Quadratic Backward Stochastic Differential Equations with Unbounded Coefficients and their Application

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Contents

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Declarations

I hereby declare that this thesis represents my own work in accordance with the regulations of the University of Liverpool. The work is original expect when indicated otherwise. This thesis has not been submitted for examination at any other university.

Abstract

We focus on solving problems on *quadratic backward stochastic differential equations* (BSDEs). We improve fundamental results on *quadratic* BSDEs by using more general conditions on the coefficients. Firstly, we consider quadratic BSDEs with possibly *unbounded* coefficients. We prove a monotonicity theorem, which gives the main argument of the existence result. Then, we give sufficient conditions for the existence of a solution pair. Secondly, we consider the general one-dimensional case of *Riccati* BSDEs. We prove that some integrability conditions, which are weaker than the existing ones, are sufficient for the existence of a solution to the *Riccati* BSDE. As an application, we obtain the existence of a solution to linear-quadratic optimal control problems with possibly *unbounded* coefficients. Thirdly, we study a certain class of Riccati BSDEs with possibly *unbounded* coefficients. Integrability conditions on the coefficients are derived that ensure the existence and uniqueness of the solution pair.

Chapter 1

Introduction

1.1 Overview

Backward stochastic differential equations (BSDEs) are a special type of stochastic differential equations. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a given complete filtered probability space on which a d-dimensional standard Brownian motion $\{W(t), 0 \leq t \leq T\}$ is defined. We assume that $\{\mathcal{F}_t\}_{t\geq 0}$ is the augmentation of $\sigma\{W(s) : 0 \leq s \leq t\}$ by all $\mathbb{P}$-null sets of $\mathcal{F}$. Consider the backward stochastic differential equation (BSDE):

$$y(t) = \xi + \int_t^T F(s, y(s), z(s))\, ds - \int_t^T z(s)\, dW(s), \quad t \in [0, T], \tag{1.1.1}$$

where ξ is a given $\mathcal{F}_T$ -measurable $\mathbb{R}$ -valued random variable, and the random generator $F : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a progressively measurable function.

BSDEs are terminal value problems of stochastic differential equations. Considering a terminal value problem in a stochastic setting is different from one in a deterministic setting (see [64] for details). In particular, for an ordinary differential equation, both the initial and terminal value problems have solutions under the Lipschitz condition. However, for backward stochastic differential equations (BSDEs), the situation is different since we are looking for a solution that is adapted to the given filtration. The solution of equation (1.1.1) is a pair of adapted stochastic processes $(y(\cdot), z(\cdot))$. The role of the second component of the solution is to correct the possible non-adaptiveness that may occur because of the backward nature of the BSDE. In other words, the presence of the second component is crucial in order to have an adapted solution.

The history of backward stochastic differential equations can be dated back to Bismut in [9]. Bismut introduced a linear BSDE with adapted solutions when he studied an optimal stochastic control problem. The general nonlinear case was introduced by Pardoux and Peng [52], which obtained the existence and uniqueness results of square integrable solutions under the Lipschitz condition of the generator. Many articles have focused on relaxing the existing conditions and finding weaker ones that are sufficient for the existence of a solution. Pardoux and Peng [53] proved the existence and uniqueness of a solution when the generator is locally Lipschitz in y and globally Lipschitz in z. Mao [47] obtained the existence and uniqueness under weaker conditions than the Lipschitz condition. Bahlali [3] weakened the global Lipschitz condition to a local Lipschitz condition. One direction of relaxing the conditions under which the BSDE has a solution is to assume that the generator is only continuous and satisfies a linear or quadratic growth condition. Lepeltier and San Martin [45] considered the BSDE when the generator is continuous and satisfies a linear growth condition, see also [36] and [44].

The theory of backward stochastic differential equations has been studied in different fields, such as stochastic control and mathematical finance (see for example, [51], [30] and [23]). In stochastic control, BSDEs can be applied to solve stochastic optimal control problems such as linear-quadratic optimal control problems. Moreover, in mathematical finance, the pricing problem of a contingent claim can be formulated in terms of a linear BSDE. In addition, BSDEs have an interesting application to partial differential equations (PDEs). In particular, some results show that there is a link between BS-

DEs and solutions of partial differential equations (see for example, [12], [20] and [37]). Therefore, from the viewpoints of stochastic control, mathematical finance and partial differential equations, studying BSDEs in more detail is useful. For a detailed discussion on the theory of BSDEs, see for example, [64] and [48]. [64] also provides an extensive literature review of stochastic optimal control problems such as linear-quadratic optimal control problems. See [48], [23] and [21] for more details about applications of BSDEs to finance and insurance.

1.2 Backward stochastic differential equation with unbounded coefficients

In the theory of BSDEs, improving the conditions under which a BSDE is solvable is an important subject. An important direction of weakening the conditions on the generator is to consider the possibility of unbounded coefficients, which is also the main subject of this thesis. Many articles have considered the problem of solvability of BSDEs with unbounded coefficients. El Karoui and Huang [22] proved the solvability of general BSDEs that satisfy the following Lipschitz condition

$$|F(t,y_1,z_1)-F(t,y_2,z_2)| \leq c_1(t)\,|y_1-y_2|+c_2(t)|z_1-z_2|, \tag{1.2.1}$$

for all $y_1,\ y_2 \in \mathbb{R},\ z_1,\ z_2 \in \mathbb{R}^d,\ a.e.\ \ t \in [0,T]\ a.s.$, for some nonnegative processes $c_1(\cdot)$ and $c_2(\cdot)$ that can be unbounded. This type of condition was also assumed in Bender and Kohlmann [6] and Wang et al. [59]. Gashi and Li [27] considered the general BSDEs that satisfy (1.2.1) with unbounded coefficients. They proved the existence of a unique solution pair, but under weaker assumptions on the coefficients than those in El Karoui and Huang [22]. The solution pairs in [27] and [22] are proved to exist in specific weighted spaces. Yong [63] considered linear BSDEs with unbounded coefficients and proved the existence of solution pairs that belong to non-weighted spaces. Bahlali et al. [4] considered BSDEs with continuous quadratic generators where unbounded generators are also considered. Gashi and Li [28] obtained some integrability results and applied them to prove the solvability of a special case of quadratic BSDEs, called Riccati BSDEs, with unbounded coefficients.

BSDE problems with possibly unbounded coefficients are not only of significant theoretical importance but are also motivated by applications in mathematical finance. For example, the modelling of interest rate gives rises to the unboundedness of the coefficients. Some important interest rate models are given by stochastic differential equations (see for example, Date and Gashi [19] and Yong [63]). In general, the solutions to these equations are unbounded processes. This generates several problems in a market with unbounded coefficients, such as the market completeness problem. Gashi and Li [28] obtained the solvability of linear BSDEs with unbounded coefficients, which in turn was used to solve the problems of market completeness. Moreover, the problem of optimal investment has been studied with possibly unbounded coefficients, see for example [7], [8], [41] and [42]. Gashi and Li [28] used their results on Riccati BSDEs to solve an optimal investment problem with power utility in a market with unbounded coefficients. Shen [57] considered a mean–variance portfolio selection problem in a complete market with unbounded coefficients. To solve the problem, he proved the existence and uniqueness of solutions to two BSDEs with unbounded coefficients, a linear BSDE and a Riccati BSDE.

1.3 Quadratic backward stochastic differential equations

Quadratic backward stochastic differential equations, whose generators have quadratic growth in z, are a special class of BSDEs. Now, we introduce the following simple example of a quadratic BSDE (see [37]). Consider the following equation

$$y(t) = \xi + \int_t^T \frac{1}{2}\, |z(s)|^2 \, ds - \int_t^T z(s)\, dW(s), \quad t \in [0, T]. \tag{1.3.1}$$

The exponential transformation $y_1 := e^y$, $z_1 := (e^y\, z)$ leads to the equation

$$y_1(t) = e^{\xi} - \int_t^T z_1(s)\, dW(s), \quad t \in [0,T]. \tag{1.3.2}$$

In this example, if $e^{\xi} \in \mathcal{L}^2_{\mathcal{F}}(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^d)$, (1.3.2) has a unique solution $(y_1(\cdot), z_1(\cdot)) \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$. The process $y_1(\cdot)$ is given by

$$y_1(t) = \mathbb{E}\big[e^{\xi}|\mathcal{F}_t\big], \text{ for all } t \in [0,T] \tag{1.3.3}$$

and $z_1(\cdot)$ is determined by the martingale representation theorem. If we define $y(t) := \log(y_1(t))$ and $z(t) := \dfrac{z_1(t)}{y_1(t)}$ for all $t \in [0,T]$, then $(y(\cdot), z(\cdot)) \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$ is a solution of (1.3.1).

Many problems on quadratic BSDEs have been solved in the literature. They were first solved by Kobylanski [36]. In [37], Kobylanski assumed that ξ is bounded, and that the generator satisfies the following condition:

$$F(t,y,z) = a_0(t,y,z)\, y + F_0(t,y,z) \tag{1.3.4}$$

$\forall\ (t,y,z) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d$, with

$$|F_0(t,y,z)| \le b + c\big(|y|\big)\, |z|^2 \ \ a.s.,$$

$$\beta_0 \le a_0(t,y,z) \le \alpha_0 \ \ a.s.,$$

where $\beta_0,\ \alpha_0,\ b \in \mathbb{R}$ and $c : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function. Under such conditions, equation (1.1.1) admits a solution pair $(y,z) \in \mathcal{L}^{\infty}_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$, where $\mathcal{L}^{\infty}_{\mathcal{F}}(0,T;\mathbb{R})$ is the space of all $\mathcal{F}_t$-progressively measurable $\mathbb{R}$-valued processes which are almost surely bounded for almost every t in $[0,T]$, and $\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$ is the space of all $\mathcal{F}_t$-progressively measurable $\mathbb{R}^d$-valued processes $\theta(\cdot)$ such that $\mathbb{E}\int_0^T |\theta(s)|^2 ds < \infty$. Lepeltier and San Martin [44] proved the existence of a solution for BSDEs when the generator is continuous and has superlinear growth in y and quadratic growth in z. In [13], Briand and Hu proved the existence result for BSDEs with quadratic generators with respect to the variable z and with unbounded terminal conditions. No uniqueness result was stated in that work. Later, in [14], the uniqueness of the solution was proved under the assumption that the generator is convex with respect to the variable z. Morlais [51] proved the existence and uniqueness of solutions for a class of quadratic BSDEs driven by some general martingale, providing applications to the utility maximisation problem. Fan [25] obtained the existence and uniqueness results for bounded solutions and p-integrable solutions of one dimensional BSDEs when the generator has superlinear growth in y and quadratic growth in z, under weaker conditions than [44]. Jamneshan *et al.* [33] considered the solvability of multidimensional quadratic BSDEs with bounded and unbounded terminal conditions, when the generators are bounded and independent of y. They give sufficient conditions under which the existence and uniqueness of solutions hold. Quadratic BSDEs have several interesting applications in mathematical finance and stochastic control. Hu et al. [30] considered the problem of utility maximisation for traders on financial markets. In order to obtain the value function and optimal trading strategy, they used a quadratic BSDE and applied a result of Kobylanski [37] to prove the existence of the BSDE solution. In addition, some optimal control problems can be solved by applying results of quadratic BSDEs. Kohlmann and Tang [38] considered two BSDEs, a linear BSDE and a Riccati BSDE, to solve the the singular linear-quadratic stochastic control problem and the mean-variance hedging problem.

1.3.1 Riccati BSDEs

Riccati BSDEs are matrix-valued nonlinear BSDEs with an additional algebraic matrix positive definiteness constraint. In particular, this constraint is part of the equation that must be satisfied by any solution. Hence, the solvability problem of general Riccati BSDEs is more challenging than solving general BSDEs. The main difficulties come from the nonlinear nature of the generator, the fact that the equation is matrix-valued, and from the algebraic constraint.

Now, let us introduce the general one-dimensional case of Riccati BSDEs:

$$\begin{cases} dK(t) = -\big[a(t)\,K(t) + c'(t)L(t) \;+ Q(t) + F(t, K(t), L(t))\big]dt \\ \qquad\qquad +L'(t)dW, \quad t \in [0,T], \\ K(T) = \; M \;\; a.s., \\ N(t) + K(t)\,D'(t)\,D(t) > 0 \;\; a.e. \;\; t \in [0,T] \;\; a.s., \end{cases} \tag{1.3.5}$$

with

$$F(t,x,y) := -\left[B(t)x + C'(t)D(t)x + y'\,D(t)\right]\left[N(t) + xD'(t)D(t)\right]^{-1} \\ \times \left[B(t)x + C'(t)D(t)x + y'\,D(t)\right]',$$

for all $t \in [0,T],\; x \in \mathbb{R},\; y := (y_1, \cdots, y_d)' \in \mathbb{R}^d$,

$$a(t) := 2A(t) + \sum_{i=1}^{d} C_i^2(t),\; c(t) := (c_1(t), ..., c_d(t))' := 2\,(C_1(t), ..., C_d(t))'\,, t \in [0,T],$$

$$C(t) := (C_1(t), ..., C_d(t))' \quad \text{and} \quad D(t) := \left(D_1'(t), ..., D_d'(t)\right)', t \in [0,T].$$

Here M is an $\mathcal{F}_T$-measurable random variable, the coefficients $A(\cdot), C_i(\cdot)$ and $Q(\cdot)$ where $i = 1, ..., d$, are $\{\mathcal{F}_t\}_{t\geq 0}$-progressively measurable $\mathbb{R}$-valued stochastic processes, the coefficients $B(\cdot)$ and $D_i(\cdot)$ are $\{\mathcal{F}_t\}_{t\geq 0}$-progressively measurable $\mathbb{R}^m$-valued stochastic processes and $N(\cdot)$ is an $\{\mathcal{F}_t\}_{t\geq 0}$-progressively measurable $\mathbb{R}^{m\times m}$-valued stochastic process. A' is the transpose of the vector or matrix A. Note that the first two equations in (1.3.5) represent a non-linear BSDE. The solution of (1.3.5) is the matrix-valued stochastic process $(K(t), L_1(t), ..., L_d(t))$ adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which must satisfy the first equation (BSDE), the terminal condition and the algebraic constraint.

Many cases of Riccati BSDEs have been solved in the literature. First, the case where all coefficients are determistic was solved by Wonham [61]. In this case, the unrandomness of the coefficients implies that $L = 0$, and thus the Riccati BSDE is reduced to a nonlinear ordinary differential equation. Wonham treated the equation as a limit of a sequence of linear differential equations. Bismut [9] was the first to study Riccati BSDEs with random coefficients. In [10], he assumed that $C = 0$ and $D = 0$, and thus the generator becomes independent of L. In [11], he only assumed that $D = 0$, and here the generator becomes linear in L. In [11], Bismut commented on pp. 220 that: "We could not prove the existence of a solution for equation (1.3.5) for the general case". Peng [54] proved the existence and uniqueness of a certain class of Riccati BSDE with random coefficients under which he treated the equation as a nonlinear backward stochastic differential equation. Later, Kohlmann and Zhou [40] studied the case when all coefficients are deterministic, and the control weighting matrix N is reduced to zero (the singular case). They additionally assumed that $C_1 = ... = C_d = 0$, $M = I_{n\times n}$ and $A + A' \geq B\,B'$. Kohlmann and Tang [38] also studied the singular multi-dimensional case when the coefficients are allowed to be random and M and $D'D$ are uniformly positive. Kohlmann and Tang [39] obtained the existence and uniqueness results for the general one-dimensional Riccati BSDE with random coefficients. They assumed that all coefficients and the terminal matrix M are bounded. The existence result was obtained using an approximation technique initially used by Kobylanski [36] and Lepeltier and San Martin [45]. Kohlmann and Tang's [39] result solves the one-dimensional case of Bismut's problem, which was introduced in [11]. Hu and Zhou [31] applied the results of quadratic BSDEs in [37] to prove that the indefinite general Riccati BSDEs admits a nonnegative solution. In [29], Guatteri and Tessitore obtained the existence and uniqueness results for Riccati BSDEs in infinite dimensions with random coefficients arising in quadratic optimal control problems with infinite dimensional stochastic differential state equations. In all cases mentioned above, the coefficients are assumed to be bounded.

Riccati BSDEs have been applied in different fields including stochastic control. We can study linear-quadratic optimal stochastic control problems (LQ problems) using Riccati BSDEs. One of the advantages of linear-quadratic optimal control problems is the ability to obtain explicit forms for the optimal feedback law and the optimal value function through Riccati BSDEs. The following connection is well known (see Bismut [10] and Kohlmann and Tang [39] for details): a stochastic LQ problem is solvable if there is a solution to the associated Riccati BSDE, and by this solution we can obtain an optimal control. In other words, the stochastic LQ problem can be reduced to that of solving the Riccati BSDE. One of the methods that gives rise to a Riccati BSDE is the completion of squares. Using this method, one can obtain an optimal control in a linear state feedback form.

1.4 Contributions

This thesis considers the solvability problems of certain classes of quadratic BSDEs with possibly unbounded generators. Considering possibly unbounded generators is an important direction to improve the conditions under which a BSDE is solvable. From the application point of view, linear-quadratic optimal stochastic control problems with possibly unbounded coefficients give rise to BSDEs with possibly unbounded generators.

We now give a summary of the contributions of this thesis. In chapter 3, we consider the solvability problem of quadratic BSDEs when the generator satisfies:

$$F(t,x,u) := c_1(t,x,u)\, x + F_0(t,x,u),$$

for all $(t,x,u) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$, with

$$|F_0(t,x,u)| \leq c_0(t) + \tfrac{1}{2}\, c_2(t)|u|^2 \quad a.e.\ t \in [0,T]\ a.s..$$

Here the processes $c_0(\cdot)$, $c_2(\cdot)$ and $c_1(\cdot)$ are possibly unbounded. Clearly, due to the processes $c_0(\cdot)$, $c_2(\cdot)$ and $c_1(\cdot)$, our conditions are weaker in comparison to [37], see (1.3.4). In particular, we give sufficient conditions for the existence of an adapted solution pair and those conditions permit for unbounded coefficients, which is not the case in [37], where the coefficients are assumed to be bounded. From the literature review, this case of quadratic

BSDEs with unbounded coefficients has not been considered. Before studying the solvability problem, we prove a monotonicity theorem which is the main reasoning behind the existence result, with the proof presented under weaker conditions than [37]. In particular, we prove that if $(F^n, \xi^n)_{n\geq 0}$ is a sequence of parameters such that

$$|F^n(t, x^n, u^n)| \leq c_0(t) + c_2(t)\,|u^n|^2,$$

for all $x^n \in \mathbb{R}$, $u^n \in \mathbb{R}^d$, *a.e.* $t \in [0,T]$ *a.s.*, and if for each n, the BSDE with (F^n, ξ^n) has a solution (x^n, u^n), then the solutions (x^n, u^n) converge to the solution (x, u) of the BSDE with parameters (F, ξ). We give sufficient conditions for this theorem, and they permit for unbounded coefficients. We expect that these results will be useful in solving some difficult problems with unbounded coefficients, such as reflected BSDEs with quadratic growth, the indefinite case of Riccati BSDEs and their application in optimal investment.

In chapter 4, we study the solvability for the general one-dimensional case of Riccati BSDEs under weaker conditions on the coefficients in comparison to the existing ones. We prove that some integrability conditions, which permit for unbounded coefficients, are sufficient for the existence of a solution pair to the Riccati BSDE. These results are expected to contribute to solve the problems of optimal investment and mean–variance portfolio selection with possibly unbounded coefficients.

As an application, we obtain the solution to the LQ optimal control problem under weaker conditions on the coeffcients in comparison to Theorem 5.2. of Kohlmann and Tang [39]. In particular, we give an explicit solution to the LQ optimal control problem with possibly unbounded coefficients, using our results in this chapter for the Riccati BSDE with unbounded coefficients.

In chapter 5, we study the solvability of a certain class of Riccati BSDEs when $D = 0$, under weaker conditions on the coefficients as compared to Theorem 5.1. of Peng [54]. In particular, we give sufficient conditions for the existence of a unique solution pair for this class of Riccati BSDEs when the coefficients are possibly unbounded, which is not the case in Peng [54] where the coefficients are assumed to be bounded. This result is expected to play an important role in solving similar problems on Riccati BSDEs, such as solving the singular case with possibly unbounded coefficients. In addition, applications such as the LQ optimal control problem with $D = 0$ can be obtained by applying this result.

1.5 Notation

The key notation used throughout this thesis are as follows:

- $|\cdot|$ is the Euclidean norm.
- $\mathbb{R}^d$ is d-dimensional real Euclidean space.
- $\mathbb{R}^+$ is the set of all non-negative real numbers.
- A' is the transpose of the vector or matrix A.
- $\langle A, B\rangle$ is the inner product of the two vectors A and B.
- $\mathbb{R}^{m\times n}$ is the set of $m \times n$ real matrices.
- $L^p(\Omega; \mathbb{R}^d)$ is the space of $\mathbb{R}^d$-valued random variables ξ with $\mathbb{E}[|\xi|^p] < \infty$, for $p \in [0, \infty)$.
- $\mathcal{L}^\infty(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^d)$ is the space of $\mathcal{F}_T$-measurable $\mathbb{R}^d$-valued bounded random variables ξ.
- $\mathcal{L}^2_{\mathcal{F}_0}(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$ is the space of $\mathcal{F}_0$-measurable $\mathbb{R}^d$-valued random variables ξ such that $\mathbb{E}\left(|\xi|^2\right) < \infty$.
- $\mathcal{L}^2_{\mathcal{F}}(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^d)$ is the space of $\mathcal{F}_T$-measurable $\mathbb{R}^d$-valued random variables ξ such that $\mathbb{E}\left(|\xi|^2\right) < \infty$.
- $\mathcal{L}^0_{\mathcal{F}}(0, T; \mathbb{R}^d)$ is the space of $\mathcal{F}_t$-adapted $\mathbb{R}^d$-valued processes.
- $\mathcal{L}^\infty_{\mathcal{F}}(0, T; \mathbb{R}^d)$ is the space of $\mathcal{F}_t$-progressively measurable $\mathbb{R}^d$-valued processes $\theta(\cdot)$ which are almost surely bounded for almost every t in $[0, T]$, i.e, $||\theta(\cdot)||_{\mathcal{L}^\infty_{\mathcal{F}}} := \sup_{t\in[0,T]} |\theta(t)| < \infty \ a.s.$
- $\mathcal{L}_{\mathcal{F}}(0, T; \mathbb{R}^d)$ (resp. $\mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^d)$) is the space of $\mathcal{F}_t$-progressively measurable $\mathbb{R}^d$-valued processes $\theta(\cdot)$ such that $\mathbb{E}\int_0^T |\theta(t)|\, dt < \infty$ (resp. $\mathbb{E}\int_0^T |\theta(t)|^2 dt < \infty$).

Chapter 2

Preliminaries

In this chapter, we present a number of basic concepts of probability theory and stochastic calculus that are to be used throughout this thesis. This chapter is presented briefly without including any proofs. More detailed information and proofs are given in textbooks on stochastic calculus and stochastic control such as those by Klebaner [35], Mao [48] and Yong and Zhou [64].

2.1 Stochastic processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A *filtration* is a family $\{\mathcal{F}_t\}_{t\geq 0}$ of increasing sub σ-algebras of $\mathcal{F}$, i.e. $\mathcal{F}_t \subset \mathcal{F}_s \subset ... \subset \mathcal{F}$, for all $0 \leq t \leq s < \infty$. A *probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be complete if for any $\mathbb{P}$-null set A (i.e. $\mathbb{P}(A) = 0$ for a set $A \in \mathcal{F}$), one has another $\mathbb{P}$-null set $B \in \mathcal{F}$ whenever $B \subseteq A$ (thus, B is also a $\mathbb{P}$-null set). A *filtered* probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is a probability space equipped with the $\{\mathcal{F}_t\}_{t\geq 0}$ of its σ-algebra $\mathcal{F}$, which consists of: a sample space of elementary events, a field of events, a probability defined on that field and a filtration of increasing subfields.

A stochastic process with state space $\mathbb{R}^d$ is a collection $\{X(t), t \geq 0\}$ of $\mathbb{R}^d$-valued random variables. The stochastic process can be regarded as a function of two variables (t, ω) from $I \times \Omega$ to $\mathbb{R}^d$, where I is an index set.

A stochastic process is said to be *continuous* if for almost all $\omega \in \Omega$, the function $X(t,\omega)$ is continuous on $t \geq 0$. A stochastic process is *integrable* if for every $t \geq 0$, $X(t)$ is an *integrable* random variable. It is said to be *measurable* if the stochastic process regarded as a function of two variables (t,ω) from $\mathbb{R}^+ \times \Omega$ to $\mathbb{R}^d$ is $\mathcal{B}(\mathbb{R}^+) \times \mathcal{F}$-measurable, where $\mathcal{B}(\mathbb{R}^+)$ is the family of all Borel subsets of $\mathbb{R}^+$. A stochastic process is *progressively measurable* (or *progressive*) if for each $t \geq 0$, $\{X(s), 0 \leq s \leq t\}$ regarded as a function of (s,ω) from $[0,t] \times \Omega$ to $\mathbb{R}^d$, is $\mathcal{B}([0,t]) \times \mathcal{F}_t$-measurable, where $\mathcal{B}([0,t])$ is the family of all Borel subsets of $[0,t]$. It is said to be $\{\mathcal{F}_t\}$-*adapted* (or *adapted*) if for every t, $X(t)$ is $\mathcal{F}_t$-measurable. Note that if $X(t)$ is $\{\mathcal{F}_t\}_{t\geq 0}$-progressively measurable, it must be measurable and $\{\mathcal{F}_t\}_{t\geq 0}$-adapted.

2.2 Convergence of random variables

In this section, we state some definitions of the main convergence concepts of a sequence of random variables. Important results on the relations between these convergence concepts are also given.

Definition 2.2.1 (Convergence in probabilty)**.** ([35], Definition 2.12)
$\{X_n\}$ converges in probability to X if for any $\epsilon > 0$, $\lim_{n\to\infty} \mathbb{P}\big(\omega : |X_n(\omega) - X(\omega)| > \epsilon\big) = 0$ as $n \to \infty$.

Definition 2.2.2 (Convergence almost surely)**.** ([35], Definition 2.13)
$\{X_n\}$ converges almost surely (a.s.) or with probability 1 to X if there exists a set of zero probability such that for every ω outside the set, the sequence $X_n(\omega)$ converges to $X(\omega)$ as $n \to \infty$, i.e. $\mathbb{P}\big(\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\big) = 1$.

Definition 2.2.3. ([35], Definition 2.14)
For any $p > 1$, we say that a random variable $X \in L^p(\Omega;\mathbb{R}^d)$ if $\mathbb{E}|X|^p < \infty$ and we can define a norm

$$||X||_p := (\mathbb{E}|X|^p)^{\frac{1}{p}}.$$

Definition 2.2.4 (Convergence in L^p)**.** ([35], Definition 2.15)
A sequence of random variables $\{X_n\}$ converges in $L^p(\Omega;\mathbb{R}^d)$ to a random variable X, if $\lim_{n\to\infty} \mathbb{E}[\,|X_n - X|^p] = 0$.

Note that since the L^p norms are increasing in p, convergence in L^p implies convergence in L^r for $r < p$.

Theorem 2.2.5. *Let $\{X_n\}$ be a sequence of random variables, and let X be some other variable. If $\{X_n\}$ converges in L^p to X for some $p \geq 1$, or if $\{X_n\}$ converges almost surely to X, then $\{X_n\}$ also converges in probability to X.*

Theorem 2.2.6. *If $\{X_n\}$ converges to X in probability, then there is a subsequence $\{X_{n_k}\}$ converging almost surely to the same limit X.*

Theorem 2.2.7 (Continuous Mapping Theorem)**.** *Let $\{X_n\}$ be a sequence of random variables and X be some other variable. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. If $\{X_n\}$ converges almost surely to X, then $f(X_n)$ converges almost surely to $f(X)$.*

2.3 Convergence of functions

We now introduce a number of theorems for the convergence of a sequence of functions. We also discuss some theorems for the convergence of expectations. The following theorem gives an equivalent definition of uniform convergence.

Theorem 2.3.1. *$f_n(x)$ converges uniformly to $f(x)$ if and only if* $\sup_x |f_n(x) - f(x)| \to 0$.

Theorem 2.3.2 (Uniform Convergence Theorem)**.** *If f_n is a sequence of continuous functions which converges uniformly to the function f, then the limit f is also continuous.*

Note that the limit of a pointwise convergent sequence of continuous functions does not have to be continuous.

Theorem 2.3.3 (Monotone Convergence Theorem)**.** ([1], Theorem 2.4.2.) *If a sequence is monotone and bounded, then it converges.*

Theorem 2.3.4 (Dominated Convergence Theorem)**.** *Suppose* $f_n : \mathbb{R} \to [-\infty, \infty]$ *are measurable functions such that the pointwise limit* $f(x) = \lim_{n\to\infty} f_n(x)$ *exists and there is an integrable* $g : \mathbb{R} \to [0, \infty]$ *with* $|f_n(x)| \le g(x)$ *for all* n *and for all* $x \in \mathbb{R}$. *Then* f *is integrable, as is* f_n *for each* n, *and*

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu = \int_{\mathbb{R}} \lim_{n\to\infty} f_n \, d\mu = \int_{\mathbb{R}} f \, d\mu.$$

In other words, under certain conditions, the dominated convergence theorem allows one to interchange limits and [Lebesgue] integrals. In particular, we can apply this theorem to integrands that change sign if there exists a dominating integrable function and a pointwise limit of the sequence of integrands.

Functions can be seen as elements of normed spaces. In the following, we introduce number of theorems for the convergence of functions in normed spaces.

Theorem 2.3.5. ([24], Theorem 1)
Let X *be a normed space and* $\{x_n\}_n$ *a sequence converging weakly to* $x \in X$. *Then*

$$||x|| \le \liminf_{n\to\infty} ||x_n||.$$

Theorem 2.3.6. ([24], Theorem 2)
Let $\{x_n\}_n$ *be a bounded sequence in a normed space* X, *then there exists a subsequence* $\{x_{n_k}\}_k$ *of* $\{x_n\}_n$ *which converges weakly to some* $x \in X$.

2.4 Martingales

Martingales represent a special class of stochastic processes. In this thesis, martingales constructed from a Brownian motion are considered.

Definition 2.4.1. *A stochastic process* $\{X(t)\}_{t\ge 0}$ *is called square-integrable if* $\mathbb{E}|X(t)|^2 < \infty$ *for every* $t \ge 0$.

Definition 2.4.2. ([48], Definition 4.1)
A (standard) one-dimensional Brownian motion is a real-valued continuous $\{\mathcal{F}_t\}$*-adapted process* $\{W(t)\}_{t\geq 0}$ *with the following properties:*

(i) $W(0) = 0$ *a.s.;*
(ii) for $0 \leq s \leq t < \infty$*, the increment* $W(t) - W(s)$ *is normally distributed with mean zero and variance* $t-s$*. This means that if* $s = 0$*, then* $W(t)-W(0)$ *has distribution* $N(0,t)$*;*
(iii) for $0 \leq s < t < \infty$*, the increment* $W(t) - W(s)$ *is independent of* $\mathcal{F}_s$*.*

Note that one can also define a Brownian motion $W(\cdot)$ over any time interval $[a, b]$ or $[a, b)$ for any $0 \leq a < b < \infty$. In particular, $W(\cdot)$ is said to be standard over $[a, b]$ if $W(a) = 0$.

A Brownian motion $\{W(t)\}$ has many important properties. It is a *continuous square integrable martingale.*

Definition 2.4.3. ([35], Definition 2.30) *A real-valued* $\{\mathcal{F}_t\}$*-adapted stochastic process* $X(t)$ *for* $t \geq 0$*, is called a martingale with respect to* $\{\mathcal{F}_t\}$ *(or simply a martingale) if for any* t*,* $X(t)$ *is integrable, i.e.* $\mathbb{E}\left[|X(t)|\right] < \infty$ *and for any* $0 \leq s < t$*,*

$$\mathbb{E}[X(t)|\mathcal{F}_s] = X(s) \quad a.s.$$

In other words, a martingale is a stochastic process for which at a certain time, given all previous values, the conditional expectation of the next value is equal to the current value. In particular, if we know the values of the process up to time s, and $X(s) = x$, then the expected future value at any future time is x.

In general, stochastic integrals with respect to martingales are local martingales but not martingales. Thus, we introduce the definition of local martingales. Before introducing this definition, we present the definition of *stopping times.*

Definition 2.4.4 (Stopping Times)**.** ([48], Definition 2.33) *A non-negative random variable* $\tau : \Omega \to [0, \infty]$*, where* τ *may be infinite, is called a stopping* *time (with respect to the filtration* $\{\mathcal{F}_t\}_{t\geq 0}$*) if for each* t *the event* $\{\tau \leq t\} \in \mathcal{F}_t$.

In other words, a random time τ is a stopping time if for any t we can decide to stop (knowing whether or not τ has occurred) based only on the information we observed up to time t and not on any future information.

Definition 2.4.5 (Local Martingales)**.** ([35], Definition 7.20) *An adapted process* $M(t)$ *is called a local martingale if there exists an increasing sequence of stopping times* τ_n*, such that for every* n *the stopped process* $M(t \wedge \tau_n)$ *is a martingale.*

The following result gives the condition under which a local martingale becomes a martingale.

Theorem 2.4.6. ([35], Corollary 7.22) *Let* $M(t)$ *for* $0 \leq t \leq \infty$*, be a local martingale such that* $\mathbb{E}\big[\sup\limits_{s\leq t} |M(s)|\big] < \infty$ *for all* t*. Then* $M(t)$ *is a martingale.*

2.4.1 Martingales inequalities

Theorem 2.4.7 (Doob's Martingale Inequality)**.** ([48], Theorem 3.8)
Let $M(t)$ *be an* $\mathbb{R}^d$*-valued martingale and* $[s, T]$ *be a bounded interval in* $\mathbb{R}^+$*. If* $p \geq 1$ *and* $M(t) \in L^p(\Omega; \mathbb{R}^d)$*, then*

$$\mathbb{P}\big(\sup_{s\leq t\leq T} |M(t)| \geq c\big) \leq \frac{1}{c^p}\mathbb{E}|M(T)|^p$$

for all $c > 0$.

Theorem 2.4.8 (Burkholder-Davis-Gundy Inequality)**.** ([64], Theorem 5.4)
Let $W(t)$ *be a* d*-dimensional standard Brownian motion. Let* $\sigma(t) : [0, T] \times \Omega \to \mathbb{R}^d$ *be an* $\{\mathcal{F}_t\}_{t\geq 0}$*-adapted stochastic process such that*

$$\int_0^T |\sigma(t)|^2 \, dt < \infty \quad a.s.$$

Then, for any $p > 0$*, there exists a constant* $K_p > 0$ *such that for any stopping time* τ,

$$\frac{1}{K_p}\mathbb{E}\left[\int_0^\tau |\sigma(s)|^2 ds\right]^p \leq \mathbb{E}\left[\sup_{0\leq t\leq \tau}\left|\int_0^t \sigma(s)\, dW(s)\right|^{2p}\right] \leq K_p\mathbb{E}\left[\int_0^\tau |\sigma(s)|^2 ds\right]^p.$$

2.5 Stochastic calculus

In this section, we introduce the stochastic integral

$$\int_0^T X(t)\, dW(t),$$

where $X(t)$ is some stochastic process and $W(t)$ is a Brownian motion.

The following theorem gives two important properties of the stochastic integral, also called the Itô integral.

Theorem 2.5.1. ([35], Theorem 4.3)
Let $X(t)$ for $0 \leq t \leq T$ be an $\{\mathcal{F}_t\}$-progressively measurable process such that $\int_0^t X^2(s)\, ds < \infty$ a.s., then the Itô Integral $\int_0^t X(s)\, dW(s)$ exists.

Moreover, if $\int_0^t \mathbb{E}\left[X^2(s)\right] ds < \infty$, then the following properties hold

(1) *Zero mean property:*

$$\mathbb{E}\left(\int_0^t X(s)\, dW(s)\right) = 0.$$

(2) *Isômetry property:*

$$\mathbb{E}\left(\int_0^t X(s)\, dW(s)\right)^2 = \int_0^t \mathbb{E}\left(X(s)\right)^2 ds.$$

Remark 2.5.2. ([35], Remark 4.1)
One of the preferable properties of the integral is that we can interchange the expectation and the integral. In order to have this property, the condition that $X(t)$ is adapted is not enough and a stronger condition, that of a progressively measurable process, is needed.

Theorem 2.5.3. ([35], Theorem 4.7)
Let $X(t)$ be an adapted process such that $\int_0^T \mathbb{E}\left[X^2(s)\right] ds < \infty$. Then,

$Y(t) = \int_0^t X(s)\,dW(s)$ *for* $0 \leq t \leq T$ *is a continuous zero mean square integrable martingale.*

Remark 2.5.4. ([35], Remark 4.6)
If $\int_0^T \mathbb{E}\left[X^2(s)\right]\, ds = \infty$*, then the stochastic integral* $\int_0^t X(s)\,dW(s)$ *may fail to be a martingale, however it is always a local martingale.*

Definition 2.5.5 (Itô process)**.** *A one-dimensional Itô process is a continuous adapted process* $X(t)$ *of the form*

$$X(t) = X(0) + \int_0^t \mu(s)\,ds + \int_0^t \sigma(s)\,dW(s), \quad 0 \leq t \leq T, \tag{2.5.1}$$

where $\mu(\cdot) : [0,T] \times \Omega \to \mathbb{R}$ *and* $\sigma(\cdot) : [0,T] \times \Omega \to \mathbb{R}^d$ *with*

$$\int_0^T |\mu(t)|\,dt < \infty \text{ a.s. and } \int_0^T |\sigma(t)|^2\,dt < \infty \text{ a.s.} \tag{2.5.2}$$

We say that $X(t)$ *has stochastic differential* $dX(t)$ *on* $t \in [0,T]$ *given by*

$$dX(t) = \mu(t)\,dt + \sigma(t)\,dW(t).$$

Theorem 2.5.6 (Itô's formula for the Itô process)**.** ([48], Theorem 6.2)
Let $X(t)$ *be given by* (2.5.1) *such that* (2.5.2) *holds. Let* $F(t,x) : [0,T] \times \mathbb{R} \to \mathbb{R}$ *be* C^1 *in* t *and* C^2 *in* x *with* F_t*,* F_x *and* F_{xx} *continuous. Then* $F(t,X(t))$ *is an Itô process with stochastic differential given by*

$$\begin{aligned} dF(t,X(t)) &= \left[F_t(t,X(t)) + F_x(t,X(t))\,\mu(t) + \frac{1}{2}\,F_{xx}(t,X(t))\ \sigma^2(t)\right] dt \\ &+ F_x(t,X(t))\ \sigma(t)\ dW(t). \end{aligned}$$

Moreover, the formula for integration by parts (stochastic product rule) in differential notation is given by

$$d\big(X(t)Y(t)\big) = X(t)\,dY(t) + Y(t)\,dX(t) + d[X,Y](t).$$

If

$$\begin{aligned} dX(t) &= \mu_X(t)\,dt + \sigma_X(t)\,dW(t), \\ dY(t) &= \mu_Y(t)\,dt + \sigma_Y(t)\,dW(t), \end{aligned}$$

then,

$$d[X,Y](t) = dX(t)\,dY(t) = \sigma_X(t)\,\sigma_Y(t)\,(dW(t))^2 = \sigma_X(t)\,\sigma_Y(t)\,dt,$$

leading to the formula

$$d\big(X(t)Y(t)\big) = X(t)\,dY(t) + Y(t)\,dX(t) + \sigma_X(t)\,\sigma_Y(t)\,dt.$$

2.6 Quadratic BSDEs

In this section, we state some results on the existence and uniqueness of quadratic BSDEs. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a given complete filtered probability space on which a d-dimensional standard Brownian motion $(W(t), 0 \leq t \leq T)$ is defined. We assume that $(\mathcal{F}_t, 0 \leq t \leq T)$ is the augmentation of $\sigma\{W(s) : 0 \leq s \leq t\}$ by all $\mathbb{P}$-null sets of $\mathcal{F}$.

The first theorem, due to Kobylanski [37], considers the solvability of one-dimensional BSDEs with quadratic growth. Let α_0, β_0, $b \in \mathbb{R}$ and $c : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous increasing function. The generator F satisfies condition *(K1)* with α_0, β_0, b and c if F is continuous and satisfies:

$$F(t,y,z) = \; a_0(t,y,z)\,y + F_0(t,y,z),$$

for all $(t,y,z) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d$, and

$$\begin{gathered} \beta_0 \leq a_0(t,y,z) \leq \alpha_0 \;\; a.s., \\ |F_0(t,y,z)| \leq \; b + \; c\big(|y|\big)\,|z|^2 \;\; a.s.. \end{gathered}$$

Theorem 2.6.1. ([37], Theorem 2.3) *Let (F,T,ξ) be a set of parameters of the following BSDE*

$$y(t) = \xi + \int_t^T \; F(s, y(s), z(s))\; ds - \int_t^T \; z(s)\; dW(s), \; t \in [0,T]. \qquad (2.6.1)$$

Suppose that the generator F satisfies (K1) with α_0, β_0, b, c and $\xi \in \mathcal{L}^\infty(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^d)$. Then the BSDE (2.6.1) *has at least one solution $(y,z) \in \mathcal{L}^\infty_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$.*

The second theorem, due to [39], considers the unique solvability of the Riccati BSDE in the regular case (i.e. $M \geq 0$, $Q \geq 0$ and $N > 0$). Assume that the coefficients A, B, C_i and D_i for $i = 1, ..., d$, are $\{\mathcal{F}_t\}_{t\geq 0}$-adapted, bounded, matrix-valued processes, defined on $\Omega \times [0,T]$, of dimension $n \times n$, $n \times m$, $n \times n$ and $n \times m$, respectively. Assume that Q and N are $\{\mathcal{F}_t\}_{t\geq 0}$-adapted, bounded, matrix-valued processes, defined on $\Omega \times [0,T]$ of dimension $n \times n$ and $m \times m$ respectively. Assume that the matrix $M : \Omega \to \mathbb{R}^{n\times n}$ is an $\mathcal{F}_T$-measurable and bounded $n \times n$ random matrix. In this theorem, the one-dimensional case has been considered. Set

$$a(t) := 2A(t) + |C(t)|^2, \quad t \in [0,T],$$

$$c(t) := (c_1(t), \cdots, c_d(t))' := 2\left(C_1(t), \cdots, C_d(t)\right)', \quad t \in [0,T],$$

$$C(t) := (C_1(t), ..., C_d(t))' \text{ and } D(t) := \left(D_1'(t), ..., D_d'(t)\right)', \quad t \in [0,T].$$

Theorem 2.6.2. ([39], Theorem 2.1) *Assume that $M \geq 0$, $Q(t) \geq 0$ a.e. $t \in [0,T]$ and $N(t) \geq \varepsilon I_{m\times m}$ a.e. $t \in [0,T]$ for some constant $\varepsilon > 0$. Then, the following equation*

$$\begin{cases} dK(t) = -\big[a(t)K(t) + c'(t)\, L(t) + Q(t) + F(t, K(t), L(t))\big]\, dt \\ \qquad\qquad +L(t)\, dW(t), \quad t \in [0,T], \\ K(T) = \; M \;\; a.s., \end{cases}$$

with

$$\begin{aligned} F(t, K(t), L(t)) = -&\left[B(t)K(t) + C'(t)D(t)K(t) + L'(t)D(t)\right] \\ &\times \left[N(t) + K(t)D'(t)D(t)\right]^{-1} \\ &\times \left[B(t)K(t) + C'(t)D(t)K(t) + L'(t)D(t)\right]', \end{aligned}$$

has a unique $\{\mathcal{F}_t\}_{t\geq 0}$-adapted solution $(K(t), L(t))$ in $\mathcal{L}^\infty_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$.

The next theorem, due to [54], considers the unique solvability of the Riccati BSDE in the regular case (i.e. $M \geq 0$, $Q \geq 0$ and $N > 0$) with the additional condition $D = 0$.

Theorem 2.6.3. ([54], Theorem 5.1) *Assume that $Q(t) \geq 0$ a.e. $t \in [0, T]$, $N(t) > 0$ a.e. $t \in [0, T]$ and $M \geq 0$. Further, assume that $D(t) = 0$. Then, the following Riccati BSDE*

$$\begin{cases} dK(t) = -\big[A'(t)\,K(t) + K(t)\,A(t) + L(t)C(t) + C'(t)L(t) + C'(t)K(t)C(t) \\ \qquad\qquad +Q(t) - K(t)\,B(t)N^{-1}(t)B'(t)\,K(t)\big]\,dt + L(t)\,dW(t), \quad t \in [0, T], \\ K(T) = \; M, \;\; a.s., \end{cases}$$

has a unique solution $(K(t), L(t))$ in $\mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^n) \times \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$ such that $K(t)$ is nonnegative and bounded.

2.7 Linear-quadratic optimal control problems

In general, stochastic optimal control problems have a number of common features: there is a mathematical system which is described by a stochastic differential equation, there are many alternative decisions that can impact the system, there are some constraints that the decisions and/or the state are subject to, and there is a function that measures the performance of the decisions. By solving such a problem, we mean to optimise (maximise or minimise) the performance function by choosing one decision between the decisions satisfying all constraints (see [64] for details). The linear-quadratic optimal control problems (LQ problems) is a special class of optimal control problems where the state equations are linear in both the state and control, and the cost functionals are quadratic in both the state and control. Now, let us introduce the stochastic LQ problem with random coefficients. Let $T > 0$ be given. For any $(s, x) \in [0, T] \times \mathbb{R}$, consider the following linear controlled stochastic differential equation

$$
\begin{cases}
dX(t) = [A(t)X(t) + B(t)u(t)]\, dt \\
\qquad\qquad + \sum_{i=1}^{d} \big[C_i(t)X(t) + D_i(t)u(t)\big] dW_i(t), \quad t \in [s,T], \\
X(s) = x,
\end{cases}
\tag{2.7.1}
$$

under the same coefficients assumptions as in section 1.3.1. A quadratic cost functional is given by

$$
J(u(\cdot); s, x) = \mathbb{E}\left[\int_s^T \big[Q(t)X^2(t) + u'(t)N(t)u(t)\big] dt + M\, X^2(T)\big|\mathcal{F}_s\right].
\tag{2.7.2}
$$

For any $s \in [0,T]$, we denote by $\mathcal{U}^{\omega}[s,T]$ the set of $u(\cdot)$ satisfying the following:

(i) $u(\cdot) \in \mathcal{L}^2_{\mathcal{F}}(s,T;\mathbb{R}^m)$ (the space of $\mathcal{F}_t$-adapted $\mathbb{R}^m$-valued processes $\theta(\cdot)$ such that $\mathbb{E}\int_s^T |\theta(t)|^2 dt < \infty$).
(ii) Under $u(\cdot)$, for any $x \in \mathbb{R}$, equation (2.7.1) has a unique solution $X(\cdot)$.
(iii) The right-hand side of (2.7.2) is well-defined under $u(\cdot)$.

We then state the stochastic linear-quadratic optimal control problem (LQ problem) as follows: for each $(s,x) \in [0,T] \times \mathbb{R}$, find $\hat{u}(\cdot) \in \mathcal{U}^{\omega}[s,T]$ such that

$$
J(\hat{u}(\cdot); s, x) = \inf_{u(\cdot)\in\mathcal{U}^{\omega}[s,T]} J(u(\cdot); s, x) = V(s,x).
$$

V is called the value function of the LQ problem. The LQ problem is said to be solvable at $(s,x) \in [0,T] \times \mathbb{R}$ if there exists a control $\hat{u}(\cdot) \in \mathcal{U}^{\omega}[s,T]$ such that $J(\hat{u}; s, x) = V(s,x)$. $\hat{u}(\cdot)$ is called an optimal control and the corresponding $\hat{X}(\cdot)$ is called an optimal state process. For example, if the processes appearing in (2.7.1) and (2.7.2) satisfy the following:

$$\begin{cases} A, C_i \in \mathcal{L}^\infty_{\mathcal{F}}(0,T;\mathbb{R}), B, D_i \in \mathcal{L}^\infty_{\mathcal{F}}(0,T;\mathbb{R}^m), \\ Q \in \mathcal{L}^\infty_{\mathcal{F}}(0,T;\mathbb{R}), N \in \mathcal{L}^\infty_{\mathcal{F}}(0,T;\mathbb{R}^{m\times m}), M \in \mathcal{L}^\infty(\Omega, \mathcal{F}_T, \mathbb{P};\mathbb{R}), \end{cases}$$

then (2.7.1) has a unique solution $X(\cdot) \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R})$ and (2.7.2) is well-defined.

The LQ problem was introduced at the beginning of optimal control theory. Many articles have considered this problem, see for example, [46], [60], [50] and [49]. Kalman [34] solved the deterministic LQ problem (see [2] and [60] for details). Stochastic LQ problems using Riccati BSDEs were first studied by Wonham [61], who solved the problem when all coefficients are deterministic. Stochastic LQ problem has been studied further in several articles (see for example, [55], [15] and [32]). Bismut [10] was the first to study stochastic LQ problems with random coefficients. He proved the existence and uniqueness of solutions to the associated Riccati BSDEs when N is positive definite and Q and M are positive semidefinite. Then, he gave sufficient conditions for the existence of a solution to the LQ problem. Chen et al. [16] proved that stochastic LQ problems with random coefficients when the cost functional is allowed to have a negative weight on the square of the control variable, i.e. N is negative definite, can be solvable. They gave conditions under which the stochastic LQ problem when $C = 0$ has a solution. Chen and Zhou in [17], obtained an optimal feedback control for the indefinite case when $C \neq 0$. Further articles considering the indefinite case include [18], [56] and [62]. Kohlmann and Tang [39] later proved that the solvability of Riccati BSDEs with random coefficients leads to the existence of solutions of the stochastic LQ problems under weaker conditions.

Chapter 3

Quadratic Backward Stochastic Differential Equations with Unbounded Coefficients

3.1 Abstract

We consider the solvability problem of quadratic backward stochastic differential equations (BSDEs) with possibly unbounded coefficients. We give sufficient conditions, which are weaker than those that already exist, for the existence of a solution pair.

3.2 Introduction

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathbb{P})$ be a given complete filtered probability space on which a d-dimensional standard Brownian motion $(W(t), t \geq 0)$ is defined. We assume that $(\mathcal{F}_t, t \geq 0)$ is the augmentation of $\sigma\{W(s) : 0 \leq s \leq t\}$ by all $\mathbb{P}$-null sets of $\mathcal{F}$. Consider the backward stochastic differential equation

(BSDE):

$$y(t) = \xi + \int_t^T F(s, y(s), z(s))\, ds - \int_t^T z'(s)\, dW(s), \quad t \in [0,T], \quad (3.2.1)$$

where ξ is a given $\mathcal{F}_T$-measurable $\mathbb{R}$-valued random variable and the random generator $F : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a progressively measurable function.

Many problems on quadratic BSDEs were solved in the literature. They have been first solved by Kobylanski, see [36] and [37]. Lepeltier and San Martin [44] proved the existence of a solution for BSDEs when the generator is continuous, has superlinear growth in y and quadratic growth in z. In [13], Briand and Hu proved the existence result for BSDEs with linear growth in y and quadratic growth in z, with unbounded terminal value. In all cases mentioned above, the coefficients are assumed to be bounded. Kobylanski [37] assumed that the terminal value is bounded and that the generator satisfies

$$F(t, y, z) = a_0(t, y, z)\, y + F_0(t, y, z), \quad (3.2.2)$$

for all $(t, y, z) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d$, with

$$\beta_0 \le a_0(t, y, z) \le \alpha_0 \;\; a.s.$$

and

$$|F_0(t, y, z)| \le b + c\big(|y|\big)\, |z|^2 \;\; a.s.,$$

where β_0, α_0, $b \in \mathbb{R}$ and $c : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function. Under such conditions, equation (3.2.1) admits a solution, however, one which is in general non-unique.

An important direction of relaxing the conditions on the generator under which equation (3.2.1) has a solution is to consider possibly unbounded coefficients. In this chapter, we first prove the following monotonicity theorem: under weaker conditions than those in [37], the sequence of solutions $\{(y^n, z^n)\}$ of the BSDEs with parameters $\{(F^n, \xi^n)\}$ converge to the solution pair (y, z) of the BSDE with parameters (F, ξ). This result provides the main argument of the existence theorem. We then show the solvability of equation (3.2.1) when F is continuous and has quadratic growth in z. In particular, we obtain the existence result when the coefficients are not assumed to be bounded.

3.3 Notation and assumptions

The key notation and assumptions used in this chapter are as follows:

- $1 < \beta$, $1 < \beta_1$ and $1 < \beta_2$ are given real constants such that $\beta_1 \leq \beta$ and $\beta_2 \leq \beta$.

- $\gamma(\cdot)$, $\hat{\gamma}(\cdot)$, $c_0(\cdot)$ and $\hat{c}_0(\cdot)$ are given $\mathbb{R}$-valued positive progressively measurable processes such that $\hat{\gamma}(\cdot) \leq \gamma(\cdot)$ *a.s.*, $c_0(\cdot) \leq \hat{c}_0(\cdot)$ *a.s.* and $\hat{c}_0(\cdot) \geq 1$ *a.s.*.

- ξ is a given $\mathcal{F}_T$-measurable $\mathbb{R}$-valued non-negative random variable.

- $c_1(\cdot)$ and $a(\cdot)$ are given $\mathbb{R}$-valued progressively measurable processes such that $a(\cdot) \geq 0$ *a.s.* and $c_1(\cdot) \leq a(\cdot)$ *a.s.*.

- $\hat{c}_2(\cdot)$ and $c_2(\cdot)$ are given $\mathbb{R}$-valued progressively measurable processes such that $1 \leq c_2(\cdot) \leq \hat{c}_2(\cdot)$ *a.s.* and $\hat{c}_2(t) := \hat{c}_2(0) + \int_0^t a(s)\, \hat{c}_2(s)\, ds$ for $0 \leq t \leq T$.

- $k_1(\cdot)$ and $k_2(\cdot)$ are given $\mathbb{R}$-valued non-negative progressively measurable processes such that $k_1(\cdot) \leq \hat{c}_2(\cdot)$ *a.s.*, $k_2^2(\cdot) \leq \frac{1}{2}\hat{c}_2(\cdot)$ *a.s.*, $k_2^2(t)$ is differentiable and $d[k_2^2(t)]/dt = k_1(t)$.

- $\varphi(\cdot) \geq 1$ is a given $\mathbb{R}$-valued progressively measurable process such that $\varphi(t) \leq \hat{c}_2(t)$ *a.e.* $t \in [0, T]$ *a.s.* and $d\varphi(t) = a(t)\, \varphi(t)\, dt$ for $t \in [0, T]$.

- $p(t) := \exp\left[\int_0^t (\gamma(s) + 4\beta \hat{c}_0^2(s)\, \hat{c}_2^2(s))\, ds\right], \quad t \in [0, T]$.

- $\hat{p}(t) := \exp\left[\int_0^t (\hat{\gamma}(s) + 2\beta_1 k_1^2(s) + 2\beta_2 k_2^2(s))\, ds\right], \quad t \in [0, T]$.

- $64\,a(t)\,\hat{c}_2^2(t) \leq \gamma(t) + 4\beta\,\hat{c}_0^2(t)\,\hat{c}_2^2(t) \quad a.e.\ \ t \in [0,T]\ a.s.$.
- $\mathcal{H}_p^2(0,T;\mathbb{R}^d)$ (resp. $\mathcal{H}_{\hat{p}}^2(0,T;\mathbb{R}^d)$) is the space of $\mathcal{F}_t$-progressively measurable $\mathbb{R}^d$-valued processes $\theta(\cdot)$ such that $\mathbb{E}\left[\sup_{t\in[0,T]}\ p(t)|\theta(t)|^2\right] < \infty$ (resp. $\mathbb{E}\left[\sup_{t\in[0,T]}\ \hat{p}(t)|\theta(t)|^2\right] < \infty$).
- $\mathcal{M}_p^2(0,T;\mathbb{R}^d)$ (resp. $\mathcal{M}_{\hat{p}}^2(0,T;\mathbb{R}^d)$) is the space of $\mathcal{F}_t$-progressively measurable $\mathbb{R}^d$-valued processes $\theta(\cdot)$ such that $\mathbb{E}\left[\int_0^T\ p(s)|\theta(s)|^2 ds\right] < \infty$ (resp. $\mathbb{E}\left[\int_0^T\ \hat{p}(s)|\theta(s)|^2 ds\right] < \infty$).
- $\mathcal{M}_\varphi^2(0,T;\mathbb{R}^d)$ is the space of $\mathcal{F}_t$-progressively measurable $\mathbb{R}^d$-valued processes $\theta(\cdot)$ such that $\mathbb{E}\left[\int_0^T\ \varphi(s)\,|\theta(s)|^2 ds\right] < \infty$.

We say that the pair (F,ξ) satisfies *conditions* H if:

(1) F is a continuous function of x and u;

(2) $F(t,x,u) := c_1(t)\,x + F_0(t,x,u)$, for all $(t,x,u) \in [0,T]\times\mathbb{R}\times\mathbb{R}^d$, with

$$|F_0(t,x,u)| \leq c_0(t) + \tfrac{1}{2}\,c_2(t)|u|^2,\ \ a.e.\ t \in [0,T]\ \ a.s.;$$

(3) there exists a positive constant M such that:

   (a) $\mathbb{E}\big[\,p(T)\,e^{8\hat{c}_2(T)M}\big] < \infty$,

   (b) $\mathbb{E}\big[e^{\int_t^T\ 2\hat{c}_0(s)\,\hat{c}_2(s)\,ds + 2\hat{c}_2(T)\xi}\big|\mathcal{F}_t\big] \leq e^{2\hat{c}_2(t)\,M},\ \ t \in [0,T] \quad a.s.$,

   (c) $\varphi(t)\,e^{32\,\hat{c}_2(t)\,M}\,c_0(t) \in \mathcal{L}_{\mathcal{F}}(0,T;\mathbb{R})$,

   (d) $\varphi(t)\,a(t)\,e^{32\,\hat{c}_2(t)\,M} \in \mathcal{L}_{\mathcal{F}}(0,T;\mathbb{R})$,

   (e) $\hat{c}_2(T)\ \int_0^T\ c_0(t)\,dt \leq e^{2\hat{c}_2(T)\,M}\ \ a.s.$,

   (f) $\hat{c}_2(T)\ \int_0^T\ a(t)\,dt \leq e^{2\hat{c}_2(T)\,M}\quad a.s.$,

   (g) $\frac{1}{2k_2^2(t)}\log\left\{\mathbb{E}\left[e^{\int_t^T\ \frac{1}{2}k_2^2(s)ds + 2k_2^2(T)\exp(\hat{c}_2(T)\xi)}\middle|\ \mathcal{F}(t)\right]\right\} \leq e^{\hat{c}_2(t)M}$, $t \in [0,T]\ \ a.s.$,

(h) $\left\{\mathbb{E}\left[e^{\int_t^T \eta(s)ds - \hat{c}_2(T)\xi}\middle| \mathcal{F}_t\right]\right\}^{-1} \geq e^{-\hat{c}_2(t)M}, \quad t \in [0,T] \;\; a.s.,$

where $\eta(t) := [a(t) - c_1(t)]\hat{c}_2(t)(M+1) + c_0(t)\hat{c}_2(t), \quad t \in [0,T].$

Here we give sufficient conditions, which permit for unbounded coefficients, for the existence of an adapted solution pair. This is not the case in [37], where the coefficients are assumed to be bounded. In particular, due to the processes $c_0(\cdot)$, $c_1(\cdot)$ and $c_2(\cdot)$, our conditions are more general. For example, if $c_0(t) = 0$, $c_1(t) = 0$, $c_2(t) := \sin(W(t)) + 1 + \exp(\int_0^t W(s)\,ds)$, $t \in [0,T]$, the known results on the existence of solutions do not apply and thus our conditions are weaker as compared to [37]. Note that we can choose $k_1(\cdot)$, $k_2(\cdot)$, $\hat{c}_0(\cdot)$, $\hat{c}_2(\cdot)$, $a(\cdot)$ and M. This provides more flexibility on the conditions. For instance, condition H-(3) can be suitable weakened by choosing large values for M. Moreover, the process $\gamma(\cdot)$ is important as we can take an arbitrary large value of $\gamma(\cdot)$ under which $64\,a(\cdot)\;\hat{c}_2^2(\cdot) \leq \gamma(\cdot) + 4\beta\;\hat{c}_0^2(\cdot)\,\hat{c}_2^2(\cdot)$ holds $a.s.$.

3.4 Special cases of quadratic BSDEs with unbounded coefficients

In this section, we give some useful results for the existence theorem, when we prove that the solution of (3.2.1) with (F,ξ) is bounded. In Lemma 3.4.1 and Lemma 3.4.2, we prove the existence and uniqueness results for two different quadratic BSDEs with possibly unbounded coefficients. In Theorem 3.4.5 and Lemma 3.4.7, we prove that if equation (3.2.1) with parameters (F,ξ) admits a solution (y,z), then we can find upper and lower bounds for y and an explicit upper bound of z in a weighted space. Define:

$$\phi(t) := \hat{c}_2^2(t)\; p(t)\, e^{4\hat{c}_2(t)\,M},$$

where M is given in condition $H(3)$. Let $\mathcal{M}_\phi^2(0,T;\mathbb{R}^d)$ be the space of $\mathcal{F}_t$-progressively measurable $\mathbb{R}$-valued processes $\theta(\cdot)$ such that $\mathbb{E}\left[\int_0^T \phi(s)|\theta(s)|^2 ds\right]$

$< \infty$.

Lemma 3.4.1. *Let* (G_1, ξ) *be a set of parameters of equation* (3.2.1) *such that:*

$$G_1(t, y_1, z_1) := \hat{c}_0(t) + a(t)\; y_1 + \hat{c}_2(t)\; |z_1|^2, \;\; (t, y_1, z_1) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d. \tag{3.4.1}$$

If conditions H hold, then equation (3.2.1) *with* (G_1, ξ) *has a unique solution pair* $(y_1(\cdot), z_1(\cdot)) \in \mathcal{L}^{\infty}_{\mathcal{F}}(0, T; \mathbb{R}) \times \mathcal{M}^2_{\phi}(0, T; \mathbb{R}^d)$ *such that*

$$y_1(t) = \tfrac{1}{2\hat{c}_2(t)}\; \log \left\{ \mathbb{E}\big[e^{\int_t^T \; 2\hat{c}_0(s)\, \hat{c}_2(s)\, ds + 2\hat{c}_2(T)\xi} \big| \mathcal{F}_t \big] \right\}, \;\; 0 \leq t \leq T. \tag{3.4.2}$$

This lemma gives an example of a BSDE with possibly unbounded coefficients that has a bounded solution. For instance, if $\hat{c}_0(t) = 0$, $\xi = 1/2$, $\hat{c}_2(t) = W(t)$, $0 \leq t \leq T$, then $y_1(t) \in \mathcal{L}^{\infty}_{\mathcal{F}}(0, T; \mathbb{R})$

Proof. We prove this lemma by transforming equation (3.2.1) with the generator G_1 into an equivalent linear BSDE. Let $v := e^{2\hat{c}_2(t) y_1}$ for $(t, y_1) \in [0, T] \times \mathbb{R}$. We have:

$$\begin{aligned} dv(t) \;\; &= \;\; 2v(t) y_1(t) a(t)\; \hat{c}_2(t)\, dt - 2\, \hat{c}_2(t) v(t)\, G_1(t, y_1(t), z_1(t))\, dt \\ &\quad + 2\, \hat{c}_2^2(t)\, v(t)\, |z_1(t)|^2\, dt + 2\hat{c}_2(t)\, v(t)\, z_1'(t)\; dW(t). \end{aligned}$$

By setting $z(t) := 2\hat{c}_2(t)\, v(t)\, z_1(t)$, we have:

$$\begin{aligned} dv(t) \;\; &= \;\; 2v(t)\, y_1(t)\, a(t)\, \hat{c}_2(t)\, dt - 2\, \hat{c}_2(t)\, v(t)\, \left[\hat{c}_0(t) + a(t)\; y_1(t) + \hat{c}_2(t)\; |z_1(t)|^2 \right] dt \\ &\quad + 2\, \hat{c}_2^2(t)\, v(t)\, |z_1(t)|^2\, dt + z'(t)\; dW(t). \end{aligned}$$

Thus, we obtain:

$$\begin{cases} dv(t) = -2\,\hat{c}_0(t)\,\hat{c}_2(t)v(t)\,dt + z'(t)\,dW(t), \quad t \in [0,T], \\ \\ v(T) = e^{2\hat{c}_2(T)\,\xi}. \end{cases} \tag{3.4.3}$$

As equation (3.4.3) is a linear BSDE, it satisfies the conditions of Theorem 2.1 of [27] and it thus has a unique solution pair $(v(\cdot), z(\cdot)) \in \mathcal{H}^2_p(0,T;\mathbb{R}) \times \mathcal{M}^2_p(0,T;\mathbb{R}^d)$. Now, consider the following equation:

$$\begin{cases} dx(t) = 2\,\hat{c}_0(t)\,\hat{c}_2(t)\,x(t)\,dt, \quad t \in [0,T], \\ \\ x(0) = 1, \end{cases} \tag{3.4.4}$$

which has a unique solution $x(t) = \exp\left[\int_0^t 2\hat{c}_0(s)\,\hat{c}_2(s)\,ds\right]$. By the Itô's product rule, the differential of $x(t)v(t)$ is:

$$\begin{aligned} d\{x(t)\,v(t)\} &= v(t)\,dx(t) + x(t)\,dv(t) \\ &= 2\,\hat{c}_0(t)\,\hat{c}_2(t)\,x(t)\,v(t)\,dt + x(t)\big[-2\hat{c}_0(t)\,\hat{c}_2(t)\,v(t)\,dt + z'(t)\,dW(t)\big] \\ &= x(t)\,z'(t)\,dW(t). \end{aligned} \tag{3.4.5}$$

Since $\hat{c}_0(t) \geq 1$, $\hat{c}_2(t) \geq 1$, $\gamma(t) > 0, \beta > 1$, we have:

$$\mathbb{E}\left[\int_0^T e^{4\int_0^t \hat{c}_0(s)\,\hat{c}_2(s)\,ds}\,|z(t)|^2\,dt\right] < \infty.$$

Then, the stochastic integral in (3.4.5) is a martingale and thus by integrating from t to T, and by taking the expectation, (3.4.5) becomes:

$$\mathbb{E}\left[x(T)\,v(T) - x(t)\,v(t)|\mathcal{F}_t\right] = 0.$$

Therefore,

$$\begin{aligned} v(t) &= x^{-1}(t)\,\mathbb{E}\left[x(T)\,v(T)|\mathcal{F}_t\right] = \mathbb{E}\left[x^{-1}(t)\,x(T)\,v(T)|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[e^{\int_t^T 2\hat{c}_0(s)\,\hat{c}_2(s)\,ds}\,e^{2\hat{c}_2(T)\,\xi}|\mathcal{F}_t\right] > 1. \end{aligned}$$

By this, together with condition $H(3)$-(b), we have $0 < y_1(t) \leq M$ *a.s.* for all $t \in [0,T]$. Thus, $\sup_{t\in[0,T]} |y_1(t)| < \infty$ *a.s.*, and therefore equation (3.2.1) with (G_1, ξ) has a unique solution pair $(y_1(\cdot), z_1(\cdot)) \in \mathcal{L}^\infty_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{M}^2_\phi(0,T;\mathbb{R}^d)$ such that:

$$y_1(t) = \frac{\log v(t)}{2\,\hat{c}_2(t)} \text{ and } z_1(t) = \frac{z(t)}{2\,\hat{c}_2(t) v(t)},\ t \in [0,T].$$

□

Lemma 3.4.2. *Let* (G_2, ξ) *be a set of parameters of equation* (3.2.1) *with:*

$$G_2(t, y_2, z_2) := -\hat{c}_0(t) + a(t)\ y_2 - \hat{c}_2(t)\ |z_2|^2,\ \ (t, y_2, z_2) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d. \tag{3.4.6}$$

If conditions H hold, then equation (3.2.1) *with* (G_2, ξ) *has a unique solution pair* $(y_2(\cdot), z_2(\cdot))$ *such that*

$$y_2(t) = \tfrac{1}{-2\hat{c}_2(t)} \log \left\{\mathbb{E}\big[e^{\int_t^T (2\hat{c}_0(s)\,\hat{c}_2(s))ds - 2\hat{c}_2(T)\xi}\big|\mathcal{F}_t\big]\right\},\quad 0 \leq t \leq T. \tag{3.4.7}$$

Proof. The proof relies on similar computations as the previous lemma. Let $\hat{v} := e^{-2\hat{c}_2(t) y_2}$, then:

$$\begin{aligned} d\hat{v}(t) &= -2\hat{v}(t)\, y_2(t)\, a(t)\ \hat{c}_2(t)\, dt + 2\,\hat{c}_2(t)\,\hat{v}(t)\, G_2(t, y_2(t), z_2(t))\, dt \\ &\quad + \frac{1}{2}\,(-2\,\hat{c}_2(t))^2\,\hat{v}(t)\,|z_2(t)|^2\, dt - 2\hat{c}_2(t)\,\hat{v}(t)\, z_2'(t)\ dW(t). \end{aligned}$$

By setting $\hat{z}(t) := -2\hat{c}_2(t)\,\hat{v}(t)\, z_2(t)$, we have:

$$\begin{aligned} d\hat{v}(t) &= -2\hat{v}(t)\, y_2(t)\, a(t)\ \hat{c}_2(t)\, dt + 2\,\hat{c}_2(t)\hat{v}(t)\,\big[-\hat{c}_0(t) + a(t)\ y_2(t) - \hat{c}_2(t)\ |z_2(t)|^2\big]\, dt \\ &\quad + 2\,\hat{c}_2^2(t)\,\hat{v}(t)\,|z_2(t)|^2\, dt + \hat{z}\,'(t)\ dW(t) \\ &= a(t)\,\hat{v}(t)\ \log\hat{v}(t)\, dt - 2\,\hat{c}_2(t)\,\hat{c}_0(t)\,\hat{v}(t)\, dt + 2\, a(t)\ y_2(t)\,\hat{c}_2(t)\,\hat{v}(t)\, dt \\ &\quad - 2\,\hat{c}_2^2(t)\,\hat{v}(t)\,|z_2(t)|^2\, dt + 2\,\hat{c}_2^2(t)\,\hat{v}(t)\,|z_2(t)|^2\, dt + \hat{z}\,'(t)\ dW(t). \end{aligned}$$

Thus, we obtain the following equation:

$$\begin{cases} d\hat{v}(t) = \; -2\,\hat{c}_0(t)\,\hat{c}_2(t)\,\hat{v}(t)\,dt + \hat{z}\,'(t)\;dW(t), \quad t\in[0,T], \\ \\ \hat{v}(T) = e^{-2\,\hat{c}_2(T)\,\xi} \quad a.s.. \end{cases} \tag{3.4.8}$$

As equation (3.4.8) is a linear BSDE, it satisfies the conditions of Theorem 2.1 of [27], and it thus has a unique solution pair $(\hat{v}(\cdot),\hat{z}(\cdot))\in\mathcal{H}^2_p(0,T;\mathbb{R})\times\mathcal{M}^2_p(0,T;\mathbb{R}^d)$. Now, applying the Itô's product rule on $x(t)\hat{v}(t)$, where x is the solution of (3.4.4), gives:

$$\hat{v}(t) = x^{-1}(t)\;\mathbb{E}\left[x(T)\,\hat{v}(T)|\mathcal{F}_t\right] = \;\mathbb{E}\left[\,e^{\int_t^T 2\hat{c}_0(s)\,\hat{c}_2(s)\,ds\;-2\hat{c}_2(T)\,\xi}|\mathcal{F}_t\right] \le e^{2\hat{c}_2(t)M}.$$

Thus equation (3.2.1) with (G_2,ξ) has a unique solution pair $(y_2(\cdot),z_2(\cdot))$ such that $y_2(t)\ge\,-M$. □

Now, we present the following useful results, Lemma 3.4.3 and Lemma 3.4.4. The proof of Lemma 3.4.3 is given in the appendix to this chapter.

Lemma 3.4.3. *Let the random function Q be defined as:*

$$Q(t,y) := \begin{cases} e^{2\hat{c}_2(t)y} - 1 - 2\hat{c}_2(t)y - 2\hat{c}_2^2(t)y^2, & \textit{for } (t,y)\in[0,T]\times[0,M], \\ 0, & \textit{for } (t,y)\in[0,T]\times[-M,0]. \end{cases}$$

Then, the following properties hold:

(i) $Q(t,y)\ge 0$ *a.s. for all* $(t,y)\in[0,T]\times[-M,M]$*, and* $Q(t,y)=0$ *for all* $t\in[0,T]$ *a.s. iff* $y\le 0$.

(ii) $\dfrac{\partial Q}{\partial y}(t,y)\ge 0$ *a.s. for all* $(t,y)\in[0,T]\times[-M,M]$,

(iii) $\dfrac{\partial Q}{\partial t}(t,y) = a(t)\,y\,\dfrac{\partial Q}{\partial y}(t,y)$ *a.s. for all* $(t,y)\in[0,T]\times[-M,M]$,

(iv) $\hat{c}_2(t)\,\dfrac{\partial Q}{\partial y}(t,y) - \dfrac{1}{2}\,\dfrac{\partial^2 Q}{\partial y^2}(t,y)\le 0$ *a.s. for all* $(t,y)\in[0,T]\times[-M,M]$,

(v) $\left[\dfrac{\partial Q}{\partial y}(t,y)\right]^2 \le 16\, e^{4M^2} e^{4\hat{c}_2^2(t)}$ *a.s. for all* $(t,y) \in [0,T] \times [-M,M]$.

Lemma 3.4.4. *If* $z(\cdot) \in \mathcal{M}_\phi^2(0,T;\mathbb{R}^d)$*, then the process*

$$\int_t^T \hat{c}_2(s)\, \frac{\partial Q}{\partial y}(s,y(s))\, z'(s)\, dW(s), \quad t \in [0,T],$$

is a martingale.

Proof. From the Burkholder-Davis-Gundy inequality, see Theorem 2.4.8, there exists a constant K such that:

$$\begin{aligned}
& \mathbb{E}\left[\sup_{t\in[0,T]} \left| \int_0^t \hat{c}_2(s) \frac{\partial Q}{\partial y}(s,y(s))\; z'(s)\, dW(s) \right|\right] \\
\le\; & K\,\mathbb{E}\left[\int_0^T \hat{c}_2^2(s) \left(\frac{\partial Q}{\partial y}(s,y(s))\right)^2 \left|z(s)\right|^2 ds\right]^{\frac{1}{2}} \\
\le\; & K\,\mathbb{E}\left[\sup_{t\in[0,T]} \frac{1}{p(t)} \left(\frac{\partial Q}{\partial y}(t,y(t))\right)^2 \int_0^T \hat{c}_2^2(s)\, p(s) \left|z(s)\right|^2 ds\right]^{\frac{1}{2}} \\
\le\; & \frac{K}{2}\,\mathbb{E}\left[\sup_{t\in[0,T]} \frac{1}{p(t)} \left(\frac{\partial Q}{\partial y}(t,y(t))\right)^2 + \int_0^T \hat{c}_2^2(s)\, p(s) \left|z(s)\right|^2 ds\right].
\end{aligned}$$

By the assumption $z(\cdot) \in \mathcal{M}_\phi^2(0,T;\mathbb{R}^d)$, the second part in the right hand side is finite. Now, we prove that the first part is finite. We have:

$$\frac{d\hat{c}_2^n(t)}{dt} = n\, \hat{c}_2^{n-1}(t)\, \frac{d\hat{c}_2(t)}{dt} = n\, a(t)\, \hat{c}_2^n(t).$$

Thus,

$$\hat{c}_2^n(t) = \hat{c}_2^n(0) + \int_0^t n\, a(s)\, \hat{c}_2^n(s)\, ds.$$

By this, together with (v) in Lemma 3.4.3, we have

$$\left[\frac{\partial Q}{\partial y}(t,y(t))\right]^2 \leq 16\, e^{4M^2}\, e^{4\left(\hat{c}_2^2(0)+\int_0^t 2a(s)\,\hat{c}_2^2(s)\,ds\right)}.$$

By this, we obtain

$$\begin{aligned}
\mathbb{E}\,\left[\sup_{t\in[0,T]}\frac{1}{p(t)}\left(\frac{\partial Q}{\partial y}(t,y(t))\right)^2\right] &\leq 16\, e^{4M^2}\,\mathbb{E}\,\left[\sup_{t\in[0,T]}\frac{1}{p(t)}\, e^{4\left(\hat{c}_2^2(0)+\int_0^t 2a(s)\,\hat{c}_2^2(s)\,ds\right)}\right] \\
&= 16\, e^{4M^2}\, e^{4\hat{c}_2^2(0)}\,\mathbb{E}\,\left[\sup_{t\in[0,T]}\frac{e^{8\int_0^t a(s)\,\hat{c}_2^2(s)\,ds}}{p(t)}\right] \\
&< \infty.
\end{aligned}$$

Therefore,

$$\mathbb{E}\,\left[\sup_{t\in[0,T]}\left|\int_0^t \hat{c}_2(s)\frac{\partial Q}{\partial y}(s,y(s))\; z'(s)\,dW(s)\right|\right] < \infty,$$

and then by Theorem 2.4.6, the stochastic integral

$$\int_t^T \hat{c}_2(s)\frac{\partial Q}{\partial y}(s,y(s))\, z'(s)\; dW(s), \quad t\in[0,T],$$

is a martingale. □

Here $c_0(\cdot)$, c_1 and $c_2(\cdot)$ are given processes, but we can choose any values for β, $\hat{c}_0(\cdot)$, $\gamma(\cdot)$, $a(\cdot)$ and $\hat{c}_2(\cdot)$ such that $c_1 \leq a(t)$, $c_2(t) \leq \hat{c}_2(0)\, e^{\int_0^t a(s)\,ds}$, and $8\, a(t)\; \hat{c}_2^2(t) - \gamma(t) - 4\beta\; \hat{c}_0^2(t)\, \hat{c}_2^2(t) \leq 0$ *a.e.* $t \in [0,T]$ *a.s.*. Note that the previous lemma is needed for Theorem 3.4.5 when we take expectation. In [37], the martingale property holds since the coefficients are bounded.

In the following theorem, we prove that if the BSDE (3.2.1) with (F,ξ) admits a solution (y,z), then y is bounded above by the solution of (3.2.1) with G_1, and bounded below by the solution of (3.2.1) with G_2.

Theorem 3.4.5. *Let* $(y(\cdot),z(\cdot)) \in \mathcal{L}^\infty_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{M}^2_\phi(0,T;\mathbb{R}^d)$ *be a solution* *of the equation:*

$$\begin{cases} dy(t) = -F(t, y(t), z(t))\, dt + z'(t)\, dW(t),\ t \in [0, T], \\ y(T) = \xi. \end{cases} \tag{3.4.9}$$

such that (F, ξ) *satisfy condition H.*
(i) If $(y_1(\cdot), z_1(\cdot)) \in \mathcal{L}^\infty_{\mathcal{F}}(0, T; \mathbb{R}) \times \mathcal{M}^2_\phi(0, T; \mathbb{R}^d)$ *is a solution of the equation:*

$$\begin{cases} dy_1(t) = -G_1(t, y_1(t), z_1(t))\, dt + z_1'(t)\, dW(t),\ t \in [0, T], \\ y_1(T) = \left[\sup\limits_{\omega\in\Omega} (\xi)\right]^+, \end{cases} \tag{3.4.10}$$

with $G_1(t, y_1, z_1)$ *that is given in* (3.4.1)*, then* $y(\cdot) \le y_1(\cdot)$ *for all* $t \in [0, T]$ *a.s..*
(ii) If $(y_2(\cdot), z_2(\cdot)) \in \mathcal{L}^\infty_{\mathcal{F}}(0, T; \mathbb{R}) \times \mathcal{M}^2_\phi(0, T; \mathbb{R}^d)$ *is a solution of the equation:*

$$\begin{cases} dy_2(t) = -G_2(t, y_2(t), z_2(t))\, dt + z_2'(t)\, dW(t),\ t \in [0, T], \\ y_2(T) = \left[\inf\limits_{\omega\in\Omega} (\xi)\right]^-, \end{cases} \tag{3.4.11}$$

with $G_2(t, y_2, z_2)$ *that is given in* (3.4.6)*, then* $y(\cdot) \ge y_2(\cdot)$ *for all* $t \in [0, T]$ *a.s..*

Proof. (i) The idea of the proof is similar to that of Proposition 2.1. in [37]. However, here it is more difficult since the coefficients are possibly unbounded. In [37], it has been proved that the solution of (3.2.1) is bounded by the solution of a given ordinary differential equation. Here, we compare the BSDE (3.4.9) with another BSDE (3.4.10). In Lemma 3.4.1, we gave sufficient conditions under which equation (3.4.10) has a unique solution. Our aim is to prove that $y(\cdot) \le y_1(\cdot)$. Let $M := \sup\limits_{\omega\in\Omega} (|y| + |y_1|)$. Define $\hat{H}(t, x) := \hat{c}_2(t)\, Q(t, x)$ for $(t, x) \in [0, T] \times [-M, M]$. The differential of

$\hat{H}(t, y(t) - y_1(t))$ is:

$$d\hat{H}(t, y(t) - y_1(t)) = d\big[\hat{c}_2(t)\, Q(t, y(t) - y_1(t))\big]$$

$$\begin{aligned}
&= d[\hat{c}_2(t)]\; Q(t, y(t) - y_1(t)) + \hat{c}_2(t)\; dQ(t, y(t) - y_1(t)) \\
&= a(t)\, \hat{c}_2(t)\; Q(t, y(t) - y_1(t))\, dt + \hat{c}_2(t)\; \bigg[\frac{\partial Q}{\partial t}(t, y(t) - y_1(t))\, dt \\
&\quad -\big[F(t, y(t), z(t)) - G_1(t, y_1(t), z_1(t))\big]\; \frac{\partial Q}{\partial y}(t, y(t) - y_1(t))\, dt \\
&\quad +\frac{1}{2}\; \frac{\partial^2 Q}{\partial y^2}(t, y(t) - y_1(t))\, |z(t) - z_1(t)|^2\; dt \\
&\quad +\frac{\partial Q}{\partial y}(t, y(t) - y_1(t))\, \big[z(t) - z_1(t)\big]'\; dW(t)\bigg].
\end{aligned}$$

By integrating from t to T, we obtain:

$$\begin{aligned}
&\hat{H}(t, y(t) - y_1(t)) = \hat{H}(T, y(T) - y_1(T)) \\
&- \int_t^T a(s)\, \hat{c}_2(s)\, Q(s, y(s) - y_1(s)))\, ds \\
&- \int_t^T \hat{c}_2(s)\, \frac{\partial Q}{\partial s}(s, y(s) - y_1(s))\, ds \\
&+ \int_t^T \hat{c}_2(s)\, \big[F(s, y(s), z(s)) - G_1(s, y_1(s), z_1(s))\big]\; \frac{\partial Q}{\partial y}\, (s, y(s) - y_1(s))\, ds \\
&- \frac{1}{2} \int_t^T \hat{c}_2(s)\, |z(s) - z_1(s)|^2\; \frac{\partial^2 Q}{\partial y^2}(s, y(s) - y_1(s))\, ds \qquad (3.4.12) \\
&- \int_t^T \hat{c}_2(s)\, \frac{\partial Q}{\partial y}(s, y(s) - y_1(s))\big[z(s) - z_1(s)\big]'\, dW(s).
\end{aligned}$$

We have:

$$\begin{aligned} F(t,y,z)-G_1(t,y_1,z_1) &= c_1(t)y+F_0(t,y,z)-G_1(t,y_1,z_1) \\ &\le c_1(t)y+|F_0(t,y,z)|-G_1(t,y_1,z_1) \\ &\le c_1(t)y+\frac{1}{2}\,c_2(t)\,|z|^2-a(t)y_1-\hat{c}_2(t)|z_1|^2. \end{aligned}$$

Substituting this in (3.4.12) and by (ii) in Lemma 3.4.3, we obtain

$$\begin{aligned} &\hat{H}(t,y(t)-y_1(t)) \\ \le\ & \hat{H}(T,y(T)-y_1(T))-\int_t^T a(s)\,\hat{c}_2(s)\,Q(s,y(s)-y_1(s)))\,ds \\ -\ & \int_t^T \hat{c}_2(s)\,\frac{\partial Q}{\partial s}(s,y(s)-y_1(s))\,ds \\ +\ & \int_t^T \hat{c}_2(s)\,\big[c_1(s)\,y(s)+\frac{1}{2}\,c_2(s)\,|z(s)|^2-a(s)y_1(s)-\hat{c}_2(s)|z_1(s)|^2\big] \\ & \times\frac{\partial Q}{\partial y}\,(s,y(s)-y_1(s))\,ds \qquad (3.4.13) \\ -\ & \frac{1}{2}\int_t^T \hat{c}_2(s)\,|z(s)-z_1(s)|^2\,\frac{\partial^2 Q}{\partial y^2}(s,y(s)-y_1(s))\,ds \\ -\ & \int_t^T \hat{c}_2(s)\,\frac{\partial Q}{\partial y}(s,y(s)-y_1(s))\big[z(s)-z_1(s)\big]'\,dW(s). \end{aligned}$$

We have:

$$\begin{aligned}\hat{c}_2(t)\ [c_1(t)\, y(t) - a(t)\, y_1(t)] &= \hat{c}_2(t)\ [c_1(t)\, y(t) - a(t)\, y_1(t) + c_1(t)\, y_1(t) - c_1(t)\, y_1(t)] \\ &= \hat{c}_2(t)\ [c_1(t)\,(y(t) - y_1(t)) + y_1(t)\ (c_1(t) - a(t))] \\ &\leq c_1(t)\, \big[y(t) - y_1(t)\big]\, \hat{c}_2(t).\end{aligned}$$

By substituting this in (3.4.13), we obtain

$$\begin{aligned}\hat{H}(t, y(t) - y_1(t)) \;\leq\; & \hat{H}(T, y(T) - y_1(T)) \\ & - \int_t^T a(s)\, \hat{c}_2(s)\, Q(s, y(s) - y_1(s))\, ds \\ & - \int_t^T \hat{c}_2(s)\, \frac{\partial Q}{\partial s}(s, y(s) - y_1(s))\, ds \\ & + \int_t^T \hat{c}_2(s)\ c_1(s)\, \big[y(s) - y_1(s)\big]\, \frac{\partial Q}{\partial y}\,(s, y(s) - y_1(s))\, ds \qquad (3.4.14) \\ & + \int_t^T \hat{c}_2(s)\ \left[\frac{1}{2}\, c_2(s)\, |z(s)|^2 - \hat{c}_2(s)\, |z_1(s)|^2\right]\ \frac{\partial Q}{\partial y}\,(s, y(s) - y_1(s))\, ds \\ & - \frac{1}{2} \int_t^T \hat{c}_2(s)\, |z(s) - z_1(s)|^2\, \frac{\partial^2 Q}{\partial y^2}(s, y(s) - y_1(s))\, ds \\ & - \int_t^T \hat{c}_2(s)\, \frac{\partial Q}{\partial y}(s, y(s) - y_1(s))\ \big[z(s) - z_1(s)\big]'\, dW(s).\end{aligned}$$

We have:

$$\begin{aligned}\hat{c}_2(t)\, c_2(t)\, |z(t)|^2 &= \hat{c}_2(t)\, c_2(t)\, |z(t) - z_1(t) + z_1(t)|^2 \\ &\leq \hat{c}_2(t)\ c_2(t)\, \big[2|z(t) - z_1(t)|^2 + 2|z_1(t)|^2\big].\end{aligned}$$

By this, and since $y(T) - y_1(T) \leq 0$ implies $\hat{H}(T, y(T) - y_1(T)) = 0$, then (3.4.14) becomes:

$$
\begin{aligned}
&\hat{H}(t, y(t) - y_1(t)) \\
\leq \quad & -\int_t^T a(s)\,\hat{c}_2(s)\,Q(s, y(s) - y_1(s))\,ds - \int_t^T \hat{c}_2(s)\,\frac{\partial Q}{\partial s}(s, y(s) - y_1(s))\,ds \\
& + \int_t^T \hat{c}_2(s)\,c_1(s)\,\big[y(s) - y_1(s)\big]\frac{\partial Q}{\partial y}\,(s, y(s) - y_1(s))\,ds \\
& + \int_t^T \hat{c}_2(s)\,\left[\,c_2(s)\,|z(s) - z_1(s)|^2 + c_2(s)\,|z_1(s)|^2 - \hat{c}_2(s)\,|z_1(s)|^2\right] \\
& \times \frac{\partial Q}{\partial y}(s, y(s) - y_1(s))ds \qquad\qquad (3.4.15) \\
& - \frac{1}{2}\int_t^T \hat{c}_2(s)\,\big|z(s) - z_1(s)\big|^2\,\frac{\partial^2 Q}{\partial y^2}(s, y(s) - y_1(s))\,ds \\
& - \int_t^T \hat{c}_2(s)\,\frac{\partial Q}{\partial y}(s, y(s) - y_1(s))\;\;\big[z(s) - z_1(s)\big]'\,dW(s).
\end{aligned}
$$

As $\hat{c}_2(t)\,\big[c_2(t) - \hat{c}_2(t)\big]\big|z_1(t)\big|^2\,\frac{\partial Q}{\partial y}\,(t, y(t) - y_1(t)) \leq 0$, and by (iv) in Lemma 3.4.3, (3.4.15) becomes:

$$
\begin{aligned}
&\hat{H}(t, y(t) - y_1(t)) \leq -\int_t^T a(s)\,\hat{c}_2(s)\,Q(s, y(s) - y_1(s))ds \\
& + \int_t^T \hat{c}_2(s)\left[c_1(s)\,\big[y(s) - y_1(s)\big]\,\frac{\partial Q}{\partial y}\,(s, y(s) - y_1(s)) - \frac{\partial Q}{\partial s}(s, y(s) - y_1(s))\right]ds \\
& - \int_t^T \hat{c}_2(s)\,\frac{\partial Q}{\partial y}(s, y(s) - y_1(s))\;\;\big[z(s) - z_1(s)\big]'\,dW(s). \qquad\qquad (3.4.16)
\end{aligned}
$$

By (iii) in Lemma 3.4.3, (3.4.16) becomes:

$$\begin{aligned}\hat{H}(t, y(t) - y_1(t)) \quad \leq \quad & -\int_t^T a(s)\, \hat{c}_2(s)\, Q(s, y(s) - y_1(s))\, ds \\ & + \int_t^T \hat{c}_2(s) \left[y(s) - y_1(s)\right] \frac{\partial Q}{\partial y}(s, y(s) - y_1(s)) \left[c_1(s) - a(s)\right] ds \\ & - \int_t^T \hat{c}_2(s)\, \frac{\partial Q}{\partial y}(s, y(s) - y_1(s)) \;\; \left[z(s) - z_1(s)\right]' dW(s).\end{aligned} \tag{3.4.17}$$

Since

$$\int_t^T \hat{c}_2(s) \left[y(s) - y_1(s)\right] \frac{\partial Q}{\partial y}(s, y(s) - y_1(s)) \left[c_1(s) - a(s)\right] ds \leq 0,$$

and $-a(t)\, \hat{c}_2(t)\, Q(t, y(t) - y_1(t)) \leq 0$, thus (3.4.17) becomes:

$$\hat{H}(t, y(t) - y_1(t)) \leq -\int_t^T \hat{c}_2(s) \frac{\partial Q}{\partial y}(s, y(s) - y_1(s)) \left[z(s) - z_1(s)\right]' dW(s). \tag{3.4.18}$$

We have:

$$\begin{aligned}\mathbb{E} \left[\int_0^T \phi(s) \left|z(s) - z_1(s)\right|^2 ds\right] \quad & \leq \quad \mathbb{E} \left[\int_0^T \phi(s) \left(|z(s)| + |z_1(s)|\right)^2 ds\right] \\ & \leq \quad 2\, \mathbb{E} \left[\int_0^T \phi(s)\, |z(s)|^2\, ds\right] \\ & + \quad 2\, \mathbb{E} \left[\int_0^T \phi(s)\, |z_1(s)|^2\, ds\right] < \infty.\end{aligned}$$

This, together with Lemma 3.4.4, implies that the process

$$\int_t^T \hat{c}_2(s) \frac{\partial Q}{\partial y}(s, y(s) - y_1(s)) \left[z(s) - z_1(s)\right]' dW(s)$$

is a martingale. By taking expectation of (3.4.18), we obtain:

$$\mathbb{E}\,\hat{H}(t, y(t) - y_1(t)) \leq \; - \mathbb{E} \int_t^T \hat{c}_2(s) \frac{\partial Q}{\partial y}(s, y(s) - y_1(s)) \big[z(s) - z_1(s)\big]' dW(s).$$

Thus, $\mathbb{E}\,\hat{H}(t, y(t) - y_1(t)) = 0$ for all $t \in [0, T]$. Since $\hat{H}(t, y(t) - y_1(t)) \geq 0$, then $\hat{H}(t, y(t) - y_1(t)) = 0$ for all $t \in [0, T]$ *a.s.*. By the definition of $\hat{H}$, $y(t) \leq y_1(t)$ for all $t \in [0, T]$ *a.s.*.

(ii) Here we compare equation (3.4.9) with equation (3.4.11). In Lemma 3.4.2, we proved that equation (3.4.11) has a unique solution. Our aim is to prove that $y(\cdot) \geq y_2(\cdot)$. Set $M := \sup_{\omega \in \Omega} (|y| + |y_2|)$. We have:

$$\begin{aligned}
d\hat{H}(t, y_2(t) - y(t)) \;&=\; d\big(\hat{c}_2(t)\, Q(t, y_2(t) - y(t))\big) \\
&=\; d(\hat{c}_2(t))\; Q(t, y_2(t) - y(t)) + \hat{c}_2(t)\; dQ(t, y_2(t) - y(t)) \\
&=\; a(t)\, \hat{c}_2(t)\; Q(t, y_2(t) - y(t))\, dt \\
&\quad + \hat{c}_2(t) \left[\frac{\partial Q}{\partial t}(t, y_2(t) - y(t))\, dt \right. \\
&\quad - \big[G_2(t, y_2(t), z_2(t)) - F(t, y(t), z(t))\big] \; \frac{\partial Q}{\partial y}(t, y_2(t) - y(t))\, dt \\
&\quad + \frac{1}{2} \; \frac{\partial^2 Q}{\partial y^2}(t, y_2(t) - y(t))\, |z(t) - z_2(t)|^2 \; dt \\
&\quad \left. + \frac{\partial Q}{\partial y}(t, y_2(t) - y(t)) \; \big[z_2(t) - z(t)\big]' \, dW(t) \right].
\end{aligned} \tag{3.4.19}$$

By the expressions of F and G_2, we have:

$$
\begin{aligned}
G_2(t,y_2,z_2)-F(t,y,z) &= G_2(t,y_2,z_2)-c_1(t)\,y-F_0(t,y,z)\\
&\le a(t)\,y_2-\hat{c}_0(t)-\hat{c}_2(t)\,|z_2|^2-c_1(t)\,y+c_0(t)\;+\frac{1}{2}\,c_2(t)\,|z|^2\\
&\le a(t)\,y_2-\hat{c}_2(t)\,|z_2|^2-c_1(t)\,y+\frac{1}{2}\,c_2(t)\,|z|^2.
\end{aligned}
$$

Substituting this in (3.4.19), and by integrating from t to T, we obtain:

$$
\begin{aligned}
\hat{H}(t,y_2(t)-y(t)) \;\le\; & \hat{H}(T,y_2(T)-y(T))-\int_t^T a(s)\,\hat{c}_2(s)\,Q(s,y_2(s)-y(s)))\,ds\\
- & \int_t^T \hat{c}_2(s)\,\frac{\partial Q}{\partial s}(s,y_2(s)-y(s))\,ds\\
+ & \int_t^T \hat{c}_2(s)\,\big[-c_1(s)\,y(s)+\tfrac{1}{2}\,c_2(s)\,|z(s)|^2+a(s)y_2(s)-\hat{c}_2(s)|z_2(s)|^2\big]\\
& \times\frac{\partial Q}{\partial y}\,(s,y_2(s)-y(s))\,ds \qquad (3.4.20)\\
- & \frac{1}{2}\int_t^T \hat{c}_2(s)\,|z_2(s)-z(s)|^2\,\frac{\partial^2 Q}{\partial y^2}(s,y_2(s)-y(s))\,ds\\
- & \int_t^T \hat{c}_2(s)\,\frac{\partial Q}{\partial y}(s,y_2(s)-y(s))\big[z_2(s)-z(s)\big]'\,dW(s).
\end{aligned}
$$

Since $c_1(t)\le a(t)$ and $y_2(t)\le 0$, we have:

$$\hat{c}_2(t)\,\left[a(t)\,y_2(t)-c_1(t)\,y(t)\right]\le c_1(t)\,\big[y_2(t)-y(t)\big]\;\hat{c}_2(t).$$

By substituting this in (3.4.20), we obtain:

$$\hat{H}(t,y_2(t)-y(t)) \;\le\; \hat{H}(T,y_2(T)-y(T))-\int_t^T a(s)\,\hat{c}_2(s)\,Q(s,y_2(s)-y(s))\,ds$$

$$
\begin{aligned}
&- \int_t^T \hat{c}_2(s)\, \frac{\partial Q}{\partial s}(s, y_2(s) - y(s))\, ds \\
&+ \int_t^T \hat{c}_2(s)\ c_1(s)\, \big[y_2(s) - y(s)\big]\ \frac{\partial Q}{\partial y}\,(s, y_2(s) - y(s))\, ds \\
&+ \int_t^T \hat{c}_2(s) \big[\tfrac{1}{2}\, c_2(s)\, |z(s)|^2 - \hat{c}_2(s)\, |z_2(s)|^2\big] \\
&\quad \times \frac{\partial Q}{\partial y}\,(s, y_2(s) - y(s))\, ds \qquad (3.4.21) \\
&- \frac{1}{2} \int_t^T \hat{c}_2(s)\, |z_2(s) - z(s)|^2\ \frac{\partial^2 Q}{\partial y^2}(s, y_2(s) - y(s))\, ds \\
&- \int_t^T \hat{c}_2(s)\, \frac{\partial Q}{\partial y}(s, y_2(s) - y(s))\ \big[z_2(s) - z(s)\big]'\, dW(s).
\end{aligned}
$$

We have:

$$
\begin{aligned}
\hat{c}_2(t)\, c_2(t)\, |z(t)|^2 &= \hat{c}_2(t)\ c_2(t)\, |z(t) - z_2(t) + z_2(t)|^2 \\
&\le \hat{c}_2(t)\ c_2(t)\, \big[2|z_2(t) - z(t)|^2 + 2|z_2(t)|^2\big].
\end{aligned}
$$

Putting this into (3.4.21), and since $y_2(T) - y(T) \le 0$, then $\hat{H}(T, y_2(T) - y(T)) = 0$, we obtain:

$$
\begin{aligned}
\hat{H}(t, y_2(t) - y(t)) \ \le\ & -\int_t^T a(s)\, \hat{c}_2(s)\, Q(s, y_2(s) - y(s))\, ds \\
& -\int_t^T \hat{c}_2(s)\, \frac{\partial Q}{\partial s}(s, y_2(s) - y(s))\, ds \\
& +\int_t^T \hat{c}_2(s)\ c_1(s)\, \big[y_2(s) - y(s)\big]\ \frac{\partial Q}{\partial y}\,(s, y_2(s) - y(s))\, ds \\
& +\int_t^T \hat{c}_2(s)\ \big[\, c_2(s)\, |z_2(s) - z(s)|^2 +\ c_2(s)\ |z_2(s)|^2 - \hat{c}_2(s)\, |z_2(s)|^2\big] \\
& \times \frac{\partial Q}{\partial y}\,(s, y_2(s) - y(s))\, ds
\end{aligned}
$$

$$-\frac{1}{2}\int_t^T \hat{c}_2(s)\,\big|z_2(s)-z(s)\big|^2\,\frac{\partial^2 Q}{\partial y^2}(s,y_2(s)-y(s))\,ds \qquad (3.4.22)$$

$$-\int_t^T \hat{c}_2(s)\,\frac{\partial Q}{\partial y}(s,y_2(s)-y(s))\;\big[z_2(s)-z(s)\big]'\,dW(s).$$

Since $\hat{c}_2(t)\big[c_2(t)-\hat{c}_2(t)\big]\big|z_2(t)\big|^2 \le 0$, and by (iv) in Lemma 3.4.3, then (3.4.22) becomes:

$$\begin{aligned}\hat{H}(t,y_2(t)-y(t)) \;\le\; & -\int_t^T a(s)\,\hat{c}_2(s)\,Q(s,y_2(s)-y(s))\,ds \\ & + \int_t^T \hat{c}_2(s)\Big[c_1(s)\big[y_2(s)-y(s)\big]\frac{\partial Q}{\partial y}(s,y_2(s)-y(s)) - \frac{\partial Q}{\partial s}(s,y_2(s)-y(s))\Big]\,ds \\ & - \int_t^T \hat{c}_2(s)\frac{\partial Q}{\partial y}(s,y_2(s)-y(s))\big[z_2(s)-z(s)\big]'\,dW(s). \qquad (3.4.23)\end{aligned}$$

Moreover, by (iii) in Lemma 3.4.3, (3.4.23) becomes:

$$\begin{aligned}\hat{H}(t,y_2(t)-y(t)) \;\le\; & -\int_t^T a(s)\,\hat{c}_2(s)\,Q(s,y_2(s)-y(s)))\,ds \qquad (3.4.24) \\ & +\int_t^T \hat{c}_2(s)\,\big[y_2(s)-y(s)\big]\,\frac{\partial Q}{\partial y}(s,y_2(s)-y(s))\,\big[c_1(s)-a(s)\big]\,ds \\ & -\int_t^T \hat{c}_2(s)\,\frac{\partial Q}{\partial y}(s,y_2(s)-y(s))\;\big[z_2(s)-z(s)\big]'\,dW(s).\end{aligned}$$

Since

$$\int_t^T \hat{c}_2(s)\,\big[y_2(s)-y(s)\big]\,\frac{\partial Q}{\partial y}(s,y_2(s)-y(s))\,\big[c_1(s)-a(s)\big]\,ds \le 0,$$

and $-a(t)\,\hat{c}_2(t)\,Q(t,y_2(t)-y(t))) \le 0$, then (3.4.24) becomes:

$$\hat{H}(t,y_2(t)-y(t)) \le -\int_t^T \hat{c}_2(s)\frac{\partial Q}{\partial y}(s,y_2(s)-y(s))\big[z_2(s)-z(s)\big]'\,dW(s). \qquad (3.4.25)$$

By Lemma 3.4.4, the stochastic integral in (3.4.25) is a martingale. Thus, taking expectation gives $\mathbb{E}\ \hat{H}(t, y_2(t) - y(t)) = 0$ for all $t \in [0,T]$. Since $\hat{H}(t, y_2(t) - y(t) \geq 0$, we have $\hat{H}(t, y_2(t) - y(t)) = 0$ for all $t \in [0,T]$ *a.s.*. By the definition of $\hat{H}$, $y_2(t) \leq y(t)$ for all $t \in [0,T]$ *a.s.*. $\square$

In the next lemma, we obtain an upper bound of $z(\cdot)$ in the space $\mathcal{M}^2_\phi(0,T;\mathbb{R}^d)$. We say that the pair (F, ξ) satisfies conditions H_1 if:

(i) conditions $H(1)$-(2) hold;

(ii) $\xi \in \mathcal{L}^\infty(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$;

(iii) there exists a positive constant M such that $\mathbb{E}\big[\, p(T)\ e^{8\hat{c}_2(T)\mu}\big] < \infty$, for $\mu := \max(M, \sup_{\omega\in\Omega} \xi)$;

(iv) $\hat{c}_2(T)\ \int_0^T\ |c_1(t)|\, dt \leq e^{2\hat{c}_2(T)\,\mu}$;

(v) $\hat{c}_2(T)\ \int_0^T\ c_0(t)\, dt \leq e^{2\hat{c}_2(T)\,\mu}$.

Before we give Lemma 3.4.7, we present the following useful lemma. The proof is given in the appendix to this chapter.

Lemma 3.4.6. *Let the random function ψ be defined as:*

$$\psi(t,x) := \frac{2\phi(t)}{\hat{c}_2^2(t)}\ \left[e^{\hat{c}_2(t)(x+M)} - 1 - \hat{c}_2(t)(x+M)\right],\quad (t,x) \in [0,T] \times [-M,M].$$

Then, the following properties hold a.s. for all $(t,x) \in [0,T] \times [-M,M]$:

(i) $\psi(t,x) \geq 0$,

(ii) $\dfrac{\partial \psi}{\partial x}(t,x) \geq 0$,

(iii) $\dfrac{\partial \psi}{\partial t}(t,x) \geq 0$,

(iv) $\dfrac{1}{2}\,\hat{c}_2(t)\ \dfrac{\partial \psi}{\partial x}(t,x) - \dfrac{1}{2}\ \dfrac{\partial^2 \psi}{\partial x^2}(t,x)\ = -\phi(t)$.

Lemma 3.4.7. *Let conditions H_1 hold and $(y(\cdot), z(\cdot)) \in \mathcal{L}^\infty_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{M}^2_\phi(0,T;\mathbb{R}^d)$ be a solution of equation* (3.2.1) *with (F, ξ). Then, there exists a constant b depends on $\gamma(t)$, β, $\hat{c}_0(t)$, $\hat{c}_2(t)$ such that*

$$\mathbb{E} \int_0^T \phi(t)\, |z(t)|^2\, dt \leq b.$$

Proof. The idea of the proof is similar to that of Proposition 2.1. in [37]. Let $M := \sup_{\omega \in \Omega} |y|$. We apply the Itô's formula to $\psi(t, y(t))$ as follows:

$$\begin{aligned} d\psi(t,y(t)) &= \frac{\partial \psi}{\partial t}(t,y(t))\, dt - \big[c_1(t)\, y(t) + F_0(t,y(t),z(t))\big] \frac{\partial \psi}{\partial y}(t,y(t))\, dt \\ &+ \frac{1}{2}\, |z(t)|^2\, \frac{\partial^2 \psi}{\partial y^2}(t,y(t))\, dt + \frac{\partial \psi}{\partial y}(t,y(t))\; z'(t)\, dW(t). \end{aligned}$$

By integrating from 0 to T, we obtain:

$$\begin{aligned} \psi(0,y(0)) &= \psi(T,y(T)) - \int_0^T \frac{\partial \psi}{\partial t}(t,y(t))\, dt \\ &+ \int_0^T \big[c_1(t)\, y(t) + F_0(t,y(t),z(t))\big] \frac{\partial \psi}{\partial y}(t,y(t))\, dt \qquad (3.4.26) \\ &- \frac{1}{2} \int_0^T |z(t)|^2\, \frac{\partial^2 \psi}{\partial y^2}(t,y(t))\, dt - \int_0^T \frac{\partial \psi}{\partial y}(t,y(t))\; z'(t)\, dW(t). \end{aligned}$$

By (i)-(iii) in Lemma 3.4.6, and substituting the expression for F_0, equation (3.4.26) becomes:

$$\begin{aligned} 0 \leq \psi(0,y(0)) &\leq \psi(T,y(T)) + \int_0^T \big[c_1(t)\, y(t) + c_0(t)\big] \frac{\partial \psi}{\partial y}(t,y(t))\, dt \\ &+ \int_0^T \left[\frac{1}{2}\, c_2(t) \frac{\partial \psi}{\partial y}(t,y(t)) - \frac{1}{2}\, \frac{\partial^2 \psi}{\partial y^2}(t,y(t))\right] |z(t)|^2\, dt \\ &- \int_0^T \frac{\partial \psi}{\partial y}(t,y(t))\; z'(t)\, dW(t). \qquad (3.4.27) \end{aligned}$$

We next prove that the stochastic integral in (3.4.27) is a martingale. By the

Burkholder-Davis-Gundy inequality, there exists a constant K such that:

$$
\begin{aligned}
& \mathbb{E}\;\left[\sup_{t\in[0,T]}\left|\int_0^t \frac{\partial\psi}{\partial y}(s,y(s))\;z'(s)\,dW(s)\right|\right] \\
\leq\quad & K\;\mathbb{E}\;\left[\int_0^T \left|\frac{\partial\psi}{\partial y}(s,y(s))\right|^2\;|z(s)|^2\;ds\right]^{\frac{1}{2}} \\
=\quad & 4\,K\;\mathbb{E}\;\left[\int_0^T \frac{\phi^2(s)}{\hat{c}_2^2(s)}\left[e^{\hat{c}_2(s)(y(s)+M)}-1\right]^2\;|z(s)|^2\;ds\right]^{\frac{1}{2}} \\
\leq\quad & 2K\;\mathbb{E}\;\left[\sup_{t\in[0,T]}\frac{\phi(t)}{\hat{c}_2^2(t)}\left[e^{\hat{c}_2(t)(y(t)+M)}-1\right]^2+\int_0^T\phi(s)\,|z(s)|^2\,ds\right] \\
\leq\quad & 2K\;\mathbb{E}\;\left[\sup_{t\in[0,T]}\frac{\phi(t)}{\hat{c}_2^2(t)}\left[2e^{2\hat{c}_2(t)(y(t)+M)}+2\right]+\int_0^T\phi(s)\,|z(s)|^2\,ds\right] \\
\leq\quad & 2K\;\mathbb{E}\;\left[\sup_{t\in[0,T]}p(t)\,e^{4\hat{c}_2(t)M}\left[2e^{4\hat{c}_2(t)\,M}+2\right]+\int_0^T\phi(s)\,|z(s)|^2\,ds\right] \\
\leq\quad & 2K\;\mathbb{E}\;\left[p(T)\,e^{4\hat{c}_2(T)M}\left[2e^{4\hat{c}_2(T)\,M}+2\right]+\int_0^T\phi(s)\,|z(s)|^2\,ds\right] \\
\leq\quad & 2K\;\mathbb{E}\;\left[4\,p(T)\,e^{8\hat{c}_2(T)\,M}+\int_0^T\phi(s)\,|z(s)|^2\,ds\right]<\infty.
\end{aligned}
$$

The last step follows from condition H_1(iii) and from $z(\cdot)\in\mathcal{M}_\phi^2(0,T;\mathbb{R}^d)$. Thus, by Theorem 2.4.6, the stochastic integral in (3.4.27) is a martingale. Moreover, by *(iv)* in Lemma 3.4.6, (3.4.27) becomes:

$$
\begin{aligned}
\int_0^T\phi(t)\;|z(t)|^2\,dt \quad\leq\quad & \psi(T,y(T))+\int_0^T\left[c_1(t)\,y(t)+c_0(t)\right]\frac{\partial\psi}{\partial y}(t,y(t))\,dt \\
- \quad & \int_0^T\frac{\partial\psi}{\partial y}(t,y(t))\;z'(t)\,dW(t).
\end{aligned}
\tag{3.4.28}
$$

Hence, by taking expectation of (3.4.28), we obtain:

$$
\begin{aligned}
& \mathbb{E} \int_0^T \phi(t)\, |z(t)|^2 \, dt \\
\leq \;& \mathbb{E}\, \psi(T, y(T)) + \mathbb{E} \left[\int_0^T \left[c_1(t)\, y(t) + c_0(t) \right] \frac{\partial Q}{\partial y}(t, y(t))\, dt \right] \\
\leq \;& 2\, \mathbb{E} \left[p(T)\, e^{4 \hat{c}_2(T) M}\, e^{\hat{c}_2(T)(\xi + M)} \right] + 2\, \mathbb{E} \left[\int_0^T c_1(t)\, y(t)\, \frac{\phi(t)}{\hat{c}_2(t)}\, e^{\hat{c}_2(t)(y(t)+M)}\, dt \right] \\
& + 2\, \mathbb{E} \left[\int_0^T c_0(t)\, \frac{\phi(t)}{\hat{c}_2(t)}\, e^{\hat{c}_2(t)(y(t)+M)} \right] dt \\
\leq \;& 2\, \mathbb{E} \left[p(T)\, e^{4 \hat{c}_2(T) M}\, e^{2 \hat{c}_2(T) \mu} \right] + 2\, \mathbb{E} \left[\int_0^T \left| c_1(t) y(t) \right| \hat{c}_2(t)\, p(t)\, e^{4 \hat{c}_2(t) M}\, e^{2 \hat{c}_2(t) M}\, dt \right] \\
& + 2\, \mathbb{E} \left[\int_0^T c_0(t)\, \hat{c}_2(t)\, p(t)\, e^{4 \hat{c}_2(t) M}\, e^{2 \hat{c}_2(t) M}\, dt \right] \\
\leq \;& 2\, \mathbb{E} \left[p(T)\, e^{6 \hat{c}_2(T) \mu} \right] + 2\, M\, \mathbb{E} \left[\hat{c}_2(T)\, p(T)\, e^{6 \hat{c}_2(T) M} \int_0^T |c_1(t)|\, dt \right] \\
& + 2\, \mathbb{E} \left[\hat{c}_2(T)\, p(T)\, e^{6 \hat{c}_2(T) M} \int_0^T c_0(t)\, dt \right] \\
\leq \;& 2\, \mathbb{E} \left[p(T)\, e^{6 \hat{c}_2(T) \mu} \right] + 2\, M\, \mathbb{E} \left[p(T)\, e^{8 \hat{c}_2(T) M} \right] + 2\, \mathbb{E} \left[p(T)\, e^{8 \hat{c}_2(T) M} \right],
\end{aligned}
$$

where the last step follows from conditions H_1(iv) and H_1(v). □

3.5 Monotonicity

Before we give the monotonicity theorem, we present the following useful lemma. The lemma proof is given in the appendix to this chapter.

Lemma 3.5.1. *Let the random function ψ_1 be defined as:*

$$\psi_1(t,x) := \frac{\varphi(t)}{8\hat{c}_2(t)}\big(e^{16\,\hat{c}_2(t)x} - 16\,\hat{c}_2(t)x - 1\big), \quad (t,x) \in [0,T] \times [0,2M].$$

Then, the following properties hold:

(i) $\psi_1(0,0) = \psi_1(T,0) = 0 \quad$ *a.s.,*

(ii) $\dfrac{\partial \psi_1}{\partial x}(t,x) \geq 0 \quad$ *a.s. for all* $(t,x) \in [0,T] \times [0,2M]$,

(iii) $\dfrac{\partial \psi_1}{\partial x}(t,0) = \dfrac{\partial \psi_1}{\partial t}(t,0) = 0$ *a.s. for all* $t \in [0,T]$.

Theorem 3.5.2. *Let the pair (F,ξ) satisfies conditions H. Also, let the sequence of pairs $\{(F^n,\xi^n)\}_{n\geq 0}$ satisfy the following assumptions:*

(1) if $\lim_{n\to\infty} y^n = y$ *and* $\lim_{n\to\infty} z^n = z$*, then* $\lim_{n\to\infty} F^n(t,y^n,z^n) = F(t,y,z)$ *a.e.* $t \in [0,T]$ *a.s.,*

(2) $|F^n(t,y,z)| \leq c_0(t) + c_2(t)\,|z|^2$*, for all* $y \in \mathbb{R}$*,* $z \in \mathbb{R}^d$*, a.e.* $t \in [0,T]$ *a.s.,*

(3) $\xi^n \in \mathcal{L}^\infty(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R})$*, and* $\lim_{n\to\infty} \xi^n = \xi$ *a.s.,*

(4) for all n, the BSDE with (F^n,ξ^n) has a solution pair $(y^n(\cdot),z^n(\cdot))$ $\in \mathcal{L}^\infty_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{M}^2_\phi(0,T;\mathbb{R}^d)$ such that the sequence $\{y^n(t)\}_{n\geq 0}$ is decreasing a.s. for all $t \in [0,T]$.

There exists a pair $(y(\cdot),z(\cdot)) \in \mathcal{L}^\infty_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{M}^2_\phi(0,T;\mathbb{R}^d)$ such that:

(i) the sequence $\{z^n(\cdot)\}_{n\geq 0}$ converges to $z(\cdot)$ in $\mathcal{M}^2_\varphi(0,T;\mathbb{R}^d)$,

(ii) there exists a subsequence of $\{y^n(\cdot)\}_{n\geq 0}$ that converges uniformly in t to $y(\cdot)$,

(iii) $(y(\cdot),z(\cdot))$ is a solution pair to equation (3.2.1) *with (F,ξ).*

Proof. The proof follows that of Proposition 2.4. in [37]. However, here the proof is more difficult since we show this result with possibly unbounded coefficients. Also, we prove the convergence of $\{z^n(\cdot)\}_{n\geq 0}$ in a weighted space. Since the sequence $\{y^n(t)\}_{n\geq 0}$ is monotone and bounded for all $t \in [0,T]$ $a.s$, thus, by the monotone convergence theorem, see Theorem 2.3.3, $\{y^n(t)\}_{n\geq 0}$ converges to a process $y(t)$ for all $t \in [0,T]$ $a.s.$.

Part 1.

In this part, we prove that $\{z^n(\cdot)\}_{n\geq 0}$ converges in $\mathcal{M}^2_\varphi(0,T;\mathbb{R}^d)$. By Lemma 3.4.7, there exists a constant b such that:

$$\mathbb{E}\int_0^T \phi(t)\,|z^n(t)|^2\,dt \leq b$$

for all n. By Theorem 2.3.6, there exists a subsequence $\{z^{n_k}(\cdot)\}_{k\geq 0}$ of $\{z^n(\cdot)\}_{n\geq 0}$ and a process $z(\cdot)$ in $\mathcal{M}^2_\phi(0,T;\mathbb{R}^d)$ such that $\{z^{n_k}(\cdot)\}_{k\geq 0}$ converges weakly to $z(\cdot)$ in $\mathcal{M}^2_\phi(0,T;\mathbb{R}^d)$. We next show that the sequence $\{z^n(\cdot)\}_{n\geq 0}$ converges to $z(\cdot)$ in $\mathcal{M}^2_\varphi(0,T;\mathbb{R}^d)$. For any $0 \leq n \leq p$, we have

$$\begin{aligned} y^n(t) - y^p(t) &= \big(y^n(T) - y^p(T)\big) \\ &+ \int_t^T \big[F^n(s,y^n(s),z^n(s)) - F^p(s,y^p(s),z^p(s))\big]\,ds \\ &- \int_t^T \big(z^n(s) - z^p(s)\big)'\,dW(s), \end{aligned}$$

The differential of $\psi_1\big(t, y^n(t) - y^p(t)\big)$ is:

$$
\begin{aligned}
& d\psi_1\big(t, y^n(t) - y^p(t)\big) \\
= \;& \frac{\partial \psi_1}{\partial t}\big(t, y^n(t) - y^p(t)\big)\, dt \\
- \;& \frac{\partial \psi_1}{\partial y}\big(t, y^n(t) - y^p(t)\big)\ \big[F^n(t, y^n(t), z^n(t)) - F^p(t, y^p(t), z^p(t))\big]\, dt \\
+ \;& \frac{1}{2}\,\frac{\partial^2 \psi_1}{\partial y^2}\big(t, y^n(t) - y^p(t)\big)\, |z^n(t) - z^p(t)|^2\, dt \\
+ \;& \frac{\partial \psi_1}{\partial y}\big(t, y^n(t) - y^p(t)\big)\, \big(z^n(t) - z^p(t)\big)'\, dW(t).
\end{aligned}
$$

By integrating from 0 to T, we obtain:

$$
\begin{aligned}
& \psi_1\big(0, y^n(0) - y^p(0)\big) \\
= \;& \psi_1\big(T, y^n(T) - y^p(T)\big) - \int_0^T \frac{\partial \psi_1}{\partial t}\big(t, y^n(t) - y^p(t)\big)\, dt \\
+ \;& \int_0^T \frac{\partial \psi_1}{\partial y}\big(t, y^n(t) - y^p(t)\big)\ \big[F^n(t, y^n(t), z^n(t)) - F^p(t, y^p(t), z^p(t))\big]\, dt \\
- \;& \frac{1}{2}\int_0^T \frac{\partial^2 \psi_1}{\partial y^2}\big(t, y^n(t) - y^p(t)\big)\, |z^n(t) - z^p(t)|^2\, dt \qquad (3.5.1) \\
- \;& \int_0^T \frac{\partial \psi_1}{\partial y}\big(t, y^n(t) - y^p(t)\big)\big(z^n(t) - z^p(t)\big)'\, dW(t).
\end{aligned}
$$

We have

$$
\begin{aligned}
& |F^n(t, y^n, z^n) - F^p(t, y^p, z^p)| \le 2c_0(t) + c_2(t)\,|z^n|^2 + c_2(t)\,|z^p|^2 \\
& \le 2\, c_0(t) + c_2(t)\,\big[2\,|z^n - z|^2 + 2\,|z|^2\big] + c_2(t)\,\big[2|z^n - z^p|^2 + 2\,|z^n|^2\big]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \quad 2\,c_0(t) + 2\,c_2(t)\;|z^n - z|^2 + 2\,c_2(t)|z|^2 + 2\,c_2(t)\,|z^n - z^p|^2 \\
&\qquad +4\,c_2(t)|z^n - z|^2 + 4\,c_2(t)|z|^2 \qquad\qquad (3.5.2) \\
&= \quad 2\,c_0(t) + 2\,c_2(t)\,|z^n - z^p|^2 + 6\,c_2(t)\left[|z^n - z|^2 + |z|^2\right] \quad a.e.\;\; t \in [0,T]\;\; a.s..
\end{aligned}
$$

By (ii) in Lemma 3.5.1 and by substituting (3.5.2) into equation (3.5.1), we obtain:

$$
\begin{aligned}
&\psi_1\big(0, y^n(0) - y^p(0)\big) \\
\leq \quad & \psi_1\big(T, y^n(T) - y^p(T)\big) - \int_0^T \frac{\partial \psi_1}{\partial t}\big(t, y^n(t) - y^p(t)\big)\,dt \\
+ \quad & \int_0^T \frac{\partial \psi_1}{\partial y}\big(t, y^n(t) - y^p(t)\big) \\
& \times \left[2c_0(t) + 2c_2(t)|z^n(t) - z^p(t)|^2 + 6c_2(t)|z^n(t) - z(t)|^2 + 6c_2(t)|z(t)|^2\right] dt \\
- \quad & \frac{1}{2}\int_0^T \frac{\partial^2 \psi_1}{\partial y^2}\big(t, y^n(t) - y^p(t)\big)|z^n(t) - z^p(t)|^2\,dt \\
- \quad & \int_0^T \frac{\partial \psi_1}{\partial y}\big(t, y^n(t) - y^p(t)\big)\big(z^n(t) - z^p(t)\big)'\,dW(t).
\end{aligned}
$$

By moving the terms in $|z^n - z^p|^2$ and $|z^n - z|^2$ to the left-hand side of the inequality, we obtain:

$$
\begin{aligned}
&\int_0^T \big|z^n(t) - z^p(t)\big|^2 \left[\frac{1}{2}\frac{\partial^2 \psi_1}{\partial y^2}\big(t, y^n(t) - y^p(t)\big) - 2c_2(t)\,\frac{\partial \psi_1}{\partial y}\big(t, y^n(t) - y^p(t)\big)\right] dt \\
&-6\int_0^T \frac{\partial \psi_1}{\partial y}\big(t, y^n(t) - y^p(t)\big)\; c_2(t)\; |z^n(t) - z(t)|^2\, dt \\
&\qquad \le \quad \psi_1\big(T, y^n(T) - y^p(T)\big) - \;\psi\big(0, y^n(0) - y^p(0)\big) \\
&\qquad - \quad \int_0^T \frac{\partial \psi_1}{\partial t}\big(t, y^n(t) - y^p(t)\big)\, dt \\
&\qquad + \quad \int_0^T \frac{\partial \psi_1}{\partial y}\big(t, y^n(t) - y^p(t)\big)\big[2c_0(t) + 6\, c_2(t)\, |z(t)|^2\big]\, dt \qquad (3.5.3) \\
&\qquad - \quad \int_0^T \frac{\partial \psi_1}{\partial y}\big(t, y^n(t) - y^p(t)\big)\big(z^n(t) - z^p(t)\big)'\, dW(t).
\end{aligned}
$$

We next show that the stochastic integral on the right-hand side in the inequality (3.5.3) is a martingale. By the Burkholder-Davis-Gundy inequality, there exists a constant K such that:

$$
\begin{aligned}
&\mathbb{E}\,\left[\sup_{t\in[0,T]} \left|\int_0^t \frac{\partial \psi_1}{\partial y}(s, y^n(s) - y^p(s))\big[z^n(s) - z^p(s)\big]'\, dW(s)\right|\right] \qquad (3.5.4) \\
\le\;& 4\,K\,\mathbb{E}\,\left[\int_0^T \varphi^2(s)\,\big(e^{16\,\hat{c}_2(s)(y^n(s) - y^p(s))} - 1\big)^2 \big|z^n(s) - z^p(s)\big|^2 ds\right]^{\frac{1}{2}} \\
\le\;& 4\,K\,\mathbb{E}\,\left[\sup_{t\in[0,T]} \frac{\varphi^2(t)\,\big(e^{32\,\hat{c}_2(t)\,M} - 1\big)^2}{\phi(t)} \int_0^T \phi(s)\big|z^n(s) - z^p(s)\big|^2\, ds\right]^{\frac{1}{2}} \\
\le\;& 2\,K\,\mathbb{E}\,\left[\sup_{t\in[0,T]} \frac{\varphi^2(t)\,\big(e^{32\,\hat{c}_2(t)\,M} - 1\big)^2}{\phi(t)} + \int_0^T \phi(s)\big|z^n(s) - z^p(s)\big|^2\, ds\right].
\end{aligned}
$$

We have

$$\begin{aligned}
\varphi^2(t)\left(e^{32\,\hat{c}_2(t)\,M}-1\right)^2 &\leq 2\varphi^2(t)\left(e^{64\,\hat{c}_2(t)\,M}+1\right)\\
&\leq 4\,\varphi^2(t)\,e^{64\,\hat{c}_2(t)\,M}\\
&\leq 4\,\varphi^2(t)\,e^{32\,\hat{c}_2^2(t)}\,e^{32M^2}\\
&= 4\,\varphi^2(t)\,e^{32M^2}\,e^{32[\hat{c}_2^2(0)+\int_0^t 2\,a(s)\,\hat{c}_2^2(s)\,ds]}.
\end{aligned}$$

Therefore,

$$\mathbb{E}\left[\sup_{t\in[0,T]}\frac{\varphi^2(t)\left(e^{32\,\hat{c}_2(t)\,M}-1\right)^2}{\phi(t)}\right] \leq \mathbb{E}\left[\sup_{t\in[0,T]}\frac{4\,\varphi^2(t)\,e^{32M^2}\,e^{32[\hat{c}_2^2(0)+\int_0^t 2\,a(s)\,\hat{c}_2^2(s)\,ds]}}{\hat{c}_2^2(t)\,e^{4\hat{c}_2(t)\,M}\,e^{\int_0^t(\gamma(s)+4\beta\,\hat{c}_0^2(s)\,\hat{c}_2^2(s))\,ds}}\right]$$

$$= 4\,e^{32M^2}\,e^{32\,\hat{c}_2^2(0)}\,\mathbb{E}\left[\sup_{t\in[0,T]}\frac{\varphi^2(t)\,e^{64\int_0^t a(s)\,\hat{c}_2^2(s)\,ds}}{\hat{c}_2^2(t)\,e^{4\hat{c}_2(t)\,M}\,e^{\int_0^t(\gamma(s)+4\beta\,\hat{c}_0^2(s)\,\hat{c}_2^2(s))\,ds}}\right] < \infty,$$

where the last step follows from $\varphi(t) \leq \hat{c}_2(t)$ and from $64\,a(t)\,\hat{c}_2^2(t) \leq \gamma(t) + 4\beta\,\hat{c}_0^2(t)\,\hat{c}_2^2(t)$. By this, and since $z^p(\cdot)$, $z^n(\cdot) \in \mathcal{M}_\phi^2(0,T;\mathbb{R}^d)$, the expectation in (3.5.4) is finite. Hence by Theorem 2.4.6, the stochastic integral in (3.5.3) is a martingale. By taking expectation of (3.5.3), we obtain:

$$\begin{aligned}
&\mathbb{E}\int_0^T \left|z^n(t)-z^p(t)\right|^2\left[\frac{1}{2}\frac{\partial^2\psi_1}{\partial y^2}\big(t,y^n(t)-y^p(t)\big)-2c_2(t)\,\frac{\partial\psi_1}{\partial y}\big(t,y^n(t)-y^p(t)\big)\right]dt\\
&-6\,\mathbb{E}\int_0^T \frac{\partial\psi_1}{\partial y}\big(t,y^n(t)-y^p(t)\big)\,c_2(t)\,|z^n(t)-z(t)|^2\,dt\\
\leq\;&\mathbb{E}\left[\psi_1\big(T,y^n(T)-y^p(T)\big)\right]-\mathbb{E}\left[\psi_1\big(0,y^n(0)-y^p(0)\big)\right] \qquad (3.5.5)\\
&-\mathbb{E}\int_0^T \frac{\partial\psi_1}{\partial t}\big(t,y^n(t)-y^p(t)\big)\,dt\\
&+\mathbb{E}\int_0^T \frac{\partial\psi_1}{\partial y}\big(t,y^n(t)-y^p(t)\big)\left[2c_0(t)+6\,c_2(t)\,|z(t)|^2\right]dt.
\end{aligned}$$

Before we take the limit as p goes to ∞ along the subsequence $\{n_k\}_{k\geq 0}$, we study the applicability of the dominated convergence theorem for (3.5.5). Firstly, by the continuity of ψ_1 and since $\lim_{p\to\infty} y^p(t) = y(t)\ a.s.$ for all $t \in [0,T]$, we have:

$$\lim_{p\to\infty} \mathbb{E}\,\psi_1(0, y^n(0) - y^p(0)) = \lim_{p\to\infty} \psi_1(0, y^n(0) - y^p(0)) = \psi_1(0, y^n(0) - y(0)).$$

Also, we have:

$$\begin{aligned}
\mathbb{E}\,\psi_1(T, y^n(T) - y^p(T)) &\leq \frac{1}{8}\,\mathbb{E}\left[\frac{\varphi(T)}{\hat{c}_2(T)}\, e^{32\,\hat{c}_2(T)M}\right] \\
&\leq \frac{1}{8}\, e^{16M^2}\,\mathbb{E}\left[e^{2\hat{c}_2(T)M}\, e^{16\,\hat{c}_2^2(T)}\right] \\
&= \frac{1}{8}\, e^{16M^2}\, e^{16\,\hat{c}_2^2(0)}\,\mathbb{E}\left[e^{2\hat{c}_2(T)\,M}\, e^{32\,\int_0^T a(s)\,\hat{c}_2^2(s)\,ds}\right] \\
&\leq \frac{1}{8}\, e^{16\,M^2}\, e^{16\,\hat{c}_2^2(0)}\,\mathbb{E}\left[e^{2\hat{c}_2(T)M}\, p(T)\right] < \infty.
\end{aligned}$$

Thus,

$$\lim_{p\to\infty} \mathbb{E}\,\psi_1(T, y^n(T) - y^p(T)) = \mathbb{E}\,\psi_1(T, y^n(T) - y(T)).$$

In terms of the last part of the right-hand side in (3.5.5), we have:

$$\begin{aligned}
& \mathbb{E}\int_0^T \frac{\partial \psi_1}{\partial y}\big(t, y^n(t) - y^p(t)\big)\big[2\,c_0(t) + 6\,c_2(t)\,|z(t)|^2\big]\,dt \\
\leq \quad & 4\,\mathbb{E}\int_0^T \varphi(t)\, e^{32\,\hat{c}_2(t)\,M}\, c_0(t)\,dt + 12\,\mathbb{E}\int_0^T \varphi(t)\, c_2(t)\, e^{32\,\hat{c}_2(t)\,M}\,|z(t)|^2\,dt,
\end{aligned}$$

where:

$$\mathbb{E}\int_0^T \varphi(t)\, c_2(t)\, e^{32\,\hat c_2(t)\,M}\, |z(t)|^2\, dt \le e^{16\,M^2}\ \mathbb{E}\int_0^T \varphi(t)\, c_2(t)\, e^{16\,\hat c_2^2(t)}\, |z(t)|^2\, dt$$

$$\le e^{16\,M^2}\, e^{16\,\hat c_2^2(0)}\, \mathbb{E}\int_0^T \hat c_2^2(t)\, e^{2\,\hat c_2(t) M}\, e^{\int_0^t 32\, a(s)\, \hat c_2^2(s)\, ds}\ |z(t)|^2\, dt$$

$$\le e^{16\,M^2}\, e^{16\,\hat c_2^2(0)}\, \mathbb{E}\int_0^T \phi(t)\, |z(t)|^2\, dt < \infty,$$

note that the last two steps follow from $64\, a(t)\ \hat c_2^2(t) \le \gamma(t) + 4\beta\ \hat c_0^2(t)\, \hat c_2^2(t)$, and from $z(\cdot) \in \mathcal{M}^2_\phi(0,T;\mathbb{R}^d)$. By the last inequality, together with condition $H(3)$-(c), we obtain:

$$\mathbb{E}\int_0^T \frac{\partial \psi_1}{\partial y}\big(t, y^n(t) - y^p(t)\big)\big[2\, c_0(t) + 6\, c_2(t)\, |z(t)|^2\big] dt < \infty.$$

Therefore, by Theorem 2.3.4, we have

$$\begin{aligned} &\lim_{p\to\infty} \mathbb{E}\int_0^T \frac{\partial \psi_1}{\partial y}\big(t, y^n(t) - y^p(t)\big)\big[2\, c_0(t) + 6\, c_2(t)\, |z(t)|^2\big] dt \\ =\ & \mathbb{E}\int_0^T \frac{\partial \psi_1}{\partial y}\big(t, y^n(t) - y(t)\big)\big[2\, c_0(t) + 6\, c_2(t)\, |z(t)|^2\big] dt. \end{aligned}$$

Also, we have:

$$\begin{aligned} &\lim_{p\to\infty} \mathbb{E}\int_0^T \frac{\partial \psi_1}{\partial y}\big(t, y^n(t) - y^p(t)\big)\ c_2(t)\ |z^n(t) - z(t)|^2\, dt \\ =\ & \mathbb{E}\int_0^T \frac{\partial \psi_1}{\partial y}\big(t, y^n(t) - y(t)\big)\ c_2(t)\ |z^n(t) - z(t)|^2\, dt \end{aligned}$$

Moreover,

$$
\begin{aligned}
&-\mathbb{E}\int_0^T \frac{\partial\psi_1}{\partial t}\big(t, y^n(t)-y^p(t)\big)\,dt \\
&= -2\,\mathbb{E}\int_0^T \varphi(t)\,a(t)\,\left(y^n(t)-y^p(t)\right)\,\left[e^{16\,\hat{c}_2(t)(y^n(t)-y^p(t))}-1\right]dt \\
&-\frac{1}{8}\,\mathbb{E}\int_0^T \left(\frac{\dot{\varphi}(t)}{\hat{c}_2(t)}-\frac{\varphi(t)a(t)}{\hat{c}_2(t)}\right)\left[e^{16\hat{c}_2(t)(y^n(t)-y^p(t))}-16\,\hat{c}_2(t)\big(y^n(t)-y^p(t)\big)-1\right]\,dt \\
&= -2\,\mathbb{E}\int_0^T \varphi(t)\,a(t)\,\left(y^n(t)-y^p(t)\right)\,\left[e^{16\,\hat{c}_2(t)(y^n(t)-y^p(t))}-1\right]dt \le 0,
\end{aligned}
$$

By this, and by letting p go to ∞ in the inequality (3.5.5), we obtain:

$$
\begin{aligned}
&\liminf_{p\to\infty,\,p\in\{n_k\}}\mathbb{E}\int_0^T \left|z^n(t)-z^p(t)\right|^2 \\
&\left[\frac{1}{2}\frac{\partial^2\psi_1}{\partial y^2}\big(t,y^n(t)-y^p(t)\big)-2c_2(t)\frac{\partial\psi_1}{\partial y}\big(t,y^n(t)-y^p(t)\big)\right]dt \\
&-6\,\mathbb{E}\int_0^T \frac{\partial\psi_1}{\partial y}\big(t,y^n(t)-y(t)\big)\,c_2(t)\,|z^n(t)-z(t)|^2\,dt \qquad (3.5.6)\\
\le\quad & \mathbb{E}\,\psi_1\big(T,y^n(T)-y(T)\big)-\,\psi_1\big(0,y^n(0)-y(0)\big) \\
&+\mathbb{E}\int_0^T \frac{\partial\psi_1}{\partial y}\big(t,y^n(t)-y(t)\big)\big[2c_0(t)+6\,c_2(t)\,|z(t)|^2\big]dt.
\end{aligned}
$$

By Theorem 2.3.5, we have:

$$
\begin{aligned}
&\liminf_{p\to\infty,\,p\in\{n_k\}}\left[-6\,\mathbb{E}\int_0^T \frac{\partial\psi_1}{\partial y}\big(t,y^n(t)-y^p(t)\big)\,c_2(t)\,|z^n(t)-z^p(t)|^2\,dt\right] \\
\le\quad & -6\,\mathbb{E}\int_0^T \frac{\partial\psi_1}{\partial y}\big(t,y^n(t)-y(t)\big)\,c_2(t)\,|z^n(t)-z(t)|^2\,dt.
\end{aligned}
$$

Substituting this in (3.5.6), we obtain:

$$\begin{aligned}
&\liminf_{p\to\infty,\, p\in\{n_k\}} \mathbb{E}\int_0^T \big|z^n(t)-z^p(t)\big|^2 \\
&\Big[\frac{1}{2}\frac{\partial^2\psi_1}{\partial y^2}\big(t,y^n(t)-y^p(t)\big)-8c_2(t)\,\frac{\partial\psi_1}{\partial y}\big(t,y^n(t)-y^p(t)\big)\Big]dt \\
\leq\quad & \mathbb{E}\,\psi_1\big(T,y^n(T)-y(T)\big)-\psi_1\big(0,y^n(0)-y(0)\big) \\
+\quad & \mathbb{E}\int_0^T \frac{\partial\psi_1}{\partial y}\big(t,y^n(t)-y(t)\big)\big[2c_0(t)+6\,c_2(t)\,|z(t)|^2\big]dt. \qquad (3.5.7)
\end{aligned}$$

By the expression of ψ_1, we have:

$$\begin{aligned}
&\frac{1}{2}\frac{\partial^2\psi_1}{\partial y^2}\big(t,y(t)\big)-8c_2(t)\,\frac{\partial\psi_1}{\partial y}\big(t,y(t)\big)\\
&=16\varphi(t)\,\hat{c}_2(t)\,e^{16\hat{c}_2(t)y(t)}-8c_2(t)\big[2\varphi(t)\,e^{16\hat{c}_2(t)y(t)}-2\varphi(t)\big]\\
&\geq 16\varphi(t)\,c_2(t)\,e^{16\hat{c}_2(t)y(t)}-8c_2(t)\big[2\varphi(t)\,e^{16\hat{c}_2(t)y(t)}-2\varphi(t)\big]\\
&\geq 16\varphi(t).
\end{aligned}$$

Hence, inequality (3.5.7) becomes:

$$\begin{aligned}
&16\liminf_{p\to\infty,\, p\in\{n_k\}} \mathbb{E}\int_0^T \varphi(t)|z^n(t)-z^p(t)|^2dt \\
\leq\quad & \mathbb{E}\,\psi_1\big(T,y^n(T)-y(T)\big)-\psi_1\big(0,y^n(0)-y(0)\big) \\
+\quad & \mathbb{E}\int_0^T \frac{\partial\psi_1}{\partial y}\big(t,y^n(t)-y(t)\big)\big[2c_0(t)+6\,c_2(t)\,|z(t)|^2\big]dt. \qquad (3.5.8)
\end{aligned}$$

By Theorem 2.3.5, we have:

$$16\ \mathbb{E}\int_0^T \varphi(t)|z^n(t)-z(t)|^2\ dt \leq 16\liminf_{p\to\infty,\, p\in(n_k)} \mathbb{E}\int_0^T \varphi(t)|z^n(t)-z^p(t)|^2\,dt.$$

Then, inequality (3.5.8) becomes

$$16\,\mathbb{E}\int_0^T \varphi(t)|z^n(t)-z(t)|^2\,dt \le \mathbb{E}\,\psi_1\big(T,y^n(T)-y(T)\big) - \psi_1\big(0,y^n(0)-y(0)\big)$$

$$+\mathbb{E}\int_0^T \frac{\partial\psi_1}{\partial y}\big(t,y^n(t)-y(t)\big)\big[2c_0(t)+6\,c_2(t)\,|z(t)|^2\big]dt.$$

By letting n go to ∞ in the previous inequality, and by Theorem 2.3.4, the right-hand side converges to 0, and we obtain:

$$\lim_{n\to\infty}\mathbb{E}\int_0^T \varphi(t)|z^n(t)-z(t)|^2\,dt = 0, \tag{3.5.9}$$

In other words, the sequence $(\{z^n(t)\}_{n\ge 0}, t\in[0,T])$ converges to $(z(t), t\in[0,T])$ in $\mathcal{M}^2_\varphi(0,T;\mathbb{R}^d)$.

Part 2.

In this part, we prove the uniform convergence in t of a subsequence of $\{y^n(\cdot)\}_{n\ge 0}$ to the process $y(\cdot)$. We first show that:

$$\lim_{j\to\infty}\int_t^T F^{n_j}(s,y^{n_j}(s),z^{n_j}(s))\,ds = \int_t^T F(s,y(s),z(s))\,ds \ \ a.s. \text{ uniformly in } t. \tag{3.5.10}$$

By (3.5.9), and by Theorem 2.2.6, there exists a subsequence $\{z^{n_j}(t)\}_{j\ge 0}$ of $\{z^n(t)\}_{n\ge 0}$ such that $\{z^{n_j}(t)\}_{j\ge 0}$ converges to $z(t)$ *a.e.* $t\in[0,T]$ *a.s.*.

Now, by assumption *(1)* in Theorem 3.5.2, together with the facts that $\{y^{nj}(t)\}_{j\ge 0}$ converges to $y(t)$ almost surely for all $t\in[0,T]$, and $\{z^{n_j}(t)\}_{j\ge 0}$ converges almost surely to $z(t)$ *a.e.* $t\in[0,T]$ *a.s*, we have:

$$\lim_{j\to\infty} F^{n_j}(t,y^{n_j}(t),z^{n_j}(t)) = F(t,y(t),z(t)) \quad a.e.\ \ t\in[0,T]\ \ a.s.$$

By assumption *(2)* in Theorem 3.5.2, and condition $H(3)$-(c), we have:

$$\begin{aligned}
&\int_0^T |F^{n_j}(s, y^{n_j}(s), z^{n_j}(s))| \, ds \\
&\leq \int_0^T \left[c_0(s) + c_2(s) \, |z^{n_j}(s)|^2\right] ds \\
&\leq \int_0^T \left[c_0(s) + \phi(s) \, |z^{n_j}(s)|^2\right] ds < \infty \;\; a.s..
\end{aligned}$$

Thus, by the dominated convergence theorem, see Theorem 2.3.4, we have:

$$\lim_{j\to\infty} \int_0^T \left|F^{n_j}(s, y^{n_j}(s), z^{n_j}(s)) - F(s, y(s), z(s))\right| ds = 0 \;\; a.s..$$

Therefore,

$$\begin{aligned}
&\lim_{j\to\infty} \sup_{t\in[0,T]} \left| \int_t^T F^{n_j}(s, y^{n_j}(s), z^{n_j}(s)) \, ds - \int_t^T F(s, y(s), z(s)) \, ds \right| \\
&\leq \lim_{j\to\infty} \sup_{t\in[0,T]} \int_t^T \left|F^{n_j}(s, y^{n_j}(s), z^{n_j}(s)) - F(s, y(s), z(s))\right| ds \\
&\leq \lim_{j\to\infty} \int_0^T \left|F^{n_j}(s, y^{n_j}(s), z^{n_j}(s)) - F(s, y(s), z(s))\right| ds = 0 \quad a.s..
\end{aligned}$$

In other words, the convergence (3.5.10) holds. Next, we prove that:

$$\lim_{j\to\infty} \int_t^T (z^{n_j}(s))' \, dW(s) = \int_t^T z'(s) \, dW(s) \;\; a.s. \text{ uniformly in } \; t \in [0,T]. \tag{3.5.11}$$

Since the stochastic integral is a martingale, by Theorem 2.4.7, for any constant $\epsilon > 0$, the following holds:

$$\begin{aligned}
&\mathbb{P}\left(\omega \in \Omega : \sup_{t\in[0,T]} \left| \int_t^T [z^{n_j}(s) - z(s)]' \, dW(s) \right| > \epsilon \right) \\
&\leq \frac{1}{\epsilon^2} \; \mathbb{E} \left| \int_0^T [z^{n_j}(s) - z(s)]' \, dW(s) \right|^2.
\end{aligned}$$

By the Isômetry property, see Theorem 2.5.1, we have

$$\mathbb{P}\left(\omega \in \Omega : \sup_{t\in[0,T]}\left| \int_t^T [z^{n_j}(s) - z(s)]' \, dW(s) \right| > \epsilon\right)$$

$$\leq \frac{1}{\epsilon^2}\, \mathbb{E} \int_0^T \left|z^{n_j}(s) - \ z(s)\right|^2 ds$$

$$\leq \frac{1}{\epsilon^2}\, \mathbb{E} \int_0^T \ \varphi(s)\ \left|z^{n_j}(s) - \ z(s)\right|^2 ds.$$

As the right-hand side of the last inequality goes to 0 as $j \to \infty$, it follows that:

$$\lim_{j\to\infty} \sup_{t\in[0,T]} \left| \int_t^T \ [z^{n_j}(s) - z(s)]' \, dW(s) \right| = 0 \quad \text{in probability.}$$

By Theorem 2.2.6, there exists a subsequence $\{z^{n_q}(\cdot)\}_{q\geq 0}$ of $\{z^{n_j}(\cdot)\}_{j\geq 0}$ such that:

$$\lim_{q\to\infty} \sup_{t\in[0,T]} \left| \int_t^T \ [z^{n_q}(s) - z(s)]' \, dW(s) \right| = 0 \quad a.s.,$$

i.e. the convergence (3.5.11) holds. Now, for any two elements n_q and n_m of the sequence $\{n_q\}_{q\geq 0}$, we have:

$$\begin{aligned}
&y^{n_q}(t) - y^{n_m}(t) \\
&= (y^{n_q}(T) - y^{n_m}(T)) + \int_t^T \ \left[F^{n_q}(s, y^{n_q}(s), z^{n_q}(s)) - F^{n_m}(s, y^{n_m}(s), z^{n_m}(s))\right] ds \\
&- \int_t^T \ \left[z^{n_q}(s) - z^{n_m}(s)\right]' dW(s), \quad \text{for all } t \in [0,T] \ \ a.s..
\end{aligned}$$

Then,

$$
\begin{aligned}
&|y^{n_q}(t) - y^{n_m}(t)| \\
&\leq |y^{n_q}(T) - y^{n_m}(T)| + \int_t^T \left|F^{n_q}(s, y^{n_q}(s), z^{n_q}(s)) - F^{n_m}(s, y^{n_m}(s), z^{n_m}(s))\right| ds \\
&+ \left| \int_t^T \left[z^{n_q}(s) - z^{n_m}(s)\right]' \, dW(s) \right|. \qquad (3.5.12)
\end{aligned}
$$

By the convergence in (3.5.11), we have:

$$
\begin{aligned}
&\lim_{m\to\infty} \left| \int_t^T [z^{n_q}(s) - z^{n_m}(s)]' \, dW(s) \right| \\
&\leq \lim_{m\to\infty} \left| \int_t^T [z^{n_q}(s) - z(s)]' \, dW(s) \right| + \lim_{m\to\infty} \left| \int_t^T [z(s) - z^{n_m}(s)]' \, dW(s) \right| \\
&= \left| \int_t^T [z^{n_q}(s) - z(s)]' \, dW(s) \right| \quad \text{for all } \ t \in [0, T] \ \ a.s..
\end{aligned}
$$

On the other hand, by the convergence in (3.5.10), we have:

$$
\begin{aligned}
&\lim_{m\to\infty} \int_t^T \left|F^{n_q}(s, y^{n_q}(s), z^{n_q}(s)) - F^{n_m}(s, y^{n_m}(s), z^{n_m}(s))\right| ds \\
&\leq \lim_{m\to\infty} \int_t^T \left|F^{n_q}(s, y^{n_q}(s), z^{n_q}(s)) - F(s, y(s), z(s))\right| ds \\
&+ \lim_{m\to\infty} \int_t^T \left|F(s, y(s), z(s)) - F^{n_m}(s, y^{n_m}(s), z^{n_m}(s))\right| ds \\
&= \int_t^T \left|F^{n_q}(s, y^{n_q}(s), z^{n_q}(s)) - F(s, y(s), z(s))\right| ds.
\end{aligned}
$$

Hence, by lettings $m \to \infty$ in the inequality (3.5.12), we obtain:

$$
\begin{aligned}
&|y^{n_q}(t) - y(t)| \\
&\leq |y^{n_q}(T) - y(T)| + \int_t^T \left|F^{n_q}(s, y^{n_q}(s), z^{n_q}(s)) - F(s, y(s), z(s))\right| ds \\
&+ \left| \int_t^T [z^{n_q}(s) - z(s)]' \, dW(s) \right|, \ \text{ for all } t \in [0, T] \ \ a.s..
\end{aligned}
$$

By taking the supremum over $t \in [0,T]$ in the previous inequality, we obtain:

$$\begin{aligned}
&0 \leq \sup_{t\in[0,T]} |y^{n_q}(t) - y(t)| \\
&\leq |y^{n_q}(T) - y(T)| + \int_0^T \left|F^{n_q}(s, y^{n_q}(s), y^{n_q}(s)) - F(s, y(s), z(s))\right| ds \\
&+ \sup_{t\in[0,T]} \left| \int_t^T [z^{n_q}(s) - z(s)]' \, dW(s \right| \;\; a.s..
\end{aligned}$$

As the right-hand side in the previous inequality goes to 0 almost surely as q goes to ∞, we obtain:

$$\lim_{q\to\infty} \sup_{t\in[0,T]} |y^{n_q}(t) - y(t)| = 0 \;\; a.s..$$

In other words, the subsequence $\{y^{n_q}(t)\}_{q\geq 0}$ converges to $(y(t), t \in [0,T])$ uniformly in t. Moreover, by the uniform convergence theorem, see Theorem 2.3.2, the process $(y(t), t \in [0,T])$ has continuous paths.

Part 3.

By letting $q \to \infty$, the following sequence of equations:

$$y^{n_q}(t) = y^{n_q}(T) + \int_t^T F^{n_q}(s, y^{n_q}(s), z^{n_q}(s)) \, ds - \int_t^T (z^{n_q}(s))' \, dW(s), \;\; t \in [0,T],$$

converges to

$$y(t) = y(T) + \int_t^T F(s, y(s), z(s)) \, ds - \int_t^T z'(s) \, dW(s), \;\; t \in [0,T],$$

which proves that $(y(\cdot), z(\cdot))$ is a solution pair to equation (3.2.1) with (F, ξ).

□

3.6 Existence

In this section, we give the existence result for equation (3.2.1) when the pair (F, ξ) satisfies condition H. The main idea here is to find an appropriate approximation of F, before applying Theorem 3.5.2. We apply an approximation technique also used by [45] and [27]. Before applying the approximation, see Step 2 below, we use an exponential transformation to control the growth of the generator with respect to z and a truncation to control the growth of the generator with respect to y. We follow the truncation technique in Theorem 2.3 of [37].

Step 1

We consider the exponential transformation

$$v := e^{\hat{c}_2(t)y}, \quad (t, y) \in [0, T] \times \mathbb{R},$$

which is essentially a random version of that in [37]. We have

$$\begin{aligned} dv(t) &= \big[a(t)\,\hat{c}_2(t)\,y(t)\,v(t) - \hat{c}_2(t)\,v(t)\,F(t, y(t), z(t)) + \tfrac{1}{2}(\hat{c}_2(t))^2\,v(t)\,|z(t)|^2\big]\,dt \\ &+ \hat{c}_2(t)\,v(t)\,z'(t)\,dW(t). \end{aligned}$$

Setting $\hat{z}(t) := \hat{c}_2(t)\,v(t)\,z(t)$, we obtain

$$dv(t) = \left[a(t)\,v(t)\log v(t) - \hat{c}_2(t)\,v(t)\,F\big(t, \tfrac{\log v(t)}{\hat{c}_2(t)}, \tfrac{\hat{z}(t)}{\hat{c}_2(t)v(t)}\big) + \frac{1}{2}\,\frac{|\hat{z}(t)|^2}{v(t)}\right]dt + \hat{z}'(t)\,dW(t).$$

Thus, the transformed generator is:

$$\begin{aligned} f(t, v(t), \hat{z}(t)) &= -a(t)\,v(t)\log v(t) + \hat{c}_2(t)\,v(t)\,F\big(t, \tfrac{\log v(t)}{\hat{c}_2(t)}, \tfrac{\hat{z}(t)}{\hat{c}_2(t)v(t)}\big) \\ &- \frac{1}{2}\,\frac{|\hat{z}(t)|^2}{v(t)}. \end{aligned} \tag{3.6.1}$$

By condition $H(2)$, we have

$$\begin{aligned} F(t, y(t), z(t)) &= c_1(t)\;y(t) + F_0(t, y(t), z(t)) \\ &\le c_1(t)\;y(t) + |F_0(t, y(t), z(t))| \\ &\le c_1(t)\;y(t) + c_0(t) + \frac{1}{2}\,c_2(t)|z(t)|^2. \end{aligned}$$

Thus,

$$F\big(t, \tfrac{\log v(t)}{\hat{c}_2(t)}, \tfrac{\hat{z}(t)}{\hat{c}_2(t) v(t)}\big) \leq c_1(t)\,\frac{\log v(t)}{\hat{c}_2(t)} + c_0(t) + \frac{1}{2}\,c_2(t)\frac{|\hat{z}(t)|^2}{(\hat{c}_2(t))^2\,v^2(t)} \quad a.e.\;\; t \in [0,T] \;\; a.s..$$

Hence, (3.6.1) becomes:

$$\begin{aligned} f(t, v(t), \hat{z}(t)) &\leq -a(t)\,v(t)\log v(t) \\ &+ \hat{c}_2(t)\,v(t)\Big[c_1(t)\,\frac{\log v(t)}{\hat{c}_2(t)} + c_0(t) + c_2(t)\frac{|\hat{z}(t)|^2}{2(\hat{c}_2(t))^2\,v^2(t)}\Big] - \frac{1}{2}\,\frac{|\hat{z}(t)|^2}{v(t)} \\ &\leq -a(t)\,v(t)\log v(t) + c_1(t)\,v(t)\,\log v(t) + c_0(t)\,\hat{c}_2(t)\,v(t) \\ &+ \frac{|\hat{z}(t)|^2}{2v(t)} - \frac{1}{2}\frac{|\hat{z}(t)|^2}{v(t)} \\ &= \big[c_1(t) - a(t)\big]v(t)\log v(t) + c_0(t)\,\hat{c}_2(t)\,v(t). \end{aligned}$$

In other words, this transformation eliminates the quadratic term in $z(\cdot)$. This is compatible with [37] if $\hat{c}_2(t)$ is a constant, for example, if $\hat{c}_2(0)$ is a constant and $a(t) = 0$.

Define the random function $\hat{f}$ as follows:

$$\hat{f}(t,x,u) := \Phi(t,x)\;f(t,x,u), \quad (t,x,u) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R}^d,$$

where the function Φ is defined for all $t \in [0,T]$ as:

$$\Phi(t,x) := \begin{cases} 1, & \text{if } \; x \in [e^{-\hat{c}_2(t)M}, e^{\hat{c}_2(t)M}], \\ 0, & \text{if } \; x \notin [e^{-\hat{c}_2(t)(M+1)}, e^{\hat{c}_2(t)(M+1)}]. \end{cases}$$

Step 2

We approximate the function $\hat{f}$ by an infinite sequence of Lipschitz functions. Each such function generates a backward stochastic differential equation.

Later, we show that the solutions to such a sequence of equations converge to the solution of the BSDE with $(\hat{f}, \xi)$. For $n \geqslant 0$, define the sequence of functions:

$$f_n(t, v, \hat{z}) := \sup_{(y,z)\in\mathbb{R}\times\mathbb{R}^d} \big\{\hat{f}(t, y, z) - (n + k_1(t))|y - v| - (n + k_2(t))|z - \hat{z}|\big\}.$$

Lemma 3.6.1. *The sequence of functions f_n have the following properties:*
(i) Monotonicity: f_n is a decreasing sequence in n;

(ii) Lipschitz condition: for any $v_1, v_2 \in \mathbb{R}$, $\hat{z}_1, \hat{z}_2 \in \mathbb{R}^d$,

$$\big|f_n(t, v_1, \hat{z}_1) - f_n(t, v_2, \hat{z}_2)\big| \leq (n + k_1(t))\big|v_1 - v_2\big| + (n + k_2(t))|\hat{z}_1 - \hat{z}_2|$$

a.e. $t \in [0, T]$ a.s.;

(iii) convergence: if $v_n \to v$ and $\hat{z}_n \to \hat{z}$ as $n \to \infty$, then

$$\lim_{n\to\infty} f_n(t, v_n, \hat{z}_n) = \hat{f}(t, v, \hat{z}) \;\; \text{a.e. } t \in [0, T] \text{ a.s.;}$$

(iv) linear growth: for any $v \in \mathbb{R}$ and $\hat{z} \in \mathbb{R}^d$,

$$|f_n(t, v, \hat{z})| \leq (n + k_1(t))\;|v| + (n + k_2(t))\;|\hat{z}| \;\; \text{a.e. } t \in [0, T] \;\; \text{a.s..}$$

Proof. (i) This property follows from the definition.

(ii) $\big|f_n(t, v_1, \hat{z}_1) - f_n(t, v_2, \hat{z}_2)\big|$

$$= \Bigg|\sup_{(y,z)\in\mathbb{R}\times\mathbb{R}^d} \big\{\hat{f}(t, y, z) - (n + k_1(t))|y - v_1| - (n + k_2(t))|z - \hat{z}_1|\big\}$$
$$- \sup_{(y,z)\in\mathbb{R}\times\mathbb{R}^d} \big\{\hat{f}(t, y, z) - (n + k_1(t))|y - v_2| - (n + k_2(t))|z - \hat{z}_2|\big\}\Bigg|.$$

By the inequality $\big|\sup_{i\in I} a_i - \sup_{i\in I} b_i\big| \leqslant \sup_{i\in I} |a_i - b_i|$, where I is an arbitrary index set, we have:

$$
\begin{aligned}
& \big|f_n(t,v_1,\hat{z}_1)-f_n(t,v_2,\hat{z}_2)\big| \\
\leq & \sup_{(y,z)\in\mathbb{R}\times\mathbb{R}^d}\big|-(n+k_1(t))|y-v_1|-(n+k_2(t))|z-\hat{z}_1| \\
+ & (n+k_1(t))|y-v_2|+(n+k_2(t))|z-\hat{z}_2|\big| \\
= & \sup_{(y,z)\in\mathbb{R}\times\mathbb{R}^d}\big|(n+k_1(t))\big[|y-v_2|-|y-v_1|\big]+(n+k_2(t))\big[|z-\hat{z}_2|-|z-\hat{z}_1|\big]\big| \\
\leq & \sup_{(y,z)\in\mathbb{R}\times\mathbb{R}^d}\big|(n+k_1(t))|y-v_2-y+v_1|+(n+k_2(t))\,|z-\hat{z}_2-z+\hat{z}_1|\big| \\
= & (n+k_1(t))\big|v_1-v_2\big|+(n+k_2(t))|\hat{z}_1-\hat{z}_2|.
\end{aligned}
$$

(iii) Consider $(v_n,\hat{z}_n)\to(v,\hat{z})$ as $n\to\infty$. By the definition of f_n, for any $n\geqslant 0$, there exists $(p_n,q_n)\in\mathbb{R}\times\mathbb{R}^d$ such that

$$
\begin{aligned}
\hat{f}(t,v_n,\hat{z}_n) &\leq f_n(t,v_n,\hat{z}_n) \\
&\leq \hat{f}(t,p_n,q_n)-(n+k_1(t))|p_n-v_n|-(n+k_2(t))|q_n-\hat{z}_n| \\
&+ (n+1)^{-1}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\hat{f}(t,v_n,\hat{z}_n) &\leq f_n(t,v_n,\hat{z}_n)+(n+k_1(t))|p_n-v_n|+(n+k_2(t))|q_n-\hat{z}_n| \\
&\leq \hat{f}(t,p_n,q_n)+(n+1)^{-1}. \qquad (3.6.2)
\end{aligned}
$$

By the definition of $\hat{f}$, we have $\hat{f}(t,p_n,q_n)=0$, if $p_n\notin[e^{-\hat{c}_2(t)(M+1)},e^{\hat{c}_2(t)(M+1)}]$

and $\hat{f}(t,p_n,q_n) \le (c_1(t)-a(t))\, p_n\, \log p_n + c_0(t)\, \hat{c}_2(t)\; p_n$, if $p_n \in [e^{-\hat{c}_2(t)M}, e^{\hat{c}_2(t)M}]$. This implies that:

$$\begin{aligned}\hat{f}(t,p_n,q_n) &\le |c_1(t)-a(t)|\; p_n\, |\log\, p_n| + c_0(t)\, \hat{c}_2(t)\; p_n \\ &\le\; |c_1(t)-a(t)|\, \hat{c}_2(t)\, M\, e^{\hat{c}_2(t)\, M} + c_0(t)\, \hat{c}_2(t)\, e^{\hat{c}_2(t)\, M}.\end{aligned}$$

Here, the upper bound of $\hat{f}(t,p_n,q_n)$ is independent of n. In order to make

$$f_n(t,v_n,\hat{z}_n) + (n+k_1(t))|p_n - v_n| + (n+k_2(t))|q_n - \hat{z}_n|$$

finite as $n \to \infty$, it is necessary to have:

$$\lim_{n\to\infty} (n+k_1)\, |p_n - v_n| < \infty, \qquad \lim_{n\to\infty} (n+k_2)\, |q_n - \hat{z}_n| < \infty.$$

In particular, $\lim_{n\to\infty} (p_n, q_n) = (v, \hat{z})$. By letting $n \to \infty$ in (3.6.2), we obtain:

$$\hat{f}(t,v,\hat{z}) \le \lim_{n\to\infty} f_n(t,v_n,\hat{z}_n) \le \hat{f}(t,v,\hat{z}),$$

which implies that:

$$\lim_{n\to\infty} f_n(t,v_n,\hat{z}_n) = \hat{f}(t,v,\hat{z}).$$

(iv) By the Lipschitz property, we have

$$\begin{aligned}|f_n(t,v,\hat{z})| &= |f_n(t,v,\hat{z}) - f_n(t,0,0) + f_n(t,0,0)| \\ &\le |f_n(t,v,\hat{z}) - f_n(t,0,0)| + |f_n(t,0,0)| \\ &\le (n+k_1(t))\; |v| + (n+k_2(t))\; |\hat{z}| + |f_n(t,0,0)|. \quad (3.6.3)\end{aligned}$$

From (i), we have $f_n(t,0,0) \le f_0(t,0,0)$. Since $\hat{f}(t,y,z)$ can take value of 0,

we have:

$$f_n(t,0,0) = \sup_{(y,z)\in\mathbb{R}\times\mathbb{R}^d} \left[\hat{f}(t,y,z) - (n+k_1(t))\,|y| - (n+k_2(t))\,|z| \right] \geq 0.$$

Therefore, $|f_n(t,0,0)| \leq f_0(t,0,0)$. Also, we have:

$$f_0(t,0,0) = \sup_{(y,z)\in\mathbb{R}\times\mathbb{R}^d} \left[\hat{f}(t,y,z) - k_1(t)\,|y| - k_2(t)\,|z| \right].$$

Here the supremum is reached for $y=0$ and $z=0$, and then:

$$f_0(t,0,0) = \hat{f}(t,0,0) = 0.$$

Hence, the inequality (3.6.3) becomes:

$$|f_n(t,v,\hat{z})| \leq (n+k_1(t))\,|v| + (n+k_2(t))\,|\hat{z}|.$$

□

Now, using the functions $\{f_n\}_{n\geqslant 0}$ as generators, we introduce the following sequence of equations for $n \geq 0$:

$$v_n(t) = e^{\hat{c}_2(T)\,\xi} + \int_t^T f_n(s, v_n(s), \hat{z}_n(s))\,ds - \int_t^T \hat{z}_n(s)\,dW(s), \quad t \in [0,T]. \tag{3.6.4}$$

Lemma 3.6.2. *Let conditions H hold. The BSDEs* (3.6.4) *have unique solution pairs* $(v_n(\cdot), \hat{z}_n(\cdot)) \in \mathcal{H}^2_{\hat{p}}(0,T;\mathbb{R}) \times \mathcal{M}^2_{\hat{p}}(0,T;\mathbb{R}^d)$*, for any* $n \geq 0$.

Proof. Here we show that the assumptions of Theorem 2.1. in [27] hold. This theorem gives the existence and uniqueness of a solution pair for BSDEs with unbounded Lipschitz generators. In Lemma 3.6.1, we already proved that condition (ii) of Theorem 2.1. in [27] holds, and condition (iii) of this theorem clearly holds. It remains to show that condition (i) of Theorem 2.1.

in [27] holds. We have:

$$\begin{aligned}
&\mathbb{E}\left[e^{\int_0^T\{\hat{\gamma}(t)+\beta_1(n+k_1(t))^2+\beta_2(n+k_2(t))^2\}dt}\, e^{2\hat{c}_2(T)\,\xi}\right]\\
\le\;& \mathbb{E}\left[e^{2\hat{c}_2(T)\,\xi}\;\, e^{\int_0^T\{2n^2(\beta_1+\beta_2)\}dt}\, e^{\int_0^T\{\hat{\gamma}(t)+2\beta_1(k_1(t))^2+2\beta_2(k_2(t))^2\}dt}\right]\\
\le\;& e^{2n^2\,T\,(\beta_1+\beta_2)}\;\mathbb{E}\left[e^{2\hat{c}_2(T)\,\xi}\;\, e^{\int_0^T\{\hat{\gamma}(t)+2\beta_1(k_1(t))^2+2\beta_2(k_2(t))^2\}dt}\right] < \infty,
\end{aligned}$$

where the last step follows from condition $H(3)$-(a). Therefore, by Theorem 2.1. in [27] the equations (3.6.4) have unique solution pairs $(v_n(\cdot), \hat{z}_n(\cdot)) \in \mathcal{H}^2_{\hat{p}}(0,T;\mathbb{R}) \times \mathcal{M}^2_{\hat{p}}(0,T;\mathbb{R}^d)$. $\square$

Since the sequence f_n is decreasing, by the comparison theorem in [27], see Theorem 2.2 (i), we obtain:

$$v_0(t) \ge v_1(t) \ge v_2(t) \ge \;\cdots, \quad \text{for all } t \in [0,T] \;\; a.s.. \tag{3.6.5}$$

Defining $y_n := \log v_n / \hat{c}_2(t)$, we have:

$$\begin{aligned}
dy_n(t) \;&=\; -\,\hat{c}_2^{-2}(t)\,a(t)\,\hat{c}_2(t)\,\log v_n(t)\,dt - [f_n(t, v_n(t), \hat{z}_n(t))]\;\hat{c}_2^{-1}(t)\,v_n^{-1}(t)\,dt\\
&-\;\frac{1}{2}\,v_n^{-2}(t)\,\hat{c}_2^{-1}(t)\,\left|\hat{z}_n(t)\right|^2 dt + \hat{c}_2^{-1}(t)\,v_n^{-1}(t)\,\hat{z}_n(t)\,dW(t).
\end{aligned}$$

Thus,

$$y_n(t) = \xi + \int_t^T F_n(s, y_n(s), z_n(s))\;ds - \int_t^T z_n'(s)\;dW(s), \qquad t \in [0,T], \tag{3.6.6}$$

with

$$F_n(t, y_n, z_n) := a(t)\,y_n + \frac{f_n(t, e^{\hat{c}_2(t)\,y_n}, \hat{c}_2(t)\,e^{\hat{c}_2(t)\,y_n}\,z_n)}{\hat{c}_2(t)\,e^{\hat{c}_2(t)\,y_n}} + \frac{1}{2}\,\hat{c}_2(t)\,|z_n|^2, \quad n \ge 0,$$

$$\forall (t, y_n, z_n) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d. \tag{3.6.7}$$

In the following lemma, we prove that the BSDEs (3.6.6) with parameters $\{(F_n, \xi)\}_{n\ge 0}$ have unique solution pairs $(y_n(\cdot), z_n(\cdot))$ such that $y_n(t)$ is bounded for all $t \in [0,T]$ $a.s.$. Let

$$\hat{\phi}(t) := \hat{c}_2^2(t)\, e^{2\hat{c}_2(t)\, M}\, \hat{p}(t),$$

and $\mathcal{M}^2_{\hat{\phi}}(0,T;\mathbb{R}^d)$ be the space of $\mathcal{F}_t$-progressively measurable processes $\theta(\cdot)$ such that $\mathbb{E}\left[\int_0^T \hat{\phi}(s)|\theta(s)|^2 ds\right] < \infty$.

Lemma 3.6.3. *Let conditions H hold. The BSDEs* (3.6.6) *have unique solution pairs* $(y_n(\cdot), z_n(\cdot)) \in \mathcal{L}^\infty_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{M}^2_{\hat{\phi}}(0,T;\mathbb{R}^d)$.

Proof. We proved that equations (3.6.4) have unique solution pairs $(v_n(\cdot), \hat{z}_n(\cdot)) \in \mathcal{H}^2_{\hat{p}}(0,T;\mathbb{R}) \times \mathcal{M}^2_{\hat{p}}(0,T;\mathbb{R}^d)$. Thus, equations (3.6.6) have unique solution pairs $(y_n(\cdot), z_n(\cdot))$ such that:

$$y_n(t) = \log v_n(t)/\hat{c}_2(t) \quad\text{and}\quad z_n(t) = \hat{z}_n(t)/\hat{c}_2(t)v_n(t), \quad t \in [0,T].$$

In order to show that $y_n(t)$ is bounded $a.s.$ for all $t \in [0,T]$ and for all $n \geq 0$, we only need to show that $v_n(t) \in [\exp(-\hat{c}_2(t)M), \exp(\hat{c}_2(t)M)]$ $a.s.$ for all $t \in [0,T]$ and for all $n \geq 0$. In order to show the upper bound on $v_n(t)$, we only need to find an upper bound on the process $(v_0(t), t \in [0,T])$, as the sequence $\{v_n(t)\}_{n\geq 0}$ is decreasing. From the linear growth property of the generator f_0, we have:

$$f_0(t,v,\hat{z}) \leq k_1(t)|v| + k_2(t)|\hat{z}| \leq k_1(t)|v| + \frac{1}{4} + k_2^2(t)|\hat{z}|^2 =: g(t,v,\hat{z}).$$

The BSDE with generator g and terminal value $\exp(\hat{c}_2(T)\xi)$ is:

$$\begin{cases} dv_g(t) = -g(t, v_g(t), z_g(t))dt + z_g'(t)dW(t), \quad t \in [0,T], \\ \\ v_g(T) = \exp(\hat{c}_2(T)\xi) \quad a.s.. \end{cases}$$

Its explicit solution is:

$$v_g(t) = \frac{1}{2k_2^2(t)} \log\left\{\mathbb{E}\left[\exp\left\{\int_t^T \frac{1}{2}k_2^2(s)ds + 2k_2^2(T)\exp(\hat{c}_2(T)\xi)\right\}\middle|\, \mathcal{F}(t)\right]\right\}, \quad t \in [0,T],$$

and its upper bound of $\exp(\hat{c}_2(t)M)$ is ensured by assumption $H(3)$-(g). As $g \geq f_0$, it follows by the comparison theorem of [27] that $v_0(t) \leq v_g(t)$ $a.s.$ for all $t \in [0,T]$.

In order to show the lower bound, we first find the following lower bound on the generator $\hat{f}$:

$$
\begin{aligned}
&\hat{f}(t,v,z) = \Phi(t,v)f(t,v,z)\\
&\geq \Phi(t,v)\left[-a(t)v\log(v) + \hat{c}_2(t)v\left(-c_0(t) + c_1(t)\frac{\log(v)}{\hat{c}_2(t)} - \frac{c_2(t)z^2}{2\hat{c}_2^2(t)v^2}\right) - \frac{|z|^2}{2v}\right]\\
&\geq \Phi(t,v)\left[-(a(t)-c_1(t))v\log(v) - c_0(t)\hat{c}_2(t)v - \left(\frac{c_2(t)}{\hat{c}_2^2(t)} + 1\right)\frac{|z|^2}{2v}\right]\\
&\geq \Phi(t,v)\left[-\eta(t)v - \frac{|z|^2}{v}\right]\\
&\geq -\eta(t)|v| - \frac{|z|^2}{|v|} =: h(t,v,z).
\end{aligned}
$$

The BSDE with generator h and terminal value $\exp(\hat{c}_2(T)\xi)$ is:

$$
\begin{cases}
dv_h(t) = -h(t, v_h(t), z_h(t))dt + z_h'(t)dW(t), \quad t \in [0,T],\\
\\
v_h(T) = \exp(\hat{c}_2(T)\xi) \quad a.s..
\end{cases}
$$

Its explicit solution is:

$$
v_h(t) = \left\{\mathbb{E}\left[e^{\int_t^T \eta(s)ds - \hat{c}_2(T)\xi}\,\middle|\, \mathcal{F}_t\right]\right\}^{-1}, \quad t \in [0,T],
$$

and its lower bound of $\exp(-\hat{c}_2(t)M)$ is ensured by assumption $H(3)$-(h). As $h \leq \hat{f} \leq f_n$ for all $n \geq 0$, it follows by the comparison theorem of [27] that $v_h(t) \leq v_n(t)$ $a.s.$ for all $t \in [0,T]$ and for all $n \geq 0$. Therefore, $v_n(t) \in [\exp(-\hat{c}_2(t)M), \exp(\hat{c}_2(t)M)]$ $a.s.$ for all $t \in [0,T]$ and for all $n \geq 0$. This implies that $-M \leq y_n(t) \leq M$ for all $t \in [0,T]$ $a.s.$ for all $n \geq 0$. Thus, the solution pairs $(y_n(\cdot), z_n(\cdot))$ of (3.6.6) belong to $\mathcal{L}^\infty_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{M}^2_{\hat{\phi}}(0,T;\mathbb{R}^d)$. □

Now, we present the main theorem in this chapter.

Theorem 3.6.4. *Let the pair (F,ξ) satisfies conditions H. The BSDE* (3.2.1) *admits a solution pair $(y(\cdot), z(\cdot)) \in \mathcal{L}^\infty_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{M}^2_{\hat{\phi}}(0,T;\mathbb{R}^d)$.*

Proof. The main idea of the proof is applying Theorem 3.5.2, and later Theorem 3.4.5. By (3.6.5), we have:

$$y_0(t) \geq y_1(t) \geq y_2(t) \geq \cdots, \quad \text{for all } t \in [0,T] \ \ a.s..$$

As the sequence $\{y_n(t)\}_{n\geq 0}$ is bounded and decreasing $a.s.$ for all $t \in [0,T]$, in order to apply Theorem 3.5.2, we only need to show that the remaining assumptions of this theorem hold. We first prove that the generators F_n have quadratic growth in $z_n(\cdot)$. From the fact that $f_n(t, v_n, \hat{z}_n)$ is decreasing sequence in n, we have $f_n(t, v_n, \hat{z}_n) \leq f_0(t, v_n, \hat{z}_n)$. By the expression of f_n, $f_n(t, v_n, \hat{z}_n) \geq 0$ $a.s.$ $a.e.$ $t \in [0,T]$. Therefore,

$$|f_n(t, v_n, \hat{z}_n)| \leq f_0(t, v_n, \hat{z}_n).$$

By (iv) in Lemma 3.6.1, we have

$$|f_0(t, v_n, \hat{z}_n)| \leq k_1(t)\ v_n\ + k_2(t)\ |\hat{z}_n|.$$

This implies that:

$$|f_n(t, v_n, \hat{z}_n)| \leq k_1(t)\ v_n\ + k_2(t)\ |\hat{z}_n|.$$

By this, together with (3.6.7), we obtain:

$$\begin{aligned}
|F_n(t, y_n, z_n)| &\leq a(t)\,|y_n| + \frac{|f_n(t, e^{\hat{c}_2(t)\,y_n}, \hat{c}_2(t)\,e^{\hat{c}_2(t)\,y_n}\,z_n|}{\hat{c}_2(t)\,e^{\hat{c}_2(t)\,y_n}} + \tfrac{1}{2}\,\hat{c}_2(t)\,|z_n|^2 \\
&\leq a(t)\,|y_n| + \frac{k_1(t)\ e^{\hat{c}_2(t)\,y_n} + k_2(t)\,|\hat{c}_2(t)\,e^{\hat{c}_2(t)\,y_n}\,z_n|}{\hat{c}_2(t)\,e^{\hat{c}_2(t)\,y_n}} + \tfrac{1}{2}\,\hat{c}_2(t)\,|z_n|^2 \\
&\leq a(t)\,M + \frac{k_1(t)}{\hat{c}_2(t)} + k_2(t)\,|z_n| + \tfrac{1}{2}\,\hat{c}_2(t)\,|z_n|^2 \\
&\leq a(t)\,M + \frac{k_1(t)}{\hat{c}_2(t)} + 1 + k_2^2(t)\,|z_n|^2 + \tfrac{1}{2}\,\hat{c}_2(t)\,|z_n|^2 \\
&\leq [a(t)\,M + 2] + \hat{c}_2(t)\,|z_n|^2. \qquad (3.6.8)
\end{aligned}$$

It remains to prove that the convergence of $\{F_n\}_{n\geq 0}$ holds. Define the function $\hat{F}$ as follows:

$$\hat{F}(t,y,z) := a(t)\,y + \frac{\hat{f}(t, e^{\hat{c}_2(t)\,y}, \hat{c}_2(t)\,e^{\hat{c}_2(t)\,y}\,z)}{\hat{c}_2(t)\,e^{\hat{c}_2(t)\,y}} + \frac{1}{2}\,\hat{c}_2(t)\,|z|^2,$$
$$\forall (t,y,z) \in [0,T]\times\mathbb{R}\times\mathbb{R}^d.$$

Assume that $(y_n, z_n) \to (y,z)$ as $n \to \infty$. By letting $n \to \infty$ in (3.6.7) and by Lemma 3.6.1 (iii), we obtain:

$$\lim_{n\to\infty} F_n(t, y_n, z_n) = \hat{F}(t,y,z) \quad a.e.\ \ t \in [0,T]\ \ a.s.. \tag{3.6.9}$$

Thus, by applying Theorem 3.5.2, the sequence $\{z_n(t)\}_{n\geq 0}$ converges to $(z(t), t \in [0,T])$ in $\mathcal{M}^2_{\varphi}(0,T;\mathbb{R}^d)$, there exists a subsequence of $\{y_n(t)\}_{n\geq 0}$ that converges uniformly in t to $(y(t), t\in[0,T])$, and $(y(\cdot), z(\cdot)) \in \mathcal{L}^{\infty}_{\mathcal{F}}(0,T;\mathbb{R})\times \mathcal{M}^2_{\hat{\phi}}(0,T;\mathbb{R}^d)$ is a solution of the BSDE (3.2.1) with $(\hat{F},\xi)$. The Final step is showing that the pair $(y(\cdot), z(\cdot))$ is also a solution to the BSDE (3.2.1) with (F,ξ). By Theorem 3.4.5, we have $-M \leq y(t) \leq M$ for all $t\in[0,T]$ $a.s.$, then $\Phi(t, e^{\hat{c}_2(t)\,y(t)}) = 1$. By the expression of $\hat{F}$, we have

$$\begin{aligned}
&\hat{F}(t,y,z)\\
&= \frac{\big[-a(t)\,\hat{c}_2(t)\,e^{\hat{c}_2(t)\,y}\,y + \hat{c}_2(t)\,e^{\hat{c}_2(t)\,y}\,F\big(t,y,z\big) - \frac{1}{2}\,\hat{c}_2^2(t)\,e^{\hat{c}_2(t)\,y}|z|^2\big]\,\Phi(t, e^{\hat{c}_2(t)\,y})}{\hat{c}_2(t)\,e^{\hat{c}_2(t)\,y}}\\
&+ a(t)\,y + \frac{1}{2}\hat{c}_2(t)\,|z|^2\\
&= F\big(t,y,z\big)\,\Phi(t, e^{\hat{c}_2(t)\,y}) + a(t)\,y\,\big(1 - \Phi(t, e^{\hat{c}_2(t)\,y})\big) + \frac{1}{2}\,\hat{c}_2(t)\,|z|^2\,\big(1 - \Phi(t, e^{\hat{c}_2(t)\,y})\big)\\
&= F\big(t,y,z\big).
\end{aligned}$$

Therefore, $(y(\cdot), z(\cdot))$ is also a solution pair to equation (3.2.1) with parameters (F,ξ). □

3.7 Conclusion

We have considered BSDEs with possibly unbounded generators that have quadratic growth in the control process. We give sufficient conditions for the existence of solution pairs. These conditions are new and weaker than the existing ones. We expect that these results will be useful in solving some difficult problems with unbounded coefficients, such as reflected BSDEs with quadratic growth, the indefinite Riccati BSDEs and their application in optimal investment.

3.8 Appendix

Proof of Lemma 3.4.3.

For $(t, y) \in [0, T] \times [-M, 0]$, all properties hold by the definition. Now, we show that they hold for $(t, y) \in [0, T] \times [0, M]$.

(i) This follows from the definition.

(ii) $\frac{\partial Q}{\partial y}(t, y) = 2\,\hat{c}_2(t)\left(e^{2\,\hat{c}_2(t)\,y} - 1 - 2\,\hat{c}_2(t)\,y\right) \geq 0$ *a.s. for all* $(t, y) \in [0, T] \times [0, M]$.

(iii)
$$\begin{aligned}\frac{\partial Q}{\partial t}(t, y) &= 2\,a(t)\,\hat{c}_2(t)\,y\,[e^{2\hat{c}_2(t)\,y} - 1 - 2\,\hat{c}_2(t)\,y] \\ &= a(t)\,y\,\frac{\partial Q}{\partial y}(t, y) \quad \textit{a.s. for all } (t, y) \in [0, T] \times [0, M].\end{aligned}$$

(iv) $\hat{c}_2(t)\,\frac{\partial Q}{\partial y}(t, y) - \frac{1}{2}\,\frac{\partial^2 Q}{\partial y^2}(t, y) = -4\,\hat{c}_2^3(t)\,y \leq 0$ *a.s. for all* $(t, y) \in [0, T] \times [0, M]$.

(v)
$$\begin{aligned}\left[\frac{\partial Q}{\partial y}(t, y)\right]^2 &= 4\hat{c}_2^2(t)\big[e^{2\hat{c}_2(t)\,y} - 1 - 2\hat{c}_2(t)\,y\big]^2 \\ &\leq 4\hat{c}_2^2(t)\big[2(e^{2\hat{c}_2(t)\,y} - 1)^2 + 4\hat{c}_2^2(t)y^2\big] \\ &\leq 4\hat{c}_2^2(t)[2(2e^{4\hat{c}_2(t)y} + 2) + 4\hat{c}_2^2\,(y)^2] \\ &\leq 16\hat{c}_2^2(t)[e^{4\hat{c}_2(t)M} + 1 + \hat{c}_2^2(t)M^2]. \\ &\leq 16\hat{c}_2^2(t)[e^{2(\hat{c}_2^2(t)+M^2)} + 1 + \hat{c}_2^2(t)M^2] \\ &\leq 8[\hat{c}_2^4(t) + e^{4(\hat{c}_2^2(t)+M^2)} + 2\hat{c}_2^2(t) + 2\hat{c}_2^4(t)M^2] \\ &\leq e^{4\hat{c}_2^2(t)} + 8e^{4(\hat{c}_2^2(t)+M^2)} + 4e^{4\hat{c}_2^2(t)} + e^{4\hat{c}_2^2(t)}e^{4M^2} \\ &\leq e^{4\hat{c}_2^2(t)}e^{4M^2} + 8e^{4(\hat{c}_2^2(t)+M^2)} + 4e^{4\hat{c}_2^2(t)}e^{4M^2} + e^{4\hat{c}_2^2(t)}e^{4M^2}\end{aligned}$$

$$\leq 16\, e^{4M^2} e^{4\hat{c}_2^2(t)} \quad \textit{a.s. for all } (t,y) \in [0,T] \times [0,M].$$

Proof of Lemma 3.4.6.

(i) This follows from the definition of $\psi(t,x)$.

(ii) $$\frac{\partial \psi}{\partial x}(t,x) = \frac{2\phi(t)}{\hat{c}_2^2(t)} \big[\hat{c}_2(t)\, e^{\hat{c}_2(t)(x+M)} - \hat{c}_2(t)\big]$$

$$= \frac{2\phi(t)}{\hat{c}_2(t)} \big[e^{\hat{c}_2(t)(x+M)} - 1\big] \geq 0$$

a.s. for all $(t,x) \in [0,T] \times [-M,M]$.

(iii) $$\frac{\partial \psi}{\partial t}(t,x) = \big[e^{\hat{c}_2(t)(x+M)} - 1 - \hat{c}_2(t)(x+M)\big]$$

$$\times \big[8\, p(t)\, \hat{c}_2(t)\, a(t)\, M\, e^{4\hat{c}_2(t)M} + 2p(t)\, (\gamma(t) + \beta\, \hat{c}_0^2(t)\, \hat{c}_2^2(t)) e^{4\hat{c}_2(t)M}\big]$$

$$+ 2\, p(t) e^{4\hat{c}_2(t)M} \big[(y+M) a(t)\, \hat{c}_2(t) e^{\hat{c}_2(t)(y+M)} - (y+M) a(t)\, \hat{c}_2(t)\big]$$

$$\geq 0 \quad \textit{a.s. for all } (t,x) \in [0,T] \times [-M,M].$$

(iv) $$\frac{1}{2}\, \hat{c}_2(t)\, \frac{\partial \psi}{\partial x}(t,x) - \frac{1}{2}\, \frac{\partial^2 \psi}{\partial x^2}(t,x)$$

$$= \phi(t)\, \big[e^{\hat{c}_2(t)(x+M)} - 1\big] - \phi(t) e^{\hat{c}_2(t)(x+M)} = -\phi(t)$$

a.s. for all $(t,x) \in [0,T] \times [-M,M]$.

Proof of Lemma 3.5.1.

(i) This follows immediately from the definition of $\psi_1(t,x)$.

(ii) $$\frac{\partial \psi_1}{\partial x}(t,x) = \frac{\varphi(t)}{8\hat{c}_2(t)} \big[16\hat{c}_2(t)\, e^{16\hat{c}_2(t)x} - 16\hat{c}_2(t)\big]$$

$$= 2\varphi(t) \big[e^{16\hat{c}_2(t)\, x} - 1\big] \geq 0 \quad \textit{a.s. for all } (t,x) \in [0,T] \times [0,2M].$$

(iii) $\dfrac{\partial \psi_1}{\partial x}(t,0) = 0$ follows from *(ii)*.

$$\frac{\partial \psi_1}{\partial t}(t,x) = \frac{\varphi(t)}{8\hat{c}_2(t)}\left[16\, a(t)\,\hat{c}_2(t)\,x\,e^{16\hat{c}_2(t)\,x} - 16\,a(t)\,\hat{c}_2(t)\,x\right] + \left[e^{16\hat{c}_2(t)\,x} - 16\hat{c}_2(t)\,x - 1\right]\left[\frac{\dot{\varphi}(t)}{8\hat{c}_2(t)} - \frac{a(t)\,\varphi(t)}{8\,\hat{c}_2(t)}\right].$$

Thus, $\dfrac{\partial \psi_1}{\partial t}(t,0) = 0$ *a.s. for all* $(t,x) \in [0,T] \times [0,2M]$.

Chapter 4

Riccati Backward Stochastic Differential Equation with Unbounded Coefficients

4.1 Abstract

We consider a class of Riccati BSDEs with possibly unbounded coefficients, giving sufficient conditions, which are weaker than those that already exist, for the existence of a solution pair. As an application, we obtain the existence of a solution to the linear-quadratic optimal control problem with possibly unbounded coefficients.

4.2 Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a given complete filtered probability space on which a d-dimensional standard Brownian motion $\{W(t), 0 \leq t \leq T\}$ is defined. We assume that $\{\mathcal{F}_t\}_{t\geq 0}$ is the augmentation of $\sigma\{W(s) : 0 \leq s \leq t\}$ by all $\mathbb{P}$-null sets of $\mathcal{F}$. We consider the general one-dimensional case of Riccati BSDE:

$$
\begin{cases}
dK(t) = -\big[a(t)\,K(t) + c\,'(t)L(t) \,+ Q(t) + F(t, K(t), L(t))\big]\,dt \\
\qquad\qquad +L'(t)\,dW, \quad t \in [0,T], \\
K(T) = \; M \quad a.s., \\
N(t) + K(t)\,D'(t)\,D(t) > 0 \; a.e.\; t \in [0,T] \;\; a.s.,
\end{cases}
\tag{4.2.1}
$$

with

$$
\begin{aligned}
F(t,x,y) := -&\left[B(t)x + C'(t)D(t)x + y'\,D(t)\right]\left[N(t) + xD'(t)D(t)\right]^{-1} \\
&\times \left[B(t)x + C'(t)D(t)x + y'\,D(t)\right]',
\end{aligned}
$$

for all $t \in [0,T],\; x \in \mathbb{R},\, y := (y_1, \cdots, y_d)' \in \mathbb{R}^d$,

$$a(t) := 2A(t) + |C(t)|^2, \;\; t \in [0,T],$$

$$c(t) := (c_1(t), ..., c_d(t))' := 2\,(C_1(t), ..., C_d(t))'\,, \;\; t \in [0,T],$$

$$C(t) := (C_1(t), ..., C_d(t))' \text{ and } D(t) := \left(D_1'(t), ..., D_d'(t)\right)', \;\; t \in [0,T].$$

Here M is an $\mathcal{F}_T$-measurable random variable and the coefficients $A(\cdot)$, $Q(\cdot)$ and $C_i(\cdot)$ for $i = 1, ..., d$, are $\{\mathcal{F}_t\}_{t\geq 0}$-progressively measurable $\mathbb{R}$-valued stochastic processes, the coefficients $B(\cdot)$ and $D_i(\cdot)$ for $i = 1, ..., d$, are $\{\mathcal{F}_t\}_{t\geq 0}$-progressively measurable $\mathbb{R}^m$-valued stochastic processes, the coefficient $N(\cdot)$ is an $\{\mathcal{F}_t\}_{t\geq 0}$-progressively measurable $\mathbb{R}^{m\times m}$-valued stochastic process. The solution of the Riccati BSDE is the pair of adapted stochastic processes $(K(t), L(t))$ that satisfies (4.2.1).

Bismut [9] was the first to study Riccati BSDEs with random coefficients. Kohlmann and Tang [39] obtained the existence and uniqueness results for the general one-dimensional Riccati BSDE with random coefficients. In [31], Hu and Zhou applied the results of quadratic BSDEs in [37] to prove that the general Riccati BSDE admits a nonnegative solution. In all of these cases, the coefficients are assumed to be bounded. An important weakening of the conditions on the coefficients under which equation (4.2.1) has a solution is considering possibly unbounded coefficients. Gashi and Li [28] proved the solvability of a certain class of Riccati BSDEs with possibly unbounded coefficients.

In this chapter, we consider the general case of equation (4.2.1) with possibly unbounded coefficients. We give sufficient conditions, weaker than the existing ones, for the existence of a solution pair. As an application, we obtain the solution to the LQ optimal control problem under weaker conditions on the coefficients than those in Theorem 5.2. of [39]. We give an explicit solution to the LQ optimal control problem using our result for the Riccati BSDE with unbounded coefficients.

4.3 Notation and assumptions

The key notation and assumptions used in this chapter are as follows:

- $M \geq 0 \ \ a.s.$.
- $Q(t) \geq 0 \quad a.e. \ \ t \in [0, T] \ a.s.$.
- $N(t) \geq \varepsilon \ I_{m \times m} \ a.e. \ \ t \in [0, T] \ \ a.s.$ for some constant $\varepsilon > 0$.
- $2 \leq \beta_1 \in \mathbb{R}$ and $2 \leq \beta_2 \in \mathbb{R}$ are given constants.
- $\alpha_1(\cdot)$ is a given positive $\mathbb{R}$-valued progressively measurable process such that $|a(\cdot)| \leq \alpha_1(\cdot)$.
- $\alpha_2(\cdot)$ is a given positive $\mathbb{R}$-valued progressively measurable process such that $|c(\cdot)| \leq \alpha_2(\cdot)$ and $\mathbb{E}\big[\, e^{12 \, \hat{\beta} \, \int_0^T \alpha_2^2(s) \, ds} \big] < \infty$ for some $\hat{\beta} > 1$.
- $\gamma(\cdot) \geq 1$, $\lambda_1(\cdot)$ and $\delta(\cdot)$ are given nonnegative $\mathbb{R}$-valued progressively measurable processes.

- $\lambda_2(\cdot) \geq 1$ is a given $\mathbb{R}$-valued progressively measurable process with the following representation $\lambda_2(t) := \lambda_2(0) + \int_0^t \delta(s)\, ds$ for $t \in [0,T]$ and $\delta(\cdot) \leq \lambda_2(\cdot)$.
- $p(t) := \exp\left[\int_0^t \big(\gamma(s) + 4\beta_1\, \alpha_1^2(s) + 4\beta_2\, \alpha_2^2(s)\big)\, ds\right],\ t \in [0,T]$.
- $24\,\delta(t)\,\lambda_2(t) \leq \gamma(t) + 4\beta_1\,\alpha_1^2(t) + 4\beta_2\,\alpha_2^2(t)$ *a.e.* $t \in [0,T]$ *a.s.*.
- $$V := \mathbb{E}\Big[M e^{\int_t^T (a(s)-\frac{1}{2}|c(s)|^2)ds + \int_t^T \langle c(s), dW(s)\rangle} + e^{-\int_0^t (a(s)-\frac{1}{2}|c(s)|^2)\,ds - \int_0^t \langle c(s), dW(s)\rangle} \int_t^T e^{\int_0^s (a(\tau)-\frac{1}{2}|c(\tau)|^2)\,d\tau + \int_0^s \langle c(\tau), dW(\tau)\rangle} Q(s)\,ds \,\big|\mathcal{F}_t\Big],\quad t \in [0,T].$$
- $\mathcal{H}_p^2(0,T;\mathbb{R}^d)$ is the space of $\mathcal{F}_t$-progressively measurable $\mathbb{R}^d$-valued processes $\varphi(\cdot)$ such that $\mathbb{E}\left[\sup_{t\in[0,T]}\ p(t)|\varphi(t)|^2\right] < \infty$.
- $\mathcal{M}_p^2(0,T;\mathbb{R}^d)$ (resp. $\mathcal{M}_{\lambda_2}^2(0,T;\mathbb{R}^d)$) is the space of $\mathcal{F}_t$-progressively measurable $\mathbb{R}^d$-valued processes $\varphi(\cdot)$ such that $\mathbb{E}\left[\int_0^T\ p(s)|\varphi(s)|^2 ds\right] < \infty$ (resp. $\mathbb{E}\left[\int_0^T\ \lambda_2(s)|\varphi(s)|^2 ds\right] < \infty$).

The coefficients and the random variable M satisfy conditions H_1 if:

(i) $\mathbb{E}\left[\, p(T)\right] < \infty$;

(ii) $\mathbb{E}\left[\displaystyle\int_0^T p(s)\, Q^2(s)\, ds\right] < \infty$;

(iii) there exists $\kappa \in \mathbb{R}^+$ such that $V \leq \kappa$ *a.e.* $t \in [0,T]$ *a.s.*.

The coefficients and the processes $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$ satisfy conditions H_2 if for κ, that is given in condition $H_1(iii)$, the following hold

(i) $3\,\varepsilon^{-1}\,\kappa^2\,\left[|B(t)|^2 + \ |C(t)|^2\,|D(t)|^2\right] \leq \lambda_1(t)$;

(ii) $3\,\varepsilon^{-1}\,|D(t)|^2 \leq \lambda_2(t)$;

(*iii*) $|a(t)|\left(e^{\kappa(12\lambda_2(t)+1)}-1\right)\in\mathcal{L}_{\mathcal{F}}(0,T;\mathbb{R})$;

(*iv*) $\lambda_1(t)\,e^{(12\lambda_2(t)+1)\kappa}\in\mathcal{L}_{\mathcal{F}}(0,T;\mathbb{R})$;

(*v*) $|c(t)|^2\left(e^{\kappa\,(12\lambda_2(t)+1)}-1\right)\in\mathcal{L}_{\mathcal{F}}(0,T;\mathbb{R})$.

Here we give sufficient conditions, which permit for unbounded coefficients, to obtain the existence of the Riccati BSDE (4.2.1). This is not the case in [39], where all the coefficients are assumed to be bounded. In particular, due to the processes $Q(\cdot)$, $N(\cdot)$, $a(\cdot)$, $c(\cdot)$, $B(\cdot)$ and $D(\cdot)$, those conditions are more general. For example, if we take $M=0$ *a.s.*, $Q(t)=0$ *a.s. a.e.* $t\in[0,T]$ where the processes $a(\cdot)$, $c(\cdot)$, $B(\cdot)$, $N(\cdot)$ and $D(\cdot)$ are possibly unbounded, the known results on the existence of solutions do not apply and thus our conditions are weaker as compared to [39]. Note that we can choose the processes $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\lambda_1(\cdot)$, $\lambda_2(\cdot)$, $\gamma(\cdot)$, $\delta(\cdot)$ and κ. This provides more flexibility on the conditions. For instance, condition H_1-(*iii*) can be suitable weakened by choosing large values for κ. In addition, the process $\gamma(\cdot)$ is important as we can take an arbitrary large value of $\gamma(\cdot)$ under which $24\,\delta(t)\,\lambda_2(t)\le\gamma(t)+4\beta_1\,\alpha_1^2(t)+4\beta_2\,\alpha_2^2(t)$ holds *a.e.* $t\in[0,T]$ *a.s.*. On the other hand, the set of conditions H_1 is sufficient for most of the results in this chapter. However, we need additional conditions for the convergence results, see conditions H_2 above, which permit for unbounded coefficients.

4.4 Solvability

In this section, we give sufficient conditions for the existence of a solution pair $(K(\cdot),L(\cdot))$ for (4.2.1). The method of proof, also used in [36], [39] and [45], is to construct a decreasing sequence of Lipschitz functions. Each such function generates a Riccati BSDE. Then, we prove that the solutions to such a sequence of Riccati BSDEs converge strongly to the solution of

(4.2.1). We introduce the following new generator $\hat{F}$:

$$\hat{F}(t,x,y) := -\left[B(t)x + C'(t)D(t)x + y'D(t)\right]\left[N(t) + |x|D'(t)D(t)\right]^{-1} \times \left[B(t)x + C'(t)D(t)x + y'D(t)\right]',$$
for all $t \in [0,T]$, $x \in \mathbb{R}$, $y \in \mathbb{R}^d$.

We have $\hat{F}(t,x,y) = F(t,x,y)$, for all $t \in [0,T]$, $x \in \mathbb{R}^+$, $y \in \mathbb{R}^d$.

4.4.1 Approximation

The main idea here is to approximate the generator $\hat{F}$ by a decreasing sequence of Lipschitz functions. For simplicity, in the following, we write (K, L) for $(K(t), L(t))$. We introduce the sequence of functions:

$$F^n(t,K,L) := \sup_{\hat{K}\in\mathbb{R},\hat{L}\in\mathbb{R}^d} [\hat{F}(t,\hat{K},\hat{L}) - (n+\alpha_1(t))|K-\hat{K}| - (n+\alpha_2(t))|L-\hat{L}|], \quad n \geqslant 0.$$

We give in the following lemma some important properties of F^n.

Lemma 4.4.1. *(i) Quadratic growth in K and L: for any $K \in \mathbb{R}$, $L \in \mathbb{R}^d$,*

$$0 \geq F^n(t,K,L) \geq -\left[3B_1(t)N^{-1}(t)B_1'(t) + B(t)N^{-1}(t)B'(t)\right]K^2 - 4L'D(t)N^{-1}(t)D'(t)L;$$

(ii) Monotonicity: F^n is decreasing in n;

(iii) Lipschitz condition: for any $K^1, K^2 \in \mathbb{R}$, $L^1, L^2 \in \mathbb{R}^d$,

$$|F^n(t,K^1,L^1) - F^n(t,K^2,L^2)| \leqslant [n+\alpha_1(t)]\,|K^1-K^2| + [n+\alpha_2(t)]\,|L^1-L^2|;$$

(iv) Strong convergence:

$$\textit{if} \lim_{n\to\infty} K^n = K \textit{ and } \lim_{n\to\infty} L^n = L, \textit{ then } \lim_{n\to\infty} F^n(t,K^n,L^n) = \hat{F}(t,K,L).$$

Proof. The method of proof is a combination of ideas from [45], [39] and [27].

(i) By the definition of $\hat{F}$, we have

$$F^n(t,K,L) = \sup_{\hat{K}\in\mathbb{R},\hat{L}\in\mathbb{R}^d}\Bigg[-\Big[B(t)\hat{K}+C'(t)D(t)\hat{K}+\hat{L}'D(t)\Big]\Big[N(t)+|\hat{K}|D'(t)D(t)\Big]^{-1}$$

$$\times\Big[B(t)\hat{K}+C'(t)D(t)\hat{K}+\hat{L}'D(t)\Big]' \qquad (4.4.1)$$

$$-\left[n+\alpha_1(t)\right]|K-\hat{K}|-\left[n+\alpha_2(t)\right]|L-\hat{L}|\Bigg].$$

Since $|\hat{K}|D'(t)D(t)\geq 0$, we have $N(t)+|\hat{K}|D'(t)D(t)\geq N(t)$, and then

$[N(t)+|\hat{K}|D'(t)D(t)]\,[N(t)]^{-1}\geq I$. Therefore, $[N(t)]^{-1}\geq[N(t)+\hat{K}|D'(t)D(t)]^{-1}$. By this, (4.4.1) becomes:

$$F^n(t,K,L)\geq \sup_{\hat{K}\in\mathbb{R},\hat{L}\in\mathbb{R}^d}\Bigg[-\Big[B(t)\hat{K}+C'(t)D(t)\hat{K}+\hat{L}'D(t)\Big]\left[N(t)\right]^{-1}$$

$$\times\Big[B(t)\hat{K}+C'(t)D(t)\hat{K}+\hat{L}'D(t)\Big]'$$

$$-\left[n+\alpha_1(t)\right]|K-\hat{K}|-\left[n+\alpha_2(t)\right]|L-\hat{L}|\Bigg].$$

By setting $B_1 := B+C'\,D$, we obtain:

$$F^n(t,K,L)\geq \sup_{\hat{K}\in\mathbb{R},\hat{L}\in\mathbb{R}^d}\Bigg[-\Big[B_1(t)\hat{K}+\hat{L}'D(t)\Big]\left[N(t)\right]^{-1}\Big[B_1(t)\hat{K}+\hat{L}'D(t)\Big]'$$

$$-\left[n+\alpha_1(t)\right]|K-\hat{K}|-\left[n+\alpha_2(t)\right]|L-\hat{L}|\Bigg]. \qquad (4.4.2)$$

If $B_1(t) \neq 0$ and $D(t) \neq 0$, we have:

$$
\begin{aligned}
F^n(t,K,L) \geq \sup_{\hat{K}\in\mathbb{R},\hat{L}\in\mathbb{R}^d} \Big[&- B_1(t)\hat{K}[N(t)]^{-1}\hat{K}B_1'(t) - B_1(t)\hat{K}\,[N(t)]^{-1}D'(t)\hat{L} \\
&- \hat{L}'\,D(t)\,[N(t)]^{-1}\,\hat{K}B_1'(t) - \hat{L}'\,D(t)\,[N(t)]^{-1}\,D'(t)\hat{L} \\
&- [n+\alpha_1(t)]\,|K-\hat{K}| - [n+\alpha_2(t)]\,|L-\hat{L}| \Big]
\end{aligned}
$$

$$
\begin{aligned}
= \sup_{\hat{K}\in\mathbb{R},\hat{L}\in\mathbb{R}^d} \Big[&- (\hat{K} - K + K)^2 B_1(t)[N(t)]^{-1}B_1'(t) \\
&- 2\, B_1(t)(\hat{K} - K + K)[N(t)]^{-1}D'(t)(\hat{L} - L + L) \\
&- (\hat{L}' - L' + L')D(t)[N(t)]^{-1}D'(t)(\hat{L} - L + L) \\
&- [n + \alpha_1(t)]\, |K - \hat{K}| - [n + \alpha_2(t)]\, |L - \hat{L}| \Big]
\end{aligned}
$$

$$
\begin{aligned}
= \sup_{\hat{K}\in\mathbb{R},\hat{L}\in\mathbb{R}^d} \Big[&- \big[(\hat{K} - K)^2 + 2(\hat{K} - K)K + K^2\big] B_1(t)[N(t)]^{-1}B_1'(t) \\
&- 2B_1(t)(\hat{K} - K)[N(t)]^{-1}D'(t)(\hat{L} - L) \\
&- 2B_1(t)K[N(t)]^{-1}D'(t)(\hat{L} - L) \\
&- 2B_1(t)(\hat{K} - K)[N(t)]^{-1}D'(t)L \\
&- 2B_1(t)K[N(t)]^{-1}D'(t)L \\
&- (\hat{L}' - L')D(t)[N(t)]^{-1}D'(t)(\hat{L} - L) \\
&- (\hat{L}' - L')D(t)[N(t)]^{-1}D'(t)L \\
&- L'D(t)[N(t)]^{-1}D'(t)(\hat{L} - L) - L'\, D(t)\, [N(t)]^{-1}D'(t)L \\
&- [n + \alpha_1(t)]\, |K - \hat{K}| - [n + \alpha_2(t)]\, |L - \hat{L}| \Big]
\end{aligned}
$$

$$
\begin{aligned}
&\geq \sup_{\hat{K}\in\mathbb{R},\hat{L}\in\mathbb{R}^d}\Bigg[-2(\hat{K}-K)^2B_1(t)[N(t)]^{-1}B_1'(t)-2K^2\,B_1(t)[N(t)]^{-1}B_1'(t)\\
&\qquad -B_1(t)(\hat{K}-K)^2[N(t)]^{-1}\,B_1'(t)\\
&\qquad -(\hat{L}-L)'\,D(t)[N(t)]^{-1}D'(t)\,(\hat{L}-L)\\
&\qquad -K^2\,B_1(t)[N(t)]^{-1}B_1'(t)-(\hat{L}-L)'\,D(t)[N(t)]^{-1}D'(t)\,(\hat{L}-L)\\
&\qquad -B_1(t)[N(t)]^{-1}B_1'(t)(\hat{K}-K)^2-L'\,D(t)\,[N(t)]^{-1}D'(t)L\\
&\qquad -B(t)[N(t)]^{-1}B'(t)K^2\\
&\qquad -L'\,D(t)\,[N(t)]^{-1}D'(t)L-(\hat{L}'-L')\,D(t)[N(t)]^{-1}D'(t)\,(\hat{L}-L)\\
&\qquad -\tfrac{1}{2}(\hat{L}'-L')\,D(t)[N(t)]^{-1}D'(t)\,(\hat{L}-L)-\tfrac{1}{2}\,L'\,D(t)[N(t)]^{-1}D'(t)\,L\\
&\qquad -\tfrac{1}{2}\,L'\,D(t)[N(t)]^{-1}D'(t)\,L-\tfrac{1}{2}(\hat{L}-L)'\,D(t)[N(t)]^{-1}D'(t)\,(\hat{L}-L)\\
&\qquad -L'\,D(t)[N(t)]^{-1}D'(t)\,L-[n+\alpha_1(t)]\,|K-\hat{K}|-[n+\alpha_2(t)]\,|L-\hat{L}|\Bigg]\\
&= -3K^2B_1(t)[N(t)]^{-1}B_1'(t)-K^2B(t)[N(t)]^{-1}B'(t)-4L'\,D(t)[N(t)]^{-1}D'(t)L
\end{aligned}
$$

$$
+ \sup_{\hat{K}\in\mathbb{R},\hat{L}\in\mathbb{R}^d} \Bigg[- 4(\hat{K}-K)^2 B_1(t)[N(t)]^{-1}B_1'(t)
$$

$$
- 4(\hat{L}-L)'D(t)[N(t)]^{-1}D'(t)(\hat{L}-L)
$$

$$
- [n+\alpha_1(t)]\,|K-\hat{K}| - [n+\alpha_2(t)]\,|L-\hat{L}| \Bigg]. \qquad (4.4.3)
$$

Now, we have:

$$
\begin{aligned}
& - 4(\hat{K}-K)^2 B_1(t)[N(t)]^{-1}B_1'(t) - [n+\alpha_1(t)]\,|K-\hat{K}| \\
&= -4B_1(t)[N(t)]^{-1}B_1'(t)\Bigg[(\hat{K}-K)^2 + \frac{[n+\alpha_1(t)]\,|K-\hat{K}|}{4B_1(t)[N(t)]^{-1}B_1'(t)} + \frac{[n+\alpha_1(t)]^2}{\big(8\,B_1(t)[N(t)]^{-1}B_1'(t)\big)^2}\Bigg] \\
&\quad + \frac{[n+\alpha_1(t)]^2}{16\,B_1(t)[N(t)]^{-1}B_1'(t)} \\
&= -4B_1(t)[N(t)]^{-1}B_1'(t)\left[(\hat{K}-K) + \frac{n+\alpha_1(t)}{8\,B_1(t)[N(t)]^{-1}B_1'(t)}\right]^2 + \frac{[n+\alpha_1(t)]^2}{16\,B_1(t)[N(t)]^{-1}B_1'(t)}.
\end{aligned}
$$

Moreover, we have:

$$
\begin{aligned}
& - 4(\hat{L}-L)'D(t)[N(t)]^{-1}D'(t)(\hat{L}-L) - [n+\alpha_2(t)]\,|L-\hat{L}| \\
&= -4D(t)[N(t)]^{-1}D'(t)\Bigg[(\hat{L}-L)'(\hat{L}-L) + \frac{[n+\alpha_2(t)]\,|L-\hat{L}|}{4D(t)[N(t)]^{-1}D'(t)} + \frac{[n+\alpha_2(t)]^2}{\big(8\,D(t)[N(t)]^{-1}D'(t)\big)^2}\Bigg] \\
&\quad + \frac{[n+\alpha_2(t)]^2}{16\,D(t)[N(t)]^{-1}D'(t)} \\
&= -4D(t)[N(t)]^{-1}D'(t)\left[|\hat{L}-L| + \frac{n+\alpha_2(t)}{8\,D(t)[N(t)]^{-1}D'(t)}\right]^2 + \frac{[n+\alpha_2(t)]^2}{16\,D(t)[N(t)]^{-1}D'(t)}.
\end{aligned}
$$

Therefore, the supremum in (4.4.3) is reached for $K=\hat{K}$ and $L=\hat{L}$, and hence

$$
\begin{aligned}
0 \geq F^n(t,K,L) \geq & - \big[3B_1(t)[N(t)]^{-1}\,B_1'(t) + B(t)[N(t)]^{-1}\,B'(t)\big]\,K^2 \\
& - 4L'D(t)[N(t)]^{-1}\,D'(t)L.
\end{aligned}
$$

For the case when $B_1(t) = D(t) = 0$, (4.4.2) becomes:

$$0 \geq F^n(t,K,L) \geq \sup_{\hat{K}\in\mathbb{R},\hat{L}\in\mathbb{R}^d} \Big[-[n+\alpha_1(t)]\,|K-\hat{K}| - [n+\alpha_2(t)]\,|L-\hat{L}| \Big].$$

Here, the supremum is reached for $K = \hat{K}$ and $L = \hat{L}$. Thus, $F^n(t,K,L) = 0$.
For the case when $B_1(t) = 0$ and $D(t) \neq 0$, (4.4.2) becomes:

$$\begin{aligned}
F^n(t,K,L) \geq \sup_{\hat{K}\in\mathbb{R},\hat{L}\in\mathbb{R}^d} \Big[& -\hat{L}'D(t)[N(t)]^{-1}D'(t)\hat{L} - [n+\alpha_1(t)]\,|K-\hat{K}| \\
& - [n+\alpha_2(t)]\,|L-\hat{L}| \Big] \\
= \sup_{\hat{K}\in\mathbb{R},\hat{L}\in\mathbb{R}^d} \Big[& -(\hat{L}'-L'+L')D(t)[N(t)]^{-1}D'(t)\hat{L} - [n+\alpha_1(t)]\,|K-\hat{K}| \\
& - [n+\alpha_2(t)]\,|L-\hat{L}| \Big] \\
= \sup_{\hat{K}\in\mathbb{R},\hat{L}\in\mathbb{R}^d} \Big[& -(\hat{L}'-L')D(t)[N(t)]^{-1}D'(t)\hat{L} - L'D(t)[N(t)]^{-1}D'(t)\hat{L} \\
& - [n+\alpha_1(t)]\,|K-\hat{K}| - [n+\alpha_2(t)]\,|L-\hat{L}| \Big] \\
= \sup_{\hat{K}\in\mathbb{R},\hat{L}\in\mathbb{R}^d} \Big[& -(\hat{L}'-L')D(t)[N(t)]^{-1}D'(t)(\hat{L}-L+L) \\
& - L'D(t)[N(t)]^{-1}D'(t)(\hat{L}-L+L) \\
& - [n+\alpha_1(t)]\,|K-\hat{K}| - [n+\alpha_2(t)]\,|L-\hat{L}| \Big]
\end{aligned}$$

$$= \sup_{\hat{K}\in\mathbb{R},\hat{L}\in\mathbb{R}^d} \Big[-(\hat{L}' - L')D(t)[N(t)]^{-1}D'(t)(\hat{L} - L)$$

$$-(\hat{L}' - L')D(t)[N(t)]^{-1}D'(t)L$$

$$-L'D(t)[N(t)]^{-1}D'(t)(\hat{L} - L) - L'D(t)[N(t)]^{-1}D'(t)L$$

$$-[n + \alpha_1(t)]\,|K - \hat{K}| - [n + \alpha_2(t)]\,|L - \hat{L}|\Big]$$

$$\geq \sup_{\hat{K}\in\mathbb{R},\hat{L}\in\mathbb{R}^d} \Big[-(\hat{L} - L)'D(t)[N(t)]^{-1}D'(t)(\hat{L} - L)$$

$$-\frac{1}{2}(\hat{L} - L)'D(t)[N(t)]^{-1}D'(t)(\hat{L} - L)$$

$$-\frac{1}{2}L'D(t)[N(t)]^{-1}D'(t)L - \frac{1}{2}L'D(t)[N(t)]^{-1}D'(t)L$$

$$-\frac{1}{2}(\hat{L} - L)'D(t)[N(t)]^{-1}D'(t)(\hat{L} - L)$$

$$-L'D(t)[N(t)]^{-1}D'(t)L$$

$$-[n + \alpha_1(t)]\,|K - \hat{K}| - [n + \alpha_2(t)]\,|L - \hat{L}|\Big]$$

$$= -2L'D(t)[N(t)]^{-1}D'(t)L + \sup_{\hat{K}\in\mathbb{R},\hat{L}\in\mathbb{R}^d} \Big[-2(\hat{L} - L)'D(t)[N(t)]^{-1}D'(t)(\hat{L} - L)$$

$$-[n + \alpha_1(t)]\,|K - \hat{K}| - [n + \alpha_2(t)]\,|L - \hat{L}|\Big].$$

We have:

$$
\begin{aligned}
&-2(\hat{L}-L)'D(t)[N(t)]^{-1}D'(t)(\hat{L}-L)-[n+\alpha_2(t)]\,|L-\hat{L}|\\
&=-2D(t)[N(t)]^{-1}D'(t)\left[|\hat{L}-L|^2+\frac{[n+\alpha_2(t)]\,|L-\hat{L}|}{2D(t)[N(t)]^{-1}D'(t)}+\frac{(n+\alpha_1(t))^2}{(4\,D(t)[N(t)]^{-1}D'(t))^2}\right]\\
&\quad+\frac{[n+\alpha_2(t)]^2}{8\,D(t)[N(t)]^{-1}D'(t)}\\
&=-2D(t)[N(t)]^{-1}D'(t)\left[|\hat{L}-L|+\frac{n+\alpha_2(t)}{4\,D(t)[N(t)]^{-1}D'(t)}\right]^2+\frac{[n+\alpha_2(t)]^2}{8\,D(t)[N(t)]^{-1}D'(t)}.
\end{aligned}
$$

Here the supremum is also reached for $K=\hat{K}$ and $L=\hat{L}$, and thus:

$$
\begin{aligned}
0\geq F^n(t,K,L)&\geq -2L'D(t)[N(t)]^{-1}D'(t)L\\
&\geq -\left[3B_1(t)[N(t)]^{-1}\,B_1'(t)+B(t)[N(t)]^{-1}\,B'(t)\right]K^2\\
&\quad-4L'D(t)[N(t)]^{-1}\,D'(t)L.
\end{aligned}
$$

Lastly, if $B_1(t)\neq 0$ and $D(t)=0$, (4.4.2) becomes:

$$
\begin{aligned}
F^n(t,K,L)&\geq \sup_{\hat{K}\in\mathbb{R},\hat{L}\in\mathbb{R}^d}\Big[-B_1(t)\hat{K}[N(t)]^{-1}\hat{K}B_1'(t)-[n+\alpha_1(t)]\,|K-\hat{K}|\\
&\qquad-[n+\alpha_2(t)]\,|L-\hat{L}|\Big]\\
&=\sup_{\hat{K}\in\mathbb{R},\hat{L}\in\mathbb{R}^d}\Big[-B_1(t)[N(t)]^{-1}B_1'(t)(\hat{K}-K+K)^2-[n+\alpha_1(t)]\,|K-\hat{K}|\\
&\qquad-[n+\alpha_2(t)]\,|L-\hat{L}|\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\geq -2B_1(t)[N(t)]^{-1}B_1'(t)K^2 \\
&+ \sup_{\hat{K}\in\mathbb{R},\hat{L}\in\mathbb{R}^d}\Big[-2B_1(t)[N(t)]^{-1}B_1'(t)|K-\hat{K}|^2 - [n+\alpha_1(t)]\,|K-\hat{K}| \\
&\qquad\qquad - [n+\alpha_2(t)]\,|L-\hat{L}|\Big] \\
&= -2B_1(t)[N(t)]^{-1}B_1'(t)K^2 \\
&+ \sup_{\hat{K}\in\mathbb{R},\hat{L}\in\mathbb{R}^d}\Big[-[n+\alpha_2(t)]\,|L-\hat{L}| - 2B_1(t)[N(t)]^{-1}\,B_1'(t) \\
&\quad \times\left(|K-\hat{K}|^2 + \frac{[n+\alpha_1(t)]\,|K-\hat{K}|}{2B_1(t)[N(t)]^{-1}B_1'(t)} + \frac{[n+\alpha_1(t)]^2}{(4\,B_1(t)[N(t)]^{-1}B_1'(t))^2}\right) \\
&\quad + \frac{[n+\alpha_2(t)]^2}{8\,B_1(t)[N(t)]^{-1}B_1'(t)}\Big] \\
&= -2B_1(t)[N(t)]^{-1}B_1'(t)K^2 + \sup_{\hat{K}\in\mathbb{R},\hat{L}\in\mathbb{R}^d}\Big[-[n+\alpha_2(t)]\,|L-\hat{L}| \\
&\quad - 2B_1(t)[N(t)]^{-1}B_1'(t)\left(|K-\hat{K}| + \frac{n+\alpha_2(t)}{4B_1(t)[N(t)]^{-1}B_1'(t)}\right)^2 \\
&\quad + \frac{[n+\alpha_2(t)]^2}{8\,B_1(t)[N(t)]^{-1}B_1'(t)}\Big].
\end{aligned}
$$

Since the supremum is reached for $K=\hat{K}$ and $L=\hat{L}$, we obtain:

$$
\begin{aligned}
0 \geq F^n(t,K,L) &\geq -2B_1(t)[N(t)]^{-1}B_1'(t)K^2 \\
&\geq -\left[3B_1(t)[N(t)]^{-1}\,B_1'(t) + B(t)[N(t)]^{-1}\,B'(t)\right]K^2 \\
&\quad - 4L'D(t)[N(t)]^{-1}\,D'(t)L.
\end{aligned}
$$

(ii) By the definition of F^n, it is a decreasing sequence in n.

(iii) $|F^n(t,K^1,L^1) - F^n(t,K^2,L^2)| =$

$$
\Big|\sup_{\hat{K}\in\mathbb{R},\hat{L}\in\mathbb{R}^d}\left\{\hat{F}(t,\hat{K},\hat{L}) - [n+\alpha_1(t)]\,|K^1-\hat{K}| - [n+\alpha_2(t)]\,|L^1-\hat{L}|\right\} -
$$
$$
\sup_{\hat{K}\in\mathbb{R},\hat{L}\in\mathbb{R}^d}\left\{\hat{F}(t,\hat{K},\hat{L}) - [n+\alpha_1(t)]\,|K^2-\hat{K}| - [n+\alpha_2(t)]\,|L^2-\hat{L}|\right\}\Big|.
$$

By the inequality: $|\sup_{i\in I} a_i - \sup_{i\in I} b_i| \le \sup_{i\in I} |a_i - b_i|$, where I is an arbitrary index set, we have:

$$\begin{aligned}
&\big|F^n(t,K^1,L^1) - F^n(t,K^2,L^2)\big| \\
&\le \sup_{\hat{K}\in\mathbb{R},\hat{L}\in\mathbb{R}^d} \Big| -[n+\alpha_1(t)]\,|K^1-\hat{K}| \\
&\qquad -[n+\alpha_2(t)]\,|L^1-\hat{L}| + [n+\alpha_1(t)]\,|K^2-\hat{K}| + [n+\alpha_2(t)]\,|L^2-\hat{L}|\ \Big| \\
&= \sup_{\hat{K}\in\mathbb{R},\hat{L}\in\mathbb{R}^d} \Big|\,[n+\alpha_1(t)]\big[|K^2-\hat{K}| - |K^1-\hat{K}|\big] + [n+\alpha_2(t)]\big[|L^2-\hat{L}| - |L^1-\hat{L}|\big]\ \Big| \\
&\le \sup_{\hat{K}\in\mathbb{R},\hat{L}\in\mathbb{R}^d} \Big|\ [n+\alpha_1(t)]\,|K^2-\hat{K}-K^1+\hat{K}| + [n+\alpha_2(t)]\,|L^2-\hat{L}-L^1+\hat{L}|\ \Big| \\
&= [n+\alpha_1(t)]\,|K^1-K^2| + [n+\alpha_2(t)]\,|L^1-L^2|.
\end{aligned}$$

(iv) Consider $(K^n, L^n) \to (K, L)$ as $n \to \infty$. By the definition of F^n, for any $n \geqslant 0$, there exists $(\hat{K}^n, \hat{L}^n) \in \mathbb{R}\times\mathbb{R}^d$ such that

$$\begin{aligned}
\hat{F}(t,K^n,L^n) &\le F^n(t,K^n,L^n) \\
&\le \hat{F}(t,\hat{K}^n,\hat{L}^n) - [n+\alpha_1(t)]\,|\hat{K}^n-K^n| - [n+\alpha_2(t)]\,|\hat{L}^n-L^n| \\
&\quad + (n+1)^{-1},
\end{aligned}$$

and then,

$$\begin{aligned}
\hat{F}(t,K^n,L^n) &\le F^n(t,K^n,L^n) + [n+\alpha_1(t)]\,|\hat{K}^n-K^n| + [n+\alpha_2(t)]\,|\hat{L}^n-L^n| \\
&\le \hat{F}(t,\hat{K}^n,\hat{L}^n) + (n+1)^{-1}. \qquad (4.4.4)
\end{aligned}$$

Since $\hat{F} \le 0$, thus

$$\begin{aligned}
\hat{F}(t,K^n,L^n) &\le F^n(t,K^n,L^n) + [n+\alpha_1(t)]\,|\hat{K}^n-K^n| + [n+\alpha_2(t)]\,|\hat{L}^n-L^n| \\
&\le (n+1)^{-1}.
\end{aligned}$$

In order to make: $F^n(t, K^n, L^n) + [n+\alpha_1(t)]\ |\hat{K}^n - K^n| + [n+\alpha_2(t)]\ |\hat{L}^n - L^n|$ finite as $n \to \infty$, it is necessary to have

$$\lim_{n\to\infty} [n + \alpha_1(t)]\ |\hat{K}^n - K^n| < \infty \text{ and } \lim_{n\to\infty} [n + \alpha_2(t)]\ |\hat{L}^n - L^n| < \infty.$$

In particular, $\lim_{n\to\infty} (\hat{K}^n, \hat{L}^n) = (K, L)$. Now, by taking the limit as $n \to \infty$ in (4.4.4), we obtain

$$\lim_{n\to\infty} F^n(t, K^n, L^n) = \hat{F}(t, K, L).$$

□

Now, we define:

$$G^n(t, K^n, L^n) := a(t)K^n + c'(t)\, L^n + Q(t) + F^n(t, K^n, L^n),$$

for all $t \in [0, T]$, $K^n \in \mathbb{R}$, $L^n \in \mathbb{R}^d$. By using the functions $\{G^n\}_{n\geq 0}$ as generators, we introduce the following sequence of Riccati BSDEs:

$$\begin{cases} dK^n(t) = -G^n(t, K^n(t), L^n(t))\ dt + L^n(t)\ dW(t), \quad t \in [0, T], \\ \\ K^n(T)\ = M, \quad a.s.. \end{cases} \tag{4.4.5}$$

Lemma 4.4.2. *Let conditions H_1(i)-(ii) hold. Then equations* (4.4.5) *have unique solution pairs* $(K^n(\cdot), L^n(\cdot)) \in \mathcal{H}^2_p(0, T; \mathbb{R}) \times \mathcal{M}^2_p(0, T; \mathbb{R}^d)$*, for any* $n \geq 0$.

Proof. The main idea of the proof is showing that the assumptions of Theorem 2.1. in [27] hold. This theorem gives the existence and uniqueness results for BSDEs with unbounded Lipschitz generators. We have:

$$|G^n(t,K^1,L^1) - G^n(t,K^2,L^2)|$$

$$= \big|a(t)K^1 + c'(t)\,L^1 + F^n(t,K^1,L^1) - a(t)K^2 - c'(t)\,L^2 - F^n(t,K^2,L^2)\big|$$

$$\leq |a(t)|\;|K^1 - K^2| + |c(t)|\;|L^1 - L^2| + |F^n(t,K^1,L^1) - F^n(t,K^2,L^2)|$$

$$\leq |a(t)|\,|K^1 - K^2| + |c(t)|\;|L^1 - L^2| + [n + \alpha_1(t)]\;|K^1 - K^2| + [n + \alpha_2(t)]\;|L^1 - L^2|$$

$$= \big[\;|a(t)| + n + \alpha_1(t)\big]\,|K^1 - K^2| + \big[\;|c(t)| + n + \alpha_2(t)\,\big]\,|L^1 - L^2|,$$

and

$$\mathbb{E}\big[e^{\int_0^T \{\gamma(s)+\beta_1[n+|a(s)|+\alpha_1(s)]^2+\beta_2[n+|c(s)|+\alpha_2(s)]^2\}ds}\quad M^2\big]$$

$$\leq \mathbb{E}\big[e^{\int_0^T \{\gamma(s)+2\beta_1 n^2+2\beta_1[|a(s)|+\alpha_1(s)]^2+2\beta_2 n^2+2\beta_2[|c(s)|+\alpha_2(s)]^2\}ds}\quad M^2\big]$$

$$= \mathbb{E}\big[e^{(2\beta_1 n^2+2\beta_2 n^2)T} e^{\int_0^T \{\gamma(s)+4\beta_1\alpha_1^2(s)+4\beta_2\alpha_2^2(s)\}ds}\;M^2\big] = e^{2(\beta_1+\beta_2)n^2 T}\;\;\mathbb{E}\big[p(T)\;M^2\big]$$

$$< \infty,$$

where the last step follows from condition $H_1(i)$. Also, since $\gamma(t) \geq 1$ and by condition $H_1(ii)$, we have

$$\mathbb{E}\int_0^T e^{\int_0^t \{\gamma(s)+\beta_1[n+|a(s)|+\alpha_1(s)]^2+\beta_2[n+|c(s)|+\alpha_2(s)]^2\}\,ds} \frac{Q^2(t)}{\gamma(t)+\beta_1[n+|a(t)|+\alpha_1(t)]^2+\beta_2[n+|c(t)|+\alpha_2(t)]^2}\;dt$$

$$\leq \mathbb{E} \int_0^T e^{2n^2 t(\beta_1+\beta_2)}\, e^{\int_0^t \{\gamma(s)+4\beta_1\alpha_1^2(s)+4\beta_2\alpha_2^2(s)\}\, ds}\, Q^2(t)\, dt$$

$$\leq e^{2n^2 T(\beta_1+\beta_2)}\, \mathbb{E} \int_0^T p(t)\, Q^2(t)\, dt < \infty.$$

Thus, by applying Theorem 2.1. in [27], equation (4.4.5) have unique solution pairs $(K^n(\cdot), L^n(\cdot)) \in \mathcal{H}_p^2(0,T;\mathbb{R}) \times \mathcal{M}_p^2(0,T;\mathbb{R}^d)$. $\square$

Moreover, in view of Theorem 2.2. in [27], and from the fact that the sequence $\{F^n\}_{n\geq 0}$ is decreasing, we have

$$K^0(t) \geq K^1(t) \geq \cdots \geq K^n(t) \geq K^{n+1}(t) \geq \cdots,$$

for all $t \in [0,T]$, *a.s.*.

4.4.2 Nonnegativity of solutions

In this section, we obtain the nonnegativity of $K^n(\cdot)$. We follow Proposition 3.1. in [39]. Our proof is different from [39] since we show this result with possibly unbounded coefficients. We have:

$$\begin{aligned} |F^n(t,K,L)| &= |F^n(t,K,L) - F^n(t,0,0) + F^n(t,0,0)| \\ &\leq |F^n(t,K,L) - F^n(t,0,0)| + |F^n(t,0,0)| \\ &\leq |F^n(t,K^n,L^n) - F^n(t,0,0)|, \end{aligned} \tag{4.4.6}$$

where the last step follows from the fact that $F^n(\cdot,0,0) = 0$. By (iii) in Lemma 4.4.1, (4.4.6) becomes:

$$|F^n(t,K,L)| \leq [n+\alpha_1(t)]\,|K| + [n+\alpha_2(t)]\,|L|. \tag{4.4.7}$$

Note that, (4.4.7) here is more genral than [39], since the terms $[n+\alpha_1(t)]$ and $[n+\alpha_2(t)]$ can be unbounded. Define:

$$\lambda^n(t,K,L) := \begin{cases} 0, & \text{if} \quad K = L_1 = \cdots = L_d = 0, \\ \dfrac{F^n(t,K,L)}{[n+\alpha_1(t)]\,|K| + [n+\alpha_2(t)]\,|L|}, & \text{otherwise.} \end{cases}$$

Note that $|\lambda^n(t,K,L)| \leq 1$. Therefore, we can write $F^n(t,K,L)$ as:

$$F^n(t,K,L) = \lambda^n(t,K,L)\Big[[n+\alpha_1(t)]\,|K| + [n+\alpha_2(t)]\,|L|\Big].$$

By using the function $|x| := S(x)\,x$, with:

$$S(x) := \begin{cases} 1, & \text{if} \quad x>0, \\ 0, & \text{if} \quad x=0, \\ -1, & \text{if} \quad x<0, \end{cases}$$

we can write:

$$\begin{aligned} F^n(t,K,L) &= \alpha_0^n(t,K,L)\,K + \alpha^n(t,K,L)\,L, \\ &= \alpha_0^n(t,K,L)\,K + \textstyle\sum_{i=1}^d \alpha_i^n(t,K,L)\,L_i, \end{aligned}$$

where

$$\begin{aligned} &\alpha_0^n(t,K,L) := \lambda^n(t,K,L)\;[n+\alpha_1(t)]\;S(K), \\ &\alpha^n(t,K,L) := (\alpha_1^n,\cdots,\alpha_d^n)', \\ &\alpha_i^n(t,K,L) := \lambda^n(t,K,L)\;[n+\alpha_2(t)]\;S(L_i),\;\; i=1,...,d. \end{aligned}$$

Here, we have written the Lipschitz generators F^n as linear functions in K and L. Note that, $\alpha_0^n(t,K,L)$ and $\alpha_i^n(t,K,L)$ are possibly unbounded. For simplicity, we denote $\alpha_0^n(\cdot,K(\cdot),L(\cdot))$ by $\alpha_0^n(\cdot)$ and $\alpha_i^n(\cdot,K(\cdot),L(\cdot))$ by $\alpha_i^n(\cdot)$. Therefore, (4.4.5) can be reformulated as

$$\begin{cases} dK^n(t) = -\left[a^n(t)K^n(t) + (c^n(t))'\,L^n(t) + Q(t)\right]dt \\ \qquad\qquad +L^n(t)\,dW(t), \qquad t\in[0,T], \\ K^n(T) \;=\; M, \quad a.s., \end{cases} \tag{4.4.8}$$

where

$$
\begin{aligned}
a^n(t) &:= a(t) + \alpha_0^n(t),\\
c^n(t) &:= (c_1^n, \ldots, c_d^n)',\\
c_i^n(t) &:= c_i(t) + \alpha_i^n(t), \quad i = 1, ..., d.
\end{aligned}
$$

Now, consider the following equations

$$
\begin{cases}
dY^n(t) = a^n(t)Y^n(t)\, dt + Y^n(t)\, \langle c^n(t), dW(t)\rangle\,, \qquad t \in [0,T],\\
\\
Y^n(0) = 1.
\end{cases} \tag{4.4.9}
$$

Lemma 4.4.3. *If conditions H_1 hold, then for each n, equations (4.4.9) have unique strong solutions $Y^n(\cdot)$.*

The existence and uniqueness of strong solutions of (4.4.9) follow from a result due to Gal'chuk [26], basic theorem on pp.756, also see Lemma 7.1. in [58]. The proof of Lemma 4.4.3 is given in the appendix to this chapter. We can find Y^n explicitly as:

$$
Y^n(t) = e^{\int_0^t [a^n(s) - \frac{1}{2}\,|c^n(s)|^2]\, ds + \int_0^t c^n(s)\, dW(s)}, \;\; t \in [0,T].
$$

The following lemma gives the nonnegativity result of K^n.

Lemma 4.4.4. *Let conditions H_1 hold. Then, for any $n \geq 0$, $K^n(t) \geq 0$, for all $t \in [0,T]$, a.s..*

Proof. By applying the Itô's product rule on $K^n(t)Y^n(t)$, we have:

$$
\begin{aligned}
d\,(K^n(t)\,Y^n(t)) &= (dK^n(t))\,Y^n(t) + K^n(t)\,dY^n(t) + (dK^n(t))\;(dY^n(t))\\
&= Y^n(t)\big[-a^n(t)K^n(t)\,dt - (c^n(t))'\,L^n(t)\,dt - Q(t)\,dt + L^n(t)\,dW(t)\big]\\
&\quad + K^n(t)\left[a^n(t)\,Y^n(t)\,dt + Y^n(t)(c^n(t))'\,dW(t)\right] + Y^n(t)\,(c^n(t))'\,L^n(t)\,dt\\
&= -Y^n(t)\,Q(t)\,dt + Y^n(t)\,L^n(t)\,dW(t) + Y^n(t)\,K^n(t)(c^n(t))'\,dW(t).
\end{aligned}
$$

By integrating from t to T, we obtain

$$K^n(t) = (Y^n(t))^{-1}\left[M\ Y^n(T) + \int_t^T Y^n(s)\,Q(s)\,ds \right.$$
$$\left. - \int_t^T \left[Y^n(s)\,L^n(s)\,dW(s) + Y^n(s)\,K^n(s)\,(c^n(s))'\,dW(s)\right]\right]. \quad (4.4.10)$$

Before taking expectation of (4.4.10), we show that the stochastic integrals on the right hand side are martingales. Here, we apply Theorem 2.4.8, then Theorem 2.3 in [28] to obtain finiteness for some expectation. From the Burkholder-Davis-Gundy inequality, see Theorem 2.4.8, there exists a constant K_1 such that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t Y^n(s)K^n(s)\,(c^n(s))'\,dW(s)\right|\right]\\
&\le K_1\,\mathbb{E}\left[\int_0^T |K^n(s)|^2\ |c^n(s)Y^n(s)|^2\,ds\right]^{\frac{1}{2}}\\
&\le K_1\,\mathbb{E}\left[\sup_{t\in[0,T]} p(t)\,|K^n(t)|^2\ \int_0^T \left|\frac{Y^n(s)\ c^n(s)}{\sqrt{p(s)}}\right|^2 ds\right]^{\frac{1}{2}}\\
&\le \frac{K_1}{2}\,\mathbb{E}\left[\sup_{t\in[0,T]} p(t)\,|K^n(t)|^2 + \int_0^T \frac{\left|Y^n(s)\ c^n(s)\right|^2}{p(s)}\ ds\right]. \qquad (4.4.11)
\end{aligned}$$

We have:

$$\begin{aligned}
&\mathbb{E}\int_0^T \frac{\left|Y^n(s)\ c^n(s)\right|^2}{p(s)}\ ds\\
&\le\ \mathbb{E}\left[\sup_{t\in[0,T]}\frac{|Y^n(t)|^2}{p(t)}\ \int_0^T \left|c^n(s)\right|^2\ ds\right]\\
&\le \frac{1}{2}\,\mathbb{E}\left(\sup_{t\in[0,T]}\frac{\left|Y^n(t)\right|^2}{p(t)}\right)^2 + \frac{1}{2}\,\mathbb{E}\left(\int_0^T \left|c^n(s)\right|^2\ ds\right)^2. \qquad (4.4.12)
\end{aligned}$$

For the first part in the right hand side of the previous inequality, we have

$$\begin{aligned}\frac{\big(Y^n(t)\big)^2}{p(t)} &= \frac{e^{2\int_0^t [a^n(s)-\frac{1}{2}|c^n(s)|^2]ds \,+2\int_0^t c^n(s)\,dW(s)}}{e^{\int_0^t [\gamma(s)+4\beta_1\alpha_1^2(s)+4\beta_2\alpha_2^2(s)]ds}}\\ &= e^{\int_0^t \big[2a^n(s)-|c^n(s)|^2-\gamma(s)-4\beta_1\alpha_1^2(s)-4\beta_2\alpha_2^2(s)\big]ds+2\int_0^t c^n(s)\,dW(s)}\\ &= e^{-\int_0^t -\big[2a^n(s)+|c^n(s)|^2-[\gamma(s)+4\beta_1\alpha_1^2(s)+4\beta_2\alpha_2^2(s)]\,\big]\,ds}\; e^{-\int_0^t \frac{1}{2}\,|2c^n(s)|^2\,ds}\; e^{-\int_0^t -2c^n(s)\,dW(s)}.\end{aligned}$$

We set $\hat{M}(t) := e^{-\int_0^t [r(s)+\frac{1}{2}|\theta(s)|^2]ds-\int_0^t \theta'(s)dW(s)}$, such that

$r(t) = -\big[2a^n(t)+|c^n(t)|^2-[\gamma(t)+4\beta_1\alpha_1^2(t)+4\beta_2\alpha_2^2(t)]\,\big]$, and $\theta(t) = -2c^n(t)$.
In order to prove that the following expectation

$$\mathbb{E}\left[\sup_{t\in[0,T]} \hat{M}^2(t)\right] = \mathbb{E}\left(\sup_{t\in[0,T]} \frac{\big(Y^n(t)\big)^2}{p(t)}\right)^2$$

is finite, we prove that the conditions of Theorem 2.3. in [28] hold. For some $\hat{\alpha}$ and $\hat{\beta}$ such that $\frac{\hat{\alpha}\,\hat{\beta}^2}{\hat{\beta}^2+2\,\hat{\alpha}\hat{\beta}-\hat{\alpha}} = 2$, we have:

$$\begin{aligned}&\mathbb{E}\Bigg[\sup_{t\in[0,T]} e^{-\hat{\alpha}\int_0^t r(s)\,ds}\Bigg]\\ &= \mathbb{E}\Bigg[\sup_{t\in[0,T]} e^{\hat{\alpha}\int_0^t \big[2a^n(s)+|c^n(s)|^2-\gamma(s)-4\beta_1\alpha_1^2(s)-4\beta_2\alpha_2^2(s)\big]\,ds}\Bigg]\\ &= \mathbb{E}\Bigg[\sup_{t\in[0,T]} e^{\hat{\alpha}\int_0^t \big[2[a(s)+\alpha_0^n(s)]+\sum_{i=1}^d (c_i^n(s))^2-\gamma(s)-4\beta_1\alpha_1^2(s)-4\beta_2\alpha_2^2(s)\,\big]ds}\Bigg]\\ &\le \mathbb{E}\Bigg[\sup_{t\in[0,T]} e^{\hat{\alpha}\int_0^t \big[2[a(s)+n+\alpha_1(s)]+\sum_{i=1}^d [c_i(s)+n+\alpha_2(s)]^2-\gamma(s)-4\beta_1\alpha_1^2(s)-4\beta_2\alpha_2^2(s)\,\big]ds}\Bigg]\\ &\le \mathbb{E}\Bigg[\sup_{t\in[0,T]} e^{\hat{\alpha}\int_0^t \big[2[a(s)+n+\alpha_1(s)]+3\sum_{i=1}^d c_i^2(s)+3n^2+3\alpha_2^2(s)-\gamma(s)-4\beta_1\alpha_1^2(s)-4\beta_2\alpha_2^2(s)\,\big]ds}\Bigg]\\ &= \mathbb{E}\Bigg[\sup_{t\in[0,T]} e^{\hat{\alpha}\int_0^t (2n+3n^2)ds}\;\; e^{\hat{\alpha}\int_0^t \big[2a(s)+2\alpha_1(s)+3|c(s)|^2+3\alpha_2^2(s)-\gamma(s)-4\beta_1\alpha_1^2(s)-4\beta_2\alpha_2^2(s)\,\big]ds}\Bigg]\\ &\le e^{\hat{\alpha}(2n+3n^2)T}\;\;\mathbb{E}\Bigg[\sup_{t\in[0,T]} e^{\hat{\alpha}\int_0^t \big[2a(s)+2\alpha_1(s)+3|c(s)|^2+3\alpha_2^2(s)-\gamma(s)-4\beta_1\alpha_1^2(s)-4\beta_2\alpha_2^2(s)\,\big]ds}\Bigg]\\ &< \infty,\end{aligned}$$

where the last step follows from $|a(\cdot)| \leq \alpha_1(\cdot)$, $|c(\cdot)| \leq \alpha_2(\cdot)$. Also, we have

$$
\begin{aligned}
\mathbb{E}\left[e^{\frac{\hat{\beta}}{2}\int_0^T |\theta(s)|^2\,ds}\right] &= \mathbb{E}\left[e^{\frac{\hat{\beta}}{2}\int_0^T |2c^n(s)|^2\,ds}\right] \\
&= \mathbb{E}\left[e^{2\hat{\beta}\int_0^T \sum_{i=1}^d (c_i^n(s))^2\,ds}\right] \\
&\leq \mathbb{E}\left[e^{2\hat{\beta}\int_0^T (\sum_{i=1}^d 3c_i^2(s)+3n^2+3\alpha_2^2(s))\,ds}\right] \\
&= \mathbb{E}\left[e^{2\hat{\beta}\int_0^T 3n^2\,ds}\;\; e^{2\hat{\beta}\int_0^T (3\sum_{i=1}^d c_i^2(s)+3\alpha_2^2(s))\,ds}\right] \\
&= \mathbb{E}\left[e^{6\hat{\beta}n^2T}\;\; e^{2\hat{\beta}\int_0^T (3|c(s)|^2+3\alpha_2^2(s))\,ds}\right] \qquad (4.4.13) \\
&\leq e^{6\hat{\beta}n^2T}\,\mathbb{E}\left[e^{12\hat{\beta}\int_0^T \alpha_2^2(s)\,ds}\right] < \infty.
\end{aligned}
$$

Therefore, by Theorem 2.3 in [28], we have

$$
\mathbb{E}\left(\sup_{t\in[0,T]} \frac{\big(Y^n(t)\big)^2}{p(t)}\right)^2 < \infty. \qquad (4.4.14)
$$

For the second part in the right hand side of (4.4.12), we have

$$
\begin{aligned}
\mathbb{E}\left(\int_0^T \left|c^n(s)\right|^2\;ds\right)^2 &= \mathbb{E}\left(\int_0^T \sum_{i=1}^d |c_i^n(s)|^2\,ds\right)^2 \\
&\leq \mathbb{E}\left(\int_0^T \sum_{i=1}^d [3c_i^2(s) + 3n^2 + 3\alpha_2^2(s)]\,ds\right)^2 \\
&= \mathbb{E}\left(\int_0^T [3|c(s)|^2 + 3n^2 + 3\alpha_2^2(s)]\,ds\right)^2 \\
&\leq 2\,\mathbb{E}\left(e^{\int_0^T [3|c(s)|^2+3n^2+3\alpha_2^2(s)]\,ds}\right) \\
&\leq 2\,\mathbb{E}\left(e^{3n^2T} + e^{\int_0^T 6\alpha_2^2(s)\,ds}\right) < \infty.
\end{aligned}
$$

By this, together with (4.4.14), the expectation in (4.4.12) is finite, and hence

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t Y^n(s)K^n(s)\,(c^n(s))'\,dW(s)\right|\right] < \infty.$$

Therefore, by Theorem 2.4.6, the process $\int_t^T Y^n(s)K^n(s)(c^n(s))'\,dW(s)$ is a martingale. Moreover, by Theorem 2.4.8, there exists a constant K_1 such that:

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t Y^n(s)L^n(s)\,dW(s)\right|\right]$$

$$\le K_1\,\mathbb{E}\left[\int_0^T\left|\frac{Y^n(s)}{\sqrt{p(s)}}\right|^2\;\left|\sqrt{p(s)}L^n(s)\right|^2\,ds\right]^{\frac{1}{2}}$$

$$\le \frac{K_1}{2}\,\mathbb{E}\left[\sup_{t\in[0,T]}\frac{(Y^n(t))^2}{p(t)} + \int_0^T p(s)\,|L^n(s)|^2\,ds\right] < \infty,$$

where the last step follows from the fact $L^n(\cdot)\in\mathcal{M}_p^2(0,T;\mathbb{R}^d)$ and from (4.4.14). Thus, the process $\int_t^T Y^n(s)L^n(s)\,dW(s)$ is a martingale. Going back to (4.4.10), by taking conditional expectation, we obtain

$$K^n(t) = (Y^n(t))^{-1}\,\mathbb{E}\left[M\,Y^n(T) + \int_t^T Y^n(s)\,Q(s)\,ds\middle|\mathcal{F}_t\right] \ge 0.$$

□

4.4.3 Uniform boundedness of solution pairs

In this section, we prove the uniform boundedness of the solution pairs $\{(K^n(\cdot),L^n(\cdot))\}_{n\ge 0}$.

Lemma 4.4.5. *Let conditions H_1 hold. Then $K^0(t)$ has the following repre-*

sentation:

$$K^0(t) = \mathbb{E}\Big[M e^{\int_t^T (a(s)-\frac{1}{2}|c(s)|^2)ds + \int_t^T c'(s)\, dW(s)} + \int_t^T e^{\int_t^s (a(\tau)-\frac{1}{2}|c(\tau)|^2)\, d\tau + \int_t^s c'(\tau)\, dW(\tau)} Q(s)\, ds \,\big| \mathcal{F}_t \Big], \quad t \in [0,T].$$

Proof. We have:

$$F^0(t, K^0, L^0) = \sup_{\hat{K}\in\mathbb{R}, \hat{L}\in\mathbb{R}^d} \big[\hat{F}(t, \hat{K}, \hat{L}) - \alpha_1(t)|K^0 - \hat{K}| - \alpha_2(t)|L^0 - \hat{L}| \big].$$

Since $\hat{F} \le 0$, the supremum here is reached for $K^0 = \hat{K}$ and $L^0 = \hat{L}$, and thus $F^0(t, K^0, L^0) = 0$. Therefore, the Riccati BSDE (4.4.5) that is generated by $G^0(t, K^0, L^0)$ becomes:

$$\begin{cases} dK^0(t) = -\big[a(t)K^0(t) + c'(t)\, L^0(t) + Q(t)\big]\, dt + L^0(t)\, dW(t), \quad t \in [0,T], \\ \\ K^0(T) = \ M \ \ a.s.. \end{cases}$$

This equation is a linear BSDE with possibly unbounded coefficients. The solution can be obtained in a similar way as in Lemma 4.4.4. We obtain

$$K^0(t) = \mathbb{E}\Big[X^{-1}(t)\, M\, X(T) \big| \mathcal{F}_t + X^{-1}(t) \int_t^T X(s)Q(s)ds \Big| \mathcal{F}_t \Big], \quad t \in [0,T].$$

□

Lemma 4.4.6. *If conditions H_1 hold, then the sequence $\{(K^n(\cdot), L^n(\cdot))\}_{n\ge 0}$ is uniformly bounded in $\mathcal{L}^\infty_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{M}^2_p(0,T;\mathbb{R}^d)$. In particular,*

$$\sup_{t\in[0,T]} \mathbb{E}\left[(K^n(t))^2\right] + \mathbb{E}\int_0^T p(t)|L^n(s)|^2 ds \le 2\,\mathbb{E}\big[p(T)\, M^2\big] + 2\,\mathbb{E}\int_0^T p(s)Q^2(s)\, ds,$$

where the upper bound is independent of n.

Proof. The idea of the proof is similar to that of Proposition 3.3. in [39]. However, different from [39], we obtain the boundedness of the sequence $\{L^n\}$ in the weighted space $\mathcal{M}^2_p(0,T;\mathbb{R}^d)$. By condition $H_1(iii)$, we have

$K^0(t) \leq \kappa$, for all $t \in [0,T]$ $a.s.$. By this, together with Lemma 4.4.2 and Lemma 4.4.4, we have:

$$\kappa \geq K^0(t) \geq K^1(t) \geq ... \geq K^n(t) \geq 0, \text{ for all } \quad t \in [0,T] \quad a.s..$$

Hence, the sequence $\{K^n\}_{n\geq 0}$ is uniformly bounded. To prove the uniform boundedness of $\{L^n\}_{n\geq 0}$ in $\mathcal{M}_p^2(0,T;\mathbb{R}^d)$, we apply the Itô's formula on $[p(t)(K^n(t))^2]$. Recall the equation:

$$\begin{cases} dK^n(t) = & -\big[a(t)K^n(t) + c'(t)\,L^n(t) + Q(t) + F^n(t, K^n(t), L^n(t))\big]\,dt \\ \\ & +L^n(t)\,dW, \qquad t \in [0,T], \\ \\ K^0(T) = & M \;\; a.s.. \end{cases}$$

We have:

$$\begin{aligned} d\big[p(t)\,(K^n(t))^2\big] &= p(t)\,\big[\gamma(t) + 4\beta_1\,\alpha_1^2(t) + 4\beta_2\,\alpha_2^2(t)\big](K^n(t))^2\,dt \\ &\quad + 2\,p(t)\;K^n(t)\;\big[-a(t)K^n(t) - c'(t)\,L^n(t) - Q(t)\; - F^n(t, K^n(t), L^n(t))\big]\;dt \\ &\quad + p(t)\,|L^n(t)|^2 dt + 2\,p(t)\;K^n(t)\,L^n(t)\,dW(t) \\ &= p(t)\,\big[\gamma(t) + 4\beta_1\,\alpha_1^2(t) + 4\beta_2\alpha_2^2(t)\big](K^n(t))^2\,dt - 2\,a(t)\,p(t)\,(K^n(t))^2\,dt \\ &\quad -\;2\,p(t)\;K^n(t)\,c'(t)\,L^n(t)\;dt - 2p(t)\,K^n(t)\,Q(t)\,dt \\ &\quad - 2p(t)\,K^n(t)\,F^n(t, K^n(t), L^n(t))\,dt \\ &\quad +\;p(t)\,|L^n(t)|^2\,dt +\;2\,p(t)\;K^n(t)\,L^n(t)\,dW(t). \end{aligned}$$

By integrating from t to T, and then taking expectation, we obtain:

$$
\begin{aligned}
\mathbb{E}\ \left[p(t)\ (K^n(t))^2\right] = \mathbb{E}\left[p(T)\,(K^n(T))^2\right] \\
& - \mathbb{E}\ \int_t^T p(s)\ \left[\gamma(s) + 4\beta_1\,\alpha_1^2(s) + 4\beta_2\alpha_2^2(s)\right](K^n(s))^2\ ds \\
& + 2\mathbb{E}\ \int_t^T a(s)\ p(s)\ (K^n(s))^2\ ds +\ 2\,\mathbb{E}\ \int_t^T\ p(s)\ K^n(s)\,c'(s)\,L^n(s)\,ds \\
& + 2\mathbb{E}\ \int_t^T p(s)\ K^n(s)\,Q(s)\ ds +\ 2\,\mathbb{E}\ \int_t^T p(s)\ K^n(s)\ F^n(s, K^n(s), L^n(s))\ ds \\
& - \mathbb{E}\ \int_t^T p(s)\ |L^n(s)|^2\ ds -\ 2\,\mathbb{E}\ \int_t^T p(s)\ K^n(s)\,L^n(s)\,dW(s).
\end{aligned}
\tag{4.4.15}
$$

Since $[p(t)\,K^n(t)\,F^n(t, K^n, L^n)] \leq 0$, then (4.4.15) becomes:

$$
\begin{aligned}
\mathbb{E}\left[p(t)\,(K^n(t))^2\right] \leq \mathbb{E}\left[p(T)\,M^2\right] \\
& - \mathbb{E}\int_t^T p(s)\ \left[\gamma(s) + 4\beta_1\,\alpha_1^2(s) + 4\beta_2\,\alpha_2^2(s)\right](K^n(s))^2\,ds \\
& + 2\,\mathbb{E}\int_t^T a(s)\,p(s)\,(K^n(s))^2\,ds +\ 2\,\mathbb{E}\ \int_t^T\ p(s)\,(K^n(s))^2\,|c(s)|^2\,ds \\
& + \frac{1}{2}\,\mathbb{E}\int_t^T\ p(s)\,|L^n(s)|^2\,ds + \mathbb{E}\ \int_t^T p(s)\,(K^n(s))^2\,ds + \mathbb{E}\ \int_t^T\ p(s)\,Q^2(s)\,ds \\
& - \mathbb{E}\int_t^T p(s)\,|L^n(s)|^2\,ds -\ 2\,\mathbb{E}\int_t^T p(s)\,K^n(s)\,L^n(s)\,dW(s)
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbb{E}\left[p(T)\,M^2\right] + \mathbb{E}\int_t^T p(s)\,(K^n(s))^2 \\
&\quad \times \left[2\,a(s) + 2\,|c(s)|^2 + 1 - \gamma(s) - 4\beta_1\,\alpha_1^2(s) - 4\beta_2\,\alpha_2^2(s)\right] ds \\
&- \frac{1}{2}\,\mathbb{E}\int_t^T p(s)\,|L^n(s)|^2\,ds \\
&+ \mathbb{E}\int_t^T p(s)\,Q^2(s)\,ds - 2\,\mathbb{E}\int_t^T p(s)\,K^n(s)\,L^n(s)\,dW(s).
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
&\mathbb{E}\left[p(t)\,(K^n(t))^2\right] + \frac{1}{2}\,\mathbb{E}\int_t^T p(s)\,|L^n(s)|^2\,ds \\
&\le \mathbb{E}\left[p(T)\,M^2\right] + \mathbb{E}\int_t^T p(s)\,Q^2(s)\,ds - 2\,\mathbb{E}\int_t^T p(s)\,K^n(s)\,L^n(s)\,dW(s).
\end{aligned} \tag{4.4.16}
$$

In the same way as in the proof of Lemma 2.1 (ii) of [27], it can be shown that the stochastic integral in (4.4.16) is a martingale. Hence:

$$
\mathbb{E}\left[p(t)\,(K^n(t))^2\right] + \frac{1}{2}\,\mathbb{E}\int_t^T p(s)\,|L^n(s)|^2\,ds \le \mathbb{E}\left[p(T)\,M^2\right] + \mathbb{E}\int_t^T p(s)\,Q^2(s)\,ds.
$$

By taking supremum over $t \in [0,T]$, we obtain

$$
\sup_{t\in[0,T]} \mathbb{E}\left[(K^n(t))^2\right] + \frac{1}{2}\,\mathbb{E}\int_0^T p(s)\,|L^n(s)|^2\,d \le \mathbb{E}\left[p(T)\,M^2\right] + \mathbb{E}\int_0^T p(s)\,Q^2(s)\,ds.
$$

□

4.4.4 Strong convergence and existence results

The following lemmas are the main reasoning behind the existence result. We give sufficient conditions, which permit for unbounded coefficients, for the strong convergence of $\{K^n(t)\}_{n\geq 0}$ and $\{L^n(t)\}_{n\geq 0}$.

Lemma 4.4.7. *If conditions H_1 hold, then $\{K^n(t)\}_{n\geq 0}$ converges almost surely to a process $(K(t), t \in [0,T])$ in $\mathcal{L}^{\infty}_{\mathcal{F}}(0,T;\mathbb{R})$.*

Proof. From Lemma 4.4.2 and Lemma 4.4.6, we know that the sequence

$\{K^n(t)\}_{n\geq 0}$ is decreasing and uniformly bounded. Thus, by Theorem 2.3.3, $\{K^n(t)\}_{n\geq 0}$ converges almost surely to a process $K(\cdot) \in \mathcal{L}^\infty_{\mathcal{F}}(0,T;\mathbb{R})$. $\square$

Before presenting the strong convergence result of $\{L^n(t)\}_{n\geq 0}$, we give the following lemma. The proof is given in the appendix to this chapter.

Lemma 4.4.8. *Let the random function Q be defined as:*

$$Q(t,x) := \frac{1}{12\lambda_2(t)+1}\big[e^{(12\lambda_2(t)+1)x} - 1\big] - x, \quad (t,x) \in [0,T] \times [0,\kappa].$$

The following properties hold:

(i) $Q(t,0) = 0$ *for all* $t \in [0,T]$ *a.s.;*

(ii) $\dfrac{\partial Q}{\partial x}(t,x) \geq 0$ *for all* $(t,x) \in [0,T] \times [0,\kappa]$ *a.s.;*

(iii) $\dfrac{\partial Q}{\partial t}(t,x) \leq \dfrac{12\,\delta(t)\,x\,e^{(12\lambda_2(t)+1)x}}{12\lambda_2(t)+1}$ *for all* $(t,x) \in [0,T] \times [0,\kappa]$ *a.s.;*

(iv) $\dfrac{\partial Q}{\partial x}(t,0) = \dfrac{\partial Q}{\partial t}(t,0) = 0$ *for all* $t \in [0,T]$ *a.s..*

Lemma 4.4.9. *Let conditions H_1 and H_2 hold. Then the sequence $\{L^n(t)\}_{n\geq 0}$ converges to a process $L(\cdot)$ in $\mathcal{M}^2_{\lambda_2}(0,T;\mathbb{R}^d)$. That is:*

$$\lim_{n\to\infty} \mathbb{E}\int_0^T \lambda_2(s)\,|L^n(s) - L(s)|^2\,ds = 0.$$

Proof. The method of proof has been used in [45], [37] and [39]. In our case, the proof is more difficult since we show this result with possibly unbounded coefficients. Also, we prove the convergence of $\{L^n(t)\}_{n\geq 0}$ in a weighted space. From Lemma 4.4.6, the sequence $\{L^n(t)\}_{n\geq 0}$ is uniformly bounded in $\mathcal{M}^2_p(0,T;\mathbb{R}^d)$. Therefore, by Theorem 2.3.6, there exists a subsequence $\{L^{n_k}(t)\}_{k\geq 0}$ of $\{L^n(t)\}_{n\geq 0}$ and a process $L(\cdot)$ in $\mathcal{M}^2_p(0,T;\mathbb{R}^d)$ such that $\{L^{n_k}(t)\}$ converges weakly to $L(\cdot)$ in $\mathcal{M}^2_p(0,T;\mathbb{R}^d)$. The next step is to prove that the whole sequence $\{L^n(t)\}_{n\geq 0}$ converges strongly to $L(\cdot)$ in

$\mathcal{M}^2_{\lambda_2}(0,T;\mathbb{R}^d)$. Let l and r be two elements of the subsequence $\{n_k\}_{k\geq 0}$ such that $l<r$. From (4.4.5), we have:

$$K^l(t)-K^r(t)=(K^l(T)-K^r(T))+\int_t^T \left[G^l(s,K^l(s),L^l(s))-G^r(s,K^r(s),L^r(s))\right]ds$$
$$-\int_t^T \left(L^l(s)-L^r(s)\right)'\,dW(s),\quad t\in[0,T],$$

where

$$\begin{aligned} G^l(t,K^l,L^l) &= a(t)K^l+c'(t)\,L^l+Q(t)+F^l(t,K^l,L^l),\\ G^r(t,K^r,L^r) &= a(t)K^r+c'(t)\,L^r+Q(t)+F^r(t,K^r,L^r). \end{aligned}$$

The the differential of $Q(t,K^l(t)-K^r(t))$ is:

$$\begin{aligned} dQ(t,K^l(t)-K^r(t)) &= \frac{\partial Q}{\partial t}(t,K^l(t)-K^r(t))\,dt\\ &- \left[G^l(t,K^l(t),L^l(t))-G^r(t,K^r(t),L^r(t))\right]\,\frac{\partial Q}{\partial K}(t,K^l(t)-K^r(t))\,dt\\ &+ \frac{1}{2}\,\frac{\partial^2 Q}{\partial K^2}(t,K^l(t)-K^r(t))\,\,|L^l(t)-L^r(t)|^2\,dt\\ &+ \frac{\partial Q}{\partial K}(t,K^l(t)-K^r(t))\,[L^l(t)-L^r(t)]'\,dW(t). \end{aligned}$$

By integrating from 0 to T, we obtain:

$$\begin{aligned} Q(0,K^l(0)-K^r(0)) &= Q(T,K^l(T)-K^r(T))\\ &- \int_0^T \frac{\partial Q}{\partial s}(s,K^l(s)-K^r(s))\,ds \qquad (4.4.17)\\ &+ \int_0^T [G^l(s,K^l(s),L^l(s))-G^r(s,K^r(s),L^r(s))]\\ &\times \frac{\partial Q}{\partial K}(s,K^l(s)-K^r(s))\,ds\\ &- \frac{1}{2}\int_0^T \frac{\partial^2 Q}{\partial K^2}(s,K^l(s)-K^r(s))\,|L^l(s)-L^r(s)|^2\,ds\\ &- \int_0^T \frac{\partial Q}{\partial K}(s,K^l(s)-K^r(s))\,[L^l(s)-L^r(s)]'\,dW(s). \end{aligned}$$

We have:

$$\begin{aligned}&G^l(t,K^l,L^l)-G^r(t,K^r,L^r)\\&=a(t)[K^l-K^r]+c'(t)\,[L^l-L^r]+[F^l(t,K^l,L^l)-F^r(t,K^r,L^r)].\end{aligned}$$

Since $F^l(t,K^l,L^l)\le 0$ and $F^r(t,K^r,L^r)\ge \hat{F}(t,K^r,L^r)$, we obtain:

$$G^l(t,K^l,L^l)-G^r(t,K^r,L^r)\le a(t)[K^l-K^r]+c'(t)\,[L^l-L^r]-\hat{F}(t,K^r,L^r).$$

By this, and by Lemma 4.4.8(*ii*), (4.4.17) becomes:

$$\begin{aligned}&Q(0,K^l(0)-K^r(0))\le\ Q(T,K^l(T)-K^r(T))-\int_0^T\frac{\partial Q}{\partial s}(s,K^l(s)-K^r(s))\,ds\\&+\int_0^T\Big[a(s)[K^l(s)-K^r(s)]+c'(s)\,[L^l(s)-L^r(s)]-\hat{F}(s,K^r(s),L^r(s))\Big]\\&\times\frac{\partial Q}{\partial K}(s,K^l(s)-K^r(s))\,ds\\&-\frac{1}{2}\int_0^T\frac{\partial^2 Q}{\partial K^2}(s,K^l(s)-K^r(s))\,|L^l(s)-L^r(s)|^2\,ds\\&-\int_0^T\frac{\partial Q}{\partial K}(s,K^l(s)-K^r(s))\,[L^l(s)-L^r(s)]'\,dW(s).\end{aligned}\tag{4.4.18}$$

By the Burkholder-Davis-Gundy inequality, see Theorem 2.4.8, there exists a constant K_1 such that:

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t\frac{\partial Q}{\partial K}(s,K^l(s)-K^r(s))\,\big[L^l(s)-L^r(s)\big]'\,dW(s)\Big|\Big]\\&\le K_1\,\mathbb{E}\Big[\int_0^T\Big[\frac{\partial Q}{\partial K}(s,K^l(s)-K^r(s))\Big]^2|L^l(s)-L^r(s)|^2\,ds\Big]^{\frac{1}{2}}\\&\le K_1\,\mathbb{E}\Big[\sup_{t\in[0,T]}\Big[\frac{\partial Q}{\partial K}(t,K^l(t)-K^r(t))\Big]^2\frac{1}{p(t)}\int_0^T p(s)\,|L^l(s)-L^r(s)|^2\,ds\Big]^{\frac{1}{2}}\\&\le\frac{K_1}{2}\,\mathbb{E}\Big[\sup_{t\in[0,T]}\frac{\big(e^{(12\lambda_2(t)+1)\,\kappa}-1\big)^2}{p(t)}\ +\ \int_0^T p(s)\ |L^l(s)-L^r(s)|^2\,ds\Big].\end{aligned}$$

We have:

$$\frac{d\lambda_2^2(t)}{dt} = 2\,\lambda_2(t)\,\frac{d\lambda_2(t)}{dt} = 2\,\delta(t)\,\lambda_2(t), \;\; \text{thus,} \;\; \lambda_2^2(t) = \lambda_2^2(0) + 2\int_0^t \delta(s)\,\lambda_2(s)\,ds.$$

Thus,

$$\begin{aligned}\left(e^{(12\lambda_2(t)+1)\,\kappa} - 1\right)^2 &\leq 2\,e^{2(12\lambda_2(t)+1)\,\kappa} + 2 \leq 4\,e^{2(12\lambda_2(t)+1)\,\kappa} \\ &= 4\,e^{24\,\lambda_2(t)\,\kappa}\,e^{2\,\kappa} \leq 4\,e^{12\,\kappa^2}\;e^{2\,\kappa}\,e^{12\,\lambda_2^2(t)} \\ &= 4\,e^{12\,\kappa^2}\;e^{2\,\kappa}\,e^{12\,[\lambda_2^2(0)+2\int_0^t \delta(s)\,\lambda_2(s)\,ds]} \\ &= 4\,e^{12\,\kappa^2}\;e^{2\,\kappa}\,e^{12\,\lambda_2^2(0)}\,e^{24\int_0^t \delta(s)\,\lambda_2(s)\,ds}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]} \frac{\left(e^{(12\lambda_2(t)+1)\,\kappa} - 1\right)^2}{p(t)}\right] \\ &\leq 4\,e^{12\,\kappa^2}\;e^{2\,\kappa}\,e^{12\,\lambda_2^2(0)}\;\mathbb{E}\;\left[\sup_{t\in[0,T]} \frac{e^{24\int_0^t \delta(s)\,\lambda_2(s)\,ds}}{p(t)}\right] \qquad (4.4.19) \\ &< \infty.\end{aligned}$$

By (4.4.19), together with the fact that $L^l(\cdot)$, $L^r(\cdot)$ belong to $\mathcal{M}_p^2(0,T;\mathbb{R}^d)$, we obtain:

$$\mathbb{E}\left[\sup_{t\in[0,T]} \;\left|\int_0^t \frac{\partial Q}{\partial K}(s, K^l(s) - K^r(s))\,\left[L^l(s) - L^r(s)\right]'\,dW(s)\right|\right] < \infty.$$

Then by Theorem 2.4.6, the stochastic integral

$$\int_t^T \frac{\partial Q}{\partial K}(s, K^l(s) - K^r(s))\,[L^l(s) - L^r(s)]'\,dW(s), \quad t\in[0,T],$$

is a martingale. Now, by taking expectation of (4.4.18), and by using the inequality $2ab \leq [a^2 + b^2]$, we obtain:

$$
\begin{aligned}
\mathbb{E}\, Q(0, K^l(0) - K^r(0)) \leq{}& \mathbb{E}\, Q(T, K^l(T) - K^r(T)) - \mathbb{E} \int_0^T \frac{\partial Q}{\partial s}(s, K^l(s) - K^r(s))\, ds \\
&+ \mathbb{E} \int_0^T a(s)[K^l(s) - K^r(s)]\, \frac{\partial Q}{\partial K}(s, K^l(s) - K^r(s))\, ds \\
&+ \frac{1}{2}\, \mathbb{E} \int_0^T |c(s)|^2\, \frac{\partial Q}{\partial K}(s, K^l(s) - K^r(s))\, ds \\
&+ \frac{1}{2}\, \mathbb{E} \int_0^T |L^l(s) - L^r(s)|^2\, \frac{\partial Q}{\partial K}(s, K^l(s) - K^r(s))\, ds \\
&- \mathbb{E} \int_0^T \hat{F}(s, K^r(s), L^r(s))\, \frac{\partial Q}{\partial K}(s, K^l(s) - K^r(s))\, ds \\
&- \frac{1}{2}\, \mathbb{E} \int_0^T \frac{\partial^2 Q}{\partial K^2}(s, K^l(s) - K^r(s))\; |L^l(s) - L^r(s)|^2\, ds.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mathbb{E} \int_0^T & |L^l(s) - L^r(s)|^2 \left[\frac{1}{2}\frac{\partial^2 Q}{\partial K^2}(s, K^l(s) - K^r(s)) - \frac{1}{2}\, \frac{\partial Q}{\partial K}(s, K^l(s) - K^r(s))\right] ds \\
&+ \mathbb{E}\, Q(0, K^l(0) - K^r(0)) \qquad\qquad (4.4.20) \\
&\leq \mathbb{E}\, Q(T, K^l(T) - K^r(T)) - \mathbb{E} \int_0^T \frac{\partial Q}{\partial s}(s, K^l(s) - K^r(s))\, ds \\
&+ \mathbb{E} \int_0^T a(s)[K^l(s) - K^r(s)]\, \frac{\partial Q}{\partial K}(s, K^l(s) - K^r(s))\, ds \\
&+ \frac{1}{2}\, \mathbb{E} \int_0^T |c(s)|^2\, \frac{\partial Q}{\partial K}(s, K^l(s) - K^r(s))\, ds \\
&- \mathbb{E} \int_0^T \hat{F}(s, K^r(s), L^r(s))\, \frac{\partial Q}{\partial K}(s, K^l(s) - K^r(s))\, ds.
\end{aligned}
$$

We have $|K^r|D'D \geq 0$, thus $N^{-1} \geq \left[N + |K^r|D'D\right]^{-1}$. By using this inequality, we have:

$$
\begin{aligned}
-\hat{F}(t, K^r, L^r) &= \big[B(t)K^r + C'(t)D(t)K^r + (L^r)'D(t)\big]\big[N(t) + |K^r|D'(t)D(t)\big]^{-1} \\
&\times \big[B(t)K^r + C'(t)D(t)K^r + (L^r)'D(t)\big]' \\
&\leq \big[B(t)K^r + C'(t)D(t)K^r + (L^r)'D(t)\big]\big[N(t)\big]^{-1} \\
&\times \big[B(t)K^r + C'(t)D(t)K^r + (L^r)'D(t)\big]' \\
&\leq \varepsilon^{-1}\, |B(t)K^r + C'(t)D(t)K^r + (L^r)'D(t)|^2 \\
&\leq \varepsilon^{-1}\, \{3|B(t)K^r|^2 + 3|C'(t)D(t)K^r|^2 + 3|(L^r)'D(t)|^2\} \\
&\leq \varepsilon^{-1}\, \{3\,\kappa^2\,|B(t)|^2 + 3\,\kappa^2\,|C'(t)D(t)|^2 + 3\,|(L^r)'D(t)|^2\} \\
&\leq \lambda_1(t) + \lambda_2(t)\,|L^r|^2 = \lambda_1(t) + \lambda_2(t)\,|L^r - L^l + L^l|^2 \\
&\leq \lambda_1(t) + \lambda_2(t)\big[\,2|L^l - L^r|^2 + 2|L^l|^2\big] \\
&= \lambda_1(t) + \lambda_2(t)\big[\,2|L^l - L^r|^2 + 2|L^l - L + L|^2\big] \\
&\leq \lambda_1(t) + 2\lambda_2(t)\big[\,|L^l - L^r|^2 + 2|L^l - L|^2 + 2|L|^2\big] \\
&= \lambda_1(t) + 2\,\lambda_2(t)\,|L^l - L^r|^2 + 4\,\lambda_2(t)|L^l - L|^2 + 4\,\lambda_2(t)|L|^2.
\end{aligned}
$$

Note that, $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$ depend on ε, κ and the coefficients $B(\cdot), C(\cdot), D(\cdot)$, and they are independent of r and l. Putting this into (4.4.20), we obtain:

$$
\begin{aligned}
& \mathbb{E} \int_0^T |L^l(s) - L^r(s)|^2 \left[\frac{1}{2} \frac{\partial^2 Q}{\partial K^2}(s, K^l(s) - K^r(s)) - \frac{1}{2} \frac{\partial Q}{\partial K}(s, K^l(s) - K^r(s)) \right] ds \\
+ \quad & \mathbb{E}\, Q(0, K^l(0) - K^r(0)) \\
\leq \quad & \mathbb{E}\, Q(T, K^l(T) - K^r(T)) - \mathbb{E} \int_0^T \frac{\partial Q}{\partial s}(s, K^l(s) - K^r(s))\, ds \\
+ \quad & \mathbb{E} \int_0^T a(s)\, [K^l(s) - K^r(s)]\ \frac{\partial Q}{\partial K}(s, K^l(s) - K^r(s))\, ds \\
+ \quad & \frac{1}{2}\, \mathbb{E} \int_0^T |c(s)|^2\ \frac{\partial Q}{\partial K}(s, K^l(s) - K^r(s))\, ds \\
+ \quad & \mathbb{E} \int_0^T \frac{\partial Q}{\partial K}(s, K^l(s) - K^r(s)) \\
\times \quad & \left[\lambda_1(s) + 2\, \lambda_2(s)\, |L^l(s) - L^r(s)|^2 + 4\, \lambda_2(s) |L^l(s) - L(s)|^2 + 4\, \lambda_2(s) |L(s)|^2 \right] ds.
\end{aligned}
$$

By moving the terms in $|L^l(t) - L^r(t)|^2$ and $|L^l(t) - L(t)|^2$ to the left-hand side of the previous inequality, we obtain:

$$
\begin{aligned}
& \mathbb{E} \int_0^T |L^l(s) - L^r(s)|^2 \Big[\frac{1}{2} \frac{\partial^2 Q}{\partial K^2}(s, K^l(s) - K^r(s)) - \frac{1}{2} \frac{\partial Q}{\partial K}(s, K^l(s) - K^r(s)) \\
-2 \quad & \lambda_2(s) \frac{\partial Q}{\partial K}(s, K^l(s) - K^r(s)) \Big] ds \\
+ \quad & \mathbb{E}\, Q(0, K^l(0) - K^r(0)) \\
- \quad & 4\, \mathbb{E} \int_0^T \lambda_2(s) |L^l(s) - L(s)|^2 \frac{\partial Q}{\partial K}(s, K^l(s) - K^r(s))\, ds \\
\leq \quad & \mathbb{E}\, Q(T, K^l(T) - K^r(T)) - \mathbb{E} \int_0^T \frac{\partial Q}{\partial s}(s, K^l(s) - K^r(s))\, ds \qquad (4.4.21) \\
+ \quad & \mathbb{E} \int_0^T a(s) \, [K^l(s) - K^r(s)] \frac{\partial Q}{\partial K}(s, K^l(s) - K^r(s))\, ds \\
+ \quad & \frac{1}{2} \mathbb{E} \int_0^T |c(s)|^2 \frac{\partial Q}{\partial K}(s, K^l(s) - K^r(s))\, ds \\
+ \quad & \mathbb{E} \int_0^T \frac{\partial Q}{\partial K}(s, K^l(s) - K^r(s)) \Big[\lambda_1(s) + 4\, \lambda_2(s) |L(s)|^2 \Big] ds.
\end{aligned}
$$

Before we take the limit as r goes to ∞ along the subsequence $\{n_k\}_{k \geq 0}$, we first study the applicability of the dominated convergence theorem for (4.4.21). Firstly, from Lemma 4.4.7, we know that $\lim_{r \to \infty} K^r(t) = K(t)$ for all $t \in [0, T]$ *a.s.*. By this, and since:

$$\mathbb{E}\, Q(0, K^l(0) - K^r(0)) = Q(0, K^l(0) - K^r(0)),$$

we obtain:

$$\lim_{r \to \infty} \mathbb{E}\, Q(0, K^l(0) - K^r(0)) = Q(0, K^l(0) - K(0)).$$

Moreover, we have:

$$\begin{aligned}
&\mathbb{E} \int_0^T a(s)[K^l(s) - K^r(s)]\, \frac{\partial Q}{\partial K}(s, K^l(s) - K^r(s))\, ds \\
&= \mathbb{E} \int_0^T a(s)\, [K^l(s) - K^r(s)]\, \left(e^{(12\lambda_2(s)+1)\,[K^l(s)-K^r(s)]} - 1\right) ds \\
&\leq \kappa\ \mathbb{E} \int_0^T |a(s)|\, \left(e^{(12\lambda_2(s)+1)\,\kappa} - 1\right) ds < \infty,
\end{aligned}$$

then, by applying Theorem 2.3.4, we have

$$\begin{aligned}
&\lim_{r\to\infty} \mathbb{E} \int_0^T a(s)[K^l(s) - K^r(s)]\, \frac{\partial Q}{\partial K}(s, K^l(s) - K^r(s))ds \\
&= \mathbb{E} \int_0^T a(s)[K^l(s) - K(s))\, \frac{\partial Q}{\partial K}(s, K^l(s) - K(s))ds.
\end{aligned}$$

Furthermore, by Lemma 4.4.8(iii), we have

$$\begin{aligned}
&\mathbb{E} \int_0^T \frac{\partial Q}{\partial s}(s, K^l(s) - K^r(s))\, ds \\
&\leq 12\, \mathbb{E} \int_0^T [K^l(s) - K^r(s)]\, \frac{\delta(s)\ e^{(12\lambda_2(s)+1)\,\kappa}}{12\lambda_2(s)+1} ds \\
&\leq 12\, e^{\kappa}\, e^{6\kappa^2}\, \mathbb{E} \int_0^T [K^l(s) - K^r(s)]\, e^{6\lambda_2^2(s)} ds \\
&= 12\, e^{\kappa}\, e^{6\kappa^2}\, \mathbb{E} \int_0^T [K^l(s) - K^r(s)]\, e^{6[\lambda_2^2(0) + 2\int_0^s \delta(u)\,\lambda_2(u)\,du]}\ ds \\
&= 12\, e^{\kappa}\, e^{6\kappa^2}\, e^{6\lambda_2^2(0)}\, \mathbb{E} \int_0^T [K^l(s) - K^r(s)]\, e^{12\int_0^s \delta(u)\,\lambda_2(u)\,du}\ ds \\
&\leq 24\, e^{\kappa}\, e^{6\kappa^2}\, e^{6\lambda_2^2(0)}\, \mathbb{E} \left[\sup_{t\in[0,T]} p(t)\, \{(K^l(t))^2 + (K^r(t))^2\}\right] < \infty,
\end{aligned}$$

where the last step follows from the fact that $\{K^n(t)\}_{n\geq 0}$ belongs to $\mathcal{H}_p^2(0,T;\mathbb{R})$. Thus by applying Theorem 2.3.4, we obtain:

$$\lim_{r\to\infty} \mathbb{E} \int_0^T \frac{\partial Q}{\partial s}(s, K^l(s) - K^r(s))\, ds = \mathbb{E} \int_0^T \frac{\partial Q}{\partial s}(s, K^l(s) - K(s))\, ds.$$

In addition, we have:

$$\mathbb{E}\int_0^T |c(s)|^2 \frac{\partial Q}{\partial K}(s, K^l(s) - K^r(s))\, ds \leq \mathbb{E}\int_0^T |c(s)|^2 \big(e^{(12\lambda_2(s)+1)\,\kappa} - 1\big)\, ds < \infty.$$

As a result, we obtain:

$$\lim_{r\to\infty} \mathbb{E}\int_0^T |c(s)|^2 \frac{\partial Q}{\partial K}(s, K^l(s) - K^r(s))\, ds = \mathbb{E}\int_0^T |c(s)|^2 \frac{\partial Q}{\partial K}(s, K^l(s) - K(s))\, ds.$$

Also, we have:

$$\begin{aligned}
&\mathbb{E} \int_0^T \big[\lambda_1(s) + 4\,\lambda_2(s)\; |L(s)|^2\big]\; \frac{\partial Q}{\partial K}(s, K^l(s) - K^r(s))\, ds \\
&\leq \mathbb{E} \int_0^T \big[\lambda_1(s) + 4\lambda_2(s)|L(s)|^2\big]\big(e^{(12\lambda_2(s)+1)\kappa} - 1\big)\, ds \qquad (4.4.22)\\
&= \mathbb{E}\int_0^T \lambda_1(s)\,\big(e^{(12\lambda_2(s)+1)\kappa} - 1\big)\, ds + 4\,\mathbb{E}\int_0^T \lambda_2(s)\,\big(e^{(12\lambda_2(s)+1)\kappa} - 1\big)|L(s)|^2\, ds.
\end{aligned}$$

For the second term in the right hand side, we have:

$$\begin{aligned}
\lambda_2(t)\,\big(e^{(12\lambda_2(t)+1)\,\kappa} - 1\big) &\leq\; e^{\lambda_2(t)}\, e^{(12\lambda_2(t)+1)\,\kappa} \;\leq\; e^{\lambda_2(t)}\, e^{\kappa}\, e^{6\lambda_2^2(t)}\, e^{6\,\kappa^2} \\
&\leq\; e^{\lambda_2^2(t)}\, e^{\kappa}\, e^{6\lambda_2^2(t)}\, e^{6\,\kappa^2} \;=\; e^{\kappa}\, e^{7\lambda_2^2(t)}\, e^{6\,\kappa^2} \\
&=\; e^{6\,\kappa^2}\, e^{\kappa}\, e^{7\,[\lambda_2^2(0) + 2\int_0^t \delta(s)\,\lambda_2(s)\,ds]} \\
&=\; e^{6\,\kappa^2}\, e^{\kappa}\, e^{7\,\lambda_2^2(0)}\, e^{14\int_0^t \delta(s)\,\lambda_2(s)\,ds}.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
& 4\,\mathbb{E}\int_0^T \lambda_2(s)\left(e^{(12\lambda_2(s)+1)\kappa}-1\right)|L(s)|^2\,ds \\
& \leq 4\,e^{6\,\kappa^2}\,e^{\kappa}\,e^{7\,\lambda_2^2(0)}\,\mathbb{E}\int_0^T e^{14\int_0^t \delta(s)\,\lambda_2(s)\,ds}\,|L(s)|^2\,ds \\
& \leq 4\,e^{6\,\kappa^2}\,e^{\kappa}\,e^{7\,\lambda_2^2(0)}\,\mathbb{E}\int_0^T p(s)\,|L(s)|^2\,ds < \infty,
\end{aligned}
$$

where the last two steps follow from $24\,\delta(t)\,\lambda_2(t) \leq \gamma(t)+4\beta_1\alpha_1^2(t)+4\beta_2\alpha_2^2(t)$, and from the fact that $L(\cdot)$ belongs to $\mathcal{M}_p^2(0,T;\mathbb{R}^d)$. By this, together with condition $H_2(iv)$, it follows that:

$$
\begin{aligned}
& \lim_{r\to\infty}\ \mathbb{E}\int_0^T \left[\lambda_1(s)+4\lambda_2(s)|L(s)|^2\right]\frac{\partial Q}{\partial K}(s,K^l(s)-K^r(s))\,ds \\
& = \mathbb{E}\int_0^T \left[\lambda_1(s)+4\lambda_2(s)|L(s)|^2\right]\frac{\partial Q}{\partial K}(s,K^l(s)-K(s))\,ds.
\end{aligned}
$$

Similarly, we have:

$$
\begin{aligned}
& \mathbb{E}\int_0^T \lambda_2(s)\,|L^l(s)-L(s)|^2\,\frac{\partial Q}{\partial K}(s,K^l(s)-K^r(s))\,ds \\
& \leq\ \mathbb{E}\int_0^T \lambda_2(s)\,|L^l(s)-L(s)|^2\left(e^{(12\lambda_2(s)+1)\kappa}-1\right)ds \\
& \leq 2\,e^{6\,\kappa^2}\,e^{\kappa}\,e^{7\,\lambda_2^2(0)}\,\mathbb{E}\int_0^T p(s)\,\{|L^l(s)|^2+|L(s)|^2\}\,ds < \infty.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& \lim_{r\to\infty}\ \mathbb{E}\int_0^T \lambda_2(s)\,|L^l(s)-L(s)|^2\,\frac{\partial Q}{\partial K}(s,K^l(s)-K^r(s))\,ds \\
& = \mathbb{E}\int_0^T \lambda_2(s)\,|L^l(s)-L(s)|^2\,\frac{\partial Q}{\partial K}(s,K^l(s)-K(s))\,ds.
\end{aligned}
$$

Finally, since $K^l(T)-K^r(T)=0$, and by Lemma 4.4.8(i), we have

$\mathbb{E}\, Q(T, K^l(T) - K^r(T)) = 0$. Now, letting $r \to \infty$ in (4.4.21) gives:

$$\liminf_{r\to\infty, r\in(n_k)} \; \mathbb{E} \int_0^T \; |L^l(s) - L^r(s)|^2 \Big[\frac{1}{2}\frac{\partial^2 Q}{\partial K^2}(s, K^l(s) - K^r(s))$$

$$-\frac{1}{2}\frac{\partial Q}{\partial K}(s, K^l(s) - K^r(s)) - 2\lambda_2(s)\frac{\partial Q}{\partial K}(s, K^l(s) - K^r(s))\Big]\, ds$$

$$+\, Q(0, K^l(0) - K(0))$$

$$-4\,\mathbb{E} \int_0^T \; \lambda_2(s)\; |L^l(s) - L(s)|^2 \; \frac{\partial Q}{\partial K}(s, K^l(s) - K(s))\, ds$$

$$\leq -\mathbb{E} \int_0^T \frac{\partial Q}{\partial s}(s, K^l(s) - K(s))\; ds \tag{4.4.23}$$

$$+\mathbb{E} \int_0^T a(s)\, [K^l(s) - K(s)] \; \frac{\partial Q}{\partial K}(s, K^l(s) - K(s))\, ds$$

$$+\frac{1}{2}\; \mathbb{E} \int_0^T \; |c(s)|^2 \; \frac{\partial Q}{\partial K}(s, K^l(s) - K(s))\, ds$$

$$+\mathbb{E} \int_0^T \frac{\partial Q}{\partial K}(s, K^l(s) - K(s))\, [\lambda_1(s) + 4\; \lambda_2(s)|L(s)|^2]\; ds.$$

The weak convergence of $\{L^{n_k}(t)\}_{k\geq 0}$ in $\mathcal{M}_p^2(0,T;\mathbb{R}^d)$, and Theorem 2.3.5, give:

$$\mathbb{E} \int_0^T p(s)\; |L^l(s) - L(s)|^2 \; ds \leq \liminf_{r\to\infty, r\in(n_k)} \; \mathbb{E} \int_0^T p(s)\; |L^l(s) - L^r(s)|^2 \; ds.$$

As a consequence of this, together with:

$$\mathbb{E} \int_0^T \lambda_2(s)\; |L^l(s) - L(s)|^2 \, \frac{\partial Q}{\partial K}(s, K^l(s) - K(s))\, ds$$
$$\leq \; e^{6\,\kappa^2}\; e^{\kappa}\; e^{7\,\lambda_2^2(0)}\, \mathbb{E} \int_0^T \; p(s)\; |L^l(s) - L(s)|^2 \, ds,$$

implies that:

$$\mathbb{E}\int_0^T \lambda_2(s)\,|L^l(s)-L(s)|^2\,\frac{\partial Q}{\partial K}(s,K^l(s)-K(s))\,ds$$
$$\leq \liminf_{r\to\infty, r\in(n_k)} \;\mathbb{E}\int_0^T \lambda_2(s)\,|L^l(s)-L^r(s)|^2\,\frac{\partial Q}{\partial K}(s,K^l(s)-K^r(s))\,ds.$$

Thus,

$$-4\,\mathbb{E}\int_0^T \lambda_2(s)\,|L^l(s)-L(s)|^2\,\frac{\partial Q}{\partial K}(s,K^l(s)-K(s))\,ds$$
$$\geq -4 \liminf_{r\to\infty, r\in(n_k)} \;\mathbb{E}\int_0^T \lambda_2(s)\,|L^l(s)-L^r(s)|^2\,\frac{\partial Q}{\partial K}(s,K^l(s)-K^r(s))\,ds.$$

By putting this in (4.4.23), we obtain:

$$\begin{aligned}
& Q(0,K^l(0)-K(0)) + \liminf_{r\to\infty, r\in(n_k)} \;\mathbb{E}\int_0^T \;|L^l(s)-L^r(s)|^2 \\
& \left[\frac{1}{2}\frac{\partial^2 Q}{\partial K^2}(s,K^l(s)-K^r(s)) - \frac{1}{2}\frac{\partial Q}{\partial K}(s,K^l(s)-K^r(s)) - 6\lambda_2(s)\frac{\partial Q}{\partial K}(s,K^l(s)-K^r(s))\right] ds \\
\leq \;& -\mathbb{E}\int_0^T \frac{\partial Q}{\partial s}(s,K^l(s)-K(s))\,ds \qquad (4.4.24) \\
+ \;& \mathbb{E}\int_0^T a(s)[K^l(s)-K(s)]\frac{\partial Q}{\partial K}(s,K^l(s)-K(s))\,ds \\
+ \;& \frac{1}{2}\,\mathbb{E}\int_0^T |c(s)|^2\;\frac{\partial Q}{\partial K}(s,K^l(s)-K(s))\,ds \\
+ \;& \mathbb{E}\int_0^T \frac{\partial Q}{\partial K}(s,K^l(s)-K(s))\;[\lambda_1(s)+4\,\lambda_2(s)|L(s)|^2\;]\,ds.
\end{aligned}$$

We have:

$$
\begin{aligned}
& \frac{1}{2}\,\frac{\partial^2 Q}{\partial K^2}(t,K) - \frac{1}{2}\frac{\partial Q}{\partial K}(t,K) - 6\,\lambda_2(t)\,\frac{\partial Q}{\partial K}(t,K) \\
= \; & \frac{1}{2}\,(12\lambda_2(t)+1)\,e^{(12\lambda_2(t)+1)K} - \frac{1}{2}\,(e^{(12\lambda_2(t)+1)K}-1) \\
- \; & 6\,\lambda_2(t)\,(e^{(12\lambda_2(t)+1)K}-1) \\
= \; & 6\,\lambda_2(t) + \frac{1}{2} \geq \lambda_2(t).
\end{aligned} \tag{4.4.25}
$$

Moreover, we have:

$$
\mathbb{E}\int_0^T \lambda_2(s)\,|L^l(s)-L(s)|^2\,ds \leq\; e^{\lambda_2^2(0)}\,\mathbb{E}\int_0^T\; p(s)\,|L^l(s)-L(s)|^2\,ds.
$$

By this, and Theorem 2.3.5, we obtain:

$$
\mathbb{E}\,\int_0^T |L^l(s)-L(s)|^2\;\lambda_2(s)\;ds \leq \liminf_{r\to\infty, r\in(n_k)}\;\mathbb{E}\;\int_0^T |L^l(s)-L^r(s)|^2\;\lambda_2(s)\;ds. \tag{4.4.26}
$$

By (4.4.25) and (4.4.26), (4.4.24) becomes:

$$
\begin{aligned}
Q(0,K^l(0)-K(0)) \quad + \quad & \mathbb{E}\,\int_0^T\; |L^l(s)-L(s)|^2\;\lambda_2(s)\,ds \\
\leq \quad & -\mathbb{E}\,\int_0^T \frac{\partial Q}{\partial s}(s,K^l(s)-K(s))\;ds \\
+ \quad & \mathbb{E}\,\int_0^T a(s)\,[K^l(s)-K(s)]\;\frac{\partial Q}{\partial K}(s,K^l(s)-K(s))\,ds \\
+ \quad & \frac{1}{2}\,\mathbb{E}\,\int_0^T |c(s)|^2\;\frac{\partial Q}{\partial K}(s,K^l(s)-K(s))\,ds \\
+ \quad & \mathbb{E}\,\int_0^T \frac{\partial Q}{\partial K}(s,K^l(s)-K(s))\Big[\lambda_1(s)+4\,\lambda_2(s)|L(s)|^2\Big]ds.
\end{aligned}
$$

By letting $l\to\infty$, and by Theorem 2.3.4 and Lemma 4.4.8$(i)-(iv)$, the right-hand side of the previous inequality converges to 0. Also, the first part of the left-hand side goes to 0. Hence, we obtain:

$$\lim_{l \to \infty} \mathbb{E} \int_0^T \lambda_2(s) \left| L^l(s) - L(s) \right|^2 \, ds = 0,$$

i.e. the whole sequence $\{L^n(t)\}_{n \geq 0}$ converges to $(L(t), t \in [0,T])$ in $\mathcal{M}^2_{\lambda_2}(0,T;\mathbb{R}^d)$.

□

Note that the way to find the difference $G^l(t, K^l, L^l) - G^r(t, K^r, L^r)$, see equation (4.4.17), is different from chapter 3. In this chapter, we go back to $\hat{F}$ from the sequence F^n. The reason is, we use the fact that $F^n(t, K, L) \geq \hat{F}(t, K, L)$, together with the fact that F^n does not change the sign. As a result, we can eliminate F^l and we have $\hat{F}$ instead of F^r. However, different from this, in chapter 3 the sequence F^n changes the sign.

In the following lemma, we prove the uniform convergence in t of a subsequence of $\{K^n(t)\}_{n \geq 0}$ to $(K(t), t \in [0,T])$.

Lemma 4.4.10. *If conditions H_1 and H_2 hold, then there exists a subsequence of $\{K^n(t)\}_{n \geq 0}$ that converges uniformly in t to $(K(t), t \in [0,T])$ almost surely.*

Proof. Since the sequence $\{L^n(t)\}_{n \geq 0}$ converges in $\mathcal{M}^2_{\lambda_2}(0,T;\mathbb{R}^d)$, by Theorem 2.2.6, there exists a subsequence $\{L^{n_j}(t)\}_{j \geq 0}$ of $\{L^n(t)\}_{n \geq 0}$ such that $\{L^{n_j}(t)\}_{j \geq 0}$ converges to $L(t)$ *a.e.* $t \in [0,T]$ *a.s.*. We next show that:

$$\lim_{j \to \infty} \sup_{t \in [0,T]} \left| \int_t^T \left[F^{n_j}(s, K^{n_j}(s), L^{n_j}(s)) - \hat{F}(s, K(s), L(s)) \right] ds \right| = 0 \ \ a.s., \tag{4.4.27}$$

$$\lim_{j \to \infty} \sup_{t \in [0,T]} \left| \int_t^T \left[L^{n_j}(s) - L(s) \right]' \, dW(s) \right| = 0 \ \ a.s.. \tag{4.4.28}$$

Now, by Lemma 4.4.1(iv), together with the facts that $\{K^{nj}(t)\}_{j \geq 0}$ converges to $K(t)$ for all $t \in [0,T]$ *a.s.*, and $\{L^{n_j}(t)\}_{j \geq 0}$ converges to $L(t)$ *a.e.* $t \in [0,T]$

$a.s.$, we have:

$$\lim_{j\to\infty} F^{n_j}(t, K^{n_j}(t), L^{n_j}(t)) = \hat{F}(t, K(t), L(t)) \;\; a.e. \;\; t \in [0,T] \;\; a.s..$$

Since $N(t) \geq \varepsilon\, I_{m\times m}$ implies that $-\varepsilon^{-1} \; I_{m\times m} \leq -N^{-1}(t)$, and from Lemma 4.4.1$(i)$, we have:

$$\begin{aligned} 0 \geq F^{n_j}(t, K^{n_j}, L^{n_j}) &\geq -\big[3B_1(t)\varepsilon^{-1} \; B_1'(t) + B(t)\; \varepsilon^{-1} \; B'(t)\big](K^{n_j})^2 \\ &\quad - 4(L^{n_j})' D(t)\; \varepsilon^{-1} \; D'(t) L^{n_j} \\ &= -\varepsilon^{-1}\left[3B_1(t)\; B_1'(t) + B(t)\; B'(t)\right](K^{n_j})^2 \\ &\quad - 4\,\varepsilon^{-1}\,(L^{n_j})' D(t)\; D'(t) L^{n_j}, \end{aligned}$$

and thus,

$$\begin{aligned} |F^{n_j}(t, K^{n_j}, L^{n_j})| &\leq 3\,\varepsilon^{-1}\,|B_1(t)|^2 \;\; (K^{n_j})^2 + \varepsilon^{-1}\,|B(t)|^2\; (K^{n_j})^2 \\ &\quad + 4\,\varepsilon^{-1}\;|D(t)|^2\,|L^{n_j}|^2. \end{aligned}$$

Since

$$\begin{aligned} 3\,\varepsilon^{-1}\,|B_1(t)|^2 \;\; (K^{n_j})^2 &\leq 3\,\varepsilon^{-1}\,|B_1(t)|^2\;\kappa^2 = 3\,\varepsilon^{-1}\,|B(t) + C'(t)\,D(t)|^2\;\kappa^2 \\ &\leq 3\,\varepsilon^{-1}\left[2|B(t)|^2 + 2|C(t)|^2\,|D(t)|^2\right]\kappa^2 \leq 2\lambda_1(t), \end{aligned}$$

we obtain:

$$\begin{aligned} |F^{n_j}(t, K^{n_j}, L^{n_j})| &\leq 2\,\lambda_1(t) + \varepsilon^{-1}\,|B(t)|^2\;\kappa^2 + 4\,\varepsilon^{-1}\;|D(t)|^2\,|L^{n_j}|^2 \\ &\leq 2\,\lambda_1(t) + \lambda_1(t) + \; 2\lambda_2(t)\;|L^{n_j}|^2 \\ &= 3\,\lambda_1(t) + \; 2\lambda_2(t)\;|L^{n_j}|^2. \end{aligned}$$

Therefore,

$$\begin{aligned}
\int_0^T \left|F^{n_j}(s, K^{n_j}(s), L^{n_j}(s))\right| ds &\le \int_0^T \left[3\,\lambda_1(s) + 2\,\lambda_2(s)\,|L^{n_j}(s)|^2\right] ds \\
&\le \int_0^T \left[3\,\lambda_1(s) + 2\,e^{\lambda_2^2(0)}\,p(s)\,|L^{n_j}(s)|^2\right] ds \\
&\le 3\,e^{\lambda_2^2(0)} \int_0^T \left[\lambda_1(s) + p(s)\;|L^{n_j}(s)|^2\right] ds \\
&< \infty \quad a.s..
\end{aligned}$$

By the dominated convergence theorem, we obtain:

$$\lim_{j\to\infty} \int_0^T F^{n_j}(s, K^{n_j}(s), L^{n_j}(s))\, ds = \int_0^T \hat{F}(s, K(s), L(s))\; ds \quad a.s.,$$

and

$$\lim_{j\to\infty} \int_0^T \left|F^{n_j}(s, K^{n_j}(s), L^{n_j}(s)) - \hat{F}(s, K(s), L(s))\right| ds = 0 \quad a.s..$$

Therefore, we have:

$$\begin{aligned}
&\lim_{j\to\infty} \sup_{t\in[0,T]} \left| \int_t^T F^{n_j}(s, K^{n_j}(s), L^{n_j}(s))\; ds - \int_t^T \hat{F}(s, K(s), L(s))\; ds \right| \\
&\le \lim_{j\to\infty} \sup_{t\in[0,T]} \int_t^T \left|F^{n_j}(s, K^{n_j}(s), L^{n_j}(s)) - \hat{F}(s, K(s), L(s))\right| ds \\
&\le \lim_{j\to\infty} \int_0^T \left|F^{n_j}(s, K^{n_j}(s), L^{n_j}(s)) - \hat{F}(s, K(s), L(s)))\right| ds = 0 \quad a.s..
\end{aligned}$$

In other words, (4.4.27) holds. Now, we prove that (4.4.28) holds. Since the stochastic integral is a martingale, by Theorem 2.4.7, for any constant $\epsilon > 0$, we have:

$$\begin{aligned}
&\mathbb{P}\left(\omega \in \Omega : \sup_{t\in[0,T]} \left| \int_t^T [L^{n_j}(s) - L(s)]'\; dW(s) \right| > \epsilon \right) \\
&\le \frac{1}{\epsilon^2}\; \mathbb{E} \left| \int_0^T [L^{n_j}(s) - L(s)]'\; dW(s) \right|^2 .
\end{aligned}$$

By the Itô Isômetry, see Theorem 2.5.1, we have

$$\mathbb{P}\left(\omega \in \Omega : \sup_{t\in[0,T]} \left| \int_t^T \; [L^{n_j}(s) - L(s)]' \, dW(s) \right| > \epsilon \right)$$
$$\leq \frac{1}{\epsilon^2}\, \mathbb{E} \int_0^T \; \left|L^{n_j}(s) - \; L(s)\right|^2 ds$$
$$\leq \frac{1}{\epsilon^2}\, \mathbb{E} \int_0^T \; \lambda_2(s)\, \left|L^{n_j}(s) - \; L(s)\right|^2 ds.$$

As the right-hand side of the last inequality goes to 0 as $j \to \infty$, it follows that:

$$\lim_{j\to\infty} \sup_{t\in[0,T]} \left| \int_t^T \; [L^{n_j}(s) - L(s)]' \; dW(s) \right| = 0 \quad \text{in probability.}$$

By Theorem 2.2.6, there exists a subsequence $\{L^{n_q}\}_{q\geq 0}$ of the sequence $\{L^{n_j}\}_{j\geq 0}$ such that:

$$\lim_{q\to\infty} \sup_{t\in[0,T]} \left| \int_t^T \; [L^{n_q}(s) - L(s)]' \, dW(s) \right| = 0 \quad a.s.,$$

i.e. (4.4.28) holds. Now, for any two elements n_q and n_m of the sequence $\{n_q\}_{q\geq 0}$, we have:

$$\begin{aligned} K^{n_q}(t) - K^{n_m}(t) = {} & (K^{n_q}(T) - K^{n_m}(T)) \\ & + \int_t^T \left[F^{n_q}(s, K^{n_q}(s), L^{n_q}(s)) - F^{n_m}(s, K^{n_m}(s), L^{n_m}(s))\right] ds \\ & - \int_t^T [L^{n_q}(s) - L^{n_m}(s)]' \, dW(s), \;\; t \in [0,T] \;\; a.s.. \end{aligned}$$

Then,

$$\begin{aligned} |K^{n_q}(t) - K^{n_m}(t)| \leq {} & \int_t^T |F^{n_q}(s, K^{n_q}(s), L^{n_q}(s)) - F^{n_m}(s, K^{n_m}(s), L^{n_m}(s))| \, ds \\ & + \left| \int_t^T [L^{n_q}(s) - L^{n_m}(s)]' \, dW(s] \right| \qquad (4.4.29) \end{aligned}$$

By (4.4.28), we have:

$$\begin{aligned}
& \lim_{m\to\infty} \left| \int_t^T [L^{n_q}(s) - L^{n_m}(s)]' \, dW(s) \right| \\
\leq & \lim_{m\to\infty} \left| \int_t^T [L^{n_q}(s) - L(s)]' \, dW(s) \right| + \lim_{m\to\infty} \left| \int_t^T [L(s) - L^{n_m}(s)]' \, dW(s) \right| \\
= & \left| \int_t^T [L^{n_q}(s) - L(s)]' \, dW(s) \right|, \ \ t \in [0,T] \ \ a.s.
\end{aligned}$$

On the other hand, by (4.4.27), we have:

$$\begin{aligned}
& \lim_{m\to\infty} \int_t^T |F^{n_q}(s, K^{n_q}(s), L^{n_q}(s)) - F^{n_m}(s, K^{n_m}(s), L^{n_m}(s))| \, ds \\
\leq \quad & \lim_{m\to\infty} \int_t^T |F^{n_q}(s, K^{n_q}(s), L^{n_q}(s)) - \hat{F}(s, K(s), L(s))| \, ds \\
+ \quad & \lim_{m\to\infty} \int_t^T |\hat{F}(s, K(s), L(s)) - F^{n_m}(s, K^{n_m}(s), L^{n_m}(s))| \, ds \\
= \quad & \int_t^T |F^{n_q}(s, K^{n_q}(s), L^{n_q}(s)) - \hat{F}(s, K(s), L(s))| \, ds.
\end{aligned}$$

Hence, by taking limits as $m \to \infty$ in (4.4.29), we obtain:

$$\begin{aligned}
|K^{n_q}(t) - K(t)| \leq & \int_t^T |F^{n_q}(s, K^{n_q}(s), L^{n_q}(s)) - \hat{F}(s, K(s), L(s))| \, ds \\
& + \left| \int_t^T [L^{n_q}(s) - L(s)]' \, dW(s) \right|, \ \ t \in [0,T] \ \ a.s.. \qquad (4.4.30)
\end{aligned}$$

By taking the supremum over $t \in [0,T]$ in the previous inequality, we obtain:

$$\begin{aligned}
0 \leq \sup_{t\in[0,T]} |K^{n_q}(t) - K(t)| \quad \leq & \quad \int_0^T |F^{n_q}(s, K^{n_q}(s), L^{n_q}(s)) - \hat{F}(s, K(s), L(s))| \, ds \\
+ & \quad \sup_{t\in[0,T]} \left| \int_t^T [L^{n_q}(s) - L(s)]' \, dW(s) \right| \ \ a.s..
\end{aligned}$$

By (4.4.27) and (4.4.28), the right hand side in the previous inequality goes to 0 almost surely as q goes to ∞, and we obtain:

$$\lim_{q\to\infty} \sup_{t\in[0,T]} |K^{n_q}(t) - K(t)| = 0 \;\; a.s.$$

In other words, $\{K^{n_q}(t)\}_{q\geq 0}$ converges uniformly in t to $(K(t), t \in [0,T])$. $\square$

Here by the uniform convergence theorem, see Theorem 2.3.2, the process $(K(t), t \in [0,T])$ has continuous paths. The following theorem gives the main result of existence.

Theorem 4.4.11. *If conditions H_1 and H_2 hold, then the backward stochastic differential equation* (4.2.1) *has a solution pair* $(K(\cdot), L(\cdot)) \in \mathcal{L}^{\infty}(0,T;\mathbb{R}) \times \mathcal{M}^2_{\lambda_2}(0,T;\mathbb{R}^d)$.

Proof. From Lemma 4.4.9 and Lemma 4.4.10, by letting q go to ∞ in the sequence of equations:

$$\begin{aligned} K^{n_q}(t) &= K^{n_q}(T) \\ &+ \int_t^T \left[a(s)\, K^{n_q}(s) + c'(s)\, L^{n_q}(s) + Q(s) + F^{n_q}(s, K^{n_q}(s), L^{n_q}(s))\right] ds \\ &- \int_t^T L^{n_q}(s)\, dW(s), \qquad t \in [0,T], \end{aligned}$$

we obtain that the pair $(K(\cdot), L(\cdot))$ is a solution of the equation

$$\begin{aligned} K(t) &= K(T) + \int_t^T \left[a(s)\, K(s) + c'(s)\, L(s) + Q(s) + \hat{F}(s, K(s), L(s))\right] ds \\ &- \int_t^T L'(s)\, dW(s), \;\; t \in [0,T]. \end{aligned}$$

By the nonnegativity of $(K(t), t \in [0,T])$, the pair $(K(\cdot), L(\cdot))$ is also a solution to equation (4.2.1). $\square$

4.5 Application to linear-quadratic optimal control problems

The known results on the existence of solutions to the linear-quadratic optimal control problems do not cover the case when the coefficients are possibly unbounded. In this section, we give an application of our results in section 4.4 to the linear-quadratic optimal control problem with possibly unbounded coefficients. Consider the following linear controlled stochastic differential equation:

$$\begin{cases} dX(t) = [A(t)\,X(t) + B(t)\,u(t)]\,dt \\ \qquad\qquad +[C(t)\,X(t) + D(t)\,u(t)]'\,dW(t), \quad t \in [s,T], \\ X(s) = x, \end{cases} \tag{4.5.1}$$

under the same setting for $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$ as in section 4.4, $(s,x) \in [0,T] \times \mathbb{R}$ are the initial time and initial state, respectively. The control $u(\cdot)$ is an $\mathbb{R}^m$-valued process. Here, we denote by $\mathcal{A}$, the set of all such admissible controls $u(\cdot)$. For any (s,x), and $u(\cdot) \in \mathcal{A}$, the quadratic cost functional is given by:

$$J(u(\cdot);s,x) = \mathbb{E}\Big[M\,X^2(T) + \int_s^T \big[Q(t)\,X^2(t) + u'(t)\,N(t)\,u(t)\big]\,dt\,\Big|\mathcal{F}_s\Big],\ t \in [s,T]. \tag{4.5.2}$$

The coefficients $Q(\cdot)$, $N(\cdot)$, $M(\cdot)$ are under the same setting as in section 4.4. We consider the following LQ optimal control problem:

$$\begin{cases} \inf_{u(\cdot)\in\mathcal{A}} J(u(\cdot);0,x), \\ \text{subject to } \ (4.5.1). \end{cases} \tag{4.5.3}$$

In other words, the target of this problem is to minimize the cost functional $J(u(\cdot);0,x)$ for a given x, over all $u(\cdot) \in \mathcal{A}$, where the admissible set $\mathcal{A}$ is defined below. The value function is defined as:

$$\hat{V} := \hat{V}(0,x) := \inf_{u(\cdot)\in\mathcal{A}} J(u(\cdot);0,x).$$

An admissible control $u^*(\cdot)$ is called optimal for the control problem (4.5.3) if $u^*(\cdot)$ achieves the infimum of $J(u(\cdot);0,x)$.

In this section, we obtain the solution to the LQ optimal control problem (4.5.3) under weaker conditions on the coefficients as compared to Theorem 5.2. in Kohlmann and Tang [39]. In particular, we give an explicit solution to the LQ optimal control problem with possibly unbounded coefficients using our results in the previous section. Throughout this section, the following additional notations will be used:

- $s_1(t)$ and $s_2(t)$ are given $\{\mathcal{F}_t\}_{t\geq 0}$-adapted $\mathbb{R}$-valued processes such that $|B(t)\left[N(t)\right]^{-1}D'(t)| \leq s_1(t)$ *a.e.* $t\in[0,T]$ *a.s.* and $B(t)\left[N(t)\right]^{-1}B'(t) \leq s_2(t)$ *a.e.* $t\in[0,T]$ *a.s.*.
- $S(t)$ is a given $\{\mathcal{F}_t\}_{t\geq 0}$-adapted $\mathbb{R}^{d\times d}$-valued process such that $D(t)\left[N(t)\right]^{-1}D'(t) \leq S(t)$ *a.e.* $t\in[0,T]$ *a.s.*.

Let $(K(\cdot),L(\cdot))$ be a solution of the Riccati BSDE (4.2.1). Define:

$$\hat{X} := \int_0^t \left[v^2\, z + 2v\, y\, [C(s)\, v + D(s)\, u]\,\right]'\, dW(s),$$

for all $(t,y,z,v,u)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^m$. The admissible set $\mathcal{A}$ is defined as:

$$\mathcal{A} := \Big\{u(\cdot)\in\mathcal{L}^0_{\mathcal{F}}(0,T;\mathbb{R}^m):\ \text{under } u(\cdot), \text{ equation (4.5.1) admits a strong unique solution } X_u(\cdot), \text{ and } \hat{X}(\cdot,K(\cdot),L(\cdot),X_u(\cdot),u(\cdot)) \text{ is a martingale}\Big\}.$$

Note that our method of solving the linear quadratic optimal control problem implies putting such a technical condition for the stochastic integral

$\hat{X}(\cdot)$. Define:

$$u^*(t,y,z,v) := -[N(t) + y\,D'(t)D(t)]^{-1}[y\,B(t) + y\,C'(t)D(t) + z'\,D(t)]'\,v,$$
$$\forall\ \ (t,y,z,v) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}. \tag{4.5.4}$$

We say that the processes $s_1(\cdot)$, $s_2(\cdot)$ and $S(\cdot)$ satisfy conditions H_3 if:

(i) $s_1(\cdot) \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R})$;
(ii) $s_2(\cdot) \in \mathcal{L}_{\mathcal{F}}(0,T;\mathbb{R})$;
(iii) $S(\cdot) \in \mathcal{L}^\infty_{\mathcal{F}}(0,T;\mathbb{R}^{d\times d})$.

The conditions H_3 are needed for Theorem 4.5.4 to prove that the control process u^* is admissible, using our results in section 4.4. Those conditions permit for unbounded coefficients which is not the case in [39]. In particular, we assume that the products of the coefficients $B(\cdot)$, $[N(\cdot)]^{-1}$ and $D(\cdot)$ to be bounded by some stochastic processes $s_1(\cdot)$ and $s_2(\cdot)$ that satisfy finite expectations, see conditions H_3(i)-(ii). On the other hand, in condition H_3(iii), we assume that the upper bound of $[D(t)\left[N(t)\right]^{-1}D'(t)\,]$ belongs to $\mathcal{L}^\infty_{\mathcal{F}}(0,T;\mathbb{R}^{d\times d})$. This condition also permit for unbounded processes $D(t)$ and $N(t)$. For example, if $D(t) = W(t)$ and $N(t) = W'(t)W(t) + 1$, H_3(iii) holds but the known results do not apply. Thus, our conditions are weaker as compared to [39].

In the following theorem, we show that if u^* is admissible, then it is the unique solution to problem (4.5.3). Later, in Theorem 4.5.4, we give sufficient conditions under which the control process u^* is admissible, using our results in section 4.4.

Theorem 4.5.1. *Let $(K(\cdot), L(\cdot))$ be a solution of the Riccati BSDE* (4.2.1)*. If u^* is admissible, then it is the unique solution (unique optimal control) of the optimal control problem* (4.5.3)*, and the corresponding optimal cost is*

$$J(u^*) = K(0)\,x^2.$$

Proof. The proof follows closely that of Theorem 3.1 in [16]. We include it for completeness. For any $u \in \mathcal{A}$, the differential of $K(\cdot)X^2(\cdot)$ is:

$$
\begin{aligned}
d[K(t)X^2(t)] &= [dK(t)]\, X^2(t) + K(t)\, dX^2(t) + [dK(t)]\, [dX^2(t)] \\
&= X^2(t)\big[- a(t)K(t)\, dt - c'(t)\, L(t)\, dt - Q(t)\, dt \\
&\quad -F(t, K(t), L(t))\, dt + L(t)\, dW(t)\big] \\
&\quad +K(t)\Big[2X(t)\, [A(t)X(t) + B(t)u(t)]\, dt + \big|C(t)X(t) + D(t)u(t)\big|^2\, dt \\
&\quad +2X(t)[C(t)X(t) + D(t)u(t)]' dW(t)\Big] \\
&\quad +2X(t)\, L'(t)[C(t)\, X(t) + D(t)u(t)]\, dt \\
&= -a(t)K(t)X^2(t)\, dt - X^2(t)\, c'(t)L(t)dt - X^2(t)\, Q(t)\, dt \\
&\quad -X^2(t)\, F(t, K(t), L(t))\, dt + X^2(t)L(t)\, dW(t) \\
&\quad +2X(t)K(t)\big[A(t)X(t) + B(t)u(t)\big] dt \\
&\quad +K(t)\big|C(t)X(t) + D(t)u(t)\big|^2\, dt \\
&\quad +2\, X(t)\, K(t)\, [C(t)X(t) + D(t)u(t)]'\, dW(t) \\
&\quad +2X(t)\, L'(t)\, [C(t)\, X(t) + D(t)\, u(t)]\, dt
\end{aligned}
$$

By integrating from 0 to T:

$$
\begin{aligned}
K(T)X^2(T) &= K(0)X^2(0) + \int_0^T \Big[-a(t)K(t)X^2(t) - X^2(t)\,c'(t)\,L(t) \\
&\quad -Q(t)\,X^2(t) - F(t,K(t),L(t))\,X^2(t) \\
&\quad +2X(t)\;K(t)\;\big[A(t)X(t)+B(t)u(t)\big] + K(t)\big|C(t)X(t)+D(t)u(t)\big|^2 \\
&\quad +2X(t)\,L'(t)\big[C(t)\,X(t)+D(t)\,u(t)\big]\Big]dt \qquad (4.5.5)\\
&\quad + \int_0^T \big[X^2(t)\,L(t) + 2\,X(t)\,K(t)\,[C(t)X(t)+D(t)u(t)]\,\big]'\,dW(t).
\end{aligned}
$$

By the admissibility of u, the stochastic integral on the right hand side is a martingale. Thus, by taking the expectation in (4.5.5) , we obtain:

$$
\begin{aligned}
\mathbb{E}\big(K(T)X^2(T)\big) &= K(0)X^2(0) + \mathbb{E}\int_0^T \Big[-a(t)K(t)X^2(t) - c'(t)L(t)X^2(t) \\
&\quad -Q(t)X^2(t) + \big[K(t)B(t) + K(t)C'(t)D(t) + L'(t)D(t)\big] \\
&\quad \times\big[N(t)+K(t)D'(t)D(t)\big]^{-1} \\
&\quad \big[K(t)B(t) + K(t)C'(t)D(t) + L'(t)D(t)\big]'\;X^2(t) \\
&\quad +2X(t)K(t)\big[A(t)X(t)+B(t)u(t)\big] \\
&\quad +K(t)\big[|C(t)X(t)|^2 + 2X(t)C'(t)D(t)u(t) + u'(t)D'(t)D(t)u(t)\big] \\
&\quad +2X(t)L'(t)\big[C(t)X(t)+D(t)u(t)\big]\Big]dt.
\end{aligned}
$$

Putting this in (4.5.2), we obtain:

$$\begin{aligned}
J(u) &= K(0)\,X^2(0) + \mathbb{E}\int_0^T \Big[u'(t)\,N(t)\,u(t) + Q(t)X^2(t) - a(t)K(t)X^2(t) \\
&\quad -c'(t)\,L(t)X^2(t) - Q(t)X^2(t) + [K(t)B(t) + K(t)C'(t)D(t) + L'(t)D(t)] \\
&\quad \times[N(t) + K(t)D'(t)D(t)]^{-1}[K(t)B(t) + K(t)C'(t)D(t) + L'(t)D(t)]'\,X^2(t) \\
&\quad +2X^2(t)\,K(t)\,A(t) + 2X(t)\,K(t)\,B(t)\,u(t) + K(t)\,|C(t)X(t)|^2 \\
&\quad +2\,K(t)\,X(t)\,C'(t)\,D(t)\,u(t) + K(t)\,u'(t)\,D'(t)\,D(t)\,u(t) + 2X^2(t)\,L'(t)\,C(t) \\
&\quad +2X(t)\,L'(t)\,D(t)\,u(t)\Big]dt \\
&= K(0)\,X^2(0) + \mathbb{E}\int_0^T \Big[u'(t)\,N(t)\,u(t) + [K(t)B(t) + K(t)C'(t)D(t) + L'(t)D(t)] \\
&\quad \times[N(t) + K(t)D'(t)D(t)]^{-1}[K(t)B(t) + K(t)C'(t)D(t) + L'(t)D(t)]'\,X^2(t) \\
&\quad +2X(t)\,K(t)\,B(t)\,u(t) + 2K(t)\,X(t)\,C'(t)\,D(t)\,u(t) \\
&\quad +K(t)\,u'(t)\,D'(t)\,D(t)\,u(t) + 2X(t)\,L'(t)\,D(t)\,u(t)\Big]dt \\
&= K(0)X^2(0) + \mathbb{E}\int_0^T \Big[u'(t)\big[N(t) + K(t)D'(t)D(t)\big]u(t) \\
&\quad +[K(t)B(t) + K(t)C'(t)D(t) + L'(t)D(t)] \\
&\quad \times[N(t) + K(t)D'(t)D(t)]^{-1}[K(t)B(t) + K(t)C'(t)D(t) + L'(t)D(t)]'\,X^2(t) \\
&\quad +2X(t)\,\big[K(t)\,B(t) + K(t)\,C'(t)\,D(t) + L'(t)\,D(t)\big]\,u(t)\Big]dt.
\end{aligned}$$

Therefore:

$$
\begin{aligned}
J(u) &= K(0)\ X^2(0) + \mathbb{E}\int_0^T \Big[u(t) + \big[N(t)+K(t)D'(t)D(t)\big]^{-1} X(t) \\
&\quad \big[K(t)B(t)+K(t)C'(t)D(t)+L'(t)D(t)\big]'\Big]' \big[N(t)+K(t)D'(t)D(t)\big] \\
&\quad \times \Big[u(t) + \big[N(t)+K(t)D'(t)D(t)\big]^{-1} X(t) \\
&\quad \big[K(t)B(t)+K(t)C'(t)D(t)+L'(t)D(t)\big]' \Big] dt \\
&\geq K(0)\ x^2,
\end{aligned}
$$

with equality if and only if $u(t) = u^*(t)$, *a.e.* $t \in [0,T]$ *a.s.*, and the corresponding optimal cost (lowest value) is $J(u^*) = K(0)\ x^2$. □

Note that Theorem 4.5.1 can be applied with different conditions on the coefficients. In particular, we only need a Riccati BSDE that has a solution under which u^* is admissible, regardless of any properties of the solution.

In the following lemma, we state a useful result which is an immediate adaption of Gal'chuk basic thorem [26], on the existence and uniqueness of a strong solution of linear stochastic differential equations, see Lemma 7.1. in [58].

Lemma 4.5.2. *Assume that the functions $f : \Omega \times [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, and $g : \Omega \times [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ satisfy the following two conditions:*
(i) for any $X \in \mathbb{R}^n$, $f(\cdot, X)$ and $g(\cdot, X)$ are $\{\mathcal{F}_t\}$-adapted processes such that

$$\int_0^T |f(t,0)|\ dt < \infty, \quad \int_0^T |g(t,0)|^2\ dt < \infty, \quad a.s.$$

(ii) there exist two positive $\{\mathcal{F}_t\}$-adapted processes $\hat{\alpha}_1$ and $\hat{\alpha}_2$ such that

$$\int_0^T \hat{\alpha}_1(t)\ dt < \infty, \quad \int_0^T |\hat{\alpha}_2(t)|^2\ dt < \infty \quad a.s.,$$

and for any $X_1, X_2 \in \mathbb{R}^n$*,*

$$|f(t,X_1) - f(t,X_2)| \leq \hat{\alpha}_1(t)|X_1 - X_2|,$$
$$|g(t,X_1) - g(t,X_2)| \leq \hat{\alpha}_2(t)|X_1 - X_2|.$$

Then, the equation

$$\begin{cases} dX(t) = f(t,X(t))\,dt \,+\, g(t,X(t))\,dW(t), \quad t \in [0,T], \\ \\ X(0) = x, \end{cases}$$

has a unique strong solution.

In the following theorem, we give sufficient conditions for equation (4.5.1) to have a unique strong solution. That is, the first requirement for u^* to be admissible.

Theorem 4.5.3. *Let the conditions of Theorem* 4.4.11 *hold, and* $(K(\cdot), L(\cdot)) \in \mathcal{L}^{\infty}_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{M}^2_{\lambda_2}(0,T;\mathbb{R}^d)$ *be a solution of equation* (4.2.1)*. If conditions* H_3 *hold, then under* u^**, the stochastic differential equation* (4.5.1) *has a unique strong solution* $X_{u^*}(\cdot)$*.*

Proof. By substituting u^* into (4.5.1), and setting $\Gamma(t) := [N(t)+K(t)D'(t)D(t)]^{-1}$, (4.5.1) becomes:

$$\begin{cases} dX_{u^*}(t) = \quad X_{u^*}(t)\Big[A(t) - B(t)\Gamma(t)[K(t)B(t) + K(t)C'(t)D(t) + L'(t)D(t)]'\Big]dt \\ \\ \qquad\qquad +X_{u^*}(t)\Big[C(t) - D(t)\Gamma(t)[K(t)B(t) + K(t)C'(t)D(t) + L'(t)D(t)]'\Big]' \\ \\ \qquad\qquad \times dW(t), \quad\quad t \in [0,T], \\ \\ X_{u^*}(0) = x. \end{cases} \tag{4.5.6}$$

This is a linear stochastic differential equation with possibly unbounded coefficients which depend on the solution pair $(K(\cdot), L(\cdot))$ of equation (4.2.1).

Therefore, we cannot apply the previous results on the solvability of linear stochastic differential equations (see, for example, Theorem 6.14. in [64]). Here, we show that the conditions of Lemma 4.5.2 hold in our case. Clearly, condition (i) of Lemma 4.5.2 holds. Now, we prove that condition (ii) of Lemma 4.5.2 holds. By using the basic inequality $2xy \leq (x^2 + y^2)$, we have:

$$
\begin{aligned}
& \mathbb{E} \int_0^T \left| A(t) - B(t)\Gamma(t)\big[K(t)B(t) + K(t)C'(t)D(t) + L'(t)D(t)\big]' \right| dt \\
\leq \; & \mathbb{E} \int_0^T |A(t)|\, dt + \mathbb{E} \int_0^T K(t)\, |B(t)\,(N(t))^{-1}\, B'(t)|\, dt \\
& + \mathbb{E} \int_0^T K(t) |B(t)\,(N(t))^{-1}\, D'(t)\, C(t)|\, dt + \mathbb{E} \int_0^T |B(t)\,(N(t))^{-1}\, D'(t)\, L(t)|\, dt \\
\leq \; & \mathbb{E} \int_0^T |A(t)|\, dt + \mathbb{E} \int_0^T K(t)\, s_2(t)\, dt \\
& + \frac{1}{2}\, \mathbb{E} \int_0^T K^2(t) |B(t)\,(N(t))^{-1}\, D'(t)|^2\, dt + \frac{1}{2}\, \mathbb{E} \int_0^T |C(t)|^2\, dt \\
& + \frac{1}{2}\, \mathbb{E} \int_0^T B(t)\,(N(t))^{-1}\, B'(t)\, dt + \frac{1}{2}\, \mathbb{E} \int_0^T |D(t)\,(N(t))^{-1}\, D'(t)|\, |L(t)|^2\, dt \\
\leq \; & \mathbb{E} \int_0^T |A(t)|\, dt + \kappa\, \mathbb{E} \int_0^T s_2(t)\, dt + \frac{1}{2}\, \kappa^2 \;\mathbb{E} \int_0^T s_1^2(t) dt \\
& + \frac{1}{2}\, \mathbb{E} \int_0^T |C(t)|^2 dt + \frac{1}{2}\, \mathbb{E} \int_0^T s_2(t)\, dt + \frac{1}{2}\, \mathbb{E} \int_0^T \lambda_2(t) |L(t)|^2\, dt \\
< \; & \infty. \qquad (4.5.7)
\end{aligned}
$$

Moreover, we have:

$$
\begin{aligned}
& \mathbb{E} \int_0^T \left| C(t) - D(t)\Gamma(t)\, \big[K(t)B(t) + K(t)C'(t)D(t) + L'(t)D(t)\big]' \right|^2 dt \\
\leq \; & 2\, \mathbb{E} \int_0^T |C(t)|^2 dt + 2\mathbb{E} \int_0^T \left| D(t)\Gamma(t)\big[K(t)B(t) + K(t)C'(t)D(t) + L'(t)D(t)\big]' \right|^2 dt
\end{aligned}
$$

$$
\begin{aligned}
\leq\quad & 2\,\mathbb{E}\int_0^T \ |C(t)|^2\,dt + 6\,\mathbb{E}\int_0^T \Big[\big|D(t)\Gamma(t)B'(t)K(t)\big|^2 + \big|D(t)\Gamma(t)D'(t)C(t)K(t)\big|^2 \\
& +\big|D(t)\Gamma(t)D'(t)L(t)\big|^2\Big]\,dt \\
\leq\quad & 2\,\mathbb{E}\int_0^T \ |C(t)|^2\,dt + 6\,\mathbb{E}\int_0^T \Big[K^2(t)\big|D(t)\big(N(t)\big)^{-1}B'(t)\big|^2 \\
& +K^2(t)\big|D(t)\big(N(t)\big)^{-1}D'(t)C(t)\big|^2 + |D(t)\big(N(t)\big)^{-1}D'(t)L(t)\big|^2\Big]\,dt \\
\leq\quad & 2\,\mathbb{E}\int_0^T \ |C(t)|^2\,dt + 6\,\mathbb{E}\int_0^T \Big[K^2(t)s_1^2(t) + K^2(t)\big|S(t)\big|^2\big|C(t)\big|^2 + \big|S(t)\big|^2\big|L(t)\big|^2\Big]\,dt \\
\leq\quad & 2\,\mathbb{E}\int_0^T \ |C(t)|^2\,dt + 6\,\kappa^2\,\mathbb{E}\int_0^T \ s_1^2(t)\,dt + 6\,\kappa^2 \sup_{t\in[0,T]}\big|S(t)\big|^2\ \mathbb{E}\int_0^T \ \big|C(t)\big|^2\,dt \\
& +6\sup_{t\in[0,T]}\big|S(t)\big|^2\ \mathbb{E}\int_0^T \lambda_2(t)\,\big|L(t)\big|^2\,dt < \infty,\quad a.s.. \qquad (4.5.8)
\end{aligned}
$$

From (4.5.7) and (4.5.8), the conditions of Lemma 4.5.2 hold. Thus, the stochastic differential equation (4.5.1) has a unique strong solution $X_{u^*}(\cdot)$.
□

In the following theorem, we prove that the second requirement for u^* to be admissible holds. Due to the unboundedness of the coefficients, we put a further assumption on the coefficients.

Theorem 4.5.4. *Let the assumptions of Theorem* 4.5.3 *hold. If the coefficients of* (4.5.1) *are such that* $\mathbb{E}\big[\sup_{t\in[0,T]} X^4_{u^*}(t)\big] < \infty$, *then* $u^* \in \mathcal{A}$.

Note that here we partially solve the linear-quadratic optimal control problem because of the condition $\mathbb{E}\big[\sup_{t\in[0,T]} X^4(t)\big] < \infty$. However, this condition is not an explicit condition and hence we can give conditions on the coefficients under which the moment $\mathbb{E}\big[\sup_{t\in[0,T]} X^4(t)\big] < \infty$ holds. For example, if the coefficients are bounded, the known results give all finite moments (see [64]). On the other hand, if we give an application of the solvability result of Riccati BSDEs when $D = 0$ to the linear-quadratic optimal control problem, the controlled stochastic differential equation (4.5.1) becomes independent of the process $L(\cdot)$. Thus, the condition

$$\mathbb{E}\left[\ \sup_{t\in[0,T]}\ e^{4\int_0^t [A(s)-1/2\,C^2(s)]ds+4\int_0^t C(s)\,dW(s)}\right] < \infty,$$

which is in terms of the processes $A(\cdot)$ and $C(\cdot)$, is sufficient to obtain the moment.

Proof. We have proved in Theorem 4.5.3, under the solution $(K(\cdot), L(\cdot)) \in \mathcal{L}^{\infty}_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{M}^2_{\lambda_2}(0,T;\mathbb{R}^d)$ of (4.2.1), that equation (4.5.1) has a unique strong solution $X_{u^*}(\cdot)$. Thus, it remains to prove that the process $\hat{X}(\cdot, K(\cdot), L(\cdot), X_{u^*}(\cdot), u^*(\cdot))$ is a martingale. By Theorem 2.4.8, there exists a constant c_1 such that:

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t X^2_{u^*}(s)L'(s)dW(s)\right|\right] &\leq c_1\mathbb{E}\left[\int_0^T X^4_{u^*}(s)|L(s)|^2ds\right]^{\frac{1}{2}} \\
&\leq c_1\mathbb{E}\left[\sup_{t\in[0,T]} X^4_{u^*}(t)\int_0^T |L(s)|^2ds\right]^{\frac{1}{2}} \qquad (4.5.9)\\
&\leq \frac{c_1}{2}\mathbb{E}\left[\sup_{t\in[0,T]} X^4_{u^*}(t) + \int_0^T \lambda_2(s)|L(s)|^2ds\right] < \infty.
\end{aligned}$$

Moreover, there exists a constant c_2 such that:

$$\begin{aligned}
& \mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t 2X_{u^*}(s)K(s)[C(s)X_{u^*}(s)+D(s)u^*(s)]'\,dW(s)\right|\right] \\
= \quad & 2\,\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t X_{u^*}(s)K(s)\big(C(s)X(s)-D(s)\,\Gamma(s)\right.\right. \\
& \left.\left.[K(s)B(s)+K(s)C'(s)D(s)+L'(s)D(s)]'\,X_{u^*}(s)\big)'dW(s)\right|\right] \\
= \quad & 2\,\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t X_{u^*}^2(s)K(s)\big(C(s)-D(s)\,\Gamma(s)\right.\right. \\
& \left.\left.[K(s)B(s)+K(s)C'(s)D(s)+L'(s)D(s)]'\big)'dW(s)\right|\right] \\
\leq \quad & 2\,c_2\,\mathbb{E}\left[\int_0^T X_{u^*}^4(s)K^2(s)\,\big|C(s)-D(s)\Gamma(s)\right. \\
& \left.\times[K(s)B(s)+K(s)C'(s)D(s)+L'(s)D(s)]'\big|^2 ds\right]^{\frac{1}{2}} \\
\leq \quad & 2c_2\,\mathbb{E}\left[\sup_{t\in[0,T]}X_{u^*}^4(t)\;K^2(t)\int_0^T\big|C(s)-D(s)\Gamma(s)\right. \qquad (4.5.10)\\
& \left.\times[K(s)B(s)+K(s)C'(s)D(s)+L'(s)D(s)]'\big|^2 ds\right]^{\frac{1}{2}} \\
\leq \quad & c_2\,\kappa^2\,\mathbb{E}\left[\sup_{t\in[0,T]}X_{u^*}^4(t)\right] \\
& +c_2\;\;\mathbb{E}\left[\int_0^T\big|C(s)-D(s)\Gamma(s)[K(s)B(s)+K(s)C'(s)D(s)+L'(s)D(s)]'\big|^2 ds\right] \\
< \quad & \infty,
\end{aligned}$$

where the last step follows from the assumption on $X_{u^*}(\cdot)$, and from (4.5.8). Therefore, from (4.5.9) and (4.5.10), and by Theorem 2.4.6, the process $\hat{X}(\cdot,K(\cdot),L(\cdot),X_{u^*}(\cdot),u^*(\cdot))$ is a martingale. Hence, $u^*\in\mathcal{A}$. □

4.6 Application to the mean–variance hedging problem

The mean–variance hedging problem is a special case of the linear-quadratic optimal stochastic control problem. The known results on the existence of solutions to this problems do not cover the case when the coefficients are possibly unbounded. In this section, we formulate the mean–variance hedging problem when the market coefficients are allowed to be unbounded. Consider a market with one bond and m stocks, the prices of which are, respectively,

$$\begin{cases} dS_0(t) = r(t)\, S_0(t)\, dt, \quad t \in [0,T], \\ \\ dS_i(t) = S_i(t)\, [\mu_i(t) \, + \sigma_i'(t)\, dW(t)], \quad i = 1, ..., m, \quad t \in [0,T]. \end{cases} \tag{4.6.1}$$

The process $r(\cdot)$ is the interest rate, the processes $\mu_i(\cdot)$, $i = 1, ..., m$, are the appreciation rates, and the processes $\sigma_i(\cdot) = [\sigma_{i1}(\cdot), ..., \sigma_{i\,d}(t)]$, $i = 1, ..., m$, are the volatilities of the stocks. We assume that $\sigma(\cdot)$ is nonsingular. That is, there exists a positive constant ε such that $\sigma\sigma'(t) \geq \varepsilon\, I_{m\times m}$ *a.e.* $t \in [0,T]$ *a.s.*. The risk premium process is given by $\eta(t) := \sigma'(t)[\sigma(t)\,\sigma'(t)]^{-1}\,\hat{\mu}(t)$, $t \in [0,T]$, where $\mathbb{1}_m = (1, ..., 1)' \in \mathbb{R}^m$, and $\hat{\mu}(t) := \mu(t) - r(t)\,\mathbb{1}_m$. For any $x \in \mathbb{R}$ and $\pi(\cdot) \in \mathcal{A}$, the wealth process $(X(t), t \in [0,T])$, with initial wealth x and with amount of wealth $\pi(\cdot)$ invested in $S(\cdot)$, is given (assuming that the investment is self-financing) by:

$$\begin{cases} dX(t) = [r(t)\, X(t) + \, \hat{\mu}'(t)\, \pi(t)\,]\, dt + \pi'(t)\, \sigma(t)\, dW(t), \quad t \in [0,T], \\ \\ X(0) = x. \end{cases} \tag{4.6.2}$$

Given a random variable ξ, the mean–variance hedging problem is a one-dimensional singular LQ optimal control problem which is given by:

$$\begin{cases} \min\limits_{\pi(\cdot)\in\mathcal{A}} \; \mathbb{E}|X(T) - \xi|^2, \\ \\ \text{subject to } \; (4.6.2). \end{cases} \tag{4.6.3}$$

The case of the mean–variance hedging problem (4.6.3) with $\xi = 0$ is a special case of the LQ optimal control problem (4.5.3). For the general mean–variance hedging problem, the solution is based on the Riccati BSDE where our results in section 4.4 can be applied, similarly to Kohlmann and Tang paper [39].

4.7 Conclusion

We have considered the solvability for the one-dimensional case of Riccati BSDEs under weaker conditions on the coefficients. We have given sufficient conditions for the existence of a solution pair when the coefficients are possibly unbounded. As an application, we have proved the solvability to the LQ optimal control problem with possibly unbounded coefficients using our results for the Riccati BSDE with unbounded coefficients.

4.8 Appendix

Proof of Lemma 4.4.3.

By setting $f(t, Y^n(t)) := a^n(t)\, Y^n(t)$ and $g(t, Y^n(t)) := c^n(t)\, Y^n(t)$, we have:

$$\int_0^T |f(t,0)|\, dt < \infty \;\; a.s., \quad \int_0^T |g(t,0)|^2\, dt < \infty \;\; a.s..$$

Moreover, we have:

$$\begin{aligned}
\int_0^T |a^n(t)|\,dt &= \int_0^T \big|a(t) + \alpha_0^n(t, K^n(t), L^n(t))\big|\,dt \\
&\leq \int_0^T |a(t)|\,dt + \int_0^T |\alpha_0^n(t, K^n(t), L^n(t))|\,dt \\
&\leq \int_0^T |a(t)|\,dt + \int_0^T |n + \alpha_1(t)|\,dt \\
&\leq n\,T + 2\int_0^T \alpha_1(t)\,dt < \infty \quad a.s..
\end{aligned}$$

Also, we have:

$$\begin{aligned}
&\int_0^T |c^n(t)|^2\,dt \\
=\;& \int_0^T \sum_{i=1}^d [c_i^n(t)]^2\,dt \\
=\;& \int_0^T \sum_{i=1}^d \big[c_i(t) + \alpha_i^n(t, K^n, L^n)\big]^2\,dt \\
=\;& \int_0^T \sum_{i=1}^d \big[c_i(t) + \lambda^n(t, K^n, L^n)\;(n + \alpha_2(t))\;S(L_i^n)\big]^2\,dt \\
\leq\;& \int_0^T \sum_{i=1}^d \big[c_i(t) + n + \alpha_2(t)\big]^2\,dt \leq \int_0^T \sum_{i=1}^d \big[3(c_i(t))^2 + 3n^2 + 3\alpha_2^2(t)\big]\,dt \\
=\;& 3n^2T + 3\int_0^T |c(t)|^2\,dt + 3\int_0^T \alpha_2^2(t)\,dt \leq 3n^2\,T + 6\int_0^T \alpha_2^2(t)\,dt < \infty \quad a.s..
\end{aligned}$$

Therefore, by [26] there exists a unique solution $Y^n(t)$ of equation (4.4.9).

Proof of Lemma 4.4.8.

(i) This follows immediately from the definition of Q.

(ii) $$\frac{\partial Q}{\partial x}(t,x) = \frac{1}{12\lambda_2(t)+1}\left[(12\lambda_2(t)+1)e^{(12\lambda_2(t)+1)x}\right] - 1$$

$$= e^{(12\lambda_2(t)+1)x} - 1 \geq 0 \quad \text{for all } (t,x) \in [0,T]\times[0,\kappa] \ \ a.s..$$

(iii) $$\frac{\partial Q}{\partial t}(t,x) = \frac{12\,\delta(t)\,x\,e^{(12\lambda_2(t)+1)x}}{12\lambda_2(t)+1} - 12\,\delta(t)\,\frac{e^{(12\lambda_2(t)+1)x}-1}{(12\lambda_2(t)+1)^2}$$

$$\leq \frac{12\,\delta(t)\,x\,e^{(12\lambda_2(t)+1)x}}{12\lambda_2(t)+1} \quad \text{for all } (t,x) \in [0,T]\times[0,\kappa] \ \ a.s..$$

(iv) This follows from *(ii) and (iii).*

Chapter 5

Further Results on Riccati Backward Stochastic Differential Equations with Unbounded Coefficients

5.1 Abstract

We consider a class of Riccati backward stochastic differential equations with possibly unbounded coefficients. Integrability conditions on the coefficients, which are weaker than those that already exist, are derived that ensure the existence and uniqueness of the solution pair.

5.2 Introduction

We consider the following one-dimensional case of Riccati BSDEs:

$$
\begin{cases}
dK(t) = -\big[a(t)\,K(t) + c'(t)\,L(t) + Q(t) + F(t, K(t), L(t))\big]\,dt \\
\qquad\qquad +L'(t)\,dW(t), \qquad t \in [0,T], \\
K(T) = \;M \quad a.s.,
\end{cases}
\tag{5.2.1}
$$

under the same settings as in chapter 4. A pair of processes $(K(\cdot), L(\cdot))$ that satisfies (5.2.1) is said to be a solution pair. Bismut was the first to study Riccati BSDEs with random coefficients. In [10], he proved the existence result for the case where $C = 0$ and $D = 0$. The problem of solving equation (5.2.1) when only $D = 0$ was introduced by Bismut [11]. Later, Peng [54] proved the existence and uniqueness of this class of Riccati BSDEs with random coefficients under which he treated the Riccati equation as a nonlinear BSDE. In all of these cases, the coefficients are assumed to be bounded. In this chapter, we study the solvability of this class of Riccati BSDEs under weaker conditions on the coefficients in comparison to Theorem 5.1. of [54]. By assuming $D = 0$, equation (5.2.1) becomes:

$$
\begin{cases}
dK(t) = -\big[a(t)\,K(t) + c'(t)\,L(t) + Q(t) \\
\qquad\qquad -B(t)N^{-1}(t)B'(t)K^2(t)\big]\,dt + L'(t)\,dW(t), \quad t \in [0,T], \\
K(T) = \;M \quad a.s..
\end{cases}
\tag{5.2.2}
$$

Here, we give sufficient integrability conditions, which permit for unbounded coefficients, for the unique solvability of (5.2.2).

5.3 Notation and assumptions

The key notation and assumptions used in this chapter are as follows:

- $M \geq 0$ *a.s.*.
- $Q(t) \geq 0$ *a.e.* $t \in [0,T]$ *a.s.* and $N(t) \geq \varepsilon\, I_{m\times m}$ *a.e.* $t \in [0,T]$ *a.s.* for some constant $\varepsilon > 0$.
- $1 \leq \beta_1 \in \mathbb{R}$ and $1 \leq \beta_2 \in \mathbb{R}$ are given constants.
- $\gamma(t)$ is a given $\mathbb{R}$-valued positive progressively measurable process.
- $p(t) := \exp\left\{\int_0^t \left[\gamma(s) + \beta_1 a^2(s) + \beta_2\, |c(s)|^2\right] ds\right\}$.
- $\hat{p}(t) := \exp\left\{\int_0^t \left[\gamma(s) + 2\beta_1 a^2(s) + 4\beta_1\, [B(s)\, N^{-1}(s)\, B'(s)\,]^4 + \beta_2\, |c(s)|^2\right]\, ds\right\}$.
- $\alpha(t) := \inf\limits_{x\in[0,\hat{K}]} [a(t) - 2\, B(t)\, N^{-1}(t)\, B'(t)\ x]^2,\ t \in [0,T]$, for some constant $\hat{K}$.
- $\gamma(t) + 2\beta_1 a^2(t) + 4\beta_1 [B(t)N^{-1}(t)B'(t)]^4 + (\beta_2 - 2)|c(t)|^2 - 2a(t) \geq 0$ *a.e.* $t \in [0,T]$ *a.s.*.
- $V_1 := \mathbb{E}\Bigg[M e^{\int_t^T (a(s) - \frac{1}{2}|c(s)|^2)\, ds + \int_t^T \langle c(s), dW(s)\rangle}$
  $+ \int_t^T e^{\int_t^s (a(\tau)-\frac{1}{2}|c(\tau)|^2)\, d\tau + \int_t^s \langle c(\tau), dW(\tau)\rangle}\ [B(s)\, N^{-1}(s)\, B'(s) + Q(s)]\, ds\, \Bigg|\mathcal{F}_t\Bigg]$,
  $t \in [0,T]$.
- $\mathcal{H}_p^2(0,T;\mathbb{R}^d)$ (resp. $\mathcal{H}_{\hat{p}}^2(0,T;\mathbb{R}^d)$ is the space of $\mathcal{F}_t$-progressively measurable $\mathbb{R}^d$-valued processes $\varphi(\cdot)$ such that
  $\mathbb{E}\left[\sup_{t\in[0,T]}\ \ p(t)|\varphi(t)|^2\right] < \infty$ (resp. $\mathbb{E}\left[\sup_{t\in[0,T]}\ \ \hat{p}(t)|\varphi(t)|^2\right] < \infty$).
- $\mathcal{M}_p^2(0,T;\mathbb{R}^d)$ (resp. $\mathcal{M}_{\hat{p}}^2(0,T;\mathbb{R}^d)$ is the space of $\mathcal{F}_t$-progressively measurable $\mathbb{R}^d$-valued processes $\varphi(\cdot)$ such that $\mathbb{E}\left[\int_0^T\ \ p(s)|\varphi(s)|^2 ds\right] < \infty$ (resp. $\mathbb{E}\left[\int_0^T\ \ \hat{p}(s)|\varphi(s)|^2 ds\right] < \infty$).

We say the coefficients and the random variable M satisfy conditions A if:

(i) $\mathbb{E}\left[\, \hat{p}(T)\right] < \infty$;

(ii) $\mathbb{E}\left[\int_0^T \hat{p}(t)\ \left([B(t)\,N^{-1}(t)\,B'(t)]^2 + \dfrac{Q^2(t)}{\gamma(t)+\beta_2\,|c(t)|^2}\right)\,dt\right] < \infty$;

(iii) there exists $\hat{K} \in \mathbb{R}^+$ such that $V_1 \leq \hat{K}$ *a.e.* $t \in [0,T]$ *a.s.*.

Here the conditions permit for unbounded coefficients which is not the case in [39]. For example, if $d = m = 1$, $\gamma(t) = 1$ *a.s. a.e.* $t \in [0,T]$, $B(t) = W(t)$, $N(t) = W^2(t) + \nu(t)$, $t \in [0,T]$ for some unbounded process $\nu(t)$, M is bounded *a.s.*, $A(t) = C(t) = 0$ *a.s. a.e.* $t \in [0,T]$, $Q(t) = B^2(t)/N(t)$ *a.s. a.e.* $t \in [0,T]$, the known results on the existence of solutions do not apply here where conditions A hold. In other words, our conditions are weaker as compared to [39]. Note that we can choose the process $\gamma(\cdot)$ and $\hat{K}$. This provides more flexibility on the conditions. For instance, condition A-(iii) can be suitable weakened by choosing large values for $\hat{K}$. .

5.4 Existence and uniqueness

In this section, we prove the existence of a unique solution pair $(K(\cdot), L(\cdot))$ for (5.2.2). Our method of proof is the quasilinearization approach of Bellman [5]. This method offers a way to solve a nonlinear ordinary differential equation as a limit of a sequence of linear differential equations. Wonham [61] applied this method to solve Riccati BSDEs when all the coefficients are deterministic. Further, Peng [54] gave a generalization of Wonham's approach to the random setting. In theorem 5.1., he proved the existence and uniqueness of the Riccati BSDE with random coefficients by using Bellman's method under which he treated this equation as a nonlinear BSDE.

Here, we construct an infinite sequence of linear BSDEs. Then, we prove that the approximating solutions of the sequence converge strongly to the solution of (5.2.2). To prove the uniqueness, we follow Peng [54] to show that the first component $(K(t), t \in [0,T])$ of the solution pair is unique.

However, our proof is diffrenet from [54] since the coefficints are possibly unbounded, and thus we apply a diffrent known result in order to obtain the solvability of the equation of differences. On the other hand, we follow [27] to prove the uniqueness of the second component $(L(t), t \in [0, T])$ of the solution pair. Before studying the solvability of equation (5.2.2), we present the following lemma. In this lemma, we apply a known result due to Gashi and Li [27] which considers general BSDEs with unbounded coefficients.

Lemma 5.4.1. *Let conditions A hold. The linear backward stochastic differential equation:*

$$\begin{cases} dy(t) = -[a(t)y(t) + c'(t)L(t) + Q(t)]\, dt + L'(t)\, dW(t), \quad t \in [0, T], \\ \\ y(T) = M \quad a.s., \end{cases} \tag{5.4.1}$$

has a unique solution pair $(y(\cdot), L(\cdot)) \in \mathcal{H}^2_p(0, T; \mathbb{R}) \times \mathcal{M}^2_p(0, T; \mathbb{R}^d)$ *with* $y(t) \geq 0$ *for all* $t \in [0, T]$ *a.s..*

Proof. As equation (5.4.1) is a linear BSDE, it satisfies the conditions of Theorem 2.1 of [27], and it thus has a unique solution pair $(y(\cdot), L(\cdot)) \in \mathcal{H}^2_p(0, T; \mathbb{R}) \times \mathcal{M}^2_p(0, T; \mathbb{R}^d)$. In order to show the non-negativity of y, we use the comparison theorem of [27], as follows. Consider the linear BSDE:

$$\begin{cases} dx(t) = -[a(t)x(t) + c'(t)z(t)]\, dt + z'(t)\, dW(t), \quad t \in [0, T], \\ \\ x(T) = 0 \quad a.s.. \end{cases} \tag{5.4.2}$$

This equation satisfies the conditions on Theorem 2.1 of [27], and it thus has a unique solution pair. Moreover, it is clear that this unique solution pair is $x(t) = 0$ for all $t \in [0, T]$ *a.s.*, and $z(t) = 0$ *a.e.* $t \in [0, T]$ a.s.. Let $f(t, y, L) := a(t)y + c'(t)L + Q(t)$ and $g(t, x, z) := a(t)x + c'(t)z$. Due to the non-negativity of Q, it holds that $f(t, y, L) \geq g(t, y, L)$ *a.s.* $\forall (t, y, L) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$, and since M is also non-negative, by Theorem 2.2 of [27], we have $y(t) \geq x(t) = 0$ $\forall t \in [0, T]$ a.s.. □

Equation (5.2.2) can be written as:

$$
\begin{cases}
dK(t) = -[F(t, K(t), L(t), \hat{u}(t, K(t))) + \hat{u}'(t, K(t))\, N(t)\, \hat{u}(t, K(t))]\; dt \\
\qquad\qquad +L'(t)\; dW(t), \quad t \in [0, T], \\
K(T) = M \quad a.s.,
\end{cases}
\tag{5.4.3}
$$

where the random functions F and $\hat{u}$ are defined as:

$$
\begin{aligned}
&F(t, x_1, y_1, z_1) := a(t)x_1 + c'(t)y_1 + Q(t) + 2x_1 B(t) z_1, \\
&\forall (t, x_1, y_1, z_1) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^m, \\
&\hat{u}(t, x_1) := -x_1 N^{-1}(t) B'(t), \quad \forall (t, x_1) \in [0, T] \times \mathbb{R}.
\end{aligned}
$$

Let $(K_0(t), L_0(t)) := (0, 0)$ for all $t \in [0, T]$. As an approximation to (5.4.3), we introduce the following infinite sequence of *linear* BSDEs:

$$
\begin{cases}
dK_{n+1}(t) = -[F(t, K_{n+1}(t), L_{n+1}(t), \hat{u}(t, K_n(t))) \\
\qquad\qquad +\hat{u}'(t, K_n(t))\, N(t)\, \hat{u}(t, K_n(t))]\, dt \\
\qquad\qquad +L'_{n+1}(t)\, dW(t), \quad t \in [0, T], \\
K_{n+1}(T) = M \quad a.s., \quad n \in \mathbb{N}_0.
\end{cases}
\tag{5.4.4}
$$

Lemma 5.4.2. *Let conditions A hold. There exists unique solution pairs* $(K_{n+1}(\cdot), L_{n+1}(\cdot))$ *to equations* (5.4.4) *with the following properties:*

(i) $(K_{n+1}(\cdot), L_{n+1}(\cdot)) \in \mathcal{H}^2_{\hat{p}}(0, T; \mathbb{R}) \times \mathcal{M}^2_{\hat{p}}(0, T; \mathbb{R}^d)$, $n \in \mathbb{N}_0$,

(ii) $K_{n+1}(t) \in [0, \hat{K}]$ *a.s. for all* $t \in [0, T]$, $n \in \mathbb{N}_0$,

(iii) $K_n(t) \geq K_{n+1}(t)$ *a.s. for all* $t \in [0, T]$, $n \in \mathbb{N}_0$.

Proof. We give a proof by induction. The first of the equations in (5.4.4), corresponding to $n = 0$, is:

$$\begin{cases} dK_1(t) = -[F(t, K_1(t), L_1(t), \hat{u}(t, K_0(t))) + \hat{u}'(t, K_0(t))\, N(t)\, \hat{u}(t, K_0(t))]\, dt \\ \qquad\qquad +L_1'(t)\, dW(t), \quad t \in [0, T], \\ K_1(T) = M \quad a.s.. \end{cases} \tag{5.4.5}$$

After substituting the expressions for F and $\hat{u}$, this equation becomes:

$$\begin{cases} dK_1(t) = -[a(t)K_1(t) + c'(t)L_1(t) + Q(t)]\, dt + L_1'(t)\, dW(t), \quad t \in [0, T], \\ K_1(T) = M \quad a.s.. \end{cases} \tag{5.4.6}$$

By Lemma 5.4.1, there exists a unique solution pair $(K_1(\cdot), L_1(\cdot))$ $\in \mathcal{H}_p^2(0, T; \mathbb{R}) \times \mathcal{M}_p^2(0, T; \mathbb{R}^d)$ such that $K_1(t) \geq 0$ *a.s.* for all $t \in [0, T]$. As this is a linear BSDE, the explicit expression for K_1 is:

$$\begin{aligned} K_1(t) \;&=\; \mathbb{E}\left\{ M \exp\left[\int_t^T (a(s) - \frac{1}{2}c'(s)c(s))ds + \int_t^T c'(s)dW(s) \right] \right. \\ &\quad \left. + \int_t^T \exp\left[\int_t^s (a(\tau) - \frac{1}{2}c'(\tau)c(\tau))d\tau + \int_t^s c'(\tau)dW(\tau) \right] Q(s)ds \,\middle|\, \mathcal{F}_t \right\}, \\ &\quad t \in [0, T]. \end{aligned} \tag{5.4.7}$$

By assumption A(iii), it follows that $K_1(t) \leq \hat{K}$ *a.s.* for all $t \in [0, T]$. We now *assume* that for some $1 < m \in \mathbb{N}$ the equation:

$$\begin{cases} dK_m(t) = -[F(t, K_m(t), L_m(t), \hat{u}(t, K_{m-1}(t))) \\ \qquad\qquad +\hat{u}'(t, K_{m-1}(t))\, N(t)\, \hat{u}(t, K_{m-1}(t))]\, dt \\ \qquad\qquad +L_m'(t)\, dW(t), \quad t \in [0, T], \\ K_m(T) = M \quad a.s., \end{cases} \tag{5.4.8}$$

has a unique solution pair $(K_m(\cdot), L_m(\cdot))$ with the claimed properties. It remains to *prove* that the equation:

$$\begin{cases} dK_{m+1}(t) = -[F(t, K_{m+1}(t), L_{m+1}(t), \hat{u}(t, K_m(t))) \\ \qquad\qquad +\hat{u}'(t, K_m(t))\, N(t)\, \hat{u}(t, K_m(t))]\, dt \\ \qquad\qquad +L'_{m+1}(t)\, dW(t), \quad t \in [0, T], \\ K_{m+1}(T) = M \quad a.s., \end{cases} \tag{5.4.9}$$

has a unique solution pair $(K_{m+1}(\cdot), L_{m+1}(\cdot))$ with the claimed properties. By substituting the expressions for F and $\hat{u}$ in (5.4.9), we obtain:

$$\begin{cases} dK_{m+1}(t) = -\{[a(t) - 2B(t)\, N^{-1}(t)\, B'(t) K_m(t)]\, K_{m+1}(t) + c'(t)\, L_{m+1}(t) \\ \qquad\qquad +Q(t) + B(t) N^{-1}(t) B'(t) K_m^2(t)\}\, dt + L'_{m+1}(t)\, dW(t), \quad t \in [0, T], \\ K_{m+1}(T) = M \quad a.s.. \end{cases}$$

We first show that this equation satisfies the requirements of Theorem 2.1 of [27] for the existence of a unique solution pair. From the basic inequality

$$\beta_1[a(t) - 2B(t)N^{-1}(t)B'(t)K_m(t)]^2 \leq 2\beta_1 a^2(t) + 4\beta_1[B(t)N^{-1}(t)B'(t)]^4 + 4\beta_1 \hat{K}^4,$$

it follows that:

$$\begin{aligned} & \mathbb{E}\big[e^{\int_0^T \{\gamma(t)+\beta_1[a(t)-2\,B(t)\,N^{-1}(t)\,B'(t)K_m(t)]^2+\beta_2\,|c(t)|^2\}dt}\,M^2\big] \\ \leq \;& \mathbb{E}\big[e^{\int_0^T \{\gamma(t)+2\,\beta_1\,a^2(t)+4\,\beta_1[\,B(t)\,N^{-1}(t)\,B'(t)]^4+4\,\beta_1\,\hat{K}^4+\beta_2\,|c(t)|^2\}dt}\,M^2\big] \\ = \;& e^{4\,\beta_1\,\hat{K}^4\,T}\,\mathbb{E}\big[e^{\int_0^T \{\gamma(t)+2\,\beta_1\,a^2(t)+4\,\beta_1[\,B(t)\,N^{-1}(t)\,B'(t)]^4+\beta_2\,|c(t)|^2\}dt}\,M^2\big] < \infty, \end{aligned}$$

where the last inequality is due to assumption A(i). By the same basic inequality, we further have:

$$
\begin{aligned}
&\mathbb{E}\bigg[\int_0^T \frac{e^{\int_0^t \{\gamma(s)+\beta_1[a(s)-2\,B(s)\,N^{-1}(s)\,B'(s)K_m(s)]^2+\beta_2\,|c(s)|^2\}ds}\,[K_m^2(t)\,B(t)\,N^{-1}(t)\,B'(t)+Q(t)]^2}{\gamma(t)+\beta_1[a(t)-2\,B(t)\,N^{-1}(t)\,B'(t)K_m(t)]^2+\beta_2\,|c(t)|^2}\,dt\bigg] \\
\leq\quad & e^{4\beta_1\hat{K}^4T}\mathbb{E}\left[\int_0^T \hat{p}(t)\,\frac{[B(t)\,N^{-1}(t)\,B'(t)\hat{K}^2+Q(t)]^2}{\gamma(t)+\beta_1\alpha(t)+\beta_2\,|c(t)|^2}\,dt\right] < \infty,
\end{aligned}
$$

where the last inequality follows from assumption A(ii). By Theorem 2.1 of [27], we now know that there exist unique solution pairs $(K_{m+1}(\cdot), L_{m+1}(\cdot))$ to equations (5.4.9) with the claimed property (i). Moreover, since $Q(t) + B(t)N^{-1}(t)B'(t)K_m^2(t) \geq 0$ *a.e.* $t \in [0,T]$ *a.s.*, by the comparison Theorem 2.2 of [27], and in the same way as in the proof of Lemma 5.4.1, we further know that $K_{m+1}(t) \geq 0 \;\; \forall t \in [0,T]$ *a.s.*.

In order to show that $K_{m+1}(t) \leq \hat{K}$ for all $t \in [0,T]$ *a.s.* and that the property (iii) holds, we consider the difference $K_m(t) - K_{m+1}(t)$, the equation of which is:

$$
\begin{cases}
d[K_m(t) - K_{m+1}(t)] = -[F(t, K_m(t), L_m(t), \hat{u}(t, K_{m-1}(t))) \\
\qquad\qquad +\hat{u}'(t, K_{m-1}(t))\,N(t)\,\hat{u}(t, K_{m-1}(t)) \\
\qquad\qquad -F(t, K_{m+1}(t), L_{m+1}(t), \hat{u}(t, K_m(t))) \\
\qquad\qquad -\hat{u}'(t, K_m(t))\,N(t)\,\hat{u}(t, K_m(t))]\,dt \\
\qquad\qquad +[L_m(t) - L_{m+1}(t)]'\,dW(t), \quad t \in [0,T], \\
K_m(T) - K_{m+1}(T) = 0 \quad a.s..
\end{cases}
$$

This equation has a unique solution pair $(K_m(\cdot) - K_{m+1}(\cdot), L_m(\cdot) - L_{m+1}(\cdot))$ (since equations (5.4.8) and (5.4.9) also have unique solutions pairs), and can rewritten as:

$$\begin{cases} d[K_m(t) - K_{m+1}(t)] = -[F(t, K_m(t), L_m(t), \hat{u}(t, K_m(t))) \\ \qquad\qquad -F(t, K_{m+1}(t), L_{m+1}(t), \hat{u}(t, K_m(t))) + Q_m(t)]\, dt \\ \qquad\qquad +[L_m(t) - L_{m+1}(t)]'\, dW(t), \quad t \in [0, T], \\ K_m(T) - K_{m+1}(T) = 0 \quad a.s., \end{cases} \tag{5.4.10}$$

where

$$\begin{aligned} Q_m(t) \; := \;& F(t, K_m(t), L_m(t), \hat{u}(t, K_{m-1}(t))) + \hat{u}'(t, K_{m-1}(t))\, N(t)\, \hat{u}(t, K_{m-1}(t)) \\ & -F(t, K_m(t), L_m(t), \hat{u}(t, K_m(t))) - \hat{u}'(t, K_m(t))\, N(t) \hat{u}(t, K_m(t)). \end{aligned}$$

Note that:

$$\begin{aligned} & F(t, K_m(t), L_m(t), \hat{u}(t, K_m(t))) - F(t, K_{m+1}(t), L_{m+1}(t), \hat{u}(t, K_m(t))) \\ = \;& a(t) K_m(t) + c'(t) L_m(t) + Q(t) + 2K_m(t) B(t)\, [-K_m(t)\, N^{-1}(t)\, B'(t)] \\ & -a(t)\, K_{m+1}(t) - c'(t) L_{m+1}(t) - Q(t) - 2K_{m+1}(t) B(t)\, [-K_m(t) N^{-1}(t) B'(t)] \\ = \;& a(t)[K_m(t) - K_{m+1}(t)] + c'(t)[L_m(t) - L_{m+1}(t)] \\ & +[K_m(t) - K_{m+1}(t)]\, 2B(t)\, [-K_m(t) N^{-1}(t) B'(t)] \\ = \;& \big[a(t) - 2B(t)\, N^{-1}(t)\, B'(t)\, K_m(t)\big]\, [K_m(t) - K_{m+1}(t)] + c'(t)[L_m(t) - L_{m+1}(t)], \end{aligned}$$

$$\begin{aligned} Q_m(t) \; = \;& a(t)\, K_m(t) + c'(t) L_m(t) + Q(t) + 2K_m(t)\, B(t)[-K_{m-1}(t)\, N^{-1}(t)\, B'(t)] \\ & +\big[-K_{m-1}(t)\, N^{-1}(t)\, B'(t)\big]'\, N(t)\, \big[-K_{m-1}(t)\, N^{-1}(t)\, B'(t)\big] \\ & -a(t)\, K_m(t) - c'(t)\, L_m(t) - Q(t) - 2K_m(t)\, B(t)\, \big[-K_m(t)\, N^{-1}(t)\, B'(t)\big] \\ & -\big(-K_m(t)\, N^{-1}(t)\, B'(t)\big)'\, N(t)\, \big(-K_m(t)\, N^{-1}(t)\, B'(t)\big) \\ = \;& -2K_m(t)\, K_{m-1}(t)\, B(t) N^{-1}(t) B'(t) + K_{m-1}^2(t)\, B(t) N^{-1}(t) B'(t) \\ & +2K_m^2(t)\, B(t) N^{-1}(t) B'(t) - K_m^2(t)\, B(t) N^{-1}(t) B'(t) \\ = \;& [K_m(t) - K_{m-1}(t)]^2 B(t) N^{-1}(t) B'(t). \end{aligned}$$

Equation (5.4.10) can thus be written as:

$$\begin{cases} d[K_m(t) - K_{m+1}(t)] = -\left\{\left[a(t) - 2B(t)\, N^{-1}(t)\, B'(t)\, K_m(t)\right][K_m(t) - K_{m+1}(t)] \right. \\ \qquad\qquad\qquad\qquad\quad \left. +c'(t)[L_m(t) - L_{m+1}(t)] + Q_m(t)\right\} dt \\ \qquad\qquad\qquad\qquad\quad +[L_m(t) - L_{m+1}(t)]'\, dW(t), \quad t \in [0,T], \\ K_m(T) - K_{m+1}(T) = 0 \quad a.s.. \end{cases}$$

This equation satisfies the requirements of the comparison Theorem 2.2 of [27], and in the same way as in the proof of Lemma 5.4.1, we further know that $K_m(t) - K_{m+1}(t) \geq 0$ for all $t \in [0,T]$ $a.s.$. This implies that $K_{m+1}(t) \leq K_m(t) \leq \hat{K}$ for all $t \in [0,T]$ $a.s.$, which concludes the proof that the pair $(K_{m+1}(\cdot), L_{m+1}(\cdot))$ has the claimed properties. □

Theorem 5.4.3. *The Riccati backward stochastic differential equation* (5.2.2) *has a unique solution pair* $(K(\cdot), L(\cdot)) \in \mathcal{L}^\infty_\mathcal{F}(0,T;\mathbb{R}) \times \mathcal{M}^2_{\hat{p}}(0,T;\mathbb{R}^d)$.

Proof of existence. As shown in Lemma 5.4.2, the sequence $\{K_n(t)\}_{n\geq 1}$ is bounded and decreasing, and thus, by the monotone convergence theorem, it converges to a process $K(t)$ for all $t \in [0,T]$ $a.s.$. Later in the proof we show that this convergence is actually uniform in t. We next show the strong convergence of the sequence $\{L_n(t)\}_{n\geq 1}$. If $n < m$, then the differential of

$\hat{p}(t)[K_n(t) - K_m(t)]^2$ is:

$$d\left\{\hat{p}(t)\,[K_n(t) - K_m(t)]^2\right\} =$$

$$[K_n(t) - K_m(t)]^2 \left[\gamma(t) + 2\beta_1 a^2(t) + 4\beta_1 (B(t)\, N^{-1}(t)\, B'(t))^4 + \beta_2\, |c(t)|^2\right]\, \hat{p}(t)\, dt$$

$$-2[K_n(t) - K_m(t)]\, \hat{p}(t) \left\{\left[a(t) - 2B(t)\, N^{-1}(t)\, B'(t)\, K_{n-1}(t)\right] [K_n(t) - K_m(t)]\right.$$

$$+\, c'(t)[L_n(t) - L_m(t)]$$

$$\left. +\, [K_{n-1}(t) - K_{m-1}(t)]\, [K_{n-1}(t) + K_{m-1}(t) - 2K_m(t)] B(t) N^{-1}(t) B'(t)\right\} dt$$

$$+\hat{p}(t)\, |L_n(t) - L_m(t)|^2 dt + 2[K_n(t) - K_m(t)]\, \hat{p}(t)\, [L_n(t) - L_m(t)]'\, dW(t).$$

Integration from t to T, and the fact that almost surely $K_n(T) = K_m(T)$, give:

$$
\begin{aligned}
&\hat{p}(t)\,[K_n(t) - K_m(t)]^2 \\
= \quad & -\int_t^T [K_n(s) - K_m(s)]^2 \\
&\times \left[\gamma(s) + 2\beta_1 a^2(s) + 4\beta_1 (B(s)\,N^{-1}(s)\,B'(s))^4 + \beta_2\,|c(s)|^2\right]\,\hat{p}(s)\,ds \\
&+ \int_t^T 2\,\left[a(s) - 2B(s)\,N^{-1}(s)\,B'(s)\,K_{n-1}(s)\right]\,\hat{p}(s)\,[K_n(s) - K_m(s)]^2\,ds \\
&+ \int_t^T 2\,\hat{p}(s)\,[K_n(s) - K_m(s)]c'(s)[L_n(s) - L_m(s)]ds \\
&+ \int_t^T 2\,\hat{p}(s)\,[K_n(s) - K_m(s)][K_{n-1}(s) - K_{m-1}(s)]\,[K_{n-1}(s) + K_{m-1}(s) - 2K_m(s)] \\
&\times \quad B(s)N^{-1}(s)B'(s)\,ds \\
&- \int_t^T \hat{p}(s)\,|L_n(s) - L_m(s)|^2\,ds - \int_t^T 2\,[K_n(s) - K_m(s)]\,\hat{p}(s)\,[L_n(s) - L_m(s)]'\,dW(s).
\end{aligned}
$$

By applying the simple inequality $2xy \leq 2x^2 + y^2/2$ to the product $2c'(t)[L_n(t)-$

$L_m(t)]$, we obtain:

$$
\begin{aligned}
& \hat{p}(t)\,[K_n(t)-K_m(t)]^2+\int_t^T \hat{p}(s)\,|L_n(s)-L_m(s)|^2\,ds/2 \\
\leq\; & -\int_t^T [K_n(s)-K_m(s)]^2 \\
& \times\quad \left[\gamma(s)+2\beta_1 a^2(s)+4\beta_1(B(s)\,N^{-1}(s)\,B'(s))^4+(\beta_2-2)|c(s)|^2\right]\,\hat{p}(s)\,ds \\
& +\int_t^T 2\,\left[a(s)-2B(s)\,N^{-1}(s)\,B'(s)\,K_{n-1}(s)\right]\,\hat{p}(s)\,[K_n(s)-K_m(s)]^2\,ds \\
& +\int_t^T 2\,\hat{p}(s)\,[K_n(s)-K_m(s)][K_{n-1}(s)-K_{m-1}(s)]\,[K_{n-1}(s)+K_{m-1}(s)-2K_m(s)] \\
& \times\quad B(s)N^{-1}(s)B'(s)\,ds \\
& -\int_t^T 2\,[K_n(s)-K_m(s)]\,\hat{p}(s)\,[L_n(s)-L_m(s)]'\,dW(s).
\end{aligned}
$$

In the same way as in the proof of Lemma 2.1 (ii) of [27], it can be shown that the stochastic integral in the above inequality is a martingale. This implies that:

$$
\begin{aligned}
& \mathbb{E}\,[K_n(0)-K_m(0)]^2+\mathbb{E}\left[\int_0^T \hat{p}(s)\,|L_n(s)-L_m(s)|^2\,ds/2\right] \\
\leq\; & -4\,\mathbb{E}\left[\int_0^T B(s)\,N^{-1}(s)\,B'(s)\,K_{n-1}(s)\;\hat{p}(s)\,[K_n(s)-K_m(s)]^2\,ds\right] \\
& +2\,\mathbb{E}\left[\int_0^T \hat{p}(s)\,[K_n(s)-K_m(s)][K_{n-1}(s)-K_{m-1}(s)]\right. \\
& \left.\times[K_{n-1}(s)+K_{m-1}(s)-2K_m(s)]\,B(s)N^{-1}(s)B'(s)\,ds\right].
\end{aligned}
$$

As the sequence $\{K_n(t)\}_{n\geq 1}$ converges to $K(t)$ for all $t\in[0,T]$ $a.s.$, it follows by the dominated convergence theorem that such a term converges to 0 as

(m,n) tends to ∞. This implies that the second term on the right-hand side of the above inequality also converges to 0 as (m,n) tends to ∞. This further implies that the sequence $\{L_n(t)\}_{n\geq 1}$ is a Cauchy sequence in the space $\mathcal{M}^2_{\hat{p}}(0,T;\mathbb{R}^d)$, and thus it converges to a process $(L(t), t\in[0,T])$ in the same space. Moreover, this implies that there exists a subsequence $\{L_{n_j}(t)\}_{j\geq 1}$ of the sequence $\{L_n(t)\}_{n\geq 1}$ that converges to $(L(t), t\in[0,T])$ for *a.e.* $t\in[0,T]$ *a.s.*.

By Theorem 2.4.7, for any constant $\epsilon > 0$, the following holds:

$$\begin{aligned}
&\mathbb{P}\left(\omega\in\Omega : \sup_{t\in[0,T]}\left|\int_t^T [L_{n_j}(s)-L(s)]'\, dW(s)\right| > \epsilon\right)\\
&\leq \frac{1}{\epsilon^2}\,\mathbb{E}\left[\left|\int_0^T [L_{n_j}(s)-L(s)]'\, dW(s)\right|^2\right]\\
&= \frac{1}{\epsilon^2}\,\mathbb{E}\left[\int_0^T |L_{n_j}(s)-L(s)|^2\, ds\right] \leq \frac{1}{\epsilon^2}\,\mathbb{E}\left[\int_0^T \hat{p}(s)|L_{n_j}(s)-L(s)|^2\, ds\right].
\end{aligned}$$

As the right-hand side of the last inequality tends to 0 as $j\to\infty$, it follows that:

$$\lim_{j\to\infty}\sup_{t\in[0,T]}\left|\int_t^T [L_{n_j}(s)-L(s)]'\, dW(s)\right| = 0 \quad \text{in probability.}$$

By Theorem 2.2.6, there exists a subsequence $\{L_{n_q}\}_{q\geq 1}$ of the sequence $\{L_{n_j}\}_{j\geq 1}$ such that:

$$\lim_{q\to\infty}\sup_{t\in[0,T]}\left|\int_t^T [L_{n_q}(s)-L(s)]'\, dW(s)\right| = 0 \quad a.s.. \tag{5.4.11}$$

For any two elements n_q and n_m of the sequence $\{n_q\}_{q\geq 1}$ we have:

$$\begin{aligned}
|K_{n_q}(t) &- K_{n_m}(t)| \leq \int_t^T |a(s) - 2K_{n_q-1}(s)B(s)N^{-1}(s)B'(s)|\, |K_{n_q}(s) - K_{n_m}(s)| ds \\
&+ \int_t^T |K_{n_q-1}(s) - K_{n_m-1}(s)|\, |K_{n_q-1}(s) + K_{n_m-1}(s) - 2K_{n_m}(s)| \\
&\times \quad B(s)N^{-1}(s)B(s)\, ds \\
&+ \int_t^T |c(s)|\, |L_{n_q}(s) - L_{n_m}(s)|\, ds + \left| \int_t^T [L_{n_q}(s) - L(s)]'\, dW(s) \right| \\
&+ \sup_{t\in[0,T]} \left| \int_t^T [L_{n_m}(s) - L(s)]'\, dW(s) \right|.
\end{aligned}$$

Due to the almost sure convergence of the sequences $\{K_{n_m}(t)\}_{m\geq 1}$ and $\{L_{n_m}(t)\}_{m\geq 1}$, and the uniform convergence (5.4.11), as $m \to \infty$ the above inequality becomes:

$$\begin{aligned}
|K_{n_q}(t) &- K(t)| \leq \int_t^T |a(s) - 2K_{n_q-1}(s)B(s)N^{-1}(s)B'(s)|\, |K_{n_q}(s) - K(s)| ds \\
&+ \int_t^T |K_{n_q-1}(s) - K(s)|^2\, B(s)N^{-1}(s)B(s)\, ds \\
&+ \int_t^T |c(s)|\, |L_{n_q}(s) - L(s)|\, ds + \left| \int_t^T [L_{n_q}(s) - L(s)]' dW(s) \right|.
\end{aligned}$$

Moreover, by taking the supremum over $t \in [0,T]$ and letting $q \to \infty$, we obtain:

$$\begin{aligned}
\lim_{q\to\infty} \sup_{t\in[0,T]} |K_{n_q}(t) - K(t)| &\leq \lim_{q\to\infty} \int_0^T |a(s) - 2K_{n_q-1}(s)B(s)N^{-1}(s)B'(s)||K_{n_q}(s) - K(s)|\, ds \\
&+ \lim_{q\to\infty} \int_0^T |K_{n_q-1}(s) - K(s)|^2 B(s)N^{-1}(s)B(s)\, ds \\
&+ \lim_{q\to\infty} \int_0^T |c(s)|\, |L_{n_q}(s) - L(s)|\, ds \\
&+ \lim_{q\to\infty} \sup_{t\in[0,T]} \left| \int_t^T [L_{n_q}(s) - L(s)]'\, dW(s) \right| = 0 \quad a.s..
\end{aligned}$$

Thus, the subsequence $\{K_{n_q}(t)\}_{q\geq 1}$ converges uniformly in t to $K(t)$, which in particular means that the process $(K(t), t \in [0,T])$ has continuous paths (by the uniform convergence theorem). This further implies that as $q \to \infty$, the sequence of equations:

$$\begin{aligned} K_{n_q+1}(t) &= K_{n_q+1}(T) + \int_t^T \big[F(s, K_{n_q+1}(s), L_{n_q+1}(s), \hat{u}(s, K_{n_q}(s))) \\ &+ \hat{u}'(s, K_{n_q}(s))\, N(s)\, \hat{u}(s, K_{n_q}(s))\big] ds \\ &- \int_t^T L'_{n_q+1}(s)\, dW(s), \end{aligned}$$

converges to

$$\begin{aligned} K(t) &= K(T) + \int_t^T \big[F(s, K(s), L(s), \hat{u}(s, K(s))) + \hat{u}'(s, K(s))\, N(s)\, \hat{u}(s, K(s))\big]\, ds \\ &\quad - \int_t^T L'(s)\, dW(s), \end{aligned}$$

which proves that $(K(\cdot), L(\cdot))$ is a solution pair to equation (5.2.2).

Proof of uniqueness. Let $(K^1(\cdot), L^1(\cdot))$ and $(K^2(\cdot), L^2(\cdot))$ be two solution pairs to equation (5.2.2) with the claimed properties. We show that these two solution pairs coincide, and thus equation (5.2.2) has a unique solution pair. To show that $K^1(t) = K^2(t)$ *a.s.* for all $t \in [0,T]$, we prove that $K^1(t) - K^2(t) \geq 0$ *a.s.* for all $t \in [0,T]$, and $K^2(t) - K^1(t) \geq 0$ *a.s.* for all $t \in [0,T]$. The equation of the difference $K^1(t) - K^2(t)$, $t \in [0,T]$, is:

$$\begin{cases} d[K^1(t) - K^2(t)] = -\big[F(t, K^1(t), L^1(t), \hat{u}(t, K^1(t))) + \hat{u}'(t, K^1(t))\, N(t)\, \hat{u}(t, K^1(t)) \\ \qquad\qquad - F(t, K^2(t), L^2(t), \hat{u}(t, K^2(t))) - \hat{u}'(t, K^2(t))\, N(t)\, \hat{u}(t, K^2(t))\big]\, dt \\ \qquad\qquad + [L^1(t) - L^2(t)]'\, dW(t), \quad t \in [0, T], \\ K^1(T) - K^2(T) = 0 \quad a.s.. \end{cases}$$

This equation can be rewritten as:

$$\begin{cases} d[K^1(t) - K^2(t)] = -\big[F(t, K^1(t), L^1(t), \hat{u}(t, K^1(t))) - F(t, K^2(t), L^2(t), \hat{u}(t, K^1(t))) \\ \qquad\qquad + Q^1(t)]dt + [L^1(t) - L^2(t)]'\, dW(t), \quad t \in [0, T], \\ K^1(T) - K^2(T) = 0 \quad a.s., \end{cases} \tag{5.4.12}$$

where

$$\begin{aligned} Q^1(t) \quad &:= \quad F(t, K^2(t), L^2(t), \hat{u}(t, K^1(t))) + \hat{u}'(t, K^1(t))\, N(t)\, \hat{u}(t, K^1(t)) \\ &- \quad F(t, K^2(t), L^2(t), \hat{u}(t, K^2(t))) - \hat{u}'(t, K^2(t))\, N(t)\, \hat{u}(t, K^2(t)). \end{aligned}$$

Note that:

$$\begin{aligned} & F(t, K^1(t), L^1(t), \hat{u}(t, K^1(t))) - F(t, K^2(t), L^2(t), \hat{u}(t, K^1(t))) \\ = \; & a(t)K^1(t) + c'(t)L^1(t) + Q(t) + 2K^1(t)B(t)[-K^1(t)\, N^{-1}(t)\, B'(t)] \\ & -a(t)K^2(t) - c'(t)L^2(t) - Q(t) - 2K^2(t)B(t)[-K^1(t)N^{-1}(t)B'(t)] \\ = \; & a(t)[K^1(t) - K^2(t)] + c'(t)[L^1(t) - L^2(t)] \\ & +[K^1(t) - K^2(t)]2B(t)[-K^1(t)N^{-1}(t)B'(t)] \\ = \; & \big[a(t) - 2B(t)\, N^{-1}(t)\, B'(t)\, K^1(t)\big]\, [K^1(t) - K^2(t)] + c'(t)[L^1(t) - L^2(t)], \end{aligned}$$

$$
\begin{aligned}
Q^1(t) &= a(t)\,K^2(t) + c'(t)L^2(t) + Q(t) + 2K^2(t)\,B(t)[-K^1(t)\,N^{-1}(t)\,B'(t)] \\
&\quad +\big[-K^1(t)\,N^{-1}(t)\,B'(t)\big]'\,N(t)\,\big[-K^1(t)\,N^{-1}(t)\,B'(t)\big] \\
&\quad -a(t)\,K^2(t) - c'(t)L^2(t) - Q(t) - 2K^2(t)\,B(t)\,\big[-K^2(t)\,N^{-1}(t)\,B'(t)\big] \\
&\quad -\big(-K^2(t)\,N^{-1}(t)\,B'(t)\big)'\,N(t)\,\big(-K^2(t)\,N^{-1}(t)\,B'(t)\big) \\
&= -2K^2(t)\,K^1(t)\,B(t)N^{-1}(t)B'(t) + (K^1(t))^2\,B(t)N^{-1}(t)B'(t) \\
&\quad +2(K^2(t))^2\,B(t)N^{-1}(t)B'(t) - (K^2(t))^2\,B(t)N^{-1}(t)B'(t) \\
&= [K^2(t) - K^1(t)]^2 B(t)N^{-1}(t)B'(t).
\end{aligned}
$$

Equation (5.4.12) can thus be written as:

$$
\begin{cases}
d[K^1(t) - K^2(t)] = -\big\{\big[a(t) - 2B(t)\,N^{-1}(t)\,B'(t)\,K^1(t)\big]\,[K^1(t) - K^2(t)] \\
\qquad\qquad +c'(t)[L^1(t) - L^2(t)] + Q^1(t)\}dt \\
\qquad\qquad +[L^1(t) - L^2(t)]'dW(t), \quad t \in [0,T], \\
K^1(T) - K^2(T) = 0 \quad a.s..
\end{cases}
$$

This equation satisfies the requirements of the comparison theorem of [27], see Theorem 2.2, and in the same way as in the proof of Lemma 5.4.1, we further know that $K^1(t) - K^2(t) \geq 0$ for all $t \in [0,T]$ $a.s.$. In the same way one can show that $K^2(t) - K^1(t) \geq 0$ for all $t \in [0,T]$ $a.s.$, and these two together imply that $K^1(t) = K^2(t)$ for all $t \in [0,T]$ $a.s.$.

We next show that $L^1(t) = L^2(t)$ $a.e.$ $t \in [0,T]$ $a.s.$. The differential of

$\hat{p}(t)[K^1(t) - K^2(t)]^2$ is:

$$
\begin{aligned}
& d\left\{\hat{p}(t)[K^1(t) - K^2(t)]^2\right\} \\
= \; & [K^1(t) - K^2(t)]^2 \left[\gamma(t) + 2\beta_1 a^2(t) + 4\beta_1 (B(t)\, N^{-1}(t)\, B'(t))^4 + \beta_2 c'(t) c(t)\right] \hat{p}(t)\, dt \\
& -2\hat{p}(t)[K^1(t) - K^2(t)] \left[a(t) - 2B(t)\, N^{-1}(t)\, B'(t)\, K^1(t)\right] [K^1(t) - K^2(t)] dt \\
& -2\hat{p}(t)[K^1(t) - K^2(t)] \; [c'(t)\, [L^1(t) - L^2(t)] + Q^1(t)] dt \\
& +\hat{p}(t)\, |L^1(t) - L^2(t)|^2\, dt + 2\, [K^1(t) - K^2(t)] \;\; \hat{p}(t)[L^1(t) - L^2(t)]' dW(t).
\end{aligned}
$$

Integration from t to T gives:

$$
\begin{aligned}
& \hat{p}(t)[K^1(t) - K^2(t)]^2 \\
= \; & -\int_t^T [K^1(s) - K^2(s)]^2 \left[\gamma(s) + 2\beta_1 a^2(s) + 4\beta_1 (B(s)\, N^{-1}(s)\, B'(s)^4 + \beta_2 c'(s) c(s)\right] \hat{p}(s)\, ds \\
& + \int_t^T 2\hat{p}(s)[K^1(s) - K^2(s)] \left[a(s) - 2B(s)\, N^{-1}(s)\, B'(s)\, K^1(s)\right] [K^1(s) - K^2(s)] ds \\
& + \int_t^T 2\hat{p}(s)[K^1(s) - K^2(s)] \; [c'(s)\, [L^1(s) - L^2(s)] + Q_1(s)] ds \\
& - \int_t^T \hat{p}(s)\, |L^1(s) - L^2(s)|^2\, ds - \int_t^T 2\, [K^1(s) - K^2(s)] \;\; \hat{p}(s)[L^1(s) - L^2(s)]' dW(s).
\end{aligned}
$$

Since $K^1(t) = K^2(t)$ for all $t \in [0, T]$ *a.s.*, this equation becomes:

$$0 = \int_t^T \hat{p}(s)\, |L^1(s) - L^2(s)|^2\, ds \quad a.s. \quad \forall t \in [0, T],$$

which implies that $L^1(t) = L^2(t)$ for *a.e.* $t \in [0, T]$ *a.s.*. □

5.5 Conclusion

We have considered a class of Riccati BSDEs with possibly unbounded coefficients. Sufficient conditions for the existence of a unique solution pair are given. These conditions are new and weaker than the existing ones. This result is expected to play an important role in solving other problems on Riccati BSDEs, such as solving the singular case with possibly unbounded coefficients. Also, some applications can be obtained by applying this result, such as the LQ optimal control problem.

Chapter 6

Conclusion

Throughout the thesis, we focus on studying the theory of Quadratic Backward Stochastic Differential Equations with possibly unbounded coefficients. We consider different classes of those equations under different sets of assumptions which are weaker than the well-known ones. Also, depending on our results, we give some possible works in the future.

In each chapter of the thesis, we improve fundamental results for the existence (and/or uniqueness) of solutions, and obtain a more general random version of the monotonicity theorem. We derive sufficient conditions, that are weaker than the existing ones, for the existence of solutions to BSDEs with unbounded generators that have quadratic growth in the control process; we derive sufficient conditions for the existence of solutions to Riccati BSDEs with unbounded generators; we additionally obtain sufficient conditions, that are weaker than the existing ones, for the existence and uniqueness of solutions to a certain class of Riccati BSDEs with unbounded generators.

Based on our results, we consider an important problem in stochastic control. As an application, we obtain the unique solution to the linear-quadratic optimal control problem with possibly unbounded coefficients under weaker conditions. In particular, we give an explicit solution to this problem using our results for the Riccati BSDE with unbounded coefficients.

Our results give a foundation for possible extensions to more general situations. Notably, we have tried to prove the existence of solutions to the quadratic BSDEs with unbounded terminal conditions. In fact, it was proved that the boundedness of the terminal condition is not necessary to get a solution. For instance, see Theorem 2 in [13], Lemma 3.4.1 and Lemma 3.4.2. Also, we expect that our results on the quadratic BSDEs in chapter 3 will contribute to tackle some difficult problems on BSDEs with unbounded coefficients, such as reflected BSDEs with quadratic growth, the indefinite case of Riccati BSDEs and their applications in optimal investment. Furthermore, our results on Riccati BSDEs in chapter 4 are expected to contribute to solve the problem of optimal investment with power utility with unbounded coefficients. Another potential extension to our results on the Riccati BSDEs in chapter 5 is solving similar problems on Riccati BSDEs, such as the singular case with possibly unbounded coefficients. Also, some applications can be obtained by applying this result, such as the linear-quadratic optimal control problem and mean-variance portfolio selection with possibly unbounded coefficients.

This thesis gives new results on the theory of quadratic BSDEs, which is still improving, under more general (or weaker) conditions. These results generalize some well-known results and give a foundation for the extension to more general circumstances.

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Appendix

We describe here the definitions of two basic convergence of a sequence of functions. Also, we introduce some concepts of convergence in normed spaces.

Definition 0.0.1 (Pointwise convergence)**.** *A sequence $f^n(x)$ of functions converges pointwisely to a function f(x) if for any x and $\epsilon > 0$, there exists a natural number $N = N(\epsilon, x)$ such that for all $n > N$, $|f^n(x) - f(x)| < \epsilon$. Note that, in this definition, N depends on x. In other words, for a given $\epsilon > 0$, a value of N that makes the statement hold for some x, may not work for different x.*

Definition 0.0.2 (Uniform convergence)**.** *A sequence $f^n(x)$ of functions converges uniformly to a function $f(x)$ if for any $\epsilon > 0$, there exists a natural number $N = N(\epsilon)$ such that for all $n > N$ and for all x, $|f^n(x) - f(x)| < \epsilon$.*

Note that the difference between pointwise and uniform convergence is the placement of "for all x" . Here, it comes after the existence of N, so the single N has to work for all x, but for pointwise convergence it comes before the existence of N, so different numbers N can be used for different values of x.

Definition 0.0.3. *A norm is a real-valued function $||\cdot|| : X \to \mathbb{R}$ defined on the vector space X over K, and has the following properties*
(i) for every vector x, $||x|| \geq 0$.
(ii) $||x|| = 0 \Leftrightarrow x = 0$.
(iii) for every vector x and scalar a, $||ax|| = |a|\ ||x||$.
(iv) for every vectors x and y, $||x + y|| \leq ||x|| + ||y||$.

Definition 0.0.4 (Vector Space)**.** ([43], Definition 2.1-1)
A vector space over K is a nonempty set X of elements (called vectors) together with two algebraic operations, called vectors addition and multiplication of vectors by scalars, that is, by elements of K.
K is called the scalar field (or coefficient field) of the vector space X, and X is called a real vector space if $K = \mathbb{R}$. An example of vector space is $\mathbb{R}^n$.

Definition 0.0.5 (Normed Vector Space)**.** ([43], Definition 2.2-1)
A normed vector space (or normed space) X is a vector space over $\mathbb{R}$ on which a norm $||\cdot||$ is defined.
In other words, the pair $(X, ||\cdot||)$ is called a normed space.
An example of normed vector space is $\mathbb{R}^n$ endowed with the Euclidean norm $|x| := \left(\sum_{j=1}^{n} x_j^2\right)^{\frac{1}{2}}$.

Definition 0.0.6 (Strong Convergence)**.** ([43], Definition 4.8-1)
The usual convergence in norm is called strong convergence.
A sequence $\{x_n\}_n$ in a normed space X converges strongly (or in norm) to $x \in X$ if

$$\lim_{n\to\infty} ||x_n - x|| = 0.$$

Definition 0.0.7 (Weak Convergence)**.** ([43], Definition 4.8-2)
A sequence $\{x_n\}_n$ in a normed space X converges weakly to $x \in X$ if

$$\lim_{n\to\infty} f(x_n) = f(x)$$

for all continuous linear mappings $f : X \to \mathbb{R}$.

Note that the strong convergence implies the weak convergence but the converse is not always true.