A probabilistic interpretation of the Gaussian binomial coefficients

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Abstract

We give a stand-alone simple proof of a probabilistic interpretation of the Gaussian binomial coefficients by conditioning a random walk to hit a given lattice point at a given time.

Introduction

The Gaussian binomial coefficients \([n \choose m]_q\) are generalizations of classical binomial coefficients and are usually defined as

\[
{n \choose m}_q = \frac{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q^{n-m+1})}{(1 - q)(1 - q^2) \cdots (1 - q^m)}.
\]

The term “generalization” is justified, e.g., by the fact that \(\lim_{q \to 1} {n \choose m}_q = \binom{n}{m}\), which becomes obvious if we divide each term in the numerator and denominator of the last display by \(1 - q\) and expand the ratio into a power series with finitely many terms. The Gaussian binomial coefficients turn out to be polynomial functions of the variable \(q\) and satisfy many analogs of the usual properties of binomial coefficients. We refer, e.g., to the textbook of Kac and Cheung [5].

Originally, they appeared in combinatorics, so it is not surprising that they are nowadays very important in random polymer models which have strong connections to algebraic combinatorics; see, for example, the recent work on the \(q\)-weighted version of the Robinson-Schensted algorithm introduced by O’Connell and Pei [6]. In the study of random graphs, Gaussian binomial coefficients are present, for instance, in the distributions of the sizes of the transitive closure and transitive reduction of node 1 in a random acyclic digraph with \(n\) nodes, see [3] and [2]. Another application is in integer-valued random matrices; see, for example, [1] where the distribution of the \(m\)-rank of a random matrix is expressed in terms of these coefficients.

The purpose of this note is to give a short proof of a probabilistic interpretation of the Gaussian coefficients which, not surprisingly, is very similar to their combinatorial interpretation, given by Pólya [7], as counting the number of nondecreasing paths in a rectangle in the 2-dimensional integer lattice that leave a fixed area below them. The probabilistic proof given below (Theorem 1) is different than Pólya’s. The note is stand-alone in that everything discussed is proved, including Heine’s formula (see (4) below) that is needed at the end of the proof of Theorem 1. The probabilistic interpretation gives a natural meaning to several identities and properties satisfied by the coefficients (see end remarks).

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The statement and proof

Consider nondecreasing paths in the standard 2-dimensional integer lattice \( \mathbb{Z}^2 \), that is, finite or infinite sequences \( x_0, x_1, \ldots \) of elements of \( \mathbb{Z}^2 \) such that \( \eta_i = x_i - x_{i-1} \) is either \( e_1 \) or \( e_2 \), where \( e_1 = (1,0) \), \( e_2 = (0,1) \), the standard unit vectors. Let \( r, s \) be nonnegative integers. By a random nondecreasing path from \((0,0)\) to \((r,s)\) we mean a finite nondecreasing path that starts at \( x_0 = (0,0) \) and ends at \( x_{r+s} = (r,s) \), and that is chosen uniformly at random among the set of all such paths. Since there are \( (r+s) \) such paths, the increments sequence \((\eta_1, \ldots, \eta_m)\) is assigned probability equal to \( (r+s)^{-1} \).

**Theorem 1.** Consider a random nondecreasing path from \((0,0)\) to \((r,s)\). This path splits the rectangle \([0,r] \times [0,s]\) into two regions. Let \( A_{r,s} \) be the area of the region under the path. Then

\[
\mathbb{E} q^{A_{r,s}} = \binom{r+s}{r} q \left( \frac{r+s}{r} \right).
\]

**Proof.** Toss a fair coin independently and let \( e_1 \) represent heads and \( e_2 \) tails. Denote by \( \xi_1, \xi_2, \ldots \) the successive outcomes, a random independent sequence with \( \mathbb{P}(\xi_t = e_i) = 1/2 \), \( i = 1, 2, t \geq 1 \). Let \( X_0 = 0, X_t = \xi_1 + \cdots + \xi_t, t \geq 1 \). If it takes \( T_{r+1} \) coin tosses until the \((r+1)\)-th head occurs for the first time then, conditional on the event that we have seen \( s \) tails up to \( T_{r+1} \), the sequence \((\xi_1, \ldots, \xi_{T_{r+1}-1})\) (of length \( T_{r+1} - 1 = s + (r+1) - 1 = s + r \), under the conditioning) has uniform distribution. Thus, conditional on the same event, the path \((X_0, X_1, \ldots, X_{T_{r+1}-1})\) is a random nondecreasing path from \((0,0)\) to \((r,s)\). Let \( N_i(t) \) be the number of heads/tails seen up to the \( t \)-th toss:

\[
N_i(t) := \sum_{k=1}^{t} 1\{\xi_k = e_i\}, \quad i = 1, 2,
\]

and consider the stopping times

\[
T_0 := 0, \quad T_m := \inf\{t \geq 1 : N_1(t) = m\}, \quad m \geq 1.
\]

Since \( T_1, T_2, \ldots \) is an increasing sequence of stopping times in i.i.d. Bernoulli trials, the random variables \( Z_i = N_2(T_{i+1}) - N_2(T_i), i = 0, 1, 2, \ldots \), are i.i.d. geometric: \( \mathbb{P}(Z_i = j) = (1/2)^{j+1}, j \geq 0 \), and so \( \mathbb{E} q^{Z_i} = \frac{1}{2} (1 - \theta/2)^{-1}, |\theta| < 2 \). Simply putting it, the \( Z_i \) count the number of up-steps of the path between two successive right-steps and so

\[
V = N_2(T_{r+1}) = \sum_{i=0}^{r} Z_i
\]

is the total number of up-steps up to \( T_{r+1} \). The distribution of \( V \) is

\[
\mathbb{P}(V = s) = \left( \frac{1}{2} \right)^{r+s+1} \binom{r+s}{r}, \quad s \geq 0.
\]  \hspace{1cm} (1)

The area \( A = A_{r,s} \) under the path \((X_0, X_1, \ldots, X_{T_{r+1}-1})\) is then

\[
A = r Z_0 + (r-1)Z_1 + \cdots + Z_{r-1}.
\]

Letting \( q, \theta \) be variable with, say, \(|q|, |\theta| < 2\), we have

\[
\mathbb{E} q^{A V} = \mathbb{E} \left[ (q^\theta)^Z_0 (q^{r-1} \theta)^Z_1 \cdots (q \theta)^Z_{r-1} \theta^{Z_r} \right]
\]

\[
= \left( \frac{1}{2} \right)^{r+1} \frac{1}{1 - q^\theta} \frac{1}{1 - q^{r-1} \theta} \cdots \frac{1}{1 - \frac{q^\theta}{2}} \frac{1}{1 - \frac{\theta}{2}} = \left( \frac{1}{2} \right)^{r+1} \sum_{s=0}^{\infty} C_{r,s}(q)(\theta/2)^s, \quad (2)
\]
where the $C_{r,s}(q)$ are defined by the right-hand side as coefficients in the Taylor expansion in the variable $\theta/2$. On the other hand,

$$\mathbb{E} q^A \theta^V = \sum_{s=0}^{\infty} \theta^s \mathbb{P}(V = s) \mathbb{E}[q^A|V = s].$$

(3)

Equating coefficients in (2) and (3), also taking into account (1), gives

$$\mathbb{E}[q^A|V = s] = C_{r,s}(q)/\binom{r+s}{r}.$$ 

It remains to show that the $C_{r,s}(q)$ are Gaussian binomial coefficients. To this end, we prove that if

$$F_r(x) := \prod_{j=0}^{r} \frac{1}{1-q^j x} = \sum_{s=0}^{\infty} C_{r,s}(q) x^s, \quad (4)$$

then the recurrence relation

$$C_{r,s}(q) = C_{r,s-1}(q) \frac{1-q^{r+s}}{1-q^s}, \quad s \geq 1, \quad (5)$$

holds. This follows quite easily from the observation that

$$(1-q^{r+1} x)F_r(q,x) = (1-x)F_r(x).$$

Indeed, if, in this identity, we replace $F_r(q,x)$ and $F_r(x)$ by their series, from the right-hand side of (4), and equate coefficients of similar powers, we obtain (5). Since, clearly, $C_{r,0}(q) = F_r(0) = 1$, we can iterate (5) to obtain

$$C_{r,s}(q) = \frac{1-q^{r+s}}{1-q^s} \frac{1-q^{r+s-1}}{1-q^{s-1}} \cdots \frac{1-q^{r+1}}{1-q} = \left[\frac{r+s}{r}q\right].$$

This completes the proof. \qed

Remarks

1. Since $\binom{r+s}{r}_q$ is proportional to $\mathbb{E} q^{A_{r,s}}$ we have that $\binom{r+s}{r}_q$ is a polynomial in $q$.
2. Formula (4) with $C_{r,s}(q)$ the Gaussian binomial coefficients is known as Heine’s formula [5]. When $q = 1$ it corresponds to the Taylor series (Newton’s formula) $(1-x)^{-r} = \sum_{s \geq 0} \binom{-r}{s} (-x)^s = \sum_{s \geq 0} \binom{r+s}{s} x^s$.
3. By symmetry, the area above the random nondecreasing path has the same distribution as the area below, i.e., the random variables $A_{r,s}$ and $rs - A_{r,s}$ have the same distribution. This is equivalent to the identity $\binom{r+s}{r}_q = q^s \binom{r+s}{r-1}_q/1$. \[ \frac{q^s}{\binom{r+s}{r-1}_q} \]
4. By the definition of the random variable $A_{r,s}$ as the area under a random nondecreasing path from $(0,0)$ to $(r,s)$ we see, by conditioning on the last edge of this path, that $A_{r,s}$ is in distribution equal to $A_{r,s-1}$ with probability $s/(r+s)$ or to $A_{r-1,s} + s$ with probability $r/(r+s)$. Using then the result of Theorem 1, the well-known recursion $\binom{r+s}{r}_q = \binom{r+s-1}{r-1}_q + q^s \binom{r+s-1}{r-1}_q$ follows.

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References


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