\alpha}_k \tag{13}$$

or

$$\mathbf{P}_k \mathbf{y} = \boldsymbol{\alpha}_k \tag{14}$$

with the obvious definition of $\mathbf{P}_k$and $\mathbf{y}$. Similarly, for the invariant poles $\{\lambda_k\}_{k=p+1}^{2n}$, Eq. (6) may be cast in a similar style to Eq. (13) as,

$$\begin{bmatrix} \lambda_k \mathbf{v}_k^T & 0 & \cdots & 0 & \mathbf{v}_k^T & 0 & \cdots & 0 \\ 0 & \lambda_k \mathbf{v}_k^T & \cdots & 0 & 0 & \mathbf{v}_k^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \mathbf{v}_k^T & 0 & 0 & \cdots & \mathbf{v}_k^T \end{bmatrix} \begin{pmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_m \\ \mathbf{g}_1 \\ \vdots \\ \mathbf{g}_m \end{pmatrix} = \mathbf{0} \tag{15}$$

or

$$\mathbf{Q}_k \mathbf{y} = 0, \; k = p+1, \cdots, 2n \tag{16}$$

Pole placement for MIMO systems with the method of receptance may thus be achieved as follows. First, choose arbitrarily the parameters $\alpha_{k,j}$, elements of the vector $\boldsymbol{\alpha}_k$, for $k = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, m$, and obtain the closed loop eigenvector $\mathbf{w}_k, k = 1, 2, \cdots, p$, by using Eq. (12). Then, solve the system of linear equations,

$$\boldsymbol{\Lambda} \begin{pmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_m \\ \mathbf{g}_1 \\ \vdots \\ \mathbf{g}_m \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \vdots \\ \boldsymbol{\alpha}_p \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix} \tag{17}$$

with

$$\boldsymbol{\Lambda} = \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \vdots \\ \mathbf{P}_p \\ \mathbf{Q}_{p+1} \\ \mathbf{Q}_{p+2} \\ \vdots \\ \mathbf{Q}_{2n} \end{bmatrix} \tag{18}$$

If the feedback gains $\mathbf{F}$ and $\mathbf{G}$ are real, and $\{\mu, \mathbf{w}\}$ is an eigenpair of the closed loop system of Eq. (3), then it may be shown that its complex conjugate $\{\bar{\mu}, \bar{\mathbf{w}}\}$ is an eigenpair too. It thus follows by Eq. (10) that each pair of vectors $\boldsymbol{\alpha}_k$, associated with the eigenvalue pair $\left\{\mu, \bar{\mu}\right\}$, must be chosen closed under conjugation too, so that $\mathbf{F}$ and $\mathbf{G}$ are real.

2.2. Problem statement

In practice, the feedback gains may be found by following the 5 steps shown in the flowchart of Fig. 1. However, each choice of the eigenvector $\mathbf{w}_k$ or $\boldsymbol{\alpha}_k$ (step 3) will lead to a different matrix $\boldsymbol{\Lambda}$,and thus to a different solution. Indeed, Eq. (17) admits an exact solution as long as the square matrix $\boldsymbol{\Lambda}$ is full rank; however, the most critical situation is when the vector $\boldsymbol{\alpha}_k$ (equivalently $\mathbf{w}_k$) is chosen in such a way the matrix $\boldsymbol{\Lambda}$ is nearly singular, which leads to high values of the feedback gains $\mathbf{F}$ and $\mathbf{G}$, and thus to a large and unfeasible control effort; this is illustrated in Fig. 2 for a simple 2 d.o.f. system. To avoid such a situation, the parameters $\boldsymbol{\alpha}_k$ (equivalently eigenvectors $\mathbf{w}_k$) may be chosen in order to minimize the control effort. This paper aims to determine a priori the optimal parameters $\boldsymbol{\alpha}_k$, or eigenvectors $\mathbf{w}_k$, which lead to the minimum norm of the feedback gains.

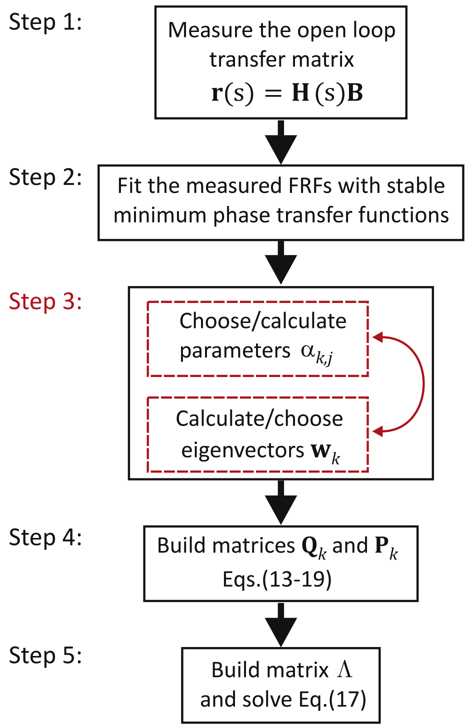

Fig. 1. Flowchart: Calculation of feedback gains by the receptance method.

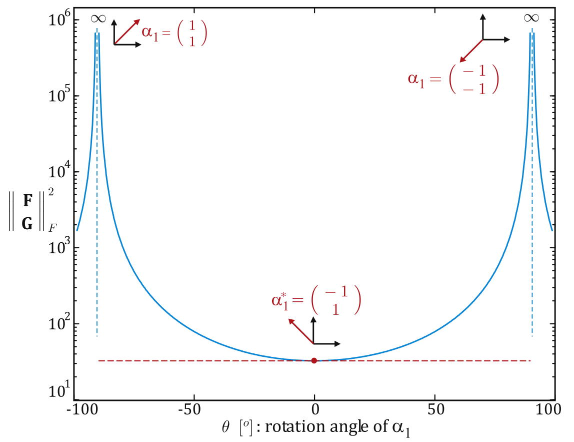

Fig. 2. Frobenius norm of the feedback gains $\mathbf{F}$ and $\mathbf{G}$ as a function of the orientation of the vector $\boldsymbol{\alpha}_1$. The minimum norm is reached when $\boldsymbol{\alpha}_1$ is chosen to minimize Eq. (32).

3. Minimum feedback norm – Optimal $\boldsymbol{\alpha}_k$

Consider pole placement with the receptance method and, without loss of generality, assume that only the first pole pair $\{\mu_1, \bar{\mu}_1\}$ is assigned, and the other poles $\{\lambda_k\}_{k=3}^{2n}$ are kept unchanged, i.e. $p = 2$. The orthogonality condition of Eq. (15) may be packed into the form,

$$\begin{bmatrix} \lambda_2 \mathbf{v}_2^T & \mathbf{v}_2^T \\ \bar{\lambda}_2 \bar{\mathbf{v}}_2^T & \bar{\mathbf{v}}_2^T \\ \lambda_3 \mathbf{v}_3^T & \mathbf{v}_3^T \\ \vdots & \vdots \\ \lambda_n \mathbf{v}_n^T & \mathbf{v}_n^T \\ \bar{\lambda}_n \bar{\mathbf{v}}_n^T & \bar{\mathbf{v}}_n^T \end{bmatrix} \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix} = \mathbf{0} \tag{19}$$

This means that the rows of the matrix $\begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix}$ must be a combination of the null space of the complex matrix $\mathbf{V} \in \mathbb{C}^{2n \times 2n-2}$, such that,

$$\mathbf{V} = \begin{bmatrix} \lambda_2 \mathbf{v}_2^T & \mathbf{v}_2^T \\ \bar{\lambda}_2 \bar{\mathbf{v}}_2^T & \bar{\mathbf{v}}_2^T \\ \lambda_3 \mathbf{v}_3^T & \mathbf{v}_3^T \\ \vdots & \vdots \\ \lambda_n \mathbf{v}_n^T & \mathbf{v}_n^T \\ \bar{\lambda}_n \bar{\mathbf{v}}_n^T & \bar{\mathbf{v}}_n^T \end{bmatrix} \tag{20}$$

If $\mathbf{z}_1$ and $\mathbf{z}_2$ denote the orthonormal basis vectors of the null space of $\mathbf{V}$, then the feedback matrix $\begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix}$ may be written as a linear combination of these two vectors,

$$\begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix} = [\mathbf{z}_1 \quad \mathbf{z}_2]\boldsymbol{\beta} \tag{21}$$

where $\boldsymbol{\beta}$ is an unknown $2 \times m$ matrix.

Then, the Frobenius norm of the feedback gain matrix $\begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix}$ is obtained as

$$\left\| \begin{matrix} \mathbf{F} \\ \mathbf{G} \end{matrix} \right\|_F^2 = \mathrm{tr}\left(\begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix}^H \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix} \right) = \mathrm{tr}\left(\boldsymbol{\beta}^H \begin{bmatrix} \bar{\mathbf{z}}_1 & \bar{\mathbf{z}}_2 \end{bmatrix}^T [\mathbf{z}_1 \quad \mathbf{z}_2]\boldsymbol{\beta} \right) \tag{22}$$

where the superscript H denote the Hermitian (i.e. the conjugate transpose). Given that the vectors $\mathbf{z}_1$ and $\mathbf{z}_2$ are orthonormal, Eq. (22) becomes,

$$\left\| \begin{matrix} \mathbf{F} \\ \mathbf{G} \end{matrix} \right\|_F^2 = \mathrm{tr}(\boldsymbol{\beta}^H \boldsymbol{\beta}) = \mathrm{tr}(\boldsymbol{\beta}\boldsymbol{\beta}^H) \tag{23}$$

The goal is to determine the vector $\boldsymbol{\alpha}_1$ which minimizes the control effort, or alternatively Eq. (23). To do so, consider Eq. (13), which may be re-arranged as,

$$\boldsymbol{\alpha}_1^T \begin{bmatrix} \mu_1 \mathbf{r}_{\mu_1}^T & \mathbf{r}_{\mu_1}^T \end{bmatrix} \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix} = \boldsymbol{\alpha}_1^T \tag{24}$$

where $\mathbf{w}_1$ has been replaced by $\mathbf{r}_{\mu_k}\boldsymbol{\alpha}_1$, as defined in Eq. (12). By incorporating Eq. (21) into Eq. (24), it is seen that,

$$\boldsymbol{\alpha}_1^T \begin{bmatrix} \mu_1 \mathbf{r}_{\mu_1}^T & \mathbf{r}_{\mu_1}^T \end{bmatrix} [\mathbf{z}_1 \quad \mathbf{z}_2]\boldsymbol{\beta} = \boldsymbol{\alpha}_1^T \tag{25}$$

Now, by introducing,

$$\mathbf{J} = \begin{bmatrix} \mu_1 \mathbf{r}_{\mu_1}^T & \mathbf{r}_{\mu_1}^T \end{bmatrix} [\mathbf{z}_1 \quad \mathbf{z}_2] \tag{26}$$

so that when each side of Eq. (25) is multiplied by its Hermitian, it is found that,

$$\boldsymbol{\alpha}_1^T \mathbf{J} \boldsymbol{\beta} \boldsymbol{\beta}^H \mathbf{J}^H \bar{\boldsymbol{\alpha}}_1 = 1 \tag{27}$$

where $\boldsymbol{\alpha}_1$ is normalized, $\boldsymbol{\alpha}_1^T \bar{\boldsymbol{\alpha}}_1 = 1$. The matrix $\boldsymbol{\beta}\boldsymbol{\beta}^H$ is of rank 2 and may be written as a function of its singular values as,

$$\boldsymbol{\beta}\boldsymbol{\beta}^H = \sigma_1^2 \mathbf{u}_1 \mathbf{u}_1^H + \sigma_2^2 \mathbf{u}_2 \mathbf{u}_2^H \tag{28}$$

where σ_1 and σ_2 are the singular values and $\mathbf{u}_1$ and $\mathbf{u}_2$ are the left eigenvectors of the matrix $\boldsymbol{\beta}$. The Frobenius norm of $\boldsymbol{\beta}$ is simply the sum of its two singular values. Thus, the optimal solution to the problem is to minimize these singular values. The combination of Eqs. (27) and (28) leads to,

$$\boldsymbol{\alpha}_1^T \mathbf{J} \mathbf{u}_1 \mathbf{u}_1^H \mathbf{J}^H \bar{\boldsymbol{\alpha}}_1 \sigma_1^2 + \boldsymbol{\alpha}_1^T \mathbf{J} \mathbf{u}_2 \mathbf{u}_2^H \mathbf{J}^H \bar{\boldsymbol{\alpha}}_1 \sigma_2^2 = 1 \tag{29}$$

One possible option is to restrict $\boldsymbol{\alpha}_1$ to real values, such that the rank of the feedback gain matrix reduces to unity with the result that $\sigma_2 = 0$. Then, as explained in the Appendix, the matrix $\boldsymbol{\beta}$ becomes rank-1, with the result, from Eq. (21), that the columns (and rows) of the matrix $\begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix}$ are proportional to each other. Thus, Eq. (29) may be simplified to obtain,

$$\sigma_1^2 = \frac{1}{\boldsymbol{\alpha}_1^T \mathbf{J} \mathbf{u}_1 \mathbf{u}_1^H \mathbf{J}^H \boldsymbol{\alpha}_1} \tag{30}$$

The term $\boldsymbol{\alpha}_1^T \mathbf{J} \mathbf{u}_1$ in the denominator is maximal when the vectors $\mathbf{u}_1$ and $\mathbf{J}^{\mathbf{H}} \boldsymbol{\alpha}_1$ are collinear, such that

$$\mathbf{u}_1 = \frac{\mathbf{J}^H \boldsymbol{\alpha}_1}{\left(\boldsymbol{\alpha}_1^T \mathbf{J} \mathbf{J}^H \boldsymbol{\alpha}_1\right)^{\frac{1}{2}}} \tag{31}$$

Finally, after substituting Eq. (31) into Eq. (30), we obtain the singular value σ_1^2, and thus the norm of feedback gains and $\boldsymbol{\beta}\boldsymbol{\beta}^H$, as a function of $\boldsymbol{\alpha}_1$,

$$\sigma_1^2 = \|\boldsymbol{\beta}\|_F^2 = \left\| \begin{matrix} \mathbf{F} \\ \mathbf{G} \end{matrix} \right\|_F^2 = \frac{1}{\boldsymbol{\alpha}_1^T \mathbf{J} \mathbf{J}^H \boldsymbol{\alpha}_1} \tag{32}$$

The ratio of Eq. (32) is simply the inverse of Rayleigh quotient (since the matrix $\mathbf{J}\mathbf{J}^H$ is Hermitian). It is minimum when $\boldsymbol{\alpha}_1$ is equal to the eigenvector associated with the maximum eigenvalue of $\mathbf{J}\mathbf{J}^H$. This is the optimal solution which minimizes the norm of the feedback gain matrix.

In order to obtain a minimum norm feedback, the arbitrary choice of $\boldsymbol{\alpha}_1$ in step 3 of the flowchart of Fig. 1 is determined systematically as follow:

1- Calculate the null space of the matrix $\mathbf{V}$ in Eq. (20);
2- Build the matrix $\mathbf{J}$, using Eq. (26);
3- Choose the eigenvector $\boldsymbol{\alpha}_1$ as the dominant eigenvector of $\mathbf{J}\mathbf{J}^H$.

Lemma

If the eigenvectors $\mathbf{v}_k$ form an orthogonal basis, then the control effort for assigning each pole pair independently with minimum effort leads to the same minimum as when assigning all the pole pairs at the same time.

Proof

As described previously, let $\begin{bmatrix} \mathbf{F}^{(j)} \\ \mathbf{G}^{(j)} \end{bmatrix}$ be the optimal feedback gain matrix for assigning the j^{th} pole pair $\{\mu_j, \bar{\mu}_j\}$, while conserving the other poles unchanged. If the proposition is assumed true, then feedback gain matrix for assigning p pole pairs is given by

$$\begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix} = \sum_{j=1}^{p} \begin{bmatrix} \mathbf{F}^{(j)} \\ \mathbf{G}^{(j)} \end{bmatrix} \tag{33}$$

and its Frobenius norm is,

$$\left\| \begin{matrix} \mathbf{F} \\ \mathbf{G} \end{matrix} \right\|_F^2 = \mathrm{tr}\left(\sum_{i=1}^{p} \sum_{j=1}^{p} \begin{bmatrix} \mathbf{F}^{(i)} \\ \mathbf{G}^{(i)} \end{bmatrix}^T \begin{bmatrix} \mathbf{F}^{(j)} \\ \mathbf{G}^{(j)} \end{bmatrix} \right) \tag{34}$$

or

$$\left\| \begin{matrix} \mathbf{F} \\ \mathbf{G} \end{matrix} \right\|_F^2 = \sum_{i=1}^{p} \left\| \begin{matrix} \mathbf{F}^{(i)} \\ \mathbf{G}^{(i)} \end{matrix} \right\|_F^2 + 2\,\mathrm{tr}\left(\sum_{i \neq j}^{p} \begin{bmatrix} \mathbf{F}^{(i)} \\ \mathbf{G}^{(i)} \end{bmatrix}^T \begin{bmatrix} \mathbf{F}^{(j)} \\ \mathbf{G}^{(j)} \end{bmatrix} \right) \tag{35}$$

Recalling from the previous analysis that the columns of $\mathbf{F}^{(i)}$ and $\mathbf{G}^{(i)}$ are proportional to each other, then, in order to satisfy the orthogonality condition of Eq. (15), the columns of the matrices $\mathbf{F}^{(i)}$ and $\mathbf{G}^{(i)}$ are seen also to be proportional to the eigenvector $\mathbf{v}_j$. Thus, since the eigenvectors $\mathbf{v}_j$ are orthogonal to each other, then

$$\mathbf{F}^{(i)T}\mathbf{F}^{(j)} = \mathbf{G}^{(i)T}\mathbf{G}^{(j)} = \mathbf{0}, \text{ for } j \neq i \tag{36}$$

and the second term of Eq. (3) vanishes, leading to

$$\left\| \begin{matrix} \mathbf{F} \\ \mathbf{G} \end{matrix} \right\|_F^2 = \sum_{i=1}^{p} \left\| \begin{matrix} \mathbf{F}^{(i)} \\ \mathbf{G}^{(i)} \end{matrix} \right\|_F^2 \tag{37}$$

The columns of the matrices $\mathbf{F}^{(i)}$ and $\mathbf{F}^{(j)}$, and $\mathbf{G}^{(i)}$ and $\mathbf{G}^{(j)}$, are orthogonal to each other, so that Eq. (33) is true and the proposition is proven.

Remark

If the procedure described above is applied to incomplete modal vectors, such that $\mathbf{v}_k$ do not form an orthogonal basis, then an optimal solution $\begin{bmatrix} \mathbf{F}^{(j)} \\ \mathbf{G}^{(j)} \end{bmatrix}$ is obtained at every step, but the overall solution $\begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix}$ is in general not optimal in term of the effort required to assign the complete system of poles.

4. Numerical illustration

In this section, through three numerical examples, we validate the findings of the previous section. First, we consider a 2 degrees of freedom system identical to the example in [13], where the poles are not complex conjugate (overdamped system with coupled eigenvectors). Then, we consider a lightly damped 3 degree of freedom mass-spring system, with a Rayleigh damping matrix, such that the eigenvectors of the system are the mode shapes of the structure. We compare the situation when all the degrees of freedom are measurable and when only 2 degrees of freedom, among 3, are measured (i.e. the eigenvectors are measured partially and, thus, are not orthogonal).

As discussed previously, each choice of the vector $\boldsymbol{\alpha}$ (and equivalently of $\mathbf{w}$) will lead to a different value of the feedback gains. In order to evaluate all the possibilities, first we consider an initial vector $\boldsymbol{\alpha}_0$, and apply a rotation operator $\mathbf{R}(\theta)$ on $\boldsymbol{\alpha}_0$, within the range $\theta = [-\pi/2, \pi/2]$. Then, for each value of θ, we evaluate the Frobenius norm of the feedback gains matrix, using Eq. (22). Finally, the Frobenius norm of the feedback gains is plotted as a function of θ (i.e. the orientation of $\boldsymbol{\alpha}$).

Example 1

Consider the open loop system with,

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix}, \ \mathbf{K} = \begin{bmatrix} 10 & -5 \\ -5 & 15 \end{bmatrix}$$

The eigenpairs of the system are,

$$\left\{ \lambda_{1,2} = \pm\sqrt{5}i, \mathbf{v}_{1,2}^T = (1 \quad 1) \right\}, \ \left\{ \lambda_3 = -2.5, \mathbf{v}_3^T = (2 \quad -1) \right\}, \ \left\{ \lambda_4 = -5, \mathbf{v}_4^T = (2 \quad -1) \right\}$$

We wish to change the eigenvalues $\lambda_{1,2}$ to $\mu_{1,2} = -1 \pm i$, and keep $\lambda_{3,4}$ unchanged by using MIMO control with the input matrix,

$$\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$$

The feedback gain matrices are found by using the receptance method described in section 2 [Eq. (17)]. Fig. 2 shows the Frobenius norm of the feedback gains as a function of $\boldsymbol{\alpha}$. Using Eq. (32), the optimal vector $\boldsymbol{\alpha}_1^\star$ which minimizes the control effort is found to be $\boldsymbol{\alpha}_1^\star = \boldsymbol{\alpha}_2^\star = (-1 \ 1)^{\mathrm{T}}$. The closed loop eigenvectors $\mathbf{w}_1^\star$ associated to the optimal vector $\boldsymbol{\alpha}_1^\star$ is given by,

$$\mathbf{w}_1^\star = \mathbf{r}_{\mu_1}\boldsymbol{\alpha}_1^\star = \begin{pmatrix} 1 \\ 0.6 + \mathrm{j} \end{pmatrix}$$

This result is obtained with the feedback gains,

$$\mathbf{F} = [\mathbf{f}_1 \quad \mathbf{f}_2] = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}, \text{ and } \mathbf{G} = [\mathbf{g}_1 \quad \mathbf{g}_2] = \begin{bmatrix} 1.5 & -1.5 \\ 3 & -3 \end{bmatrix},$$

If $\boldsymbol{\alpha}_1$ had been chosen equal to $(1 \quad 0.5)^T$, as in [13], then the feedback gains would be,

$$\mathbf{F} = [\mathbf{f}_1 \quad \mathbf{f}_2] = \begin{bmatrix} -4 & -2 \\ -8 & -4 \end{bmatrix}, \text{ and } \mathbf{G} = [\mathbf{g}_1 \quad \mathbf{g}_2] = \begin{bmatrix} 6 & 3 \\ 12 & 6 \end{bmatrix},$$

such that the Frobenius norm of the feedback gain matrix is 3.5 times higher than the optimal gains.

Observe, that although the open loop eigenvector is real, the closed loop eigenvector which minimizes the control effort is complex. Such a solution would not be predicted heuristically, e.g. by conserving the eigenvectors unchanged. If closed loop eigenvector had been conserved unchanged, i.e. $\mathbf{w}_1^\star = \mathbf{v}_1$, then the feedback gains would be only 10% higher than the optimal solution. However, the difference is more significant when the eigenvectors are only available partially, as shown in example 3 below.

Example 2

Consider the spring-mass system of Fig. 3, with

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}, \mathbf{C} = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}, \mathbf{K} = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The stiffness of the system is normalized with respect to the first resonance frequency (i.e. $\omega_1 = 1$ Hz). The mode shapes of the system are also illustrated in Fig. 3. Rayleigh damping is considered, such that the mode shapes of the system are real, and the modal damping is 1%. The eigenvectors of the open loop system $\mathbf{v}_k$ are also real and equal to the mode shapes $\boldsymbol{\phi}_k$, such that

$$\mathbf{v}_1 = \mathbf{v}_2 = \boldsymbol{\phi}_1 = (0.33 \quad 0.6 \quad 0.74)^T, \lambda_{1,2} = 2\pi\left(-0.01 \pm i\sqrt{1 - 0.01^2}\right)$$

$$\mathbf{v}_3 = \mathbf{v}_4 = \boldsymbol{\phi}_2 = (0.74 \quad 0.33 \quad -0.6)^T, \lambda_{3,4} = 5.6\pi\left(-0.01 \pm i\sqrt{1 - 0.01^2}\right)$$

$$\mathbf{v}_5 = \mathbf{v}_6 = \boldsymbol{\phi}_3 = (0.6 \quad -0.74 \quad 0.33)^T, \lambda_{3,4} = 8.2\pi\left(-0.01 \pm i\sqrt{1 - 0.01^2}\right)$$

We assume only two actuators are available, u_1 and u_2, and wish to increase the second mode damping to 10% and its resonance frequency from 2.8 Hz to 3.2 Hz.

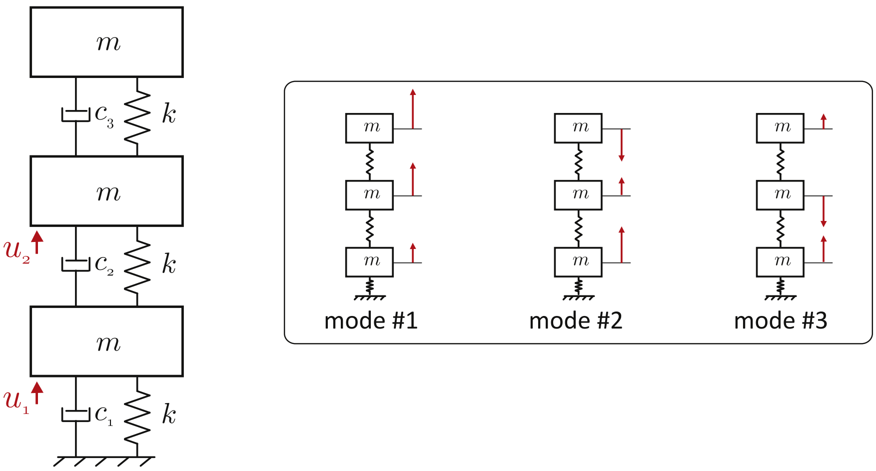

Fig. 3. Three degrees of freedom system. The dampers are tuned to obtain a modal damping of 1%.

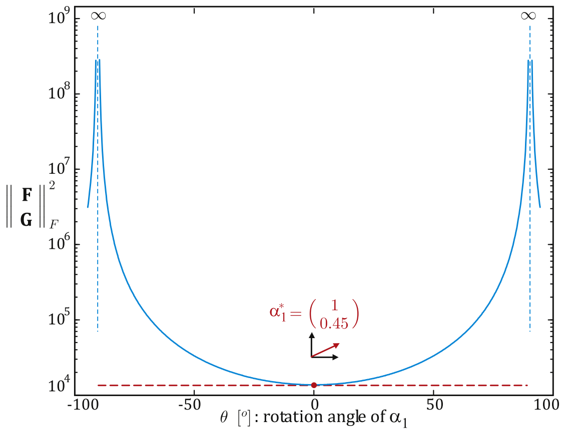

Fig. 4. Frobenius norm of the feedback gains **F** and **G** as a function of the orientation of the vector $\boldsymbol{\alpha}_3$. The minimum is reached when $\boldsymbol{\alpha}_3$ is chosen to minimize Eq. (32).

By following the same steps as in the previous example, where we vary the orientation of the vector $\boldsymbol{\alpha}_3$, the results shown in Fig. 4 are similar to those of the previous example. The minimum Frobenius norm of the feedback gain is obtained with $\boldsymbol{\alpha}_3^\star = \boldsymbol{\alpha}_4^\star = (1 \quad 0.45)^T$, which corresponds to the eigenvector pair

$$\mathbf{w}_3^\star = \bar{\mathbf{w}}_4^\star = \mathbf{r}_{\mu_3}\boldsymbol{\alpha}_3^\star = \begin{pmatrix} -0.1 - 0.73j \\ -0.23 - 0.44j \\ 0.04 + 0.46j \end{pmatrix}$$

The closed loop eigenvector $\mathbf{w}_3^\star$ is complex, its projection on the open loop eigenvector $\mathbf{v}_1$ is also complex, it is given by (after normalizing the two vectors),

$$\mathbf{v}_3^T\mathbf{w}_3^\star = -0.173 + 0.95j$$

and $|\mathbf{v}_3^T\mathbf{w}_3^\star| = 0.97 \approx 1$, which means that the two vectors are almost parallel. Once again, the optimal eigenvector $\mathbf{w}_3^\star$ is very close to the open loop eigenvector $\mathbf{v}_3$.

Finally, for a relatively small variation of the poles (i.e. the placed poles are close to the open loop poles), the optimal eigenvector $\mathbf{w}_k^\star$ is very close to the open loop eigenvector $\mathbf{v}_k$; for this example, the difference in the control gains norm would be only 3%, if $\mathbf{w}_3$ would have been chosen equal to the open loop eigenvector $\mathbf{v}_3$. However, if the pole variation is significant, particularly when the resonance frequencies are placed close to other poles (thus the closed loop stiffness matrix changes significantly), then the feedback gains arising from $\mathbf{w}_3 = \mathbf{v}_3$ would be significantly higher than those arising from the optimal solution. For example, for reducing the resonance frequency of the second mode by half, from 2.8 Hz to 2 Hz, the optimal solution $\mathbf{w}_3^\star$ leads to feedback gains 2.5 times smaller than the gains obtained with $\mathbf{w}_3 = \mathbf{v}_3$.

Example 3

Consider again Example 2, where the number of sensors n is reduced from 3 to 2. The sensor on the second mass is discarded, and the eigenvectors become:

$$\mathbf{v}_1 = \mathbf{v}_2 = \boldsymbol{\phi}_1 = (0.33 \quad 0.74)^T$$

$$\mathbf{v}_3 = \mathbf{v}_4 = \boldsymbol{\phi}_2 = (0.74 \quad -0.6)^T$$

$$\mathbf{v}_5 = \mathbf{v}_6 = \boldsymbol{\phi}_3 = (0.6 \quad 0.33)^T$$

The measured eigenvectors are no longer orthogonal (e.g. $\mathbf{v}_1$ and $\mathbf{v}_5$ are almost parallel, $\mathbf{v}_5^T\mathbf{v}_1 = 0.8$). We wish to keep the damping of the first mode unchanged and increase the resonance frequency by only 10%, from 1 Hz to 1.1 Hz. Since there are only two sensors, the maximum number of pole pairs which can be controlled is 2; we keep the third mode unchanged, while the second mode is not controlled.

The Frobenius norm of the feedback gains as a function of $\boldsymbol{\alpha}_1$ curve is similar to Fig. 4, we omit to show it. Similarly, it exhibits a minimum at the optimal $\boldsymbol{\alpha}_1^\star = (1 \quad 1.69)^T$, obtained using Eq. (32). The norm of the feedback gains obtained with $\mathbf{w}_1 = \mathbf{v}_1$ is,

$$\left\| \begin{matrix} \mathbf{F} \\ \mathbf{G} \end{matrix} \right\|_F^2 = 1090$$

which is almost twice the optimal value obtained with $\boldsymbol{\alpha}_1^\star$,

$$\left\| \begin{matrix} \mathbf{F}^\star \\ \mathbf{G}^\star \end{matrix} \right\|_F^2 = 570.$$

This example shows that the theory presented in Section 3 remains valid in the practical case when the number of sensors is fewer than the number of degrees of freedom in the system.

5. Conclusion

In this paper, the problem of minimizing the control effort by MIMO receptance-based control is solved in closed form. The analysis presented herein completes the method as described initially in the seminal work of Ram and Mottershead [13], and offers a straightforward way to choose the eigenvectors associated with the placed poles. The paper demonstrates the importance of this choice on the control effort.

The following remarks concerning the energy-conserving receptance method can be made:

1) The method is applied sequentially, one pole pair $\left(\mu_j, \bar{\mu}_j\right)$at a time while the remaining poles are retained unchanged, and at each step the Frobenius norm $\left\| \begin{matrix} \mathbf{F}^{(j)} \\ \mathbf{G}^{(j)} \end{matrix} \right\|_F$ is minimized. The pole pair is therefore assigned with minimum control energy.
2) The feedback gain vectors $\mathbf{f}_i$ and $\mathbf{g}_i$ are found to be proportional to each other, such that

   $\alpha_{k,j}\mathbf{f}_j = \alpha_{k,l}\mathbf{f}_l,$ and $\alpha_{k,j}\mathbf{g}_j = \alpha_{k,l}\mathbf{g}_l,$

   This appears in the calculated gains in Example 1 (and also in the other examples). Thus, the elements of the vector α_k may be seen as weighting factors for the control input contribution. If $\alpha_{k,j} = 0$,then the j^{th} actuator will not contribute to the control of the k^{th} pole; conversely, if $\alpha_{k,j} \gg \alpha_{k,i}(i \neq j)$, then the k^{th} pole will be principally controlled with the j^{th} actuator. Alternatively, if the parameters $\alpha_{k,j}$ are equal, then the feedback gains of each input j, $\mathbf{f}_j$ and $\mathbf{g}_j$, will be identical for all the control loops. This may be useful, for example, in configurations where the actuators are not identical, and instead of choosing the vector α_k to be optimal, it can be chosen in order to suit the capability of the actuators (e.g. stroke, bandwidth and control authority over the targeted mode).
3) When all the degrees of freedom are available:
   a. the control effort required to assign each pole pair independently with minimum effort leads to the same minimum as when assigning all the pole pairs at the same time.
   b. choosing the closed eigenvector equal to the open loop one leads to nearly optimal feedback gains, provided that the open- and the closed loop poles are not too far away from each other.
   c. while the modes occur in pairs with real orthogonal eigenvectors (as in Example 2), then one possible solution to satisfy the orthogonality condition of Eq. (19) is that the rows of the feedback gain matrices $\mathbf{F}^T$ and $\mathbf{G}^T$ are a combination of the open loop eigenvectors $\mathbf{v}_j^T = \boldsymbol{\phi}_j^T$. Thus, the feedback gain matrices project the outputs into the modal coordinates. Such a situation is very similar to the classical "Independent Modal Space Control", formulated by Meirovich [20].
4) The same method may be applied in the case of incomplete measured modes (eigenvectors $\mathbf{v}_k$ not forming an orthogonal basis). This will lead to an optimal solution at each step (as in (1) above) but in general will not lead to an overall minimum-energy solution for $\begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix}$.

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Appendix. – Rank of $\boldsymbol{\beta}$

The problem of Eq. (17) may be rearranged in the form,

$$\begin{bmatrix} \mu_1 \mathbf{w}_1^T & \mathbf{w}_1^T \\ \bar{\mu}_1 \bar{\mathbf{w}}_1^T & \bar{\mathbf{w}}_1^T \\ \lambda_2 \mathbf{v}_2^T & \mathbf{v}_2^T \\ \bar{\lambda}_2 \bar{\mathbf{v}}_2^T & \bar{\mathbf{v}}_2^T \\ \lambda_3 \mathbf{v}_3^T & \mathbf{v}_3^T \\ \vdots & \vdots \\ \lambda_n \mathbf{v}_n^T & \mathbf{v}_n^T \\ \bar{\lambda}_n \bar{\mathbf{v}}_n^T & \bar{\mathbf{v}}_n^T \end{bmatrix} \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\alpha}_k^T \\ \bar{\boldsymbol{\alpha}}_k^T \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \tag{A1}$$

and the feedback gains are found as,

$$\begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix} = \begin{bmatrix} \mu_1 \mathbf{w}_1^T & \mathbf{w}_1^T \\ \bar{\mu}_1 \bar{\mathbf{w}}_1^T & \bar{\mathbf{w}}_1^T \\ \lambda_2 \mathbf{v}_2^T & \mathbf{v}_2^T \\ \bar{\lambda}_2 \bar{\mathbf{v}}_2^T & \bar{\mathbf{v}}_2^T \\ \lambda_3 \mathbf{v}_3^T & \mathbf{v}_3^T \\ \vdots & \vdots \\ \lambda_n \mathbf{v}_n^T & \mathbf{v}_n^T \\ \bar{\lambda}_n \bar{\mathbf{v}}_n^T & \bar{\mathbf{v}}_n^T \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{\alpha}_k^T \\ \bar{\boldsymbol{\alpha}}_k^T \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \tag{A2}$$

The left multiplying $2n \times 2n$ matrix is full rank (as long as the eigenvector $\mathbf{w}_k$ has been chosen properly), the rank of the feedback gains matrix is then,

$$rank\left(\begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix}\right) \le 2,$$

since the matrix $\begin{bmatrix} \boldsymbol{\alpha}_k & \bar{\boldsymbol{\alpha}}_k & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}^T$ is rank 2. In the case where the parameters $\boldsymbol{\alpha}_k$ are real, then $\boldsymbol{\alpha}_k = \bar{\boldsymbol{\alpha}}_k$ and the rank of this matrix reduces to unity. Thus, the rank of $\begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix}$ and $\boldsymbol{\beta}$ is also 1.

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