

Six dimensional ultraviolet completion of the $\mathbb{CP}(N)$ σ model at two loops

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Abstract. We extend the recent one loop analysis of the ultraviolet completion of the $\mathbb{CP}(N)$ nonlinear σ model in six dimensions to two loop order in the $\overline{\text{MS}}$ scheme for an arbitrary covariant gauge. In particular we compute the anomalous dimensions of the fields and β -functions of the four coupling constants. We note that like Quantum Electrodynamics (QED) in four dimensions the matter field anomalous dimension only depends on the gauge parameter at one loop. As a non-trivial check we verify that the critical exponents derived from these renormalization group functions at the Wilson-Fisher fixed point are consistent with the ϵ expansion of the respective large N exponents of the underlying universal theory. Using the Ward-Takahashi identity we deduce the *three* loop $\overline{\text{MS}}$ renormalization group functions for the six dimensional ultraviolet completeness of scalar QED.

1 Introduction.

There has been wide interest in recent years in studying the ultraviolet completion of quantum field theories beyond their critical dimension. For instance, $O(N)$ ϕ^4 theory, which is renormalizable in four dimensions, has been completed to six dimensions and is related to $O(N)$ ϕ^3 theory. This has been verified in detail in [1, 2] as well as for other related field theories, [3]. Briefly a common core interaction between the matter and force fields present in both theories is responsible for the dynamics at the Wilson-Fisher critical point in d -dimensions, [4]. Through that interaction the canonical dimensions of the fields are defined and thereby determine the relevant operators of the respective theories in their critical dimensions. In other words there is a universal theory built with an infinite number of operators constructed from all the fields, [5], a finite subset of which are relevant in successive critical dimensions. Accessing the properties of this theory allows one to connect $O(N)$ ϕ^4 and ϕ^3 theories in their respective critical dimensions. One calculational tool to achieve this is the large N expansion which provides the d -dimensional critical exponents available at several orders in the parameter $1/N$ through the pioneering papers [6, 7, 8]. This parameter acts as a dimensionless perturbative coupling constant in all dimensions in this limit. Expanding such exponents in an ϵ expansion about each critical dimension the coefficients of the Taylor series are in one-to-one agreement with the ϵ expansion of the renormalization group functions of the respective theories at each of their Wilson-Fisher fixed points. Indeed the seven loop $O(N)$ ϕ^4 renormalization group functions, [9, 10, 11, 12, 13, 14, 15, 16, 17, 18], and those of ϕ^3 theory at four loops, [1, 2, 19, 20, 21, 22], have been shown to be in precise agreement with the exponents of [6, 7, 8].

Having confirmed this connection through high order computations for a well-studied set of scalar theories, other universality classes have subsequently been probed. Recently this has been undertaken for another class, similar to the scalar case already mentioned, which is that of the nonlinear $\mathbb{C}\mathbb{P}(N)$ σ model, [23, 24]. The critical dimension of this field theory is two and the model is parallel to the $O(N)$ nonlinear σ model which serves as the base theory in the tower of theories that includes $O(N)$ ϕ^4 and ϕ^3 theory. In even dimensions one can construct a renormalizable Lagrangian with the same core symmetries as the base two dimensional theory so that each of these higher dimensional models is a member of the same tower. In the $\mathbb{C}\mathbb{P}(N)$ case, as the scalar fields are complex conjugates, a $U(1)$ spin-1 field is also present in two dimensions in addition to the model's eponymous spin-0 scalar field. This $U(1)$ spin-1 field of two dimensions becomes a gauge field in the tower of theories above two dimensions. Both the scalar σ and $U(1)$ fields, which we notionally regard as force fields, couple to the complex matter fields and it is these two interactions that drive the critical point dynamics in the d -dimensional universal theory. We note that when the coupling constant of the σ field to matter is formally switched off the universality class corresponds to that of scalar QED, [23, 24, 25, 26, 27, 28]. In [23, 24] the ultraviolet completion to six dimensions was considered and a comprehensive Landau gauge one loop computation of renormalization group functions was carried out. Unlike the $O(N)$ counterpart the six dimensional extended $\mathbb{C}\mathbb{P}(N)$ σ model has four interactions. However the resulting renormalization group functions were shown to be consistent with the large N critical exponents of the underlying universal theory computed in [28, 29].

Given the establishment of this class and its perturbative analysis at one loop, it is the purpose of this article to extend the renormalization group functions of the six dimensional theory to two loop order. This is not a trivial task. For instance, of necessity when constructing the ultraviolet completion the force fields have propagators that have an additional power of the momentum in the momentum space representation. In the case of the $U(1)$ gauge field this means that it has a dipole propagator structure. Therefore this complicates the evaluation of all the two loop Feynman graphs that need to be computed. Therefore we had to appeal to various

modern ways of carrying out the renormalization. In computing the anomalous dimensions and β -functions in an arbitrary linear covariant gauge we will establish the connection to the exponents of the universal theory at a new loop order. En route we will partially check the result by carrying out the *three* loop field anomalous dimensions. An interesting corollary to this is that we will be able to deduce the full three loop renormalization group functions of the ultraviolet completion of scalar QED in six dimensions. This follows trivially since the β -function of the gauge field, which is the only core coupling constant in this class in six dimensions and drives the critical dynamics, can be deduced from the Ward-Takahashi identity similar to the one widely known in standard fermionic QED.

The article is organized as follows. The background to the $\mathbb{CP}(N)$ universality class and the six dimensional Lagrangian are briefly reviewed in the next section. Subsequently section 3 is devoted to recording the results for the renormalization group functions including discussion on the various checks undertaken to ensure their credibility. This includes reconciling the ϵ -expansion of the critical exponents with their known large N counterparts. We present concluding remarks in section 4.

2 Background.

Briefly the background to the universality class which includes scalar QED begins with the two dimensional theory which serves as the foundation for the tower we consider here. In order to have a conserved charge one has to have complex scalar fields and that determines the two dimensional theory to be the $\mathbb{CP}(N)$ nonlinear σ model which has the Lagrangian

$$L^{(2)} = \overline{D_\mu \phi^i} D^\mu \phi^i + \sigma \left(\overline{\phi^i} \phi^i - \frac{1}{g_2} \right) \quad (2.1)$$

where $1 \leq i \leq N$ and D_μ is the usual covariant derivative involving the field A_μ . At this stage we do not refer to it as a gauge field since on dimensional grounds it has no kinetic term and therefore corresponds to an auxiliary field. We will always denote the coupling constant ordinarily associated with the gauge field appearing in the covariant derivative by g_1 for reasons that will become clear later. The remaining coupling constant g_2 has been scaled out of the interaction involving the scalar field σ since it is this interaction as well as the cubic one of the ϕ^i kinetic term that drives the universality class across all dimensions via the Wilson-Fisher fixed point. In other words the canonical dimensions of σ and A_μ are respectively 2 and 1 in d -dimensions given that ϕ^i has canonical dimension $(\frac{1}{2}d - 1)$. Thus all the terms in (2.1) have the same dimension and the theory is renormalizable.

With these canonical scaling dimensions and the universal interactions of the universality class the Lagrangians for the theories in the same class that are renormalizable in higher dimensions are straightforward to write down. The method is to construct all possible independent interaction terms consistent with the critical dimension of spacetime dimension of interest and associate separate coupling constants with each. The only caveat is that one must ensure that the construction is consistent with the underlying symmetries. In this case these are the $U(1)$ symmetry due to the complex scalar and the $\mathbb{CP}(N)$ symmetry. In addition the former symmetry now becomes a gauge symmetry beyond two dimensions. Therefore in four dimensions the next theory in the tower has the Lagrangian

$$L^{(4)} = \overline{D_\mu \phi^i} D^\mu \phi^i + \frac{1}{2} \sigma^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} (\partial^\mu A_\mu)^2 + g_2 \sigma \overline{\phi^i} \phi^i \quad (2.2)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and α is the gauge parameter. In another sense one can regard α as a coupling constant of a 2-point interaction. Setting $g_2 = 0$ corresponds to scalar QED.

Repeating the argument but with six as the critical dimension one arrives at the renormalizable Lagrangian

$$\begin{aligned}
L^{(6)} = & \overline{D}_\mu \phi^i D^\mu \phi^i + \frac{1}{2} \partial^\mu \sigma \partial_\mu \sigma - \frac{1}{4} \partial_\mu F_{\nu\sigma} \partial^\mu F^{\nu\sigma} - \frac{1}{2\alpha} (\partial_\mu \partial^\nu A_\nu) (\partial^\mu \partial^\sigma A_\sigma) \\
& + g_2 \sigma \bar{\phi}^i \phi^i + \frac{g_3}{6} \sigma^3 + \frac{g_4}{2} \sigma F_{\mu\nu} F^{\mu\nu}
\end{aligned} \tag{2.3}$$

which was first given in [23] for the next Lagrangian in the $\mathbb{CP}(N)$ tower of theories. In this dimension the σ field becomes propagating for the first time and the gauge condition remains as the usual Lorenz one with $\partial^\mu A_\mu = 0$ but contained in the Lagrangian in a dimensionally consistent way. We note that we have defined our new coupling constants differently to [23] and more in keeping with previous work, [29]. Indeed like [29] the gauge field has a double pole propagator that has also been studied in a more general six dimensional gauge theory in [30].

3 Results.

Having reviewed the context in which the six dimensional extension of the $\mathbb{CP}(N)$ σ model sits in the tower of theories of the universality class we now turn to establishing this at the two loop level by explicit computation of all the renormalization group functions. The method we have followed to achieve this has been documented in [22, 29] and we refer the reader to those articles for technical details. Though we note that to determine the β -functions we had to compute each of the 3-point functions for the off-shell symmetric point configuration. By contrast in [22] the four loop renormalization group functions of scalar ϕ^3 theory in six dimensions were determined by solely considering 2-point functions. In that case the 3-point functions that needed to be renormalized were generated by a simple mapping of the propagator that was infrared safe. While it appears that the same technique could be applied to (2.3) due to the cubic interactions, it is not possible since there is a quartic interaction in addition. A contribution from such a vertex cannot be generated from the mapping construction given in [22] which is the reason why we have had to compute the two loop vertex functions directly. Finally we note that all our computations used the Laporta algorithm, [31], and specifically its REDUZE encoding, [32]. The overall computation was carried out automatically using the symbolic manipulation language FORM, [33, 34], where (2.3) was dimensionally regularized in $d = 6 - 2\epsilon$ dimensions. The Feynman diagrams were generated with QGRAF, [35]. For example there were 155, 122, 94 and 122 two loop graphs for the 3-point vertex functions associated with g_1 to g_4 respectively.

Having outlined the method of computation we now present our results. First the renormalization group functions of the fields are

$$\begin{aligned}
\gamma_A(g_i) = & - \frac{Ng_1^2}{30} \\
& + \frac{1}{1080} [10Ng_1^2g_2^2 - 370Ng_1^4 + 6Ng_1^2g_4^2 + 30Ng_2^2g_4^2 + 15g_3^2g_4^2 - 120g_3g_4^3 + 660g_4^4] \\
& + \frac{1}{1944000} [358000Ng_1^4g_2^2 - 40680N^2g_1^6 - 2384000Ng_1^6 - 15000N^2g_1^4g_2^2 \\
& \quad - 2130N^2g_1^4g_2g_4 - 537000Ng_1^4g_2g_4 + 354N^2g_1^4g_4^2 \\
& \quad - 207000Ng_1^4g_4^2 - 10500N^2g_1^2g_2^4 + 16000Ng_1^2g_2^4 + 28500Ng_1^2g_2^3g_3 \\
& \quad - 5250N^2g_1^2g_2^3g_4 + 36000Ng_1^2g_2^3g_4 - 5250Ng_1^2g_2^2g_3^2 \\
& \quad + 54000Ng_1^2g_2^2g_3g_4 - 1620N^2g_1^2g_2^2g_4^2 + 739500Ng_1^2g_2^2g_4^2 \\
& \quad - 2625Ng_1^2g_2g_3^2g_4 + 55500Ng_1^2g_2g_3g_4^2 - 964500Ng_1^2g_2g_4^3]
\end{aligned}$$

$$\begin{aligned}
& - 810N g_1^2 g_3^2 g_4^2 - 10500N g_1^2 g_3 g_4^3 + 249540N g_1^2 g_4^4 - 14250N^2 g_2^4 g_4^2 \\
& + 31500N g_2^4 g_4^2 + 157500N g_2^3 g_3 g_4^2 - 207000N g_2^3 g_4^3 - 37875N g_2^2 g_3^2 g_4^2 \\
& + 168000N g_2^2 g_3 g_4^3 - 231000N g_2^2 g_4^4 + 24000g_3^4 g_4^2 - 19500g_3^3 g_4^3 \\
& - 929250g_3^2 g_4^4 + 2820000g_3 g_4^5 + 4089000g_4^6] + O(g_i^8) \\
\gamma_\phi(g_i) = & \frac{1}{6}[3\alpha g_1^2 - 10g_1^2 + g_2^2] \\
& + \frac{1}{2160}[-196N g_1^4 + 2750g_1^4 + 420g_1^2 g_2^2 - 4560g_1^2 g_2 g_4 + 1200g_1^2 g_4^2 - 110N g_2^4 \\
& + 130g_2^4 + 240g_2^3 g_3 - 55g_2^2 g_3^2 - 780g_2^2 g_4^2] \\
& + \frac{1}{3888000} [2648N^2 g_1^6 + 2592000\zeta_3 N g_1^6 - 5723500N g_1^6 + 5832000\zeta_3 g_1^6 \\
& + 2066000g_1^6 - 3888000\zeta_3 N g_1^4 g_2^2 + 5255900N g_1^4 g_2^2 + 4536000\zeta_3 g_1^4 g_2^2 \\
& - 5595000g_1^4 g_2^2 - 2598000N g_1^4 g_2 g_4 + 31806000g_1^4 g_2 g_4 \\
& + 341700N g_1^4 g_4^2 + 7776000\zeta_3 g_1^4 g_4^2 - 34440000g_1^4 g_4^2 \\
& + 1296000\zeta_3 N g_1^2 g_2^4 - 3215000N g_1^2 g_2^4 - 1944000\zeta_3 g_1^2 g_2^4 \\
& + 3086000g_1^2 g_2^4 + 201000g_1^2 g_2^3 g_3 - 363000N g_1^2 g_2^3 g_4 + 2856000g_1^2 g_2^3 g_4 \\
& + 243750g_1^2 g_2^2 g_3^2 - 1476000g_1^2 g_2^2 g_3 g_4 - 1285500N g_1^2 g_2^2 g_4^2 \\
& - 15552000\zeta_3 g_1^2 g_2^2 g_4^2 + 8889000g_1^2 g_2^2 g_4^2 - 133500g_1^2 g_2 g_3^2 g_4 \\
& + 7776000\zeta_3 g_1^2 g_2 g_3 g_4^2 - 13026000g_1^2 g_2 g_3 g_4^2 + 15552000\zeta_3 g_1^2 g_2 g_4^3 \\
& - 45186000g_1^2 g_2 g_4^3 - 258750g_1^2 g_3^2 g_4^2 + 1710000g_1^2 g_3 g_4^3 \\
& - 2925000g_1^2 g_4^4 - 6500N^2 g_2^6 + 58000N g_2^6 - 648000\zeta_3 g_2^6 \\
& + 1133000g_2^6 - 661500N g_2^5 g_3 + 408000g_2^5 g_3 + 96500N g_2^4 g_3^2 \\
& - 648000\zeta_3 g_2^4 g_3^2 + 1470250g_2^4 g_3^2 - 2172000N g_2^4 g_4^2 + 879000g_2^4 g_4^2 \\
& - 117750g_2^3 g_3^3 + 63000g_2^3 g_3 g_4^2 - 9540000g_2^3 g_4^3 - 40875g_2^2 g_3^4 \\
& + 123000g_2^2 g_3^2 g_4^2 - 2256000g_2^2 g_3 g_4^3 - 28626000g_2^2 g_4^4] + O(g_i^8) \\
\gamma_\sigma(g_i) = & \frac{1}{12}[2N g_2^2 + g_3^2 + 60g_4^2] \\
& + \frac{1}{2160} [3820N g_1^2 g_2^2 - 2400N g_1^2 g_2 g_4 + 1656N g_1^2 g_4^2 + 20N g_2^4 + 480N g_2^3 g_3 \\
& - 110N g_2^2 g_3^2 + 65g_3^4 - 780g_3^2 g_4^2 + 12000g_3 g_4^3 + 12960g_4^4] \\
& + \frac{1}{7776000} [2570600N^2 g_1^4 g_2^2 + 45360000\zeta_3 N g_1^4 g_2^2 + 8144000N g_1^4 g_2^2 \\
& - 849600N^2 g_1^4 g_2 g_4 - 118344000N g_1^4 g_2 g_4 + 107856N^2 g_1^4 g_4^2 \\
& - 85536000\zeta_3 N g_1^4 g_4^2 + 190980000N g_1^4 g_4^2 + 2592000\zeta_3 N g_1^2 g_2^4 \\
& + 420000N g_1^2 g_2^4 + 16584000N g_1^2 g_2^3 g_3 + 18216000N g_1^2 g_2^3 g_4 \\
& + 2592000\zeta_3 N g_1^2 g_2^2 g_3^2 - 7405000N g_1^2 g_2^2 g_3^2 + 23328000\zeta_3 N g_1^2 g_2^2 g_3 g_4 \\
& - 15336000N g_1^2 g_2^2 g_3 g_4 + 62208000\zeta_3 N g_1^2 g_2^2 g_4^2 - 143424000N g_1^2 g_2^2 g_4^2 \\
& - 192000N g_1^2 g_2 g_3^2 g_4 - 7824000N g_1^2 g_2 g_3 g_4^2 - 4608000N g_1^2 g_2 g_4^3 \\
& - 1536000N g_1^2 g_3^2 g_4^2 + 8079360N g_1^2 g_3 g_4^3 + 5757840N g_1^2 g_4^4 \\
& + 1381000N^2 g_2^6 - 1296000\zeta_3 N g_2^6 + 2140000N g_2^6 - 576000N^2 g_2^5 g_3 \\
& - 264000N g_2^5 g_3 - 1500N^2 g_2^4 g_3^2 - 3240000\zeta_3 N g_2^4 g_3^2 + 6661500N g_2^4 g_3^2 \\
& + 4530000N g_2^4 g_4^2 + 390000N g_2^3 g_3^3 + 864000N g_2^3 g_3 g_4^2 \\
& + 45216000N g_2^3 g_4^3 - 238000N g_2^2 g_3^4 - 5436000N g_2^2 g_3^2 g_4^2]
\end{aligned}$$

$$\begin{aligned}
& - 11664000N g_2^2 g_3 g_4^3 - 11838000N g_2^2 g_4^4 - 324000\zeta_3 g_3^6 + 649375g_3^6 \\
& - 21000g_3^4 g_4^2 - 6816000g_3^3 g_4^3 - 77760000\zeta_3 g_3^2 g_4^4 + 110151000g_3^2 g_4^4 \\
& + 154968000g_3 g_4^5 - 59460000g_4^6] + O(g_i^8) \tag{3.1}
\end{aligned}$$

where g_i denotes each of the possible four coupling constants, ζ_z is the Riemann zeta function and the order symbol represents all combinations of the couplings at that order. Also all our results are given in the $\overline{\text{MS}}$ scheme with the scheme dependence first arising at two loops in all these expressions including the β -functions since (2.3) has more than one coupling constant. Next the β -functions are

$$\begin{aligned}
\beta_1(g_i) &= -\frac{1}{30}N g_1^3 \\
&+ \frac{g_1}{1080} [10N g_1^2 g_2^2 - 370N g_1^4 + 6N g_1^2 g_4^2 + 30N g_2^2 g_4^2 + 15g_3^2 g_4^2 - 120g_3 g_4^3 + 660g_4^4] \\
&+ O(g_i^7) \\
\beta_2(g_i) &= \frac{1}{12} [-40g_1^2 g_2 + 120g_1^2 g_4 + 2N g_2^3 - 8g_2^3 - 12g_2^2 g_3 + g_2 g_3^2 + 60g_2 g_4^2] \\
&+ \frac{1}{2160} [2832N g_1^4 g_4 - 6032N g_1^4 g_2 - 13400g_1^4 g_2 - 31200g_1^4 g_4 + 3820N g_1^2 g_2^3 \\
&\quad - 6480g_1^2 g_2^3 - 960g_1^2 g_2^2 g_3 - 2400N g_1^2 g_2^2 g_4 - 36240g_1^2 g_2^2 g_4 \\
&\quad - 19440g_1^2 g_2 g_3 g_4 + 1656N g_1^2 g_2 g_4^2 + 88800g_1^2 g_2 g_4^2 + 18480g_1^2 g_3 g_4^2 \\
&\quad + 8160g_1^2 g_4^3 - 860N g_2^5 - 2680g_2^5 + 1320N g_2^4 g_3 - 1800g_2^4 g_3 - 110N g_2^3 g_3^2 \\
&\quad - 3140g_2^3 g_3^2 - 4080g_2^3 g_4^2 - 120g_2^2 g_3^3 - 5040g_2^2 g_3 g_4^2 + 38880g_2^2 g_4^3 + 65g_2 g_4^4 \\
&\quad - 780g_2 g_3^2 g_4^2 + 12000g_2 g_3 g_4^3 + 12960g_2 g_4^4] + O(g_i^7) \\
\beta_3(g_i) &= \frac{1}{4} [-8N g_2^3 + 2N g_2^2 g_3 - 3g_3^3 + 60g_3 g_4^2 - 160g_4^3] \\
&+ \frac{1}{720} [3820N g_1^2 g_2^2 g_3 - 12720N g_1^2 g_2^3 - 19440N g_1^2 g_2^2 g_4 - 2400N g_1^2 g_2 g_3 g_4 \\
&\quad + 22560N g_1^2 g_2 g_4^2 + 1656N g_1^2 g_3 g_4^2 - 5664N g_1^2 g_4^3 - 240N g_2^5 - 3220N g_2^4 g_3 \\
&\quad - 600N g_2^3 g_3^2 + 310N g_2^2 g_3^3 - 625g_3^5 - 3300g_3^3 g_4^2 + 50880g_3^2 g_4^3 - 104160g_3 g_4^4 \\
&\quad - 61440g_4^5] + O(g_i^7) \\
\beta_4(g_i) &= \frac{1}{60} [20N g_1^2 g_2 - 4N g_1^2 g_4 + 10N g_2^2 g_4 + 5g_3^2 g_4 - 40g_3 g_4^2 + 220g_4^3] \\
&+ \frac{1}{10800} [37400N g_1^4 g_2 + 400N g_1^4 g_4 - 200N g_1^2 g_2^3 + 2100N g_1^2 g_2^2 g_3 \\
&\quad + 33700N g_1^2 g_2^2 g_4 - 400N g_1^2 g_2 g_3 g_4 - 10800N g_1^2 g_2 g_4^2 - 296N g_1^2 g_3 g_4^2 \\
&\quad + 7216N g_1^2 g_4^3 + 100N g_2^4 g_4 + 2400N g_2^3 g_3 g_4 - 1200N g_2^2 g_4^2 \\
&\quad - 550N g_2^2 g_3^2 g_4 + 2000N g_2^2 g_3 g_4^2 + 2600N g_2^2 g_4^3 + 325g_3^4 g_4 \\
&\quad + 400g_3^3 g_4^2 - 13800g_3^2 g_4^3 + 4000g_3 g_4^4 + 205200g_4^5] + O(g_i^7) . \tag{3.2}
\end{aligned}$$

These two loop β -functions complete the renormalization of (2.3) to this order. In terms of being confident that the results are correct we note that we have implemented the automatic renormalization algorithm of [36]. In other words we evaluate all the contributing graphs in terms of the bare parameters which are the coupling constants and gauge parameter. Then their renormalized counterparts are introduced by a multiplicative rescaling without having to follow the method of subtractions. This means that all the double poles of the two loop renormalization constants are already determined by their one loop simple poles and therefore we have verified that these correctly emerge. By the same token we have been able to check the two loop β -functions by computing the field anomalous dimensions to *three* loops. In this

case double and triple poles of the three loop renormalization constants are fixed by lower loop information including the two loop coupling constant renormalization constants. Again we confirm that the results of (3.1) are consistent with this check. This is also the reason for the large expressions in (3.1) compared with (3.2). For completeness we note that to determine the anomalous dimensions (3.1) the number of three loop graphs computed were 561, 428 and 572 for the A_μ , σ and ϕ^i 2-point functions respectively. Another independent check on our computations rests in the Ward-Takahashi identity associated with the $U(1)$ gauge field. As in QED the gauge field anomalous dimension is not independent and is related to the gauge β -function. In other words this identity implies

$$\beta_1(g_i) = g_1 \gamma_A(g_i) \quad (3.3)$$

in our notation here and we note that it is clearly satisfied to two loops in the $\overline{\text{MS}}$ scheme from comparing (3.1) and (3.2). By the same token we now know $\beta_1(g_i)$ to three loops but the remaining β -functions need to be computed explicitly which is beyond the scope of this article.

As a final comment on our renormalization group functions we note that we carried out our computations in an arbitrary linear covariant gauge. While in the $\overline{\text{MS}}$ scheme this means that the β -functions do not depend on the gauge parameter α , the anomalous dimensions are in fact gauge parameter dependent. However in (3.1) the only place where α appears is in the one loop term of the ϕ^i anomalous dimension. We note that in [23] the one loop computations were performed solely in the Landau gauge. Although this dependence on α may appear to be peculiar by contrast it now seems to be a feature of any $U(1)$ gauge theory in the $\overline{\text{MS}}$ scheme, independent of dimension, since the same property is present in four dimensional QED from explicit computations, [37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47], as well as in higher dimensional versions of QED, [48, 49, 29]. An interesting and novel insight into understanding the underlying reasons for this property using graphical methods that transcends the spacetime dimension has been developed in [50, 51, 52, 53].

While these represent the main internal checks on any perturbative multiloop renormalization one also has to connect with the underlying universal theory that (2.2) and (2.3) are partners to. To achieve this we note that information on the universal structure is accessed through the d -dependent critical exponents that define the properties of the Wilson-Fisher fixed point and are renormalization group invariants. These can be deduced through the large N expansion approach of [6, 7, 8] where $1/N$ acts as a dimensionless coupling constant in d -dimensions. Expanding the exponents in an ϵ expansion near the critical dimension of the theory then they will be in one-to-one correspondence with the large N and ϵ expansion of (3.1) at the Wilson-Fisher fixed point. Therefore we now record the details of this exercise but first recall that (2.3) contains two main universality classes of interest depending on which of the fields A_μ and σ are active, [23]. One corresponds to the full $\mathbb{CP}(N)$ class when both are present. When only A_μ is active then one is in the scalar QED universality class which provides us with another set of exponents to compare with available large N exponents. Strictly there is a third universality class in (2.3) when A_μ is inactive. This corresponds to a complexified scalar and lies in the same universality class as the $O(2N)$ nonlinear σ which also contains ϕ^4 theory in four dimensions as well as six dimensional ϕ^3 theory studied in [1, 19, 20, 21, 22]. However we will not present any connections here for this fixed point since the corresponding large N analysis has been given elsewhere [1, 22]. Instead we merely note that when g_1 and g_4 are set to zero the same renormalization group functions for six dimensional $O(2N)$ ϕ^3 theory emerge consistent with [1, 19, 20, 21, 22].

First we concentrate on the full $\mathbb{CP}(N)$ universality class represented by (2.3) in six dimen-

sions and note that at the Wilson-Fisher fixed point the critical couplings are

$$\begin{aligned}
g_1^* &= i\sqrt{\frac{30\epsilon}{N}} \left[\frac{1}{N} + 155\frac{\epsilon}{N^2} + O\left(\epsilon^2; \frac{1}{N^3}\right) \right] \\
g_2^* &= -\sqrt{\frac{30\epsilon}{N}} \left[\frac{1}{5N} + \frac{336}{5N^2} - 67\frac{\epsilon}{N^2} + O\left(\epsilon^2; \frac{1}{N^3}\right) \right] \\
g_3^* &= -\sqrt{\frac{30\epsilon}{N}} \left[\frac{6}{5N} + \frac{8736}{5N^2} - 1494\frac{\epsilon}{N^2} + O\left(\epsilon^2; \frac{1}{N^3}\right) \right] \\
g_4^* &= -\sqrt{\frac{30\epsilon}{N}} \left[\frac{1}{N} - \frac{224}{N^2} + 743\frac{\epsilon}{N^2} + O\left(\epsilon^2; \frac{1}{N^3}\right) \right]
\end{aligned} \tag{3.4}$$

where the leading orders agree with those of [23]. Given these we find

$$\begin{aligned}
\gamma_\phi(g_i^*) &= \left[51\epsilon - \frac{167}{2}\epsilon^2 + O(\epsilon^3) \right] \frac{1}{N} + O\left(\frac{1}{N^2}\right) \\
\gamma_\sigma(g_i^*) &= \epsilon + [1440\epsilon - 3456\epsilon^2 + O(\epsilon^3)] + O\left(\frac{1}{N^2}\right) \\
\gamma_A(g_i^*) &= \epsilon + O\left(\epsilon^3; \frac{1}{N^2}\right)
\end{aligned} \tag{3.5}$$

at leading order in large N in the Landau gauge. The absence of $O(1/N)$ corrections for the gauge field dimension derives from the way the universal theory (2.1) is formulated and the Ward-Takahashi identity. In particular the coupling constant in the 3-point interaction of the gauge field with the matter field ϕ^i is absent in the definition of the underlying universal theory as is clear in (2.1), (2.2) or (2.3). Consequently in the critical point approach used in [26] the gauge field has no anomalous dimension. This is similar to what has been observed in the large N expansion of other abelian gauge theories. Expanding the d -dimensional expressions for the Landau gauge large N exponents of the universal theory, [26], in $d = 6 - 2\epsilon$ we find exact agreement. The reason why the checks are carried out in the Landau gauge is that the gauge parameter in effect acts as a second coupling constant. Therefore since we are considering a fixed point one has to find the critical value of the gauge parameter akin to finding (3.4). In the case of α its critical value is zero.

Before repeating the same exercise for the scalar QED universality class we note that the *three* loop $\overline{\text{MS}}$ renormalization group functions are

$$\begin{aligned}
\gamma_\psi(g_1) &= \frac{[3\alpha - 10]}{6} g_1^2 - \frac{[98N - 1375]}{1080} g_1^4 \\
&\quad + [662N^2 + 648000N\zeta_3 - 1430875N + 1458000\zeta_3 + 516500] \frac{g_1^6}{972000} + O(g_1^8) \\
\gamma_A(g_1) &= -\frac{N}{30} g_1^2 - \frac{37N}{108} g_1^4 - \frac{N}{48600} [1017N + 59600] g_1^6 + O(g_1^8) \\
\beta_1(g_1) &= -\frac{N}{30} g_1^3 - \frac{37N}{108} g_1^5 - \frac{N}{48600} [1017N + 59600] g_1^7 + O(g_1^9)
\end{aligned} \tag{3.6}$$

where g_1 is the only active coupling and we have used the Ward-Takahashi identity (3.3) to deduce $\beta_1(g_1)$ here. Therefore expanding the fixed point in large N from (3.2) we have

$$\begin{aligned}
g_1^* &= i\sqrt{\frac{30\epsilon}{N}} \left[\frac{1}{N} + \frac{925\epsilon}{6N^2} - \frac{565\epsilon^2}{2N^2} + O\left(\epsilon^2; \frac{1}{N^3}\right) \right] \\
g_2^* &= g_3^* = g_4^* = O\left(\epsilon; \frac{1}{N^3}\right)
\end{aligned} \tag{3.7}$$

and find

$$\begin{aligned}\gamma_\phi(g_i^*) &= \left[50\epsilon - \frac{245}{3}\epsilon^2 - \frac{331}{18}\epsilon^3 + O(\epsilon^4) \right] \frac{1}{N} + O\left(\frac{1}{N^2}\right) \\ \gamma_A(g_i^*) &= \epsilon + O\left(\epsilon^4, \frac{1}{N^2}\right).\end{aligned}\tag{3.8}$$

These are also in agreement with the expression for the field critical exponents also available in [26] and again the gauge field has no large N corrections.

4 Discussion.

We have extended the one loop analysis of [23, 24] to two loops and established the ultraviolet completion of the six dimensional $\mathbb{CP}(N)$ σ model which also contains the scalar QED universality class as a sub-theory to this new order. As a gauge theory it shares similar features to the non-abelian gauge theory in six dimensions studied in [29, 30]. For instance at one loop the gauge β -function depends only on the gauge coupling and moreover like six dimensional QED the gauge coupling is asymptotically free as was shown in [48, 49]. That the same feature emerges in the scalar case is a consequence of the underlying gauge symmetry. Indeed there are other general structural similarities with four dimensional QED. One of these is that the gauge parameter is only present at one loop and not two loops in the ϕ^i field anomalous dimension. Not only is this feature present in QED but it would appear that the graphical proof of this given in [52, 53] for QED could be simply adapted to show this to all orders in perturbation theory. In terms of other future work in this universality class one thing that is lacking is higher order large N critical exponents for both the field dimensions and the critical β -function slopes. This would require the extension of the original formalism developed in [6, 7] for the $O(N)$ ϕ^4 universality class that produced $O(1/N^2)$ and $O(1/N^3)$ exponents in d -dimensions. Given that $O(1/N^2)$ exponents are available for QED, [54, 55, 56], the application to the $\mathbb{CP}(N)$ case ought not to be problematic. From another direction the next theory in the tower of the universality class will become active in eight dimensions. It should have a Lagrangian of the form

$$\begin{aligned}L^{(8)} &= \overline{D_\mu \phi^i} D^\mu \phi^i + \frac{1}{2} (\square \sigma)^2 - \frac{1}{4} (\partial_\mu \partial_\nu F_{\sigma\rho}) (\partial^\mu \partial^\nu F^{\sigma\rho}) - \frac{1}{2\alpha} (\square \partial^\mu A_\mu) (\square \partial^\nu A_\nu) \\ &+ g_2 \sigma \bar{\phi}^i \phi^i + \frac{g_3}{6} \sigma^2 \square \sigma + \frac{g_4}{2} (\square \sigma) F_{\mu\nu} F^{\mu\nu} + g_5 \sigma F_{\mu\nu} \square F^{\mu\nu} + \frac{g_6^2}{24} \sigma^4 \\ &+ \frac{g_7^2}{32} F_{\mu\nu} F^{\mu\nu} F_{\sigma\rho} F^{\sigma\rho} + \frac{g_8^2}{8} F_{\mu\nu} F^{\mu\sigma} F_{\nu\rho} F^{\sigma\rho} + \frac{g_9^2}{4} \sigma^2 F_{\mu\nu} F^{\mu\nu}\end{aligned}\tag{4.1}$$

which includes a different set of what are termed spectator interactions that are independent. The theory relevant for the scalar QED universality class involves the spectator couplings g_7 and g_8 , in addition to g_1 . The two associated operators are also present in the eight dimensional version of QED, [29]. To repeat the two loop analysis carried out here for (4.1) is beyond the scope of the present article. However having information on its renormalization group functions would additionally complement any future determination of the d -dimensional $O(1/N^2)$ critical exponents.

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