ESCAPING SETS ARE NOT SIGMA-COMPACT

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Abstract. Let f be a transcendental entire function. The *escaping set* $I(f)$ consists of those points that tend to infinity under iteration of f. We show that $I(f)$ is not σ-compact, resolving a question of Rippon from 2009.

1. Introduction

Let $f\colon \mathbb{C} \to \mathbb{C}$ be a transcendental entire function. The set

$$I(f) := \{z \in \mathbb{C}\colon \lim_{n\to\infty} f^n(z) = \infty\}$$

is called the *escaping set* of f, where

$$f^n = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}$$

denotes the n-th iterate of f. The escaping set was first studied by Eremenko [Ere89] and has been the subject of intensive research in recent years; see e.g. [Obe09, RRRS11, Ber18, RS19] and their references. The topological study of $I(f)$ turns out to be surprisingly intricate. For example Eremenko [Ere89, p. 343] asked whether, for every transcendental entire function, every connected component of $I(f)$ is unbounded. This question has become known as *Eremenko's conjecture*, and is one of the most famous open problems in transcendental dynamics. Strengthened versions of the conjecture have since been proved false in general; compare [RRRS11, Theorem 1.1] and [R16, Theorem 1.6]. On the other hand, Rippon and Stallard [RS11b] have shown that $I(f) \cup \{\infty\}$ is always connected, so any counterexample to Eremenko's original conjecture would have rather subtle topological properties.

The *fast escaping set* is a certain subset of the escaping set, first introduced by Bergweiler and Hinkkanen [BH99]. This set has also received considerable attention recently; in part because it appears to be more tractable than the full escaping set $I(f)$. In particular, the analogue of Eremenko's conjecture holds for $A(f)$ [RS05]. One difference between the two objects lies in the topological complexity of their definitions. Indeed, $A(f)$ can be written as a countable union of closed sets [RS12, Formula (1.3)]; so it is an F_σ set. On the other hand, by definition

$$\begin{aligned} I(f) &= \{z \in \mathbb{C}\colon \forall M \geq 0\, \exists N \geq 0\colon |f^n(z)| \geq M \text{ for } n \geq N\} \\ &= \bigcap_{M=0}^{\infty} \bigcup_{N=0}^{\infty} \bigcap_{n=N}^{\infty} f^{-n}(\mathbb{C} \setminus D(0,M)), \end{aligned}$$

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where $D(0,M)$ denotes the disc of radius M around 0. So $I(f)$ is $F_{\sigma\delta}$: a countable intersection of countable unions of closed sets. This raises the following question, posed by Rippon in 2009 [Obe09, Problem 8, p. 2960].

1.1. Question. Is there a transcendental entire function f for which $I(f)$ itself is F_σ?

Since any closed subset of $\mathbb{C}$ is a countable union of compact sets, it is equivalent to ask whether $I(f)$ is ever *σ-compact* (a countable union of compact sets); compare also [Lip20b]. It is well-known that $I(f)$ cannot be a G_δ (a countable intersection of open sets); see Lemma 2.4 below.

As far as we are aware, there is no function f for which Question 1.1 has been previously resolved; the goal of this paper is to give a negative answer in general. In fact, we prove a stronger result, as follows. Let $\mathrm{UO}(f)$ consist of all $z \in \mathbb{C}$ whose orbit $\{f^n(z)\colon n \geq 0\}$ is unbounded, and let $\mathrm{BU}(f) = \mathrm{UO}(f) \setminus I(f)$ denote the "bungee set"; see [OS16]. Recall that the *Julia set* $J(f)$ is the set of non-normality of the iterates of f.

1.2. Theorem. *Let f be a transcendental entire function. Then every σ-compact subset of $\mathrm{UO}(f)$ omits some point of $I(f) \cap J(f)$ and some point of $\mathrm{BU}(f) \cap J(f)$.*

In particular, the sets $I(f)$, $\mathrm{UO}(f)$, $\mathrm{BU}(f)$ and their intersections with $J(f)$ are not σ-compact.

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2. Proof of the Theorem

We require a result on the existence of arbitrarily slowly escaping points, which follows from work of Rippon and Stallard [RS15]; see also [RS11a, Theorem 1].

2.1. Theorem. *Let f be a transcendental entire function, and let $R_0 \geq 0$. Then there exists $M_0 > R_0$ with the following property. If $(a_m)_{m=0}^\infty$ is a sequence with $a_m \to \infty$ and $a_m \geq M_0$ for all m, then there are $\zeta \in J(f) \cap I(f)$ and $\omega \in J(f) \cap \mathrm{BU}(f)$ with*

$$R_0 \leq |f^m(\zeta)| \leq a_m \quad \text{and} \quad R_0 \leq |f^m(\omega)| \leq a_m \quad \text{for } m \geq 0.$$

Proof. The result is an easy consequence of [RS15], which studies "annular itineraries" of entire functions. Fix some $R > 0$ such that $M(r) > r$ for $r \geq R$, where $M(r)$ denotes the maximum modulus of f on the disc $\overline{D(0,r)}$. Then the *annular itinerary* of a point z is the sequence $(s_m)_{m=0}^\infty$ such that $s_m = 0$ if $|f^m(z)| < R$, and

$$M^{s_m - 1}(R) \leq |f^m(z)| < M^{s_m}(R)$$

otherwise. (Here M^k is the k-th iterate of $r \mapsto M(r)$.)

Theorem 1.2 of [RS15] implies that for suitable $R \geq R_0$, for an entry $s_m = n \geq 0$ in an annular itinerary, there is a subset $X_n \subset \{0, \dots, n+1\}$ of allowable next entries s_{m+1}, and that, for infinitely many n, all entries but at most one are allowable. More precisely,

(a) any sequence $(s_m)_{m=0}^\infty$ with $s_{m+1} \in X_{s_m}$ for all m is the annular itinerary of some $z \in J(f)$;

(b) $n + 1 \in X_n$ for all n;

(c) there is an increasing sequence $(n_j)_{j=1}^\infty$ such that $\#X_{n_j} \geq n+1$.

(With the notation of [RS15], $X_{n_j} = \{0,\dots,n_j+1\}\setminus I_j$, where $\#I_j \leq 1$, and $X_n = \{n+1\}$ when $n \neq n_j$ for all j.) We may suppose that the sequence (n_j) is chosen such that $n_1 \geq 2$.

Set $M_0 := M^{n_1}(R)$ and let $(a_m)_{m=0}^\infty$ be a sequence as in the statement of the theorem. We may assume that (a_m) is non-decreasing. Similarly as in the proof of [RS15, Corollary 1.3 (d)], we can construct an annular itinerary $(s_m)_{m=0}^\infty$ satisfying (a) such that $s_m \to \infty$ and

$$M^{s_m}(R) \leq a_m \qquad \text{for } m \geq 0. \tag{2.1}$$

Indeed, for each j, either $n_j \in X_{n_j}$ or $n_j - 1 \in X_{n_j}$; let us denote this element of X_{n_j} by $\tilde{n}_j$. Then any sequence of the form

$$(s_m)_{m=0}^\infty = \underbrace{n_1, \tilde{n}_1, n_1, \tilde{n}_1, \dots, n_1}_{\text{length } N_1}, n_1+1, n_1+2, \dots, \underbrace{n_2, \tilde{n}_2, \dots, n_2}_{\text{length } N_2}, n_2+1, n_2+2, \dots$$

satisfies (a). If N_j is chosen sufficiently large that

$$a_m \geq M^{n_{j+1}}(R) \quad \text{for } m \geq N_j, \tag{2.2}$$

then (2.1) holds for $m \geq N_1$. It also holds for $m \leq N_1$, since $s_m \leq n_1$ for these values of m, and $a_m \geq M_0 = M^{n_1}(R)$ by assumption.

Let $\zeta \in J(f)$ have annular itinerary $(s_m)_{m=0}^\infty$; then $\zeta \in I(f)$ and

$$R_0 \leq R \leq M^{s_m - 1}(R) \leq |f^m(\zeta)| < M^{s_m}(R) \leq a_m \quad \text{for all } m.$$

Hence ζ has the desired properties. To obtain ω, we instead use an unbounded but non-escaping sequence of the form

$$(\tilde{s}_m)_{m=0}^\infty = B_1, n_1+1, n_1+2, \dots, n_2, \hat{n}_2, B_2, n_1+1, n_1+2, \dots, n_3, \hat{n}_3, B_3, \dots,$$

where $\hat{n}_j \in X_{n_j}$ is either n_1 or $n_1 - 1$, and B_j is a block alternating between entries n_1 and $\tilde{n}_1$. To ensure that the sequence satisfies (a), B_j should end in n_1; it should begin with n_1 if $\hat{n}_j = n_1 - 1$, and with $\tilde{n}_1$ if $\hat{n}_j = n_1$. The length N_j of the block B_j is again chosen to satisfy (2.2). □

Proof of Theorem 1.2. If $D \subset \mathbb{C}$ is closed and $z \in \mathbb{C}$, we set

$$n_D(z) := \min\{n \geq 0\colon f^n(z) \notin D\} \leq \infty,$$

with the convention that $n_D(z) = \infty$ if no such n exists. Observe that n_D depends upper semicontinuously on z, as the infimum of upper semicontinuous functions. Indeed,

$$n_D(z) = \min\{\chi_n(z)\colon n \geq 0\}, \qquad \text{where} \qquad \chi_n(z) = \begin{cases} \infty & \text{if } z \in f^{-n}(D) \\ n & \text{if } z \notin f^{-n}(D), \end{cases}$$

and each χ_n is upper semicontinuous since $f^{-n}(D)$ is closed. Note also that $z \in \mathrm{UO}(f)$ if and only if $n_D(z) < \infty$ for every compact $D \subset \mathbb{C}$.

Now let $R_0 \geq 0$ be arbitrary, and choose M_0 as in Theorem 2.1. Let $X \subset \mathrm{UO}(f)$ be σ-compact; say $X = \bigcup_{j=0}^\infty K_j$ where each K_j is compact. Define $M_j := M_0 + j$ and $D_j := \overline{D(0, M_j)}$. Then $n_{D_j}(z) < \infty$ for every $z \in K_j$.

Since K_j is compact and n_{D_j} is upper semicontinuous,

$$n_j := \max_{z \in K_j} n_{D_j}(z)$$

exists for every j. Set $N_0 := n_0$ and $N_{j+1} := \max\{n_{j+1}, N_j + 1\}$. For $n \geq 0$, define $j(n) := \min\{j \colon N_j \geq n\}$ and

$$a_n := M_{j(n)} = \min\{M_j \colon N_j \geq n\}.$$

Then $a_n \geq M_0$ for all n and $\lim_{n\to\infty} a_n = \infty$. So by Theorem 2.1, there are $\zeta \in I(f) \cap J(f)$ and $\omega \in \mathrm{BU}(f) \cap J(f)$ such that $|f^n(z)| \leq a_n$ for $z \in \{\zeta, \omega\}$ and all $n \geq 0$.

Let $j \geq 0$. Then, for $n \leq n_j \leq N_j$, we have $|f^n(z)| \leq a_n \leq M_j$, and hence $f^n(z) \in D_j$. So $n_{D_j}(z) > n_j$, and $z \notin K_j$ by choice of n_j. Thus $z \notin X = \bigcup_{j=0}^{\infty} K_j$, as claimed. □

By the expanding property of the Julia set, we also obtain the following answer to a question of Lipham (personal communication).

2.2. Corollary. *Let f be a transcendental entire function and let Y be one of the sets $I(f) \cap J(f)$, $\mathrm{UO}(f) \cap J(f)$ and $\mathrm{BU}(f) \cap J(f)$. Then Y is nowhere σ-compact. That is, if $X \subset Y$ is σ-compact, then X is nowhere dense in Y.*

Proof. We prove a slightly stronger statement. Let Y be as in the statement and let $X \subset \mathrm{UO}(f) \cap J(f)$ be σ-compact. Then $X \cap Y$ is nowhere dense in Y; i.e., if $U \subset \mathbb{C}$ with $U \cap Y \neq \emptyset$, then X omits some point of $U \cap Y$.

Let R_0 and M_0 be as in Theorem 2.1, and set $L_0 := J(f) \cap \overline{D(0, M_0)} \setminus D(0, R_0)$. If R_0 is chosen sufficiently large, then L_0 contains no Fatou exceptional point of f. (Recall that a Fatou exceptional point is a point with finite backward orbit, and that f has at most one finite exceptional point [Ber93, p. 156].) Since $U \cap J(f) \neq \emptyset$, it follows that there is some n such that $f^n(U) \supset L_0$.

The proof of Theorem 1.2 shows that no σ-compact subset of $\mathrm{UO}(f) \cap J(f)$ contains $L_0 \cap Y$. (Note that we may take $a_0 = M_0$ in the proof to ensure that $|\zeta|, |\omega| \leq M_0$.) Moreover, $f^n(X)$ is σ-compact as the image of a σ-compact set under a continuous function. So $f^n(X)$ omits some points of $L_0 \cap Y$. Since Y is backward-invariant, X omits some points of $U \cap Y$, as claimed. □

We remark that the proof of Theorem 1.2 establishes the following very general principle:

2.3. Proposition (Abstract version of the main theorem)**.** *Let $f \colon U \to V$ be continuous, where U, V are non-empty topological spaces and U is σ-compact. Let $\mathrm{UO}(f)$ denote the set of $z \in U$ such that $f^n(z)$ is defined and in U for all $n \geq 0$, but the orbit $\{f^n(z)\}$ is not contained in any compact subset of U.*

Suppose that $X \subset \mathrm{UO}(f)$ is σ-compact and $\Delta \subset U$ is compact. Then there is a sequence $(\Delta_n)_{n=0}^{\infty}$ of compact subsets $\Delta_n \subset U$ with $\Delta \subset \Delta_0 \subset \Delta_1 \subset \dots$ and $\bigcup_{n=0}^{\infty} \Delta_n = U$ such that X contains no point $\zeta \in \mathrm{UO}(f)$ with $f^n(\zeta) \in \Delta_n$ for all $n \geq 0$.

Remark. Note that we are not assuming any relation between the spaces U and V. However, the statement is vacuous when $\mathrm{UO}(f) = \emptyset$, and in particular when $U \cap V \neq \emptyset$.

Sketch of proof. Let $(D_j)_{j=0}^{\infty}$ be an increasing sequence of compact subsets $D_j \subset U$ with $\bigcup_{j=0}^{\infty} D_j = U$. Define $j(n)$ as in the proof of Theorem 1.2; then the sets $\Delta_n := D_{j(n)}$ have the desired property. □

It follows that the set of escaping points is not σ-compact in any setting where an analogue of Theorem 2.1 holds. This includes:

(a) transcendental meromorphic functions $\mathbb{C} \to \hat{\mathbb{C}}$ [RS11a];
(b) transcendental self-maps of the punctured plane [MP18, Theorem 1.2];
(c) quasiregular self-maps $f\colon \mathbb{R}^d \to \mathbb{R}^d$ of transcendental type [Nic16];
(d) continuous functions $\varphi\colon [0,\infty) \to [0,\infty)$ with $\varphi(t) \not\to \infty$ as $t \to \infty$, and such that $I(\varphi) \neq \emptyset$ [ORS19, Theorem 2.2].

A more general setting than both (a) and (b) is provided by the *Ahlfors islands maps* of Epstein; see e.g. [RR12]. These are maps $f\colon W \to X$, where X is a compact one-dimensional manifold, $W \subset X$ is open, and f satisfies certain transcendence conditions near ∂W. We may define the escaping set $I(f)$ as the set of points $z \in W$ with $f^n(z) \in W$ for all n and $\operatorname{dist}(f^n(z), \partial W) \to 0$ as $n \to \infty$. It is plausible that an analogue of Theorem 2.1 holds for Ahlfors islands maps with $W \neq X$, using a similar proof as in [RS11a]. This would mean, by Proposition 2.3, that $I(f)$ is not σ-compact for such functions.

For completeness, we conclude by giving the simple proof that $I(f)$ is never a G_δ set; compare also [Lip20a, Corollary 3.2].

2.4. Lemma. *Let f be a transcendental entire function. Then $I(f)$ and $I(f) \cap J(f)$ are not G_δ sets.*

Proof. Every closed subset of $\mathbb{C}$ (or any metric space) is G_δ; so $J(f)$ is G_δ. The intersection of two G_δ sets is again G_δ, so it is enough to prove the claim for $I(f) \cap J(f)$. By [Ere89, Theorem 2], $I(f) \cap J(f)$ is nonempty, and hence dense in $J(f)$ by Montel's theorem. By Baire's theorem, any two dense G_δ subsets of $J(f)$ must intersect. Hence it is enough to observe that $\mathrm{BU}(f) \cap J(f)$ contains a dense G_δ by Montel's theorem, namely the set of points whose orbits are dense in $J(f)$. (See [BD00, Lemma 1].) ☐

Lipham has pointed out the following reformulation of Corollary 2.2.

2.5. Corollary. *Any G_δ set $A \subset Y := J(f) \setminus I(f)$ is nowhere dense in Y.*

In particular, Y is $G_{\delta\sigma}$ but not strongly σ-complete*; that is, it cannot be written as a countable union of relatively closed G_δ subsets.*

Proof. Suppose, by contradiction, that $\overline{D(\zeta,\varepsilon)} \cap Y \subset A$ for some $\zeta \in J(f)$ and $\varepsilon > 0$. The complement of a G_δ set is F_σ, so $B := J(f) \cap \overline{D(\zeta,\varepsilon)} \setminus A$ is an F_σ set with

$$D(\zeta,\varepsilon) \cap I(f) \cap J(f) \subset B \subset I(f) \cap J(f),$$

which contradicts Corollary 2.2.

The set Y is $G_{\delta\sigma}$ by definition. As mentioned in the proof of Lemma 2.4, Y contains a G_δ subset U that is dense in $J(f)$. On the other hand, if $(A_k)_{k=0}^\infty$ is a sequence of relatively closed G_δ subsets of Y, then $\overline{A_k}$ is closed and nowhere dense in $J(f)$. Hence

$$\bigcup_{k=0}^\infty A_k \subset \bigcup_{k=0}^\infty \overline{A_k} \not\supset U \subset Y,$$

by Baire's theorem, and Y is indeed strongly σ-complete. ☐

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