

The treewidth and pathwidth of graph unions

Bogdan Alecu* Vadim Lozin† Daniel A. Quiroz‡ Roman Rabinovich§
 Igor Razgon¶ Viktor Zamaraev||

Abstract

For two graphs G_1 and G_2 on the same vertex set $[n] := \{1, 2, \dots, n\}$, and a permutation φ of $[n]$, the *union of G_1 and G_2 along φ* is the graph which is the union of G_2 and the graph obtained from G_1 by renaming its vertices according to φ . We examine the behaviour of the treewidth and pathwidth of graphs under this “gluing” operation. We show that under certain conditions on G_1 and G_2 , we may bound those parameters for such unions in terms of their values for the original graphs, regardless of what permutation φ we choose. In some cases, however, this is only achievable if φ is chosen carefully, while yet in others, it is always impossible to achieve boundedness. More specifically, among other results, we prove that if G_1 has treewidth k and G_2 has pathwidth ℓ , then they can be united into a graph of treewidth at most $k + 3\ell + 1$. On the other hand, we show that for any natural number c there exists a pair of trees G_1 and G_2 whose every union has treewidth more than c .

1 Introduction

A k -tree is a graph that can be obtained by starting with K_k and repeatedly adding vertices and connecting them to a clique of size k . A *partial k -tree* is a (not necessarily induced) subgraph of a k -tree. The *treewidth*, $\text{tw}(G)$, of a graph G is the least k such that G is a partial k -tree.

Nash-Williams’ theorem [14, 15] states that a graph $G = (V, E)$ can be edge-covered by at most k trees if and only if for every non-empty set $U \subseteq V$ the number of edges in the subgraph of G induced by U is at most $k(|U| - 1)$. Using the above definition of a partial k -tree, it is easy to see that this classical result implies that every partial 2-tree can be edge-covered by at most two trees. But, given any pair of trees, can we always find a 2-tree that is edge covered by it? We shall see that for certain types of trees we can guarantee this, however it is natural to guess that in general this is probably too much to ask for. On the other hand, it seems reasonable to ask the following: is there some constant $c \geq 2$ such that for every pair

*School of Computing, University of Leeds, UK. Email: b.alecu@leeds.ac.uk

†Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK. Email: V.Lozin@warwick.ac.uk

‡Instituto de Ingeniería Matemática-CIMFAV, Universidad de Valparaiso, Chile. Email: daniel.quiroz@uv.cl

§ArangoDB Inc. Email: roman.rabinovich@arangodb.com

¶Department of Computer Science and Information Systems, Birkbeck University of London. Email: igor@dcs.bbk.ac.uk

||Department of Computer Science, University of Liverpool, Liverpool, UK. Email: viktor.zamaraev@liverpool.ac.uk

of trees there is some partial c -tree that is edge-covered by it? In order to state the answer to this question and our other related results, we first need to introduce some notation.

Let n be a positive integer. We denote by $[n]$ the set $\{1, \dots, n\}$, and by S_n the symmetric group of all permutations of $[n]$. For a permutation $\varphi \in S_n$ and sets $U \subseteq [n]$ and $E \subseteq \binom{[n]}{2}$, we write $\varphi(U) := \{\varphi(i) : i \in U\}$ and $\varphi(E) := \{\{\varphi(i), \varphi(j)\} : \{i, j\} \in E\}$. Given two graphs $G_1 = ([n], E_1)$ and $G_2 = ([n], E_2)$, the *union of G_1 and G_2 along φ* is the graph $([n], \varphi(E_1) \cup E_2)$. A *gluing of G_1 and G_2* is the union of G_1 and G_2 along some permutation. We may think of this operation as first relabeling the vertices of G_1 according to φ and then taking the union of the resulting graph with G_2 (see Figure 1 for illustration).

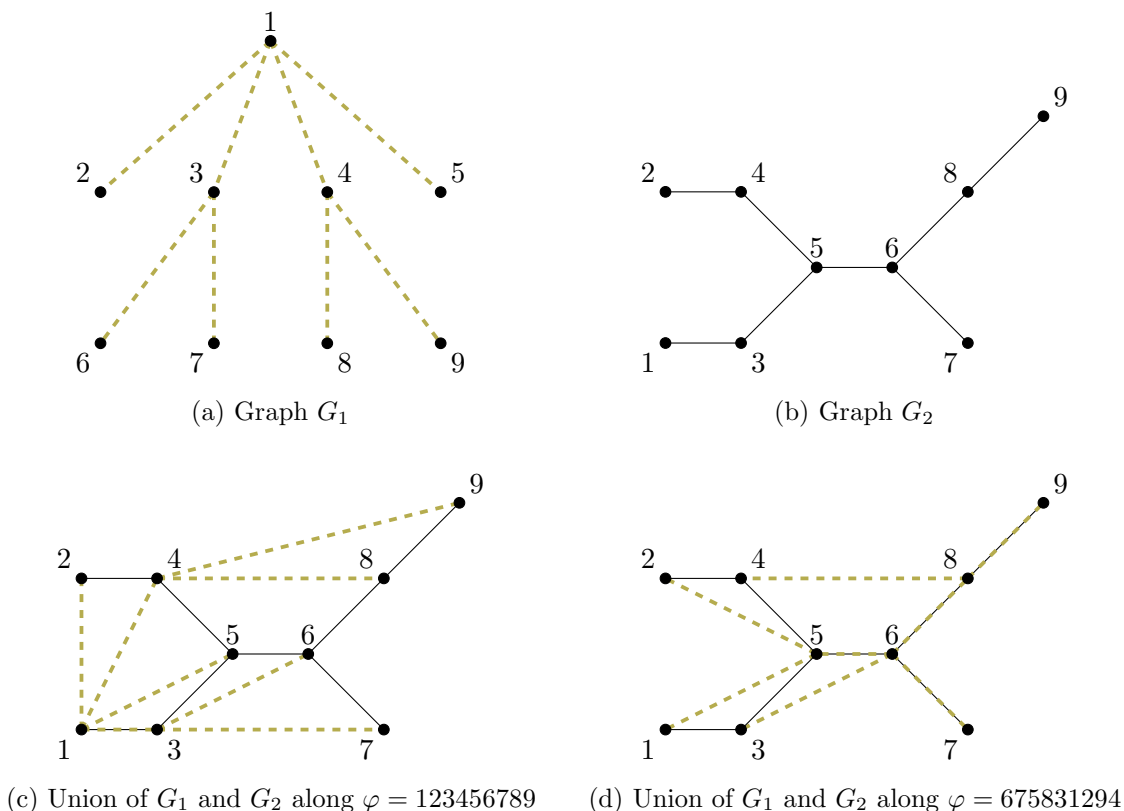


Figure 1: The union of two trees along different permutations.

In this terminology our first question can be formulated as follows. Given two n -vertex trees, can they always be glued to a graph of treewidth 2? Our first observation (see Section 2) is that any gluing of a star and a tree has treewidth at most 2. Thinking about trees as graphs of treewidth 1 and about stars as graphs of vertex cover number 1, we generalize this observation as follows.

Lemma 1. *Any gluing of an n -vertex graph of vertex cover number at most k and an n -vertex graph of treewidth at most t has treewidth at most $k + t$.*

The fact that the union of a star and a tree along any permutation is a partial 2-tree distinctly sets apart our questions from the classical graph packing problems, where unions are required

to be edge-disjoint (see e.g. [9, 7]). Indeed, while stars cannot be packed with any other tree, they are among the easiest trees to glue with.

Clearly, two paths can always be glued to a path. However, in contrast to stars, taking a right permutation is crucial to preserve bounded treewidth. Indeed, two long paths can be glued together to a large square grid, which is known to have a large treewidth. Thinking about paths as graphs of pathwidth 1, we generalize this observation as follows. (See Section 2 for a definition of pathwidth.)

Lemma 2. *Let G_1 and G_2 be n -vertex graphs of pathwidth k and t respectively. Then there is a gluing of G_1 and G_2 of pathwidth at most $k + t$.*

This lemma implies that two caterpillars, i.e. trees that become a path after removing all their leaves, can always be glued into a graph of treewidth at most 2. As we shall observe in Section 2, a tree and a path can also be glued to a graph of treewidth at most 2. However, we do not know if there is always a way of gluing together a caterpillar and an arbitrary tree to a graph of treewidth at most 2.

This brings us into the realm of our second and more general question, which asks if there exists a constant c such that any two n -vertex trees can be glued to a graph of treewidth at most c . First, we note that a graph of bounded treewidth can always be glued with a path in such a way that treewidth remains bounded. Indeed, it was shown in [13] (see also [3]) that a graph of treewidth k and a path can be glued to a graph of treewidth at most $k + 5$. This latter bound was improved to $k + 3$ in [16]¹.

We generalize these results by proving that a graph of bounded treewidth and a graph of bounded pathwidth can always be glued to a graph of bounded treewidth.

Theorem 3. *Let G_1 be an n -vertex graph of treewidth at most k , and G_2 an n -vertex graph of pathwidth at most ℓ . Then there exists a gluing of G_1 and G_2 that has treewidth at most $k + 3\ell + 1$.*

In particular, the above theorem implies that a tree and a caterpillar can be glued to a graph of treewidth at most 5. Despite this result, and all the positive results we had for the more restrictive question, we show that the answer for the second question is negative, i.e. there are pairs of trees whose any gluing has large treewidth.

Theorem 4. *For any $c > 0$, there exists $n \in \mathbb{N}$, and n -vertex trees T_1 and T_2 such that any gluing of T_1 and T_2 has treewidth at least c .*

Theorem 4 follows from a more general statement that T_1 and T_2 cannot be glued to a graph of bounded clique-width.

We note that bounded vertex cover number implies bounded pathwidth, and therefore Theorem 3 generalizes Lemma 1 in the sense that two graphs can be glued into a graph whose treewidth is linearly bounded in terms of the treewidth of one of the graphs and the pathwidth (respectively vertex cover number) of the other graph. This generalization is best possible in the following sense. For two classes of graphs \mathcal{X} and \mathcal{Y} a *gluing of \mathcal{X} and \mathcal{Y}* is a minimal class that for every pair of n -vertex graphs $G_1 \in \mathcal{X}$ and $G_2 \in \mathcal{Y}$ contains a gluing of G_1 and G_2 of minimum possible treewidth. A graph parameter is said to be bounded for a class \mathcal{X} if there exists a constant c such that for every graph in \mathcal{X} the graph parameter

¹Both [13] and [16] state results with better bounds, but their proofs actually yield the results here stated.

does not exceed c ; otherwise the graph parameter is said to be unbounded in \mathcal{X} . Our results imply the following characterization

Theorem 5. *A gluing of two classes of graphs \mathcal{X} and \mathcal{Y} has bounded treewidth if and only if both \mathcal{X} and \mathcal{Y} have bounded treewidth and one of them has bounded pathwidth.*

Proof. Clearly, if \mathcal{X} or \mathcal{Y} has unbounded treewidth, any gluing of the two classes has unbounded treewidth. Furthermore, since taking minors increases neither treewidth nor pathwidth, we can assume that both classes are minor-closed. Under this assumption, a gluing of \mathcal{X} and \mathcal{Y} has bounded treewidth if and only if at least one of the classes has bounded pathwidth. This follows from Theorems 3 and 4, and a result of Robertson and Seymour saying that a minor-closed class has bounded pathwidth if and only if the class excludes a forest [17]. \square

The results of this paper can be interpreted in the context of *multilayer networks*. Many real-world systems can be modeled as networks, where the entities of the system are represented by the network’s nodes and interactions between the entities are modeled by the edges. In many complex systems the entities can exhibit multiple types of interactions. For example, nodes of a transportation network can be connected via two types of edges, one type representing train connections and the other type representing flight connections. The abstraction of multilayer networks is a means of modeling complex systems more precisely, by capturing the different “layers” of information [11]. A natural direction in the research of multilayer networks is investigating the extent to which good properties of individual layers can be exploited in the treatment of the network formed by combining the layers.

Of particular relevance to our work is the study of Enright, Meeks, and Ryan [6], who investigate whether desirable structural properties of the individual layers translate into good algorithmic behaviour of the whole multilayer network. They show that when the layers are combined adversarially, good properties of individual layers are lost (unless the structure of the layers is severely restricted to begin with). An immediate follow-up question is what happens when the layers are not combined adversarially; what if, say, they are combined randomly? As demonstrated in [5], even this is not good enough: with high probability, the desirable properties are lost. More specifically, Enright et al. [5] considered treewidth as a measure of well-behavedness. They showed that if two very simple layers each consisting of an n -vertex path are combined randomly, then the treewidth of the resulting network grows with n with high probability.

The results of these works naturally raise the following question: since neither an adversarial, nor a random combination of layers preserves good properties, can we do better if we are free to choose how the layers are combined? Theorem 4 can be seen as an instance of this investigation: when restricting ourselves to two simple layers, each being a tree, it is impossible to achieve bounded treewidth on the whole network, even when attempting to minimise it.

The paper is organised as follows. In Section 2, we show that certain graphs of bounded treewidth can be glued into a graph of bounded treewidth. In Section 3, we prove that there are pairs of trees that cannot be glued into a graph of small treewidth. In Section 4, we conclude the paper with some open questions.

2 Positive results: achieving boundedness

In this section we will show that trees of certain types can be glued to a graph of treewidth at most 2. We will also show that more general graphs can be glued to a graph of bounded treewidth. We start with some of the definitions that we will require.

For a graph G we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G respectively. A vertex $v \in V(G)$ is a *neighbor* of another vertex $u \in V(G)$ if $\{u, v\}$ is an edge of G . The *neighborhood* of u , denoted $N_G(u)$, is the set of all neighbors of u . For a vertex set $U \subseteq V(G)$ the subgraph of G induced by U is denoted by $G[U]$.

A *tree decomposition* of a graph G is a pair $(\mathcal{T}, (X_i)_{i \in V(\mathcal{T})})$, where \mathcal{T} is a tree and $X_i \subseteq V(G)$ for each $i \in V(\mathcal{T})$, such that

- (I) $\bigcup_{i \in V(\mathcal{T})} X_i = V(G)$;
- (II) for every edge $\{u, v\} \in E(G)$, there is a $i \in V(\mathcal{T})$ such that $u, v \in X_i$; and
- (III) for every $v \in V(G)$ the subgraph \mathcal{T}_v of \mathcal{T} induced by $\{i \in V(\mathcal{T}) \mid v \in X_i\}$ is connected, i.e. \mathcal{T}_v is a tree.

So as to avoid confusion with the vertices of G , we say that the elements of $V(\mathcal{T})$ are the *nodes* of \mathcal{T} . For a node i we say that the corresponding set X_i is the *bag* of i . The width of the tree decomposition $(\mathcal{T}, (X_i)_{i \in V(\mathcal{T})})$ is $\max_{i \in V(\mathcal{T})} |X_i| - 1$. The *treewidth of G* , which we defined in the introduction, can be equivalently defined as the smallest width of a tree decomposition of G (see e.g. [2]).

It is known and easy to see that the treewidth of any tree is 1. We observe that the union H of a star $K_{1, n-1}$ and an n -vertex tree T along an arbitrary permutation φ has treewidth at most 2. Indeed, in H all the edges that are coming from the star are incident with the center of the star. Hence, if \mathcal{T} is an optimal tree decomposition of T and the center of the star is identified with vertex i of T , then by adding i to every bag of \mathcal{T} we obtain a tree decomposition of H of width at most 2. This argument immediately generalizes to graphs of bounded vertex cover number and graphs of bounded treewidth. Recall that the *vertex cover number* of a graph G is the minimum number of vertices in G such that every edge of G is incident with at least one of these vertices.

Lemma 1. *Any gluing of an n -vertex graph of vertex cover number at most k and an n -vertex graph of treewidth at most t has treewidth at most $k + t$.*

Proof. Let G_1 be a graph with a vertex cover $C \subseteq V(G_1)$ of size k , and let G_2 be a graph with a tree decomposition \mathcal{T} of width t . Let also $\varphi \in S_n$ be an arbitrary permutation and H be the union of G_1 and G_2 along φ . Then it is routine to check that by adding $\varphi(C)$ to every bag of \mathcal{T} we obtain a tree decomposition of H of width at most $k + t$. \square

Next, we will prove a similar statement for graphs of bounded pathwidth. For this we first introduce some notions and an auxiliary result that characterizes pathwidth in terms of vertex separation number.

A *path decomposition* of a graph G is a tree decomposition $(\mathcal{P}, (X_i)_{i \in V(\mathcal{P})})$ in which the tree \mathcal{P} is a path. The *pathwidth* of G is equal to the smallest width of any path decomposition

of G . A *layout* of a graph is a linear ordering of its vertices. Let $G = (V, E)$ be a graph and let π be a layout of G . The *vertex separation number of G with respect to π* is defined as

$$\text{vs}_\pi(G) = \max_{v \in V} |\{w \in V : \exists x \in N_G(w) \text{ such that } \pi(w) < \pi(v) \leq \pi(x)\}|.$$

The *vertex separation number* $\text{vs}(G)$ of G is the minimum of $\text{vs}_\pi(G)$ over all possible layouts π of G .

Theorem 6 (Kinnersley [10]). *The vertex separation number of a graph equals its pathwidth.*

We now prove that two graphs of bounded pathwidth can always be glued to a graph of bounded pathwidth.

Lemma 2. *Let G_1 and G_2 be n -vertex graphs of pathwidth k and t respectively. Then there is a gluing of G_1 and G_2 of pathwidth at most $k + t$.*

Proof. We apply Theorem 6. Let π_{G_1} and π_{G_2} be layouts of G_1 and G_2 respectively, that witness their separation numbers. Without loss of generality, assume that the vertices of G_1 are labeled according to the layout π_{G_1} , and the vertices of G_2 are labeled according to the layout π_{G_2} . Then the union of G_1 and G_2 along the identity permutation is a graph of separation number at most $k + t$, witnessed by the identity layout. \square

Caterpillars have pathwidth 1, so the above lemma implies that two caterpillars can always be glued to a graph of pathwidth (and hence treewidth) at most 2.

It is also possible to glue a path P with an arbitrary tree T to a graph of treewidth at most 2. Informally, this can be seen as follows. Consider a planar embedding of T . Then unite the vertices of P with the vertices of T following a depth-first search (DFS) ordering for T and embed the edges of P in such a way that the resulting embedding is outerplanar, i.e. all vertices belong to the outer face. This shows that P and T can be glued to an outerplanar graph, and it is known that outerplanar graphs have treewidth 2. This type of argument was used in previous work on gluing paths to graphs of bounded treewidth [16], and the proof of our main positive result in the next subsection is a generalization of the argument.

The same result can also be shown using the following book embedding argument, which we use again in Section 4, where we raise questions about gluings of three or more graphs. A *book* is a collection of half-planes, called the *pages*, all having the same line as their boundary, which is called the *spine*. A *book embedding* of graph is a generalization of a planar embedding in which the vertices of the graph are mapped to the spine and the edges are embedded in the pages without crossings. The *book thickness* of a graph is the smallest possible number of pages in a book embedding of the graph. Graphs of book thickness 1 are exactly outerplanar graphs. In particular, any tree has book thickness 1. To see that a path P and an arbitrary tree T can be glued into a graph of treewidth at most 2 we will show that there is a gluing of book thickness 1. Let us fix a book embedding of T into a single-page book. This embedding induces a linear order of the vertices as they appear in the spine. We unite T with P in such a way that the edges of P connect consecutive vertices along the spine and therefore can be embedded in the page very close to the spine without causing any edge intersections. Consequently, the union has book thickness 1. Therefore it is outerplanar and hence its treewidth is at most 2.

We do not know if a caterpillar and a tree can always be glued in a graph of treewidth at most 2. However, in the next section we show that a graph of bounded treewidth and a graph of bounded pathwidth can be glued into a graph of bounded treewidth.

2.1 Union of graphs of bounded treewidth and bounded pathwidth

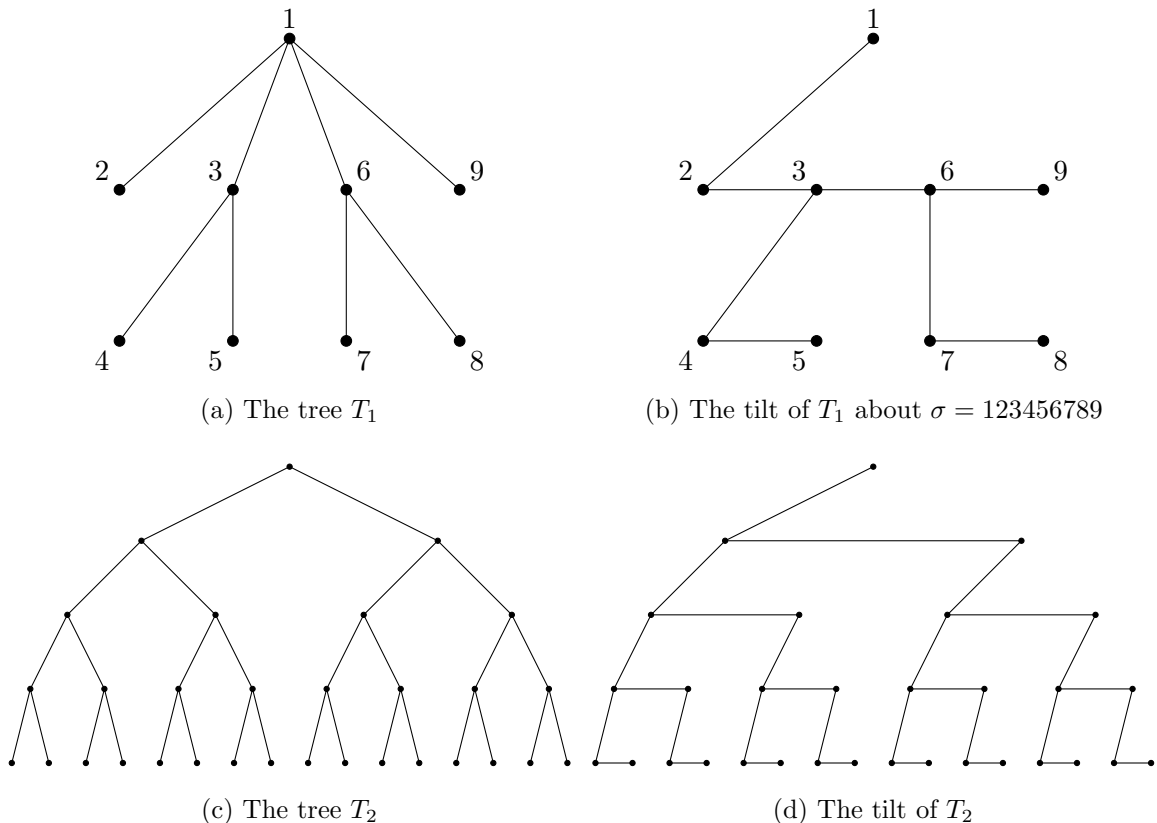


Figure 2: An illustration of two trees and their tilts

In this section, we show that any pair of n -vertex graphs such that one has treewidth at most k and the other pathwidth at most ℓ can be glued together so that the resulting graph has treewidth at most $k + 3\ell + 1$. This implies that we can glue together a tree and a caterpillar to a graph of treewidth at most 5.

In our proof we will use so-called smooth tree decompositions. A *smooth* tree decomposition of width k is a tree decomposition in which all bags have $k + 1$ vertices and adjacent bags share exactly k vertices. It is known that every n -vertex graph of treewidth k has a smooth tree decomposition of width k ; and any smooth tree decomposition has exactly $n - k$ nodes [1].

We will also need to define a *tilt* of a tree, and for this we briefly discuss the conventions we use for depth-first searches. For our purposes, the standard depth-first search tree traversal algorithm produces two outputs: a *DFS preordering* – the linear ordering of the vertices which records the order in which they were first visited by the algorithm; and a *DFS walk* – the sequence of vertices visited by the algorithm, with repetitions, that starts and ends in the root. By convention, we will assume that DFS searches always start at the root vertex.

Let T be an n -vertex tree rooted at v_1 , and let $\sigma = (v_1, \dots, v_n)$ be a DFS preordering of $V(T)$. The *tilt of T about σ* is the tree rooted at v_1 that is obtained from T as follows. For every non-root vertex x , if x is not the leftmost (with respect to σ) child of its parent

y , then we remove the edge $\{y, x\}$ and add the edge $\{x', x\}$, where x' is the child of y that immediately precedes x . For an illustration, see Figure 2, where we assume that the root vertex is at the top and the DFS algorithm visits children from left to right.

Tilts have some properties that will be useful to us. Let T' be the tilt of T about some DFS preordering σ . Then every vertex of T' has degree at most 3. Moreover, a vertex together with its children in T induce a path in T' . Finally, we note that T' admits a DFS traversal with the DFS preordering σ .

Theorem 3. *Let G_1 be an n -vertex graph of treewidth at most k , and G_2 an n -vertex graph of pathwidth at most ℓ . Then there exists a gluing of G_1 and G_2 that has treewidth at most $k + 3\ell + 1$.*

Proof. We first relabel the vertices of G_1 . Let $(\mathcal{T}, (X_y)_{y \in V(\mathcal{T})})$ be a smooth tree decomposition of G_1 of width k . We set an arbitrary node r of \mathcal{T} to be its root, and fix a DFS preordering $\sigma = (a_1 = r, a_2, \dots, a_{n-k})$ of $V(\mathcal{T})$. Next, we assign unique labels from $[k + 1]$ to the $k + 1$ vertices in bag X_{a_1} in an arbitrary way, and assign label $k + i$ to the unique vertex in $X_{a_i} \setminus X_{p_i}$ for every $2 \leq i \leq n - k$, where p_i is the parent of a_i in \mathcal{T} .

We now relabel the vertices of G_2 . For this we fix an ordering π of $V(G_2)$ that witnesses the graph's separation number and assign increasing labels from $[n]$ according to π . In other words, for every $u, v \in V(G_2)$ we have $u < v$ if and only if $\pi(u) < \pi(v)$. Let H be the union of G_1 and G_2 along the identity permutation. In the rest of the proof we construct a tree decomposition of H of width at most $k + 3\ell + 1$.

For every vertex $v \in V(G_2)$, we let

$$S(v) = \{w \in V(G_2) : \exists x \in N_{G_2}(w) \text{ such that } \pi(w) < \pi(v) \leq \pi(x)\}.$$

By definition, we have that $\max_{v \in V(G_2)} |S(v)|$ is the separation number of G_2 , and thus, by Theorem 6, we have $|S(v)| \leq \ell$ for every $v \in V(G_2)$. We observe that the definition of $S(v)$ implies the following

Claim 1. $S(i + 1) \subseteq \{i\} \cup S(i)$ for every $1 \leq i \leq n - 1$.

Let $(\mathcal{T}^*, (Z_y)_{y \in V(\mathcal{T}^*)})$ be such that the tree \mathcal{T}^* is the tilt of \mathcal{T} about σ , and $Z_{a_1} = X_{a_1}$ and $Z_{a_i} = X_{a_i} \cup X_{p_i}$ for every $2 \leq i \leq n - k$.

Claim 2. $(\mathcal{T}^*, (Z_y)_{y \in V(\mathcal{T}^*)})$ is a tree decomposition of G_1 of width at most $k + 1$.

Proof. First, it is not hard to verify that if in a tree decomposition we add to a bag all vertices of its parent bag, then all three properties of tree decompositions are preserved. Therefore, $(\mathcal{T}, (Z_y)_{y \in V(\mathcal{T})})$ is a tree decomposition of G_1 .

Next, we argue that replacing \mathcal{T} with \mathcal{T}^* in the tree decomposition $(\mathcal{T}, (Z_y)_{y \in V(\mathcal{T})})$ also preserves the properties. Clearly, properties (I) and (II) are preserved as we do not change the bags, so we only need to show that for every $v \in V(G_1)$ the subgraph \mathcal{T}_v^* is connected. Since \mathcal{T}_v is connected and $V(\mathcal{T}_v) = V(\mathcal{T}_v^*)$, it is enough to show that for every every $\{a_i, a_j\} \in E(\mathcal{T}_v)$ there is a path from a_i to a_j in \mathcal{T}_v^* . The latter means that v belongs to Z_{a_r} for every a_r on the path from a_i to a_j in \mathcal{T}^* . Without loss of generality, assume that a_i is the parent of a_j in \mathcal{T} , and let $a_{s_1}, a_{s_2}, \dots, a_{s_t}$ be the children of a_i in \mathcal{T} that precede a_j in σ . Then $a_i, a_{s_1}, a_{s_2}, \dots, a_{s_t}, a_j$ is the path from a_i to a_j in \mathcal{T}^* . Now, since $v \in Z_{a_i} \cap Z_{a_j} = X_{a_i}$ and X_{a_i} is a subset of all $Z_{a_{s_1}}, Z_{a_{s_2}}, \dots, Z_{a_{s_t}}$, we conclude that v belongs to all these bags, as required.

To finish the proof of the claim, we recall that $(\mathcal{T}, (X_y)_{y \in V(\mathcal{T})})$ is a smooth tree decomposition. Hence $|X_{p_i} \setminus X_{a_i}| = 1$ for every $2 \leq i \leq n - k$, and therefore $|Z_{a_i}| \leq k + 2$ for every $i \in [n - k]$, i.e. the width of $(\mathcal{T}^*, (Z_y)_{y \in V(\mathcal{T}^*)})$ is at most $k + 1$. \square

Next, we will iteratively extend the bags of $(\mathcal{T}^*, (Z_y)_{y \in V(\mathcal{T}^*)})$ in such a way that it satisfies properties (I) and (III) of tree decompositions after every iteration, and it also satisfies property (II) with respect to graph H after the final iteration. Note that because the vertex set of G_1 and H is the same and we will only extend the bags, property (I) will always hold.

If $k + 1 < \ell$ we set $r = \ell - k$, otherwise we set $r = 1$. Note that $\cup_{i=1}^r Z_{a_i} = \{1, 2, \dots, k + r\}$. In the first iteration, for every $i \in [r]$ we extend Z_{a_i} to be equal $\{1, 2, \dots, k + r\}$. Since σ is a DFS preordering for \mathcal{T}^* , the subgraph $\mathcal{T}^*[\{a_1, \dots, a_r\}]$ is connected and therefore this extension preserves property (III). Observe that for every $i \in [r]$, the bag Z_{a_i} contains $k + i$ and $S(k + i) \subseteq Z_{a_i}$. In particular, together with Claim 1 this implies that $S(k + r + 1) \subseteq \{k + r\} \cup S(k + r) \subseteq Z_{a_r}$.

Now, for every $r + 1 \leq i \leq n - k$ we perform the following iteration: we add $S(k + i)$ to every bag Z_{a_t} , where a_t is a vertex of the path from a_{i-1} to a_i in \mathcal{T}^* . We will prove by induction on i that after the iteration corresponding to i we have $S(k + i + 1) \subseteq Z_{a_i}$, and for every $v \in S(k + i)$ the subgraph \mathcal{T}_v^* is connected, i.e. property (III) is preserved. Indeed, since after the iteration corresponding to i the set $S(k + i)$ is a subset of Z_{a_i} and $k + i \in Z_{a_i}$ by the initial definition of the bags, we conclude from Claim 1 that $S(k + i + 1) \subseteq Z_{a_i}$. To show the second part, we observe that before the iteration, by the induction hypothesis, $S(k + i) \subseteq Z_{a_{i-1}}$ and for every $v \in S(k + i)$ the subgraph \mathcal{T}_v^* is connected. Since the extension added v only to the bags of the path from a_{i-1} to a_i , the subgraph \mathcal{T}_v^* remains connected after the iteration. Consequently, after all the iterations we have that \mathcal{T}^* satisfies properties (I) and (III) of tree decompositions and also for every $i \in [n - k]$ we have that $\{k + i\} \cup S(k + i) \subseteq Z_{a_i}$.

We will show next that the latter fact implies property (II) for H . From Claim 2 we already know that every edge of G_1 belongs to some bag of \mathcal{T}^* , so it remains to show the same for every edge of G_2 . Let $\{i, j\} \in E(G_2)$, where $i < j$, be an arbitrary edge of G_2 . If $j \leq k + 1$, then $\{i, j\} \subseteq Z_{a_r}$. If $j > k + 1$, then $\{i, j\} \subseteq Z_{j-k}$ as $i \in S(j)$ and $\{j\} \cup S(j) \subseteq Z_{a_{j-k}}$.

To finish the proof we observe that we can think that the bag updates (including the updates of the first r bags) are done when we move from the current vertex a_{i-1} of \mathcal{T}^* to the next unvisited vertex a_i along a DFS walk. In this way, since the maximum degree of \mathcal{T}^* is at most 3, every vertex is visited by the DFS walk at most 3 times. Each time the corresponding bag is extended by a set of size at most ℓ . Therefore the width of the final tree decomposition is at most $k + 1 + 3\ell$. \square

3 Unions of trees have unbounded treewidth

One might hope that the positive examples from the previous section generalize, and that graphs of bounded treewidth can always be glued together to a graph of bounded treewidth. As it turns out, this fails to be true even for graphs of treewidth 1. Namely, there are trees for which any gluing has large tree-width, and even clique-width. Proof of this result is the main aim of this section. We start with some preliminaries required for this result.

3.1 Trees and cuts

Let $G = (V, E)$ be a graph. For a vertex set $U \subseteq V$ we denote by \overline{U} the set $V \setminus U$. The partition (U, \overline{U}) of the vertex set of G is called the U -cut of G . The U -cut-set in G is the set of edges of G that have one endpoint in U and the other endpoint in \overline{U} . The edges in the U -cut-set are called the *crossing edges* of the U -cut. We denote by $e_G(U)$ the number of crossing edges of the U -cut in G . The U -cut is called *balanced* if $\frac{n}{3} \leq |U| \leq \frac{2n}{3}$, where n is the number of vertices in G .

Now let $T = (V, E)$ be a rooted tree. The *level of a vertex* v in T , denoted by $\text{lvl}(v)$, is the distance from v to the root of T . In particular, the level of the root is 0. We denote by T^v the subtree of T rooted at v , and by n_v the number of vertices in T^v . By convention, the vertices of an edge $e = (u, v)$ of T are ordered so that $\text{lvl}(u) = \text{lvl}(v) - 1$. Given a set $U \subseteq V$, we write $\mathbb{1}_U$ for the indicator function given by $\mathbb{1}_U(v) = 1$ if $v \in U$, and 0 otherwise.

Lemma 7. *Let $T = (V, E)$ be a tree on n vertices rooted at r . Let $U \subseteq V$ be a vertex set and let $e_i = (u_i, v_i)$, $i = 1, \dots, k$ be the edges of the U -cut-set. Then*

$$|U| = \mathbb{1}_U(r) \cdot n + \sum_{i=1}^k (-1)^{\mathbb{1}_U(u_i)} \cdot n_{v_i}.$$

Proof. We prove the lemma by induction on k . For $k = 1$, there is a unique edge $e_1 = (u_1, v_1)$ in the U -cut-set. It is easy to see that u_1 belongs to U if and only if r belongs to U . If they both belong to U , then $|U| = n - n_{v_1}$. Otherwise, $|U| = n_{v_1}$. In either case, $|U| = \mathbb{1}_U(r) \cdot n + (-1)^{\mathbb{1}_U(u_1)} \cdot n_{v_1}$, as required.

Let now $k > 1$ and assume, without loss of generality, that the edge $e_k = (u_k, v_k)$ is a *minimal* edge of the U -cut-set, i.e., no edge of the subtree T^{v_k} belongs to the U -cut-set.

Suppose first that $u_k \in U$ and $v_k \in \overline{U}$. This assumption and the minimality of e_k implies that $V(T^{v_k}) \subseteq \overline{U}$. By moving $V(T^{v_k})$ from one side of the cut to the other, we will remove exactly one edge, namely e_k , from the cut-set. More formally, let $U' = U \cup V(T^{v_k})$. Then clearly the U' -cut-set is equal to $\{e_1, e_2, \dots, e_{k-1}\}$. By the induction hypothesis, and since $\mathbb{1}_{U'}(w) = \mathbb{1}_U(w)$ for every $w \in \{r, u_1, \dots, u_{k-1}\}$ we have

$$|U'| = \mathbb{1}_U(r) \cdot n + \sum_{i=1}^{k-1} (-1)^{\mathbb{1}_U(u_i)} \cdot n_{v_i},$$

and therefore $|U| = |U'| - n_{v_k} = |U'| + (-1)^{\mathbb{1}_U(u_k)} \cdot n_{v_k} = \mathbb{1}_U(r) \cdot n + \sum_{i=1}^k (-1)^{\mathbb{1}_U(u_i)} \cdot n_{v_i}$, as required.

Assume now that $u_k \in \overline{U}$ and $v_k \in U$. Then, similarly to the above argument, $V(T^{v_k}) \subseteq U$ and if we define $U' = U \setminus V(T^{v_k})$, then the U' -cut-set is equal to $\{e_1, e_2, \dots, e_{k-1}\}$. Therefore, $|U| = |U'| + n_{v_k} = |U'| + (-1)^{\mathbb{1}_U(u_k)} \cdot n_{v_k} = \mathbb{1}_U(r) \cdot n + \sum_{i=1}^k (-1)^{\mathbb{1}_U(u_i)} \cdot n_{v_i}$, which completes the proof. \square

3.2 Balanced trees

For a natural number $b \geq 2$, a *b-ary tree* is a rooted tree in which each vertex has at most b children. For $\ell \in \mathbb{N}$, the ℓ -th level of a b -ary tree is the set of vertices with level ℓ . The ℓ -th level of the tree is called *last* if it is non-empty and the $(\ell + 1)$ -th level is empty. The

ℓ -th level is said to be *filled*, or *full*, if it contains b^ℓ vertices. In particular, if a level is filled, so are all the levels before it. A b -ary tree is *perfect* if all its non-empty levels are filled. A rooted b -ary tree T is called *balanced* if:

1. every non-empty level of T , except possibly the last one, is filled;
2. if x and y are two vertices on the same level of T , then $|n_x - n_y| \leq 1$.

Lemma 8. *For any integers $b \geq 2$ and $n \geq 1$, there exists a balanced b -ary tree with n vertices.*

Proof. The statement is obvious when $n = \frac{b^\ell - 1}{b - 1}$ for some natural $\ell \geq 1$, since perfect b -ary trees with ℓ levels are balanced. In general, suppose $\frac{b^\ell - 1}{b - 1} \leq n < \frac{b^{\ell+1} - 1}{b - 1}$, and assume there is a balanced b -ary tree T on n vertices. The bounds on n imply that T has ℓ filled levels. The $(\ell + 1)$ -th level of T is non-filled and possibly empty. We think of this final level as consisting of b^ℓ slots, $n - \frac{b^\ell - 1}{b - 1}$ of which are already filled with leaves. We describe how to add a leaf to one of the empty slots in this level, in order to obtain a balanced b -ary tree on $n + 1$ vertices.

Let $x_0, x_1, \dots, x_{\ell-1}$ be a path from the root x_0 to a vertex at level $\ell - 1$ (i.e., the lowest filled level) such that for every $i \in [\ell - 1]$ vertex x_i is a child of x_{i-1} with the minimum number of descendants. First, we claim that every vertex x_i of the path has the minimum number of descendants among the vertices at level i . This is clearly true for x_0 as there is only one vertex at level 0. Assume the claim is true for x_{i-1} , $i \geq 1$, and suppose, towards a contradiction, there is a vertex v at level i such that $n_v < n_{x_i}$. The choice of x_i implies that the parent $p(v)$ of v is distinct from the parent x_{i-1} of x_i . Since the tree is balanced and x_i is the child of x_{i-1} with the least number of descendants, we conclude that $n_x = n_v + 1$ for every child x of x_{i-1} , and $n_y \in \{n_v, n_v + 1\}$ for every child y of $p(v)$. Consequently, as x_{i-1} and $p(v)$ both have b children, it follows that $p(v)$ has fewer descendants than x_{i-1} does, which contradicts the induction assumption.

Now, to complete the proof, we add the new vertex as a child of $x_{\ell-1}$. This extension of the tree increases n_{x_i} by exactly 1 for every $i = 0, 1, \dots, \ell - 1$, and does not affect the number of descendants of any other vertex in the tree. Since x_i is a vertex with the minimum number of descendants at level i , it is easy to see that the balancedness property is preserved in the new tree. \square

Figure 3 provides an illustration of balanced binary and ternary trees; adding the leaves in the order given by their labels preserves balancedness at each step.

Remark 9. It is not hard to see that up to isomorphism, there is a unique balanced b -ary tree on n vertices. For convenience, we will denote by $T_b(n)$ some fixed balanced b -ary tree with vertex set $[n]$ from the isomorphism class.

The key fact we use about balanced trees is that the number n_v of vertices in the tree rooted at v only depends on n and on the level of v , up to a small error:

Lemma 10. *Let T be a balanced b -ary tree. Then for any vertex v of T*

$$n_v = \frac{n - ((b^{\text{lvl}(v)} - 1)/(b - 1))}{b^{\text{lvl}(v)}} + \beta,$$

where $|\beta| \leq 1$.

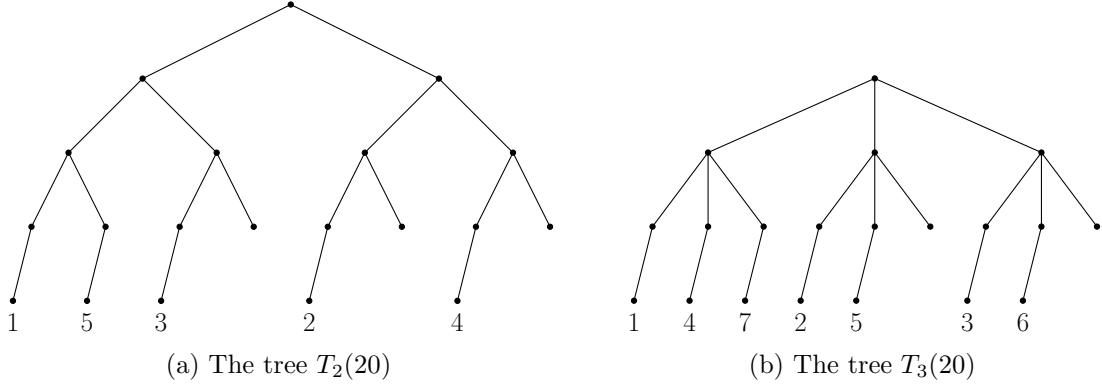


Figure 3: Balanced binary and ternary trees

Proof. If v is on the last level, the approximation holds trivially. Otherwise, the number $\frac{n - ((b^{\text{lvl}(v)} - 1)/(b - 1))}{b^{\text{lvl}(v)}}$ is what we obtain if the vertices of T without the top $\text{lvl}(v)$ levels are evenly divided among all $b^{\text{lvl}(v)}$ trees rooted at the same level as v . Balancedness of the tree ensures that the vertices are as evenly divided as possible. In particular, for every vertex u at level $\text{lvl}(v)$ the number n_u is within 1 of the above average. \square

3.3 Unbounded clique-width

The purpose of this section is to prove that any gluing of large enough balanced binary and ternary trees has large clique-width. As large clique-width implies large treewidth, this result will imply Theorem 4. We denote by $\text{cw}(G)$ the clique-width of graph G . Since we will not actually use clique-width directly, we omit its definition for brevity. For a permutation $\varphi \in S_n$, we denote by $G_\varphi(n)$ the union of $T_2(n)$ and $T_3(n)$ along φ . Formally, we will prove the following

Theorem 11. *For any $c > 0$, there exists $n \in \mathbb{N}$ such that $\text{cw}(G_\varphi(n)) > c$ for any $\varphi \in S_n$.*

Our starting point is a result from [12] that gives a lower bound for clique-width. Let $U \subseteq V(G)$ and $x, y \in U$. We say that x and y are U -similar if their sets of neighbors outside U coincide. It is not difficult to see that U -similarity is an equivalence relation on U ; we denote the number of equivalence classes by $\mu_G(U)$ and define

$$\mu(G) := \min_{\frac{|V(G)|}{3} \leq |U| \leq \frac{2|V(G)|}{3}} \mu_G(U).$$

Lemma 12 ([12], Lemma 4). *For any graph G , $\mu(G) \leq \text{cw}(G)$.*

We will apply Lemma 12 to bound below the clique-width of the graphs $G_\varphi(n)$. First, we observe that the vertices in $G_\varphi(n)$ have degree at most 7, which allows us to prove the following auxiliary lemma.

Lemma 13. *Let $G = G_\varphi(n)$, and $U \subseteq V(G)$. We have $\frac{e_G(U)}{49} \leq \mu_G(U) \leq e_G(U) + 1$.*

Proof. Let $x_1, \dots, x_{\mu_G(U)} \in U$ be representatives of the U -similar equivalence classes. Since these representatives have pairwise different neighborhoods in \overline{U} , at most one of them has no

neighbors in \bar{U} . Therefore there are at least $\mu_G(U) - 1$ edges between U and \bar{U} , from which $\mu_G(U) \leq e_G(U) + 1$.

On the other hand, note that, since degree in G is bounded above by 7, there must be at least $\frac{e_G(U)}{7}$ vertices in \bar{U} that are incident with at least one edge between U and \bar{U} . For the same reason, each of the representatives has at most 7 neighbors in \bar{U} . This implies that $\mu_G(U) \geq \frac{1}{7} \cdot \frac{e_G(U)}{7} = \frac{e_G(U)}{49}$, as claimed. \square

Let $T_b := T_b(n)$ for $b = 2, 3$, and let $G = G_\varphi(n)$ for some $\varphi \in S_n$. Any U -cut of G induces the $\varphi^{-1}(U)$ -cut in T_2 and the U -cut in T_3 . Clearly, $\max\{e_{T_2}(\varphi^{-1}(U)), e_{T_3}(U)\} \leq e_G(U)$. Our aim is to bound below $e_G(U)$, for all balanced U -cuts of G . At the heart of our argument lies the following idea: by allowing the U -cut in G to have few crossing edges, we are putting restrictions on the ratio $\frac{|U|}{n}$. We show that the restrictions coming from the $\varphi^{-1}(U)$ -cut in T_2 and those coming from the U -cut in T_3 cannot be simultaneously satisfied, provided n is large enough. In particular, since the restriction will only depend on $|U|$, and since $|\varphi^{-1}(U)| = |U|$, the permutation φ loses its importance. We will prove that, given a number k , there exists $n(k)$ such that any balanced U -cut of G with $n \geq n(k)$ vertices induces at least k crossing edges.

Theorem 14. *For any $k \geq 1$, there exists an integer $n(k)$ such that for any $n \geq n(k)$, any $\varphi \in S_n$, the graph $G_\varphi(n)$ has no balanced U -cut with at most k crossing edges.*

Theorem 14 implies Theorem 11. Indeed, Theorem 14 says that for n large enough any balanced U -cut in G has at least k crossing edges. This bounds below by k the parameter $e(G) := \min_{\frac{n}{3} \leq |U| \leq \frac{2n}{3}} e_G(U)$, and then Lemma 13 and Lemma 12 imply Theorem 11.

Let us sketch how we prove Theorem 14. For a set $U \subseteq V(G)$ we will denote by $r(U)$ the ratio $\frac{|U|}{n}$. We aim to derive estimates for $r(U)$ using the U -cut-sets in each of the two trees T_2 and T_3 , and show that the two estimates cannot agree for sufficiently large n . To summarise the intuition behind our argument, let us, for a moment, imagine that the ratio n_v/n for every vertex v at level i in T_b is exactly b^{-i} . Then, from Lemma 7, we could derive that $r(U)$ belongs to the set $[\frac{1}{3}, \frac{2}{3}] \cap \{\frac{a}{b^c} : a \in \mathbb{N}\}$ for an appropriate choice of c . Since two such sets for $b = 2$ and $b = 3$ respectively have empty intersection (as we will show in Lemma 15), this would produce a contradiction. Of course, our assumption about the ratio n_v/n is not true. However, using Lemma 10 and a carefully prepared set-up, we are able to show that $r(U)$ must simultaneously be close to the two sets $[\frac{1}{3}, \frac{2}{3}] \cap \{\frac{a}{2^c} : a \in \mathbb{N}\}$ and $[\frac{1}{3}, \frac{2}{3}] \cap \{\frac{a}{3^c} : a \in \mathbb{N}\}$ – close enough to still yield the desired contradiction.

In what follows, we write $R_{b,c}$ for the set of b -adic rationals between $\frac{1}{3}$ and $\frac{2}{3}$ with denominator at most b^c , i.e., $R_{b,c} := [\frac{1}{3}, \frac{2}{3}] \cap \{\frac{a}{b^c} : a \in \mathbb{Z}\}$. Let $r \in \mathbb{R}$, and $X, Y \subseteq \mathbb{R}$. We denote by $d(X, Y)$ the infimum of the distances between a point $x \in X$ and a point $y \in Y$. When an argument of $d(\cdot, \cdot)$ consists of single point, we will omit $\{\}$ in the notation. We denote by $\mathbb{B}(X, r)$ the set of points that are distance less than r from some point in X . Of central importance in our argument is a simple, yet useful lower bound on the distance between the sets $R_{2,i}, R_{3,j}$:

Lemma 15. $d(R_{2,i}, R_{3,j}) \geq 3^{-(i+j)}$.

Proof. First, note that $R_{2,i} \cap R_{3,j} = \emptyset$. Indeed, if $x = \frac{p}{2^i} = \frac{q}{3^j}$, then $p3^j = q2^i$; hence, since 2^i and 3^j are coprime, we have that 2^i divides p , which implies that x is an integer. But $R_{2,i} \subseteq [\frac{1}{3}, \frac{2}{3}]$ by definition.

Now put all numbers in the two sets over the common denominator $2^i 3^j$. Since they are all distinct, they must differ by at least $2^{-i} 3^{-j} \geq 3^{-(i+j)}$ as claimed. \square

For a rooted tree T and a natural number c , we define the *layer c of T* to be the set of edges of T with one endpoint at level $c - 1$ and the other endpoint at level c . In the next lemma, we assume that we have a balanced U -cut in $T_b(n)$ with at most k crossing edges, and fix a layer index c . We may then partition the U -cut-set into “top” edges that belong to the first c layers, and the “bottom” edges which are all the other edges in the U -cut-set. Assuming there is a gap of d layers between the top crossing edges and the bottom ones, and using Lemma 7, we estimate the distance from $r(U)$ to $R_{b,c}$.

Lemma 16. *Let $c, d, k, b \in \mathbb{N}$ and $b \geq 2$. Suppose that $T_b(n)$ has $\ell \geq c + d$ full levels, that U is a balanced cut in $T_b(n)$ with at most k crossing edges, and that the layers $c + 1, c + 2, \dots, c + d$ have no crossing edges. Then $r(U) \in \mathbb{B}(R_{b,c}, k \cdot (b^{-(c+d+1)} + \frac{4}{n}))$.*

Proof. Let $e_1, \dots, e_{k'}$ denote the edges of the U -cut-set, where $k' \leq k$, and let $e_i = (u_i, v_i)$ for $i \in [k']$. Assume, without loss of generality, that e_1, \dots, e_s belong to the first c layers, while $e_{s+1}, \dots, e_{k'}$ are in layers with indices at least $c + d + 1$. By Lemma 7, we have

$$|U| = \mathbb{1}_U(r) \cdot n + \sum_{i=1}^{k'} (-1)^{\mathbb{1}_U(u_i)} \cdot n_{v_i}.$$

We split this sum into two terms

$$S_1 := \mathbb{1}_U(r) \cdot n + \sum_{i=1}^s (-1)^{\mathbb{1}_U(u_i)} \cdot n_{v_i} \quad \text{and} \quad S_2 := \sum_{i=s+1}^{k'} (-1)^{\mathbb{1}_U(u_i)} \cdot n_{v_i},$$

and write $r_i = \frac{S_i}{n}$ for $i \in \{1, 2\}$, so that $r(U) = r_1 + r_2$. To get an upper bound for $d(R_{b,c}, r(U))$, we will estimate $d(R_{b,c}, r_1)$ and $d(r_1, r(U))$ separately, using Lemma 10, and then apply the triangle inequality. We have

$$r_1 = \alpha_0 + \sum_{i=1}^s \alpha_i \frac{n_{v_i}}{n},$$

where $\alpha_i \in \{0, \pm 1\}$. Directly from Lemma 10,

$$\frac{n_{v_i}}{n} = \frac{1}{b^{\text{lvl}(v_i)}} - \frac{b^{\text{lvl}(v_i)} - 1}{n(b-1)b^{\text{lvl}(v_i)}} + \varepsilon,$$

where $|\varepsilon| \leq \frac{1}{n}$. In particular, the sum of the last two terms is smaller in absolute value than $\frac{2}{n}$. Since r_1 is a sum of $\alpha_0 \in \{\frac{a}{b^c} : a \in \mathbb{Z}\}$ and at most k terms each within $\frac{2}{n}$ from a number of the form $\frac{1}{b^{\text{lvl}(v)}} = \frac{b^{c-\text{lvl}(v)}}{b^c} \in \{\frac{a}{b^c} : a \in \mathbb{Z}\}$, we get

$$d(R_{b,c}, r_1) < \frac{2k}{n}.$$

To estimate $d(r_1, r(U))$, we observe that it is equal to $|r_2|$, and note that S_2 is a sum of the values n_v for some vertices v lying at levels below $c + d$. Hence, another application of

Lemma 10 and a similar calculation as the one above then give

$$|r_2| < k \cdot \left(\frac{1}{b^{c+d+1}} + \frac{2}{n} \right),$$

and an application of the triangle inequality finishes the proof of the claim. \square

Before we proceed to the proof of Theorem 14 we need a small auxiliary lemma.

Lemma 17. *Let $A, M, K \in \mathbb{N}^+$ and $M \geq 2$. Suppose $L > A \cdot M^{K+2}$, and assume that at most K elements of $[L]$ are coloured red and the rest are black. Then there exists C with $A < C < L/M$ such that the MC elements after it are all coloured black.*

Proof. For $0 \leq i \leq K+1$, write $X_i := \{A \cdot M^i + 1, \dots, A \cdot M^{i+1}\}$. At least one of the intervals X_1, \dots, X_{K+1} has no red elements. Let $j \geq 1$ be the smallest index such that X_j has no red elements. Let C be the largest element in X_{j-1} . Then C satisfies the statement of the lemma. \square

We are now ready to prove Theorem 14.

Theorem 14. *For any $k \geq 1$, there exists an integer $n(k)$ such that for any $n \geq n(k)$, any $\varphi \in S_n$, the graph $G_\varphi(n)$ has no balanced U -cut with at most k crossing edges.*

Proof. Let $k \geq 1$ and $\ell > \lceil \log_3 2k \rceil \cdot 4^{2k+2}$. We define $n(k) = 3^{5\ell/4+1}$ and let $n \geq n(k)$. We fix an arbitrary $\varphi \in S_n$ and let $G = G_\varphi(n)$. Observe that the choice of n guarantees that all levels up to and including level $5\ell/4$ in both $T_2(n)$ and $T_3(n)$ are filled.

Assume that there exists a vertex set $U \subseteq V(G)$ such that the U -cut is balanced and has at most k crossing edges. Notice that the U -cut-set in G induces the $\varphi^{-1}(U)$ -cut-set in $T_2(n)$ and the U -cut-set in $T_3(n)$, each with at most k edges. An edge e in the U -cut-set of G comes either from $T_2(n)$ (if $\varphi^{-1}(e)$ is an edge in $T_2(n)$), or from $T_3(n)$ (if e is an edge in $T_3(n)$), or from both. Let $R(e)$ be the set that consists of the index of the layer of $T_2(n)$ that contains $\varphi^{-1}(e)$, if $\varphi^{-1}(e)$ is an edge in $T_2(n)$, and the index of the layer of $T_3(n)$ that contains e , if e is an edge in $T_3(n)$. Let $R = \bigcup R(e)$, where the union is over all edges e in the U -cut-set of G . Clearly R has at most $2k$ elements. By coloring red the numbers in $R \cap [\ell]$ and black the numbers in $[\ell] \setminus R$ and applying Lemma 17 with $A = \lceil \log_3 2k \rceil$, $M = 4$, and $K = 2k$, we conclude that there is a c with $\lceil \log_3 2k \rceil < c < \ell/4$ such that the layers $c+1, c+2, \dots, 5c$ of $T_2(n)$ contain no $\varphi^{-1}(U)$ -cut-set edges, and the layers $c+1, c+2, \dots, 5c$ of $T_3(n)$ contain no U -cut-set edges.

Since $|\varphi^{-1}(U)| = |U|$ and hence $r(\varphi^{-1}(U)) = r(U)$, by applying Lemma 16 to $T_2(n)$ and $T_3(n)$, we derive that on the one hand $r(U) \in \mathbb{B}(R_{2,c}, k \cdot (2^{-(5c+1)} + \frac{4}{n}))$, and on the other hand $r(U) \in \mathbb{B}(R_{3,c}, k \cdot (3^{-(5c+1)} + \frac{4}{n}))$. Therefore, by the triangle inequality, we obtain an upper bound on the distance between $R_{2,c}$ and $R_{3,c}$:

$$d(R_{2,c}, R_{3,c}) \leq k \cdot \left(2^{-(5c+1)} + \frac{4}{n} \right) + k \cdot \left(3^{-(5c+1)} + \frac{4}{n} \right) < k \cdot 2^{-5c} + \frac{8k}{n}.$$

Next, we will show that each of the latter two summands is smaller than $3^{-2c}/2$. This will imply a contradiction to Lemma 15 and thus prove the theorem. We start with the first

summand. Since $c > \log_3 2k$, we have that $k < 3^c/2$, and therefore

$$k \cdot 2^{-5c} < \frac{3^c \cdot 2^{-5c}}{2} < \frac{3^c \cdot 3^{-3c}}{2} = \frac{3^{-2c}}{2}.$$

To bound the second summand, we recall that $n \geq n(k) = 3^{5\ell/4+1} > 3^{5\ell/4}$ and $\log_3 2k < c < \ell/4$. The latter implies that $k < \frac{3^{\ell/4}}{2}$, and therefore we have

$$\frac{8k}{n} < \frac{4 \cdot 3^{\ell/4}}{3^{5\ell/4}} = \frac{4}{3^\ell} < \frac{4}{3^{4c}} = \frac{4}{3^{2c}} \cdot 3^{-2c} < \frac{3^{-2c}}{2},$$

where the last inequality follows from the fact that $c > 1$, as $\lceil \log_3 2k \rceil < c$ and $k \geq 1$. \square

As discussed earlier, Theorem 14 together with Lemma 13 and Lemma 12 imply Theorem 11. Now, since clique-width of any graph of treewidth k is at most $3 \cdot 2^{k-1}$ [4], Theorem 11 implies Theorem 4.

4 Conclusion and outlook

Our main result shows that graphs of bounded treewidth cannot always be glued into a graph of bounded treewidth, and that this is true even for graphs of treewidth 1. Yet we also showed that certain graphs of bounded treewidth can be glued to a graph of bounded treewidth. In particular, we observed that two caterpillars and also a path and a tree can be glued to a graph of treewidth at most 2. We do not know if it is always possible to achieve the same for arbitrary tree and caterpillar.

Question 1. Is it always possible to glue a tree and a caterpillar to a graph of treewidth at most 2?

By Theorem 3 we nevertheless know that any caterpillar and any tree can be glued into a graph of small treewidth. Indeed, a graph of bounded treewidth and a graph of bounded pathwidth can always be glued to a graph of bounded treewidth.

A tempting direction for further investigation is the gluing of three or more graphs. For the case of trees we ask two specific questions which could guide this endeavor. As with the questions that motivate our paper, these are inspired by arboricity results.

It is a well-known corollary of Nash-Williams' theorem that any planar graph can be decomposed into three forests, i.e. any planar graph is an edge-disjoint union of three forests.

Question 2. Given any three n -vertex trees is it always possible to glue them into a planar graph?

Since every graph of book thickness 2 is planar, one can use the book embedding argument from Section 2 to show that two arbitrary trees and a path can be glued into a planar graph. However, for three arbitrary trees we do not even know if they can always be glued into a graph that excludes some fixed clique as a minor.

Gonçalves proved that every planar graph can be decomposed into 4 forests of caterpillars [8].

Question 3. Is it always possible to glue arbitrary 4 caterpillars into a planar graph?

Is it possible to do this for any 3 caterpillars? From Lemma 2 we know that any k caterpillars can be glued into a graph of pathwidth at most k , and therefore into a graph that excludes some fixed clique minor.

References

- [1] Hans L Bodlaender. A linear-time algorithm for finding tree-decompositions of small treewidth. *SIAM Journal on Computing*, 25(6):1305–1317, 1996.
- [2] Hans L Bodlaender. A partial k -arboretum of graphs with bounded treewidth. *Theoretical Computer Science*, 209(1-2):1–45, 1998.
- [3] Yijia Chen and Jörg Flum. On the ordered conjecture. In *2012 27th Annual IEEE Symposium on Logic in Computer Science*, pages 225–234. IEEE, 2012.
- [4] Derek G Corneil and Udi Rotics. On the relationship between clique-width and treewidth. *SIAM Journal on Computing*, 34(4):825–847, 2005.
- [5] Jessica Enright, Kitty Meeks, William Pettersson, and John Sylvester. Tangled paths: A random graph model from mallows permutations. *arXiv preprint arXiv:2108.04786*, 2021.
- [6] Jessica Enright, Kitty Meeks, and Jessica Ryan. Two dichotomies for model-checking in multi-layer structures. *arXiv preprint arXiv:1710.08758*, 2017.
- [7] Markus Geyer, Michael Hoffmann, Michael Kaufmann, Vincent Kusters, and Csaba D Tóth. The planar tree packing theorem. *Journal of Computational Geometry*, 8(2):109–177, 2017.
- [8] Daniel Gonçalves. Caterpillar arboricity of planar graphs. *Discrete Mathematics*, 307(16):2112–2121, 2007.
- [9] Sandra M Hedetniemi, Stephen T Hedetniemi, and Peter J Slater. A note on packing two trees into K_n . *Ars Combinatoria*, 11:149–153, 1981.
- [10] Nancy G Kinnersley. The vertex separation number of a graph equals its path-width. *Information Processing Letters*, 42(6):345–350, 1992.
- [11] Mikko Kivelä, Alex Arenas, Marc Barthelemy, James P Gleeson, Yamir Moreno, and Mason A Porter. Multilayer networks. *Journal of Complex Networks*, 2(3):203–271, 2014.
- [12] Vadim Lozin, Igor Razgon, and Viktor Zamaraev. Well-quasi-ordering versus clique-width. *Journal of Combinatorial Theory, Series B*, 130:1–18, 2018.
- [13] Johann A Makowsky. Coloured Tutte polynomials and Kauffman brackets for graphs of bounded tree width. *Discrete Applied Mathematics*, 145(2):276–290, 2005.
- [14] Crispin St JA Nash-Williams. Edge-disjoint spanning trees of finite graphs. *Journal of the London Mathematical Society*, 1(1):445–450, 1961.
- [15] Crispin St JA Nash-Williams. Decomposition of finite graphs into forests. *Journal of the London Mathematical Society*, 1(1):12–12, 1964.
- [16] Daniel A Quiroz. *Chromatic and structural properties of sparse graph classes*. PhD thesis, The London School of Economics and Political Science (LSE), 2017.

- [17] Neil Robertson and Paul D Seymour. Graph minors. I. excluding a forest. *Journal of Combinatorial Theory, Series B*, 35(1):39–61, 1983.