

## RENORMALIZATION GROUP FUNCTIONS OF QCD IN LARGE $N_f$ \*

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We review the application of the critical point large  $N_f$  self-consistency method to QCD. In particular we derive the  $O(1/N_f)$   $d$ -dimensional critical exponents whose  $\epsilon$ -expansion determines the perturbative coefficients in  $\overline{\text{MS}}$  of the field dimensions,  $\beta$ -function and various twist-2 operators which occur in the operator product expansion of deep inelastic scattering.

### 1 Introduction

The renormalization group equation, (RGE), plays an important role in comparing predictions made in a quantum field theory with observations of nature. The fundamental ingredients in the RGE are the renormalization group functions. Since these are rarely known exactly even for the simplest of field theories one has to be content with approximate perturbative solutions; the accuracy being dependent upon how many orders in the perturbative coupling constant one can compute the RGE functions. This is a highly technical and tedious exercise partly because the number of Feynman diagrams at even one loop can sometimes be excessive. Also the results depend on how one removes the ultra-violet infinities. For theories which particle physicists are interested in such as quantum chromodynamics, (QCD), which is the gauge theory describing the strong interactions, most high order calculations of these functions are performed in the  $\overline{\text{MS}}$  scheme.<sup>1-4</sup> For instance, the  $\beta$ -function of QCD has been deduced at third order in this scheme. Recently information on various scattering amplitudes has been produced at the same level in an impressive set of papers.<sup>5</sup> Due to the complexity of such calculations, having independent and alternative methods to check the high order structure of the RGE functions is important.

One such method has been made available through the properties of the RGE in the neighbourhood of a fixed point which is defined to be a non-trivial zero of the  $\beta$ -function. There it is known that the critical exponents

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which characterize the phase transition correspond to the functions of the RGE evaluated at the critical coupling. So if one can compute exponents directly then information on the RGE functions is obtainable.<sup>6</sup> This has been achieved in impressive articles by Vasil'ev et al for the  $O(N)$   $\sigma$  model.<sup>7</sup> There critical exponents were determined in arbitrary dimensions order by order in powers of  $1/N$  when  $N$  is large. Those results are in total agreement with the  $\epsilon$ -expansion at the fixed point of the same exponents deduced explicitly at 5-loops in  $\overline{\text{MS}}$ . Garnered by that success it is therefore a worthwhile exercise to develop the  $1/N_f$  method for QCD, where  $N_f$  is the number of quark flavours, in relation to the present state of the art calculations.

## 2 Basic ideas

We recall the basic ideas for deducing arbitrary dimensional critical exponents in the  $1/N_f$  expansion. First from the two loop  $\beta$ -function of QCD in  $d$ -dimensions,<sup>1,2</sup> there is a fixed point at

$$g_c = \frac{3\epsilon}{T(R)N_f} + \frac{1}{4T^2(R)N_f^2} \left[ 33C_2(G)\epsilon - (27C_2(R) + 45C_2(G))\epsilon^2 + \left( \frac{99}{4}C_2(R) + \frac{237}{8}C_2(G) \right)\epsilon^3 + O(\epsilon^4) \right] + O\left(\frac{1}{N_f^3}\right) \quad (1)$$

where  $d = 4 - 2\epsilon$ . If, for example, a general RGE function takes the form

$$\gamma(g) = c_1g + (c_2N_f + d_1)g^2 + (c_3N_f^2 + d_2N_f + e_1)g^3 + O(g^4) \quad (2)$$

where the coefficients  $\{c_i, d_i, e_i \dots\}$  are independent of  $N_f$ , then the associated exponent at leading order in  $1/N_f$  is

$$\gamma(g_c) = \frac{1}{N_f} \sum_{r=1}^{\infty} c_r [3\epsilon/T(R)]^r + O(1/N_f^2) \quad (3)$$

So provided  $\gamma(g_c)$  can be computed directly in the large  $N_f$  limit its  $\epsilon$ -expansion gives the leading order sequence of coefficients  $\{c_i\}$  of  $\gamma(g)$ .

The exponents are defined with reference to the action of the theory one is interested in. For QCD this takes the form

$$L = i\bar{\psi}^{iI} \not{D}\psi^{iI} - \frac{(G_{\mu\nu}^a)^2}{4e^2} \quad (4)$$

where  $\psi^{iI}$  is the quark field,  $A_\mu^a$  is the gluon field,  $D_\mu = \partial_\mu + T^a A_\mu^a$ ,  $G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c$ ,  $T_{IJ}^a$  is the generator of the colour group whose

structure constants are  $f^{abc}$ ,  $1 \leq i \leq N_f$ ,  $1 \leq I \leq N_c$  and  $1 \leq a \leq (N_c^2 - 1)$ . The canonical dimensions of the fields of Eq. 4 at  $g_c$  are defined by demanding that the action is dimensionless. The anomalous dimensions are defined to be the extra portion of the full dimension of the field or operator and essentially are a measure of the effect of radiative corrections. For instance, in the scaling region where the propagators of Eq. 4 behave in the limit  $k^2 \rightarrow \infty$ , as,<sup>8</sup>

$$\psi(k) \sim \frac{A k}{(k^2)^{\mu-\alpha}} \quad , \quad A_{\mu\nu}(k) \sim \frac{B}{(k^2)^{\mu-\beta}} \left[ \eta_{\mu\nu} - (1-b) \frac{k_\mu k_\nu}{k^2} \right] \quad (5)$$

where  $A$  and  $B$  are momentum independent amplitudes and  $b$  is the covariant gauge parameter, we define

$$\alpha = \mu - 1 + \frac{1}{2}\eta \quad , \quad \beta = 1 - \eta - \chi \quad (6)$$

with  $d = 2\mu$ . Here  $\chi$  is the dimension of the quark gluon vertex operator and  $\eta$  is the quark anomalous dimension. Expressions for these anomalous dimensions are deduced from studying the scaling dimensions of the next to leading order corrections to the 2 and 3 point Green's function using Eq. 5.<sup>7</sup> For an arbitrary gauge parameter, the leading order results are,<sup>8</sup>

$$\eta = \frac{C_2(R)[(2\mu-1)(\mu-2) + \mu b] \eta_1^0}{(2\mu-1)(\mu-2)T(R)N_f} \quad (7)$$

$$\eta + \chi = - \frac{C_2(G)[(2\mu-1) + b(\mu-1)] \eta_1^0}{2(2\mu-1)(\mu-2)T(R)N_f} \quad (8)$$

where  $\eta_1^0 = - (2\mu-1)(2-\mu)\Gamma(2\mu)/[4\Gamma^2(\mu)\Gamma(\mu+1)\Gamma(2-\mu)]$ .

In computing these results, which agree with 3-loop perturbative calculations in the Landau gauge,<sup>4</sup> we made use of another well known feature of critical point theory. Ordinarily more than one model can be used to deduce exponents at a fixed point and such models are said to be in the same universality class. A well known example is the equivalence of the  $O(N)$   $\sigma$  model and  $O(N)$   $\phi^4$  theory in three dimensions. For the present case QCD is equivalent<sup>9</sup> at leading order in  $1/N_f$  to a non-abelian version of the Thirring model, (NATM), which is renormalizable in strictly two dimensions. Its lagrangian is

$$L = i\bar{\psi}^{iI} \not{D}\psi^{iI} - \frac{(A_\mu^a)^2}{2\lambda} \quad (9)$$

where  $\lambda$  is the coupling constant which is dimensionless in 2 dimensions. Eliminating the auxiliary spin-1 field  $A_\mu^a$  yields a 4-fermi term. The benefit of using this model, Eq. 9, is that it has a simpler structure to Eq. 4 as the 3 and 4

point gluon self interactions are absent. So one need only consider diagrams built with the quark gluon interaction. It was shown, though,<sup>9</sup> that in the  $1/N_f$  limit the 4-fermi model correctly reproduced the 3 and 4 point gluon Feynman rules in the approach to four dimensions. In other words with Eq. 9 the effect of the 3-point gluon interaction is contained in the graphs with a quark loop. This feature occurs implicitly in the calculations we report on later. Further in using a covariant gauge, ghost fields have to be included in each lagrangian but they give no contribution at leading order.

### 3 $\beta$ -function

With this basic formalism the  $O(1/N_f)$  correction to the QCD  $\beta$ -function can be computed.<sup>10</sup> Ordinarily this is the first step in determining  $O(1/N_f^2)$  information as it will encode the next order correction to  $g_c$  to all orders in  $\epsilon$ . To determine this we compute the related exponent  $\omega = -\beta'(g_c)/2$ . It is deduced from the last term of Eq. 4 which gives the scaling law

$$\omega = \eta + \chi + \chi_G \quad (10)$$

where  $\chi_G$  is the critical dimension of the composite operator  $G = (G_{\mu\nu}^a)^2$  when computed as an insertion in a Green's function in the non-abelian Thirring model. For QED  $\omega$  was originally deduced in  $1/N_f$  by explicitly performing the  $\overline{\text{MS}}$  renormalization with an infinite chain of electron bubbles.<sup>11</sup> The extension to the non-abelian case is simpler in the critical approach. Three 2-loop and one 3-loop graphs need to be evaluated which are illustrated in Fig. 1. The

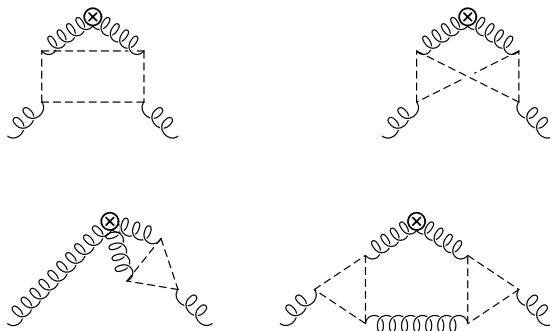


Figure 1: Graphs for  $O(1/N_f)$  contribution to  $\omega$ .

first two graphs correspond to the QED sector, whilst the remaining two would

be absent by Furry's theorem in QED as their colour group factor is  $C_2(G)$ . Consequently, using the critical propagators we find

$$\omega = (\mu - 2) - \left[ \frac{(2\mu - 3)(\mu - 3)C_2(R) - (4\mu^4 - 18\mu^3 + 44\mu^2 - 45\mu + 14)C_2(G)}{4(2\mu - 1)(\mu - 1)} \right] \frac{\eta_1^0}{T(R)N_f} \quad (11)$$

The  $\epsilon$ -expansion of Eq. 11 correctly reproduces the  $O(1/N_f)$  coefficients of the 3-loop  $\overline{\text{MS}}$   $\beta$ -function.<sup>1-4</sup> With this agreement we can deduce several new higher order coefficients. Using the notation

$$\beta(g) = (d - 4)g + \left( \frac{2}{3}T(R)N_f - \frac{11}{6}C_2(G) \right) g^2 + \sum_{r=2}^{\infty} a_r [T(R)N_f]^{r-2} g^{r+1} \quad (12)$$

for the large  $N_f$  leading order part of the  $\beta$ -function, then<sup>10</sup>

$$\begin{aligned} a_4 &= - [154C_2(R) + 53C_2(G)]/3888 \\ a_5 &= [(288\zeta(3) + 214)C_2(R) + (480\zeta(3) - 229)C_2(G)]/31104 \\ a_6 &= [(864\zeta(4) - 1056\zeta(3) + 502)C_2(R) \\ &\quad + (1440\zeta(4) - 1264\zeta(3) - 453)C_2(G)]/233280 \\ a_7 &= [(3456\zeta(5) - 3168\zeta(4) - 2464\zeta(3) + 1206)C_2(R) \\ &\quad + (5760\zeta(5) - 3792\zeta(4) - 848\zeta(3) - 885)C_2(G)]/1679616 \end{aligned} \quad (13)$$

#### 4 Twist-2 operators

With the impressive progress that has been made at 3-loops in  $\overline{\text{MS}}$  in the renormalization of the twist-2 operators of the operator product expansion used to understand processes in deep inelastic scattering<sup>5</sup> it is important to have some large  $N_f$  results available for comparison. Similar to the  $\beta$ -function calculation the critical exponents corresponding to the anomalous dimensions of such operators are deduced by inserting the operator into a Green's function in the NATM and determining the scaling behaviour of the integrals. The operators which we consider are,

$$\begin{aligned} \mathcal{O}_{\text{ns}}^{\mu_1 \dots \mu_n} &= i^{n-1} \mathcal{S} \bar{\psi}^I \gamma^{\mu_1} D^{\mu_2} \dots D^{\mu_n} T_{IJ}^a \psi^J - \text{trace terms} \\ \mathcal{O}_{\text{s}}^{\mu_1 \dots \mu_n} &= i^{n-1} \mathcal{S} \bar{\psi}^I \gamma^{\mu_1} D^{\mu_2} \dots D^{\mu_n} \psi^I - \text{trace terms} \\ \mathcal{O}_{\text{g}}^{\mu_1 \dots \mu_n} &= \frac{i^{n-2}}{2} \mathcal{S} \text{tr} G^{a \mu_1 \nu} D^{\mu_2} \dots D^{\mu_{n-1}} G^a_{\nu}{}^{\mu_n} - \text{trace terms} \end{aligned}$$

where  $\mathcal{S}$  denotes symmetrization on the Lorentz indices.

For the fermionic twist-2 flavour nonsinglet and singlet operators,  $\mathcal{O}_{\text{ns}}$  and  $\mathcal{O}_{\text{s}}$ , we deduce at leading order in  $1/N_f$  respectively<sup>12</sup>

$$\begin{aligned}\eta_{\text{ns}}^{(n)} &= \frac{2C_2(R)(\mu-1)^2\eta_1^0}{(2\mu-1)(\mu-2)T(R)N_f} \left[ \frac{(n-1)(2\mu+n-2)}{(\mu+n-1)(\mu+n-2)} + \frac{2\mu}{(\mu-1)}\Psi(n) \right] \\ \eta_{\text{s}}^{(n)} &= \frac{(\mu-1)C_2(R)\eta_1^0}{(2\mu-1)(\mu-2)T(R)N_f} \left[ \frac{2(\mu-1)(n-1)(2\mu+n-2)}{(\mu+n-1)(\mu+n-2)} + 4\mu\Psi(n) \right. \\ &\quad - \mu\Gamma(n-1)[(n^2+n+2\mu-2)^2 + 2(\mu-2)(n(n-1)(2\mu-3+2n) \\ &\quad \left. + 2(\mu-1+n))]\Gamma(2\mu)/[(\mu+n-1)(\mu+n-2)\Gamma(2\mu-1+n)] \right] \quad (14)\end{aligned}$$

where  $n$  is the operator moment,  $\Psi(n) = \psi(\mu-1+n) - \psi(\mu)$  and  $\psi(x)$  is the logarithmic derivative of the  $\Gamma$ -function. One feature of the singlet sector is that the operators do not mix since the gluonic and fermionic operators have different canonical dimensions at  $g_c$ . By contrast in the perturbative calculation there is mixing and one has to compute a matrix of anomalous dimensions. To compare the  $\epsilon$  expansion of Eq. 14 with perturbative results one realises that in the large  $N_f$  calculation the result contained in Eq. 14 is in fact the anomalous dimension of the predominantly fermionic eigenoperator of the perturbative mixing matrix. Therefore if one computes the eigenvalues of the mixing matrix<sup>5,13</sup> and evaluates them at  $g_c$  the coefficients of both  $\epsilon$  expansions ought to be in agreement. We record this occurs exactly at the 3-loop level at leading order in  $1/N_f$ .

More explicitly we present the  $n$ -dependence of the coefficient  $c_3$ , in the notation of Eq. 2, of both the nonsinglet and singlet leading order large  $N_f$  part of the anomalous dimensions. Having the explicit dependence is important since the inverse Mellin transform of the anomalous dimensions with respect to  $n$  determine the Altarelli Parisi splitting functions. These are a function of the conjugate variable,  $x$ , which is the momentum fraction carried by the partons contained in the nucleons, and are in effect a measure of the probability that a parton fragments into other partons. First, we have for the nonsinglet case,<sup>12</sup>

$$\begin{aligned}c_3^{\text{ns}} &= \frac{2}{9}S_3(n) - \frac{10}{27}S_2(n) - \frac{2}{27}S_1(n) + \frac{17}{72} \\ &\quad - \frac{[12n^4 + 2n^3 - 12n^2 - 2n + 3]}{27n^3(n+1)^3} \quad (15)\end{aligned}$$

where  $S_l(n) = \sum_{r=1}^n 1/r^l$ . To compare with the results of the explicit 3-loop  $\overline{\text{MS}}$  calculation for the first few moments,<sup>5</sup> we have evaluated Eq. 15 for various  $n$  and presented them in Table 1. For the singlet sector we can deduce

$n$	$c_3^{\text{ns}}$
2	$-\frac{28}{243}$
4	$-\frac{384277}{1944000}$
6	$-\frac{80347571}{333396000}$
8	$-\frac{38920977797}{144027072000}$
10	$-\frac{27995901056887}{95850016416000}$
12	$-\frac{65155853387858071}{210582486065952000}$
14	$-\frac{68167166257767019}{210582486065952000}$
16	$-\frac{5559466349834573157251}{16553468064672354816000}$
18	$-\frac{19664013779117250232266617}{56770118727793840841472000}$
20	$-\frac{6730392290450520870012467}{18923372909264613613824000}$
22	$-\frac{16759806821032136669044226177}{46048135637404510767879321600}$

Table 1: Values of  $c_3^{\text{ns}}$  for various  $n$ .

the  $n$ -dependence of the 3-loop coefficient of the anomalous dimension of the predominantly fermionic eigenoperator. It is,<sup>12</sup>

$$\begin{aligned}
c_3^{\text{s}} = & \frac{2}{9}S_3(n) - \frac{10}{27}S_2(n) - \frac{2}{27}S_1(n) + \frac{17}{72} \\
& - \frac{2(n^2 + n + 2)^2[S_2(n) + S_1^2(n)]}{3n^2(n+2)(n+1)^2(n-1)} \\
& - 2S_1(n)[16n^7 + 74n^6 + 181n^5 + 266n^4 + 269n^3 + 230n^2 \\
& \quad + 44n - 24]/[9(n+2)^2(n+1)^3(n-1)n^3] \\
& - [100n^{10} + 682n^9 + 2079n^8 + 3377n^7 + 3389n^6 \\
& \quad + 3545n^5 + 3130n^4 + 118n^3 - 940n^2 - 72n \\
& \quad + 144]/[27(n+2)^3(n+1)^4n^4(n-1)] \tag{16}
\end{aligned}$$

Similar to  $c_3^{\text{ns}}$  we have evaluated Eq. 16 for low moments and presented the results in Table 2. These are in exact agreement with the first four moments of the explicit three loop  $\overline{\text{MS}}$  results after diagonalizing the mixing matrix<sup>5</sup> and extracting the leading order large  $N_f$  piece corresponding to the dimension of the predominantly fermionic eigenoperator.

$n$	$c_3^{\text{S}}$
2	0
4	$-\frac{121259}{720000}$
6	$-\frac{3166907}{13891500}$
8	$-\frac{1328467729}{5038848000}$
10	$-\frac{304337312935261}{1054350180576000}$
12	$-\frac{842357166098254633}{2737572318857376000}$
14	$-\frac{42512567719680559}{131614053791220000}$
16	$-\frac{755896148277147625515451}{2251271656795440254976000}$
18	$-\frac{1121815282809553973842772849}{3235896767484248927963904000}$
20	$-\frac{78640886458671664340562623}{220772683941420492161280000}$
22	$-\frac{4248342909129791924572989157741}{11650178316263341224273468364800}$

Table 2: Values of  $c_3^{\text{S}}$  for various  $n$ .

Aside from agreeing with explicit perturbative results up to three loops, there are several other checks on the exponents arising from general principles. First, as the operators are physical their anomalous dimensions are gauge independent. We have therefore computed Eq. 14 with a non-zero covariant gauge parameter  $b$  and observed its cancellation in assembling the contributions from the relevant Feynman diagrams in each exponent. Second, for certain values of  $n$  the corresponding operators reduce to conserved physical currents. Provided the conservation of these currents is not spoiled by an anomaly then



their anomalous dimensions must be zero to all orders in perturbation theory. For the nonsinglet sector the  $n = 1$  case relates to charge conservation, whilst the singlet operator with  $n = 2$  corresponds to the energy momentum tensor. Therefore for both these respective values the critical exponents of Eq. 14 must vanish. It is an easy exercise to verify this. Indeed the zero entry for  $n = 2$  in Table 2 is a reflection of this general result in the three loop case.

## 5 Conclusions

The critical renormalization group ideas<sup>7</sup> have proved useful in giving some insight into the structure of the  $\overline{\text{MS}}$  perturbative coefficients at higher orders in QCD. Although we have concentrated on the four dimensional theory the results have been expressed as functions of  $d$ . Therefore we can also obtain information on the three dimensional model. For example, from Eq. 11

$$\omega = -\frac{1}{2} - \frac{10C_2(G)}{3\pi^2 T(R)N_f} + O\left(\frac{1}{N_f^2}\right) \quad (17)$$

Higher order  $1/N_f$  calculations are possible too. For instance, in the abelian sector the dimension of the mass operator,  $\bar{\psi}\psi$ , is available in  $d$ -dimensions. So when  $d = 3$  the gauge independent electron mass anomalous dimension is,<sup>14</sup>

$$\gamma_m(g_c) = -\frac{32}{3\pi^2 N_f} - \frac{64[3\pi^2 - 28]}{9\pi^4 N_f^2} + O\left(\frac{1}{N_f^3}\right) \quad (18)$$

Such results will be useful for comparing with numerical results for the same quantity computed by other methods. Indeed exponents which are known to similar orders in other models like the  $O(N)$  4-fermi model and evaluated for low  $N$  have been in good agreement with lattice results.<sup>15</sup>

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## References

1. D.J. Gross and F.J. Wilczek, *Phys. Rev. Lett.* **30**, 1343 (1973); H.D. Politzer, *Phys. Rev. Lett.* **30**, 1346 (1973).
2. W.E. Caswell, *Phys. Rev. Lett.* **33**, 244 (1974); D.R.T. Jones, *Nucl. Phys.* **B75**, 531 (1974).

3. O.V. Tarasov, A.A. Vladimirov and A.Yu. Zharkov, *Phys. Lett.* **B93**, 419 (1980).
4. S.A. Larin and J.A.M. Vermaseren, *Phys. Lett.* **B303**, 334 (1993).
5. S.A. Larin, T. van Ritbergen and J.A.M. Vermaseren, *Nucl. Phys.* **B427**, 41 (1994); S.A. Larin, P. Nogueira, T. van Ritbergen and J.A.M. Vermaseren, hep-ph/9605317.
6. J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon Press, Oxford, 1989).
7. A.N. Vasil'ev, Yu.M. Pis'mak and J.R. Honkonen, *Theor. Math. Phys.* **46**, 157 (1981); *Theor. Math. Phys.* **47**, 291 (1981); *Theor. Math. Phys.* **50**, 127 (1982).
8. J.A. Gracey, *Phys. Lett.* **B318**, 177 (1993).
9. A. Hasenfratz and P. Hasenfratz, *Phys. Lett.* **B297**, 166 (1992).
10. J.A. Gracey, *Phys. Lett.* **B373**, 178 (1996).
11. A. Palanques-Mestre and P. Pascual, *Commun. Math. Phys.* **95**, 277 (1984).
12. J.A. Gracey, *Phys. Lett.* **B322**, 141 (1994); hep-ph/9509276.
13. E.G. Floratos, D.A. Ross and C.T. Sachrajda, *Nucl. Phys.* **B129**, 66 (1977); *Nucl. Phys.* **B152**, 493 (1979).
14. J.A. Gracey, *Phys. Lett.* **B317**, 415 (1993).
15. J.A. Gracey, *Int. J. Mod. Phys.* **A9**, 567 (1994).