WALL-CROSSING OF UNIVERSAL BRILL-NOETHER CLASSES

ALEX ABREU AND NICOLA PAGANI

Abstract. We give an explicit graph formula, in terms of decorated boundary strata classes, for the wall-crossing of universal Brill-Noether classes.

More precisely, fix $n > 0$ and $d < g$, and two stability conditions ϕ^+ and ϕ^- for degree d compactified universal (over $\overline{\mathcal{M}}_{g,n}$) Jacobians that lie on opposite sides of a stability hyperplane. Our main result is a formula for the difference between $\mathsf{w}_d(\phi^+)$ and the pullback of $\mathsf{w}_d(\phi^-)$ along the (rational) identity map $\mathsf{Id}\colon \overline{\mathcal{J}}^d_{g,n}(\phi^+) \dashrightarrow \overline{\mathcal{J}}^d_{g,n}(\phi^-)$. The calculation involves constructing a resolution of the identity map by means of subsequent blow-ups.

Contents

1. Introduction

The Brill–Noether theory of line bundles on nonsingular algebraic curves is a classical pillar of XIX century algebraic geometry, which has been rediscovered and reused to prove important contemporary results. Broadly speaking, the theory is about studying the space of line bundles of a fixed degree having a fixed number of linearly independent global sections (see [ACGH85] and references therein for a survey of the classical results).

For fixed integers g, n (we will assume for uniformity of notation, that $g \geq 2$ and $n \geq 1$), and d there exists a universal Jacobian $\mathcal{J}^d_{g,n} \to \mathcal{M}_{g,n}$, a moduli space that parameterizes isomorphism classes of degree d line bundles over smooth, n-pointed curves of genus g. From now on we assume $d < g$ and define the universal Brill–Noether class w_d as the fundamental class in $\mathcal{J}^d_{g,n}$ of the locus W_d of line bundles that admit a nonzero global section. This locus has fiberwise codimension $g - d$ over $\mathcal{M}_{g,n}$ and it is empty for $d < 0$. In this paper we study extensions of this class to different compactifications of the universal Jacobian.

The moduli space $\mathcal{M}_{g,n}$ admits a natural, modular and well-studied compactification $\overline{\mathcal{M}}_{g,n}$ obtained by adding (Deligne–Mumford) *stable* pointed curves. On the other hand, there are several natural compactifications of $\mathcal{J}^d_{g,n}$ over $\overline{\mathcal{M}}_{g,n}$. In the words of Oda–Seshadri [OS79], this should not be seen as a drawback of the theory, but rather a merit.

In [KP19] Kass–Pagani constructed an affine space of stability conditions $V^d_{g,n}$ with an explicit hyperplane arrangement, with the property that every $\phi \in V^d_{g,n}$ produces a compactification $\overline{\mathcal{J}}^d_{g,n}(\phi)$ of the universal Jacobian, with good properties (it is a nonsingular DM stack) when ϕ is not on a hyperplane. This space comes with a natural origin — a canonical stability — and so far most of the attention has been devoted to compactified Jacobians corresponding to this particular value (or to its perturbations when the latter belongs to some hyperplanes), see [GZ14], [HMP$^+$].

In this paper we study how the Brill–Noether classes, suitably extended to classes $\mathsf{w}_d(\phi)$ on $\overline{\mathcal{J}}^d_{g,n}(\phi)$, vary in ϕ. What we mean by this is the following: for different stability conditions ϕ_1, ϕ_2, the identity on the common open set $\mathcal{J}^d_{g,n}$ of line bundles on smooth curves defines a rational map

$$\mathsf{Id}\colon \overline{\mathcal{J}}^d_{g,n}(\phi_1) \dashrightarrow \overline{\mathcal{J}}^d_{g,n}(\phi_2),$$

and we can then compute the difference $\mathsf{w}_d(\phi_2) - \mathsf{Id}^*\mathsf{w}_d(\phi_1)$. By "compute", we mean produce an explicit "graph formula", as in the case of tautological classes on the moduli space of curves $\overline{\mathcal{M}}_{g,n}$, which can all be expressed as linear combinations of "decorated boundary strata classes" (see [Pan18]). While an established theory of a tautological ring for $\overline{\mathcal{J}}^d_{g,n}(\phi)$ is not yet available (a large literature is available for the case of a single curve, the case of the universal moduli space has recently been the subject of important results in [Yin16], [BHP$^+$], [HMP$^+$]), there are several natural classes on each compactified universal Jacobian, and "decorated boundary strata classes", supported on the boundary of $\overline{\mathcal{J}}^d_{g,n}(\phi)$, may be defined in complete analogy with the case of $\overline{\mathcal{M}}_{g,n}$. In fact, an important underlying motivation for our work is to develop a categorical and wall–crossing framework for a theory of tautological classes over compactified universal Jacobians.

We now discuss what we mean by "a suitable extension" for the class $\mathsf{w}_d(\phi)$. One possible approach is to take the Zariski closure, but this is very hard to control, and it does not have good formal properties (for example, it does not commute with base change). Another approach is to consider sheaves in $\overline{\mathcal{J}}^d_{g,n}(\phi)$ that admit a nonzero global section, but that locus is, in general, not of the expected dimension and not equidimensional. Our extension instead is by means of the Thom–Porteous' formula. By virtue of its universal property, there is a tautological (or Poincaré) sheaf $\mathcal{L}_{\mathrm{tau}}(\phi)$ on the universal curve $\pi\colon \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{J}}^d_{g,n}(\phi)$. We define the extension as the degeneracy class

$$\mathsf{w}_d(\phi) := c_{g-d}(-R^\bullet\pi_*\mathcal{L}_{\mathrm{tau}}(\phi)), \tag{1.1}$$

as in [Ful98, Chapter 14]. By the Thom–Porteous formula (see *loc.cit.*), the restriction of $\mathsf{w}_d(\phi)$ to $\mathcal{J}^d_{g,n}$ equals the (Poincaré dual of the) original Brill–Noether class w_d. We compare (1.1) with the class of the Zariski closure in Proposition 3.38. The class (1.1) is supported on the universal Brill–Noether locus, but in general the latter does not have the expected codimension, hence its fundamental class does not coincide with (1.1) (more details in Proposition 4.18).

The class (1.1) is the formal analogue of the λ_{g-d} class on $\overline{\mathcal{M}}_{g,n}$ (the Hodge bundle $R^\bullet\pi_*(\omega_\pi)$ being replaced by $-R^\bullet\pi_*\mathcal{L}$). Given the important role that the λ-classes have played in the enumerative geometry of curves / intersection theory for moduli of curves, it is legitimate to expect that the same will be true of $\mathsf{w}_d(\phi)$. Also, as observed in Remark 3.32, the class (1.1) is independent of the choice of a tautological line bundle $\mathcal{L}_{\mathrm{tau}}(\phi)$. This is not the case for other natural classes, *e.g.* the first Chern class of the pull-back of the tautological line bundle via some section, or the pushforward of a power of the latter under a forgetful morphism.

In this paper we assume that ϕ^+ and ϕ^- are on opposite sides of a stability hyperplane (Definition 5.1), and we give an explicit graph formula for the difference

$$\mathsf{w}_d(\phi^+) - \mathsf{Id}^*\mathsf{w}_d(\phi^-).$$

In order to achieve this, we first produce a nonsingular resolution of the identity

$$\begin{tikzcd}
 & \widetilde{\mathcal{J}}^d_{g,n}(\phi^+,\phi^-) \arrow[dl, "p"'] \arrow[dr, "p_-"] & \\
\overline{\mathcal{J}}^d_{g,n}(\phi^+) \arrow[rr, dashed, "\mathsf{Id}"] & & \overline{\mathcal{J}}^d_{g,n}(\phi^-)
\end{tikzcd}$$

by an explicit sequence of blow-ups of $\overline{\mathcal{J}}^d_{g,n}(\phi^+)$. We use this resolution to give, in Theorem 7.4, an explicit and closed graph formula for the difference $p^*(\mathsf{w}_d(\phi^+)) - p_-^*(\mathsf{w}_d(\phi^-))$ in the cohomology of $\widetilde{\mathcal{J}}^d_{g,n}(\phi^+,\phi^-)$. Finally, we calculate the push-forward of that formula via p to write a formula (again a graph formula, explicit and closed) for the difference $\mathsf{w}_d(\phi^+) - \mathsf{Id}^*\mathsf{w}_d(\phi^-)$.

Our construction of $\widetilde{\mathcal{J}}^d_{g,n}(\phi^+,\phi^-)$ and our formulas are complicated by the fact that, for some of the hyperplanes, the locus where the identity is undefined fails to be irreducible. In those cases, the space $\widetilde{\mathcal{J}}^d_{g,n}(\phi^+,\phi^-)$ is constructed as an explicit sequence of blow ups along centers that have transversal self-intersection, and this construction plays an important part in our paper.

In this introduction we describe the particular case of our construction and formula when the indeterminacy locus is irreducible (this occurs in many cases, and in some sense in most cases as long as $n > 1$). Then the indeterminacy locus $\mathcal{J}'_\beta \subset \overline{\mathcal{J}}^d_{g,n}(\phi^+)$ generically parameterizes curves with 2 nonsingular components of genus, say, g_X and g_Y, carrying markings S and S^c, and joined at a certain number of nodes, say t, together with line bundles of some fixed bidegree, say, $(d-d_Y, d_Y)$. The locus $\mathcal{J}'_\beta$ can be parameterized by a "resolved stratum"

$$f_\beta \colon \mathcal{J}_\beta \to \mathcal{J}'_\beta \hookrightarrow \overline{\mathcal{J}}^d_{g,n}(\phi^+)$$

(which we simply call "a stratum" in the main body of the paper), where the t nodes are parameterized: a general point of $\mathcal{J}_\beta$ is a triple of a $(|S|+t)$-pointed curve of genus g_X, a $(|S^c|+t)$-pointed curve of genus g_Y, and a line bundle of bidegree $(d-d_Y, d_Y)$. The conormal bundle to f_β has rank t and it splits as a direct sum of line bundles, whose first Chern classes we call $\Psi_1, \ldots, \Psi_t$ (see Remark 7.30 for more details on how these relate to the "classical" ψ-classes in $\overline{\mathcal{M}}_{g,n}$). The base change to $\mathcal{J}_\beta$ of the universal family $\pi \colon \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{J}}^d_{g,n}(\phi^+)$ consists of two irreducible components, say $\pi_X \colon X \to \mathcal{J}_\beta$ and $\pi_Y \colon Y \to \mathcal{J}_\beta$, of genus g_X and g_Y respectively, each carrying a tautological sheaf L_X and L_Y (obtained by pulling back $\mathcal{L}_{\mathrm{tau}}(\phi^+)$).

In this particular case, our main result becomes:

Theorem. *(Corollary 7.33 with $m = 1$.) If ϕ^+ and ϕ^- are on opposites sides of a stability hyperplane (Definition 5.1) and the indeterminacy locus of the identity morphism $\mathsf{Id}\colon \overline{\mathcal{J}}^d_{g,n}(\phi^+) \dashrightarrow \overline{\mathcal{J}}^d_{g,n}(\phi^-)$ is irreducible, the difference $\mathsf{w}_d(\phi^+) - \mathsf{Id}^*\mathsf{w}_d(\phi^-)$ equals*

$$\sum_{s+j+\lambda=g-d-t} \binom{g_Y - d_Y - j - 1}{g - d - j - s} \frac{f_{\beta *}}{t!} \left(c_s(-R^{\bullet}\pi^X_* L_X(-X \cap Y)) \cdot c_j(-R^{\bullet}\pi^Y_* L_Y) \cdot h_\lambda(\Psi_1, \ldots, \Psi_t) \right),$$

where h_λ is the complete homogeneous polynomial of degree λ in t variables.

The special case $d = g - 1$ of the above formula, when the Brill–Noether class is a divisor (the theta divisor), was discovered in [KP17, Theorem 4.1]. In that case the calculation was massively simplified by the fact that the classes have codimension 1, and therefore, because the total space is nonsingular, no blowup is required.

Theorem 7.4 is the main result in this paper. It computes the pull-back to the resolution $\widetilde{\mathcal{J}}^d_{g,n}(\phi^+, \phi^-)$ of the difference $\mathsf{w}_d(\phi^+) - \mathsf{Id}^*\mathsf{w}_d(\phi^-)$ in terms of decorated boundary strata classes. The formula for the difference in $\overline{\mathcal{J}}^d_{g,n}(\phi^+)$ is obtained by pushing the latter forward along a blow-down morphism, which generates more complicated coefficients.

The starting point to construct the resolution $\widetilde{\mathcal{J}}^d_{g,n}(\phi^+, \phi^-)$ is the observation that the tautological sheaf for ϕ^+ is not ϕ^--stable, and the locus $\mathcal{J}'_\beta$ where it fails ϕ^- stability generically parameterizes curves with 2 nonsingular irreducible components (throughout called a "vine curve") and a fixed bidegree. When the locus $\mathcal{J}'_\beta$ is irreducible, the resolution is constructed by blowing up $\overline{\mathcal{J}}^d_{g,n}(\phi^+)$ at $\mathcal{J}'_\beta$. The two components $X' \cup Y'$ of the pull-back of the universal curve to the exceptional divisor E are now divisors ("universal twistors") in the blowup of the universal curve, and after suitably tensoring by one of them, the sheaf $\mathcal{L}_{\mathrm{tau}}(\phi^+)$ becomes ϕ^--stable. The latter sheaf defines the other morphism $\widetilde{\mathcal{J}}^d_{g,n}(\phi^+, \phi^-) \to \overline{\mathcal{J}}^d_{g,n}(\phi^-)$ by the universal property. The main technical difficulty is to suitably identify a sequence of blowups at centers that have transversal self-intersection, which allows one to generalize the above reasoning to the case when the base locus $\mathcal{J}'_\beta$ is not irreducible.

Note that other resolutions of the identity map may also be constructed following existing literature ([AP21], [Hol21], [MW20], [HMP$^+$], [CGH$^+$]), but those constructions yield *singular* spaces.

In **Section 3** we introduce the objects we work with, compactified universal Jacobians and Brill–Noether classes. In **Section 4** we write axioms for "resolved" strata of normal crossing stratifications, and then prove some general intersection theory results that are valid in this context. The main geometric ideas here are not entirely new, but we could not find a suitable reference in this generality, and we believe that this axiomatic point of view will prove helpful in the current research landscape. In **Section 5** we discuss the combinatorial aspects that arise from a wall-crossing situation where there are stability conditions $\phi^{\pm}$ are on opposite sides of a given stability hyperplane (Definition 5.1). Our paper is concerned with the case of rank 1 sheaves on nodal curves, and the combinatorics of Section 5 should be the shadow of a theory for higher dimension and rank. The central definition is that, for each graph G and divisor D on G and choice of stability conditions $\phi^{\pm}$ on opposite sides of a hyperplane, of a poset $\operatorname{Ext}(G, D)$ of "extremal" subsets of the vertices of G. **Section 6** gives the construction of the resolution $\widetilde{\mathcal{J}}^d_{g,n}(\phi^+, \phi^-)$. Finally, in **Section 7** we are then ready to employ intersection theory techniques and calculate the wall-crossing term. At the end of Section 7 we explain how the pullback of the wall-crossing term via an Abel–Jacobi section can be explicitly calculated in terms of decorated boundary strata classes in $\overline{\mathcal{M}}_{g,n}$ by employing the main result of [PRvZ20].

In the background of this work, we produce two results that we believe are of independent interest. The first is Theorem 3.29, where we interpret the universal quasistable family over $\overline{\mathcal{J}}^d_{g,n}(\phi)$ (also known in the literature as Caporaso's family from [Cap94], see also [MMUV22] and [EP16]) as a fine compactified universal Jacobian $\overline{\mathcal{J}}^d_{g,n+1}(\phi')$ with one extra point.

Secondly, as part of Proposition 3.38, we describe the collection of stability conditions for $d < 0$ such that $\mathsf{w}_d(\phi) = 0$. One can choose a suitable Abel–Jacobi section $\sigma\colon \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{J}}^d_{g,n}(\phi)$ and obtain a zero class $\sigma^*\mathsf{w}_d(\phi)$. A different formula for the latter as a linear combination of standard tautological classes was given in [PRvZ20] by means of the GRR formula. This gives tautological relations in $\overline{\mathcal{M}}_{g,n}$ (see Remark 3.42 for the details). Note that these relations are in degree larger than g (the degree is $g-d$ for negative d), the same range of Pixton's double ramification relations (proven by Clader-Janda in [CJ18]).

1.a. **Related work.** An important motivation that we have not mentioned in the above discussion, is its relation with the (possibly twisted) double ramification cycle. For a review of the latter and related literature, we address the reader to [BHP$^+$] and [HMP$^+$, Section 1.1]. We refer to [PRvZ20, Section 3.3] for how the double ramification cycle relates to the Brill–Noether classes discussed here. As pointed out in *loc.cit.*, the theory on how these classes are extended to the boundary and then pulled back to $\overline{\mathcal{M}}_{g,n}$ via some Abel–Jacobi section is trivial for nodal curves of compact type (i.e. the moduli space of multidegree zero line bundles is compact) and for curves with 1 node, and the complement of the locus of all such curves is generically parameterized by vine curves.

In [BHP$^+$] the authors discuss the theory of a "universal double ramification cycle" as an operational class of degree g in the Artin stack of families of line bundles on families of nodal curves, which correspond to "universally intersecting with the closure of the zero section". Our extensions (1.1) could also be described in that language, and in fact the construction of an operational class would avoid a lot of technical difficulties owing to the fact that the classes (1.1) obviously commute with base change.

In this paper we do not discuss a modular description of our resolution $\widetilde{\mathcal{J}}^d_{g,n}(\phi^+, \phi^-)$. We expect that one such description should be possible following the recent work [Mol22b] by Molcho. The same author has also recently proved in [Mol22a] that the pull-back of the Brill–Noether classes $\mathsf{w}_d(\phi)$ via all Abel–Jacobi *rational* sections is tautological in $\overline{\mathcal{M}}_{g,n}$ (this was conjectured in [PRvZ20, Section 4.1]).

1.b. **Acknowledgments.** To be added after the refereeing process.

2. Notation and preliminaries

2.a. **Posets.** In this paper we will work with many posets (typically, the one underlying some category of stratifications, and some of its subposets). Here we recollect the relevant notation.

Definition 2.1. Let P be a finite partially ordered set (or a poset).

A subset C of P is called a *chain*, if the partial order on C induced by P is a total order on C.

A poset is *ranked* if for every element a, all maximal chains having a as the largest element have the same length (called the *rank* of a).

The poset P is called a *forest*, if for every $a \in P$ the lower set $\{b \leq a\}$ is a chain. More generally, we say that a subset $F \subseteq P$ is a forest if F together with the partial order induced by P is a forest.

If $a > b$ and there exists no c such that $a > c > b$, then we say that a *covers* b, and write $a \gtrdot b$.

2.b. **Graphs.** By a *graph* we mean a finite, connected, undirected multigraph, decorated with a genus function and markings (see for example [CCUW20, Section 3.1] and [MMUV22, Section 2.1] for a precise definition).

If G is a graph, we write $V(G)$ for its set of vertices and $E(G)$ for its set of edges, we write $g\colon V(G) \to \mathbb{N}$ for the genus function and $\mathrm{leg}\colon \{1, \ldots, n\} \to V(G)$ for the markings function.

If $S \subseteq V(G)$, we write $G(S)$ for the complete subgraph of G on the vertices S, and say that $G(S)$ is the subgraph of G induced by S.

Given $V_1, V_2 \subseteq V(G)$, we write $E(V_1, V_2)$ for the edges that have one endpoint in V_1 and another in V_2 (if the edge is a loop, we include it if and only if its adjacent vertex is in both V_1 and V_2).

If G is a graph and $E \subseteq E(V(G))$, we denote by G^E the graph obtained from G by adding exactly 1 vertex, denoted v_e, in the "interior" of each edge $e \in E$. We call each such v_e an *exceptional vertex* of G^E.

A graph G is *stable* if

$$2g(v) - 2 + |E(\{v\}, \{v\}^c)| + |\operatorname{leg}^{-1}(v)| > 0$$

for every vertex $v \in V(G)$.

2.c. **Families of curves and sheaves.** A *nodal curve* C is a reduced and connected projective scheme of dimension 1 over some fixed algebraically closed field, with singularities that are at worst ordinary double points. The (arithmetic) *genus* of C is $p_a(C) = h^1(C, \mathcal{O}_C)$. A *subcurve* X of C is a connected union of irreducible components of C. Its complement X^c is the union of the other components of C.

A *n-pointed curve* is a tuple $(C, p_1, \ldots, p_n)$ where C is a nodal curve, and $p_1, \ldots, p_n$ are pairwise distinct nonsingular points of C. Its *dual graph* $G(C)$ has the irreducible components of C as vertices, the nodes of C as edges, the geometric genus (resp. the marked points) of each component as the genus (resp. the markings) decoration.

A morphism $f\colon C' \to C$ of nodal curves is *a semistable modification* if it is obtained by contracting some subcurves, not necessarily irreducible, $E \subset C'$ such that $g(E) = 0$ and $|E \cap E^c| = 2$. Every subcurve $E \subset C'$ contracted by f is called *an exceptional curve of f*. A semistable modification such that every exceptional curve is irreducible is called a *quasistable modification.*

A coherent sheaf on a nodal curve C has *rank* 1 if its localization at each generic point of C has length 1. It is *torsion-free* if it has no embedded components. If the stalk of a torsion-free sheaf F over C fails to be locally free at a point $P \in C$, which must necessarily be a node, we will say that F is *singular* at P. If F is a rank 1 torsion-free sheaf on C we say that F is *simple* if its automorphism group is $\mathbb{G}_m$ or, equivalently, if removing from C the singular points of F does not disconnect X.

A *family of nodal curves* over a $\mathbb{C}$-scheme S is a proper and flat morphism $\mathcal{C} \to S$ whose fibers are nodal curves. (Throughout, all families $\mathcal{C}/S$ will admit a distinguished section in the S-smooth locus of $\mathcal{C}$). A *semistable (resp. a quasistable) modification* of the family $\mathcal{C}/S$ is another family $\mathcal{C}'/S$ with a S-morphism $f\colon \mathcal{C}' \to \mathcal{C}$ that is a semistable (resp. a quasistable) modification (as defined above) on all geometric points $s \in S$.

If T is a S-scheme, a *family of rank* 1 *torsion-free simple sheaves* parameterized by T over a family of curves $\mathcal{C} \to S$ is a coherent sheaf F of rank 1 on $\mathcal{C} \times_S T$, flat over T, whose fibers over the geometric points are torsion-free and simple.

If F is a rank 1 torsion-free sheaf on a nodal curve C, the *(total) degree* of F is $\deg_C(F) := \chi(F) - 1 + p_a(C)$.If $X \subseteq C$ is a subcurve, we denote by F_X the maximal torsion-free quotient of $F \otimes \mathcal{O}_X$. The total degree and the degree of F_X and F_{X^c} are related by the formula

$$\deg_C(F) = \deg_X F + \deg_{X^c} F + \delta_{X \cap X^c}(F), \tag{2.2}$$

where $\delta_S(F)$ is the number of points in S where the stalk of F fails to be locally free.

A line bundle F' on a semistable modification $f\colon C' \to C$ is called *positively admissible* (see [EP16]) if $\deg_E(F')$ is either 0 or 1 on every exceptional subcurve of f. The following results follow from [EP16, Section 5].

Proposition 2.3. *Let $\pi\colon \mathcal{C} \to S$ and $\pi'\colon \mathcal{C}' \to S$ be families of nodal curves and $f\colon \mathcal{C}' \to X$ be a semistable modification. Let F' be a positively admissible sheaf on $\mathcal{C}'$ and set $F = f_*(F')$.*

1. *The sheaf F is a torsion free rank-1 sheaf and $R^1 f_*(F') = 0$, in particular $f_*(F')$ commutes with base change. Moreover, we have that $R^\bullet \pi_*(F) = R^\bullet \pi'_*(F')$.*
2. *The sheaf $f_*(F')$ is invertible if and only if $\deg_E(F') = 0$ on every exceptional subcurve of f. Moreover, in this case, $F' = f^* f_*(F')$.*
3. *If f is a quasistable modification and $\deg_E(F') = 1$ for every exceptional subcurve, then $\mathcal{C}' = \mathbb{P}_{\mathcal{C}}(F^\vee)$ and F' is isomorphic to the tautological line bundle $\mathcal{O}_{\mathbb{P}_{\mathcal{C}}(F^\vee)}(1)$.*
4. *More generally, we have that f factors as $X' \xrightarrow{g} \mathbb{P}_{\mathcal{C}}(F^\vee) \to X$, and $\mathcal{O}(1) \cong g_*(F')$ and $F' \cong g^*(\mathcal{O}(1))$.*

In particular, we have the following.

Corollary 2.4. *Let $\mathcal{C} \to S$ be a family of nodal curves. Taking the direct image under the quasistable modification gives a bijection between isomorphism classes of positively admissible line bundles on quasistable modifications of $\mathcal{C}/S$, and isomorphism classes of families of rank 1 torsion free sheaves on $\mathcal{C}$.*

We now define the multidegree of a sheaf on a nodal curve as the multidegree of the unique positively admissible line bundle as in the above corollary.

A degree d *pseudodivisor* on a graph G is a pair (E, D) where $E \subseteq E(G)$ and $D \in \mathrm{Div}^d(G^E)$ satisfies $D(v') = 1$ for each exceptional vertex v'. When E is empty, we simply write D in place of the pair $(\varnothing, D)$.

Given a degree-d rank 1 torsion free sheaf F on a curve C, we define the multidegree $\underline{\deg}(F)$ of F as the pseudodivisor (E, D) on the dual graph $G(C)$ of C as follows. The set E is the set of edges of $G(C)$ that correspond to nodes of C where F is not locally free. The divisor D on $G(C)^E$ is defined by $D(v) = \deg_{C_v}(F_{C_v})$ if $v \in V(G(C)) \subseteq V(G(X)^E)$ and $D(v) = 1$ for every exceptional vertex v. By Equation (2.2), we have that (E, D) is a degree-d pseudodivisor.

Note also that a rank 1 torsion free sheaf on C is simple if and only if its multidegree (E, D) has the property that E does not disconnect the graph $G(C)$.

2.d. **Moduli spaces and graphs.** Here we discuss some general notation on moduli spaces of curves. We refer the reader to [ACG11] for more details on nodal curves and their dual graphs.

A n-pointed curve $(C, p_1, \ldots, p_n)$ is *stable* if $|\operatorname{Aut}(C, p_i)| < \infty$. We will sometimes abuse notation and write C for (C, p_i). For example, we will say that the genus of (C, p_i) is g to mean that the $h^1(C, \mathcal{O}_C) = g$, the arithmetic genus of the underlying curve C is g.

We will denote by $\overline{\mathcal{M}}_{g,n}$ the moduli spaces of stable n-pointed curves of genus g. The moduli space comes admits a stratification by dual graphs, which we now discuss.

2.d.1. *Stable graphs.* We denote by $G_{g,n}$ the small category of stable, n-pointed graphs of genus g (where we have fixed a choice of 1 object for each isomorphism class). Morphisms $G \to G'$ are given by an edge contraction followed by an isomorphism. (More details in [ACG11] and [MMUV22]). There is a natural functor $G_{g,n+1} \to G_{g,n}$ that forgets the last point and stabilizes the graph.

2.d.2. *Stratification of moduli of stable curves.* For $G \in G_{g,n}$ there is a gluing morphism

$$f_G\colon \overline{\mathcal{M}}_G := \prod_{v \in V(G)} \overline{\mathcal{M}}_{g(v),n(v)} \to \Big[\prod_{v \in V(G)} \overline{\mathcal{M}}_{g(v),n(v)} / \operatorname{Aut}(G) \Big] \to \overline{\mathcal{M}}'_G \hookrightarrow \overline{\mathcal{M}}_{g,n}.$$

We say that G, or $\overline{\mathcal{M}}_G$, or f_G, is a (resolved) stratum of $\overline{\mathcal{M}}_{g,n}$. We regard $\overline{\mathcal{M}}_G$ as a "resolved stratum" and its image $\overline{\mathcal{M}}'_G$ as the corresponding "embedded stratum".

The codimension 1 strata are the following divisors generically parameterizing curves with 1 node:

(1) the divisor Δ_{irr}, generically parameterizing irreducible curves
(2) for $0 \leq i \leq g$ and $S \subseteq [n]$ (except $i = 0$ and $|S| < 2$ and $i = g$ and $|S| > n - 2$), the divisor $\Delta_{i,S} = \Delta_{g-i,S^c}$ generically parameterizing curves with 2 components, of which one of genus i carrying the marked points in S.

On the (resolved) stratum the normal bundle to f_G splits as a direct sum of line bundles

$$N_{f_G} = \bigoplus_{e \in E(G)} \mathbb{L}_e.$$

We denote by $\Psi_e = -c_1(\mathbb{L}_e)$. (Recall that, if e is the edge whose half edges $h(e), h'(e)$ are based at $v, v' \in V(G)$, then the cotangent line bundles to $h(e)$ and $h'(e)$ are denoted by $\mathbb{L}_{h(e)}$ and $\mathbb{L}_{h'(e)}$ and its first Chern classes $\psi_{h(e)}$ and $\psi_{h'(e)}$. We then have $\mathbb{L}_e = \mathbb{L}^\vee_{h(e)} \boxtimes \mathbb{L}^\vee_{h'(e)}$ and so $\Psi_e = \psi_{h(e)} + \psi_{h'(e)}$, but this will not play a role.)

In Section 4 we will define what the category of (resolved) strata induced by normal crossing divisors on a DM stack, and will interpret the category $G_{g,n}$ as the category of strata of the nonsingular DM-stack $\overline{\mathcal{M}}_{g,n}$ induced by the normal crossing divisor $\Delta = \Delta_{\mathrm{irr}} \cup \bigcup_{i,S} \Delta_{i,S}$.

3. Compactified Jacobians and Universal Brill–Noether Classes

In this chapter we introduce the basic objects of study in this paper, compactified universal Jacobians, and extensions of universal Brill–Noether classes by means of Thom–Porteous formula. We also recall the results on the stability space of compactified universal Jacobians that we will need later.

3.a. **The universal stability space.** Here we recall the definition and first results on the stability space of a single curve and on the universal stability space $V_{g,n}^d$ from [KP19].

Definition 3.1. For a fixed graph G, we define the space of polarizations

$$V_{\mathrm{stab}}^d(G) := \left\{ \phi \in \mathbb{R}^{V(G)} : \sum_{v \in V(G)} \phi(v) = d \right\} \subset \mathbb{R}^{V(G)}.$$

For $V \subseteq V(G)$, we write $\phi(V)$ for $\sum_{v\in V} \phi(v)$.

Every morphism $f\colon G \to G'$ of graphs induces a morphism $f_*\colon V_{\mathrm{stab}}^d(G) \to V_{\mathrm{stab}}^d(G')$ by setting

$$f_*\phi(v') = \sum_{f(v)=v'} \phi(v) \tag{3.2}$$

and we define the space of universal polarizations as the limit (or inverse limit)

$$V_{g,n}^d := \varprojlim_{G \in G_{g,n}} V_{\mathrm{stab}}^d(G),$$

i.e. as the space of assignments $\big(\phi(G) \in V_{\mathrm{stab}}^d(G)\colon\ G \in G_{g,n}\big)$ that are compatible with all graph morphisms.

We now present a simple description of the universal stability space $V_{g,n}^d$ that follows from [KP19, Corollary 4.3]. The result requires that we introduce some notation for graphs of "vine curves".

Definition 3.3. A *vine curve triple* (i, t, S) consists of two natural numbers i, t and a subset $S \subseteq [n]$, such that $0 \le i \le g$, $1 \le t$, $i + t \le g + 1$, and such that if $(i, t) = (0, 1)$ then $|S| \ge 2$, if $(i, t) = (0, 2)$ then $|S| \ge 1$, if $(i, t) = (g, 1)$ then $|S^c| \ge 2$ and if $(i, t) = (g - 1, 2)$ then $|S^c| \ge 1$.

A *vine curve* is a stable graph $G(i, t, S)$ associated to a vine curve triple, which consists of two vertices of genus i and $g - i$ respectively connected by t edges, and with marking S on the first vertex and S^c on the second vertex. We will always assume that S contains the first marked point.

The stability space $V_{\mathrm{stab}}^d(G(i, t, S))$ is an affine subspace of $\mathbb{R}^2$. We can parameterize it by means one variable $x_{i,t,S}$ by taking the inverse image under the projection onto the first factor. That means, we describe

$$V_{\mathrm{stab}}^d(G(i, t, S)) = \{(x_{i,t,S}, d - x_{i,t,S}) : x_{i,t,S} \in \mathbb{R}\} \subset \mathbb{R}^2.$$

We now introduce the stability space of "vine curves" using the previous definition.

Definition 3.4. We let

$$T^d_{g,n} := \prod_{\substack{(i,t,S) \\ \text{a vine curve triple}}} V^d_{\text{stab}}(G(i,t,S)).$$

Then we define:

(1) The vector space $C^d_{g,n}$ as the quotient of $T^d_{g,n}$ obtained as the product of all vine curve triples of the form $V^d_{\text{stab}}(G(i,1,S))$.
(2) The vector space $D^d_{g,n}$ is the quotient of $T^d_{g,n}$ obtained as the product of all $V^d_{\text{stab}}(G(0,2,\{j\})$ for $j = 1\dots,n$.

Throughout we will use the coordinates $x_{i,t,S}$ introduced in the end of Definition 3.3 on the spaces $T^d_{g,n}$ and on its quotients $C^d_{g,n}$ and $D^d_{g,n}$.

There are natural restriction affine linear maps:

$$\tau_d \colon V^d_{g,n} \to T^d_{g,n}, \quad \rho_d \colon V^d_{g,n} \to C^d_{g,n} \times D^d_{g,n}$$

One of the main results of [KP19, Section 3] is that the universal stability space embeds into the "vine curves" stability space, and that ρ_d is an isomorphism.

Proposition 3.5. *(*[KP19, Lemma 3.8, Corollary 3.4]*) The affine linear map τ_d is injective. The vector space homomorphism ρ_0 is an isomorphism. Each morphism ρ_d is an isomorphism of affine spaces.*

3.b. **The stability hyperplanes.** We will later see in Section 3.c that for every universal stability condition $\phi \in V^d_{g,n}$ there exists a compactified universal Jacobian parameterizing ϕ-stable (rank 1, torsion free) sheaves on every (flat) family of n-pointed stable curves of genus g. Here we combinatorially introduce the degenerate locus of $V^d_{g,n}$, which will later be seen to be the locus of ϕ's such that there exist strictly semistable sheaves on some stable curves. We will introduce the degenerate locus as a union of hyperplanes (which one could think of as a finite, non-centered, toric hyperplane arrangement). This explicit description is taken from [KP19, Section 5].

Definition 3.6. We say that a polarization $\phi \in V^d_{\text{stab}}(G)$ is *degenerate* if for some subset $\varnothing \subsetneq V_0 \subsetneq V(G)$ the quantity

$$\frac{|E(V_0, V_0^c)|}{2} + \sum_{v \in V_0} \phi(v) \tag{3.7}$$

is an integer.

We say that a universal stability condition $\phi \in V^d_{g,n}$ is *degenerate* if for some $G \in G_{g,n}$, the G-component $\phi(G)$ is degenerate in $V^d_{\text{stab}}(G)$.

The degenerate locus is a locally finite union of affine hyperplanes, and we will soon describe these hyperplane explicitly. Let us start with a simple example.

Example 3.8. (Vine curves). If G is a vine curve, after identifying $V^d_{\mathrm{stab}}(G) = \mathbb{R}$ by projecting onto the first factor (as done in the end of Definition 3.3), we have that the degenerate locus is a locally finite collection of points that only depends on the parity of the number of nodes t. If t is even, the degenerate locus corresponds to the $\mathbb{Z} \subset \mathbb{R}$. If t is odd, the degenerate locus corresponds to the $\frac{1}{2} + \mathbb{Z} \subset \mathbb{R}$.

We now give an explicit description of the degenerate locus in $V^d_{g,n}$, based on [KP19, Section 5]. By Proposition 3.5, we have that $V^d_{g,n} \subset T^d_{g,n}$, where the latter is the stability space of vine curves (one for each topological type), with coordinates $x_{i,t,S}$ for each vine curve triple (i, t, S) (see Definition 3.3).

For each vine curve triple (i, t, S) and integer k, define the (translate of the coordinate) hyperplane

$$T^d_{g,n} \supset H(i,t,S;k) := \begin{cases} \{x_{i,t,S} = k\} & \text{for even } t, \\ \{x_{i,t,S} = \frac{1}{2} + k\} & \text{for odd } t. \end{cases} \tag{3.9}$$

One main result of [KP19, Section 5] is that the degenerate locus in the universal stability space is the pull-back of translates of coordinate hyperplanes in the stability space of vine curves. More precisely:

Proposition 3.10. *([KP19, Lemma 5.8]) The degenerate locus in $V^d_{g,n}$ is a union of hyperplanes. Each hyperplane is the inverse image via the affine linear embedding $\tau_d \colon V^d_{g,n} \subset T^d_{g,n}$ of a hyperplane of the form $H(i,t,S;k)$.*

This description hides the difficulty that the embedding τ_d has, in general, a very high codimension.

A more explicit description of the degenerate locus can be obtained via the isomorphism $V^d_{g,n} \cong C^d_{g,n} \times D^d_{g,n}$. When expressing the hyperplanes of (3.9) in terms of the coordinates $x_{i,1,S}$ and the coordinates $x_j := x_{0,2,\{j\}}$, by [KP19, Theorem 2] we have[1]

$$x_{i,t,S} = \frac{2g-2i-t}{2g-2} \cdot \sum_{j \in S} x_j + \frac{2i-2+t}{2g-2} \cdot \left(d - \sum_{j \notin S} x_j \right) \tag{3.11}$$

whenever $t \geq 2$. Therefore, the stability hyperplanes take the following form

$$H(i,1,S;k) = \left\{ x_{i,1,S} = k + \frac{1}{2} \right\} \tag{3.12}$$

for all vine curve triples (Definition 3.3) of the form $(i, 1, S)$ (the boundary divisors in $\overline{\mathcal{M}}_{g,n}$ that generically parameterize curves with 2 components)

$$H(i,t,S;k) = \left\{ \frac{2g-2i-t}{2g-2} \cdot \sum_{j \in S} x_j + \frac{2i-2+t}{2g-2} \cdot \left(d - \sum_{j \notin S} x_j \right) = k + \frac{t}{2} \right\} \tag{3.13}$$

[1]Note that the formula in loc.cit is translated by the coordinates of a "degree-d canonical stability condition" – a choice of an origin in $V^d_{g,n}$ that we do not discuss here.

for all vine curve triples (i, t, S) with $t \geq 2$.

Note that the degenerate locus parameterized by the hyperplanes in (3.12) and (3.13) may come with multiplicities. In other words, there exist different $(i_1, t_1, S_1; k_1), (i_2, t_2, S_2; k_2)$ such that $H(i_1, t_1, S_1; k_1) = H(i_2, t_2, S_2; k_2)$. We will now analyse these hyperplanes and study when they may coincide.

It is immediate to observe that a necessary condition for two hyerplanes of this form to coincide, is that their corresponding subset of marked points must also coincide:

Proposition 3.14. *If any two hyperplanes $H(i_1, t_1, S_1; k_1)$ and $H(i_2, t_2, S_2; k_2)$ coincide, then $S_1 = S_2$.*

Proof. Straightforward. □

First we deal with the hyperplanes of (3.12), occurring on compact type vine curves (or divisorial vine curves). Those are all simple:

Proposition 3.15. *The hyperplanes in (3.12) are pairwise distinct and each of them is distinct from any of the hyperplanes in (3.13).*

Proof. Straightforward. □

The next proposition is about hyperplanes of the form (3.13) with $S \neq [n]$. As we shall discuss in Section 5.c, a stability hyperplane of this type witnesses a change of stability on loci of vine curves that are *disjoint*.

Proposition 3.16. *If $S \neq [n]$ and $(i_1, t_1; k_1) \neq (i_2, t_2; k_2)$ are such that $H(i_1, t_1, S; k_1)$ and $H(i_2, t_2, S; k_2)$ are equal, then $2i_1 + t_1 = 2i_2 + t_2$.*

Proof. Straightforward. □

The most interesting vine curves from the point of view of the stability decomposition are those with $S = [n]$. Over those vine curves it can occur that two stability hyperplanes of the form (3.13) coincide. For example, if $d = 0$, by fixing $\sum_{j \in [n]} x_j = g - 1$ one sees that all hyperplanes of the form $H(i, t, [n]; k)$ with $i + \lceil t/2 \rceil + k = g$ coincide (note that this is a finite collection, because of the constraints $i + t \leq g + 1$, $i \geq 0$ and $t \geq 2$).

3.c. **Compactified Jacobians, universal and semistable family.** Here we define, for every nondegenerate $\phi \in V^d_{g,n}$, a fine compactified universal Jacobian $\overline{\mathcal{J}}^d_{g,n}(\phi)$, parameterizing ϕ-stable sheaves. The construction is taken from [KP19, Section 4], in the language of pseudodivisors from [AP20, Section 4]. Each fine compactified Jacobian will come with a normal crossing stratification category (an abstract definition of this notion will be given and discussed in the next section).

We also describe (Theorem 3.29) a quasistable modification of the universal curve $\overline{\mathcal{C}}_{g,n}(\phi) \to \overline{\mathcal{J}}^d_{g,n}(\phi)$ as a certain $(n+1)$-universal Jacobian $\overline{\mathcal{J}}^d_{g,n+1}(\alpha(\phi))$.

Definition 3.17. For $\phi \in V^d_{\mathrm{stab}}(G)$ we say that a pseudodivisor (E, D) is *ϕ-semistable* if

$$\phi(V_0) - \deg_{V_0}(D) + \frac{|E(V_0, V_0^c)|}{2} \geq 0 \tag{3.18}$$

for every $V_0 \subseteq V(G^E)$. We say that (E, D) is *ϕ-stable* if the inequality above is strict for every V_0 such that $V_0 \neq V(G^E)$ and V_0 is not contained in the set of exceptional vertices. Given $v_0 \in V(G)$, we say that (E, D) is *(ϕ, v_0)-quasistable* if the inequality is strict for every V_0 such that $V_0 \neq V(G^E)$ and $v_0 \in V_0$.

As stipulated in Section 2.c, when $E = \varnothing$, we will simply write D for $(\varnothing, D)$.

Remark 3.19. By [AP20, Proposition 4.6] if a pseudodivisor (E, D) on G is (ϕ, v_0)-quasistable for some (ϕ, v_0), then $E \subseteq E(G)$ does not disconnect G.

Remark 3.20. We have introduced the degenerate locus of $V^d_{\mathrm{stab}}(G(X))$ and of $V^d_{g,n}$ in Definition (3.6). We claim that, in both cases, an element ϕ is nondegenerate if and only if all semistable pseudodivisors are stable. The "only if" is immediate. The other implication is proved in [KP19, Section 5].

We now define stability for rank 1 torsion free sheaves on curves.

Definition 3.21. ([KP19, Definition 4.2]) Let C be a nodal curve with dual graph $G(C)$ and let $\phi \in V^d_{\mathrm{stab}}(G(C))$. A rank 1 torsion-free sheaf F of degree d on C is *ϕ-(semi)stable* if its multidegree $\underline{\deg}(F)$ is a ϕ-(semi)stable pseudodivisor.

If $P \in C^{\mathrm{sm}}$ is a nonsingular point of C in the component C_{v_0}, we say that F is *(ϕ, P)-quasistable* if $\underline{\deg}(F)$ is (ϕ, v_0)-quasistable.

If $C' \to C$ is a semistable modification of C and C' is a positively admissible line bundle on C', we say that F' is ϕ-(semi)stable or (ϕ, P)-quasistable if so is $f_*(F')$.

For $\phi \in V^d_{\mathrm{stab}}(G(C))$ and $P \in C$, we define $\overline{\mathcal{J}}^d_{\phi,P}(C)$ to be the subscheme of $\mathrm{Simp}^d(C)$ parameterizing (ϕ, P)-quasistable sheaves.

Note that if F is a rank 1 torsion free sheaf on C then (1) if F is (ϕ, P)-quasistable then it is simple, and (2) the sheaf F is simple if and only if its multidegree (E, D) has the property that $E \subseteq G(C)$ is nondisconnecting.

Remark 3.22. Let $\phi \in V^d_{\mathrm{stab}}(G(C))$ and $p \in C^{\mathrm{sm}}$ be as above. Let $\phi' \in V^d_{\mathrm{stab}}(G(C))$ be a small perturbation of ϕ obtained by subtracting a small $\epsilon > 0$ to ϕ on the vertex of $G(C)$ containing P, and by subtracting a small positive amount on all other components (so that $\sum \phi'(v) = \sum \phi(v) = d$).

Then (ϕ, P)-quasistability coincides with ϕ'-stability which in turn coincides with ϕ'-semistability, We are now ready to introduce the notion of *universal* polarizations and compactified Jacobians. Each universal polarization will give rise to a fine compactified Jacobian, and to a stratification category. Recall that, for any $1 \leq i \leq n$, we denote by $\sigma_i \colon \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{C}}_{g,n}$ the i-th smooth section.

Definition 3.23. Let $\phi \in V^d_{g,n}$ be a universal polarization.

We define $\mathfrak{C}_{g,n}(\phi)$ to be the category whose objects are $(G, (E_G, D_G))$ where G is an object of $G_{g,n}$ and (E_G, D_G) is a ϕ-semistable pseudodivisor on G. A morphism $(G, (\varnothing, D_G)) \to (G', (\varnothing, D_{G'}))$ in $\mathfrak{C}_{g,n}(\phi)$ is a morphism $f \in \mathrm{Mor}_{G_{g,n}}(G, G')$ such that the induced homomorphism $f_* \colon \mathrm{Div}(G) \to \mathrm{Div}(G')$ on divisors satisfies $f_*(D) = D'$. We refer to [AP20, Section 2.1] for the notion of a morphism $(G, (E_G, D_G)) \to (G', (E_{G'}, D_{G'}))$ when $E_G, E_{G'}$ are nonempty.

Similarly, we define $\mathfrak{C}_{g,n}(\phi, \sigma_i)$ to be the category whose objects are $(G, (E_G, D_G))$ and (E_G, D_G) is (ϕ, σ_i)-quasistable. (By abuse of notation, σ_i gives the choice of the element $\mathrm{leg}_G(i) \in V(G)$ for each stable graph G).

We say that a family of rank 1 torsion-free simple sheaves of degree d on a family of stable curves is *ϕ-(semi)stable* or (ϕ, σ_i)-quasistable if that property holds on all geometric fibers. We define $\overline{\mathcal{J}}^d_{g,n}(\phi)$ to be the moduli stack parameterizing ϕ-semistable sheaves on families of stable curves. We define $\overline{\mathcal{J}}^d_{g,n}(\phi, \sigma_i)$ to be the moduli stack parameterizing (ϕ, σ_i)-quasistable sheaves on families of stable curves.

Notation 3.24. If $\phi \in V^d_{g,n}$ is nondegenerate then, by Remark 3.20 we have that all semistable sheaves are stable. It follows that for every $1 \leq i \leq n$ we have the equalities $\mathfrak{C}_{g,n}(\phi) = \mathfrak{C}_{g,n}(\phi, \sigma_i)$ and $\overline{\mathcal{J}}^d_{g,n}(\phi) = \overline{\mathcal{J}}^d_{g,n}(\phi, \sigma_i)$, where $\sigma_i \colon \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{C}}_{g,n}$ is the i-th section.

Remark 3.25. For $\phi \in V^d_{g,n}$ degenerate and for all $1 \leq i \leq n$ we can describe $\mathfrak{C}_{g,n}(\phi, \sigma_i)$ (resp. $\overline{\mathcal{J}}^d_{g,n}(\phi, \sigma_i)$) as $\mathfrak{C}_{g,n}(\phi)$ (resp. $\overline{\mathcal{J}}^d_{g,n}(\phi'_i)$) for some nondegenerate perturbation ϕ'_i of ϕ. (As done in Remark 3.22 for a single curve).

In order to achieve this, we let ϕ'_i by subtracting from ϕ an arbitrarily small $\epsilon > 0$ on each curve on its irreducible component containing the section σ_i, and by subtracting a small quantity on all other components (so for all curves, the sum over all irreducible components of the values of ϕ and of ϕ'_i coincide). The fact that such ϕ'_i can be constructed in a way that is compatible with graph morphisms follows by using Proposition 3.5.

The following guarantees the existence of universal moduli spaces.

Theorem 3.26. *(*[KP19, Corollary 4.4] *and* [Est01]*/*[Mel19]*) For all $\phi \in V^d_{g,n}$ and $1 \leq i \leq n$ the stack $\overline{\mathcal{J}}^d_{g,n}(\phi, \sigma_i)$ is a nonsingular Deligne–Mumford stack, and the forgetful morphism $\overline{\mathcal{J}}^d_{g,n}(\phi, \sigma_i) \to \overline{\mathcal{M}}_{g,n}$ is representable, proper and flat.*

The moduli stacks of Theorem 3.26 are called *fine compactified universal Jacobians.*

As observed in [KP19, Remark 4.6], the fine compactified (universal) Jacobians produced by this construction are the same as those defined by Esteves and Melo [Est01, Mel19].

By virtue of its universal property, the universal family $\pi\colon \overline{\mathcal{C}}_{g,n}(\phi) \to \overline{\mathcal{J}}^d_{g,n}(\phi)$ carries some tautological (or Poincaré) sheaves $F_{\text{tau}}(\phi)$. These are of fiberwise total degree d and ϕ-stable. They are not unique, but the difference of any two of them is the pullback of a line bundle from $\overline{\mathcal{J}}^d_{g,n}(\phi)$. One way to make a definite choice of a tautological sheaf is to assume that it is trivial along a given smooth section.

Note that, as described in [CMKV15], the total space $\overline{\mathcal{C}}_{g,n}(\phi)$ is singular. A natural desingularization of $\overline{\mathcal{C}}_{g,n}(\phi)$, carrying a tautological line bundle $L_{\text{tau}}(\phi)$, was provided by Esteves–Pacini in [EP16] by using a semistable modification of the universal family. Here we will give an alternative description of it using a compactified universal Jacobian with one extra point.

Remark 3.27. We observe that there is a natural map $\alpha\colon V^d_{g,n} \to V^d_{g,n+1}$, with image in the degenerate locus, defined as follows.

If G is the stable graph obtained as the stabilization of the $(n+1)$-pointed graph G' after the $n+1$ marking is removed, then there is a natural bijection between the vertices of G' and those of G, except possibly for 1 extra genus 0 vertex of G'. Then define $\alpha(\phi)$ as the assignment on G' that is defined by this bijection and that is 0 on the extra genus 0 vertex of G' (when that exists). The extra genus 0 vertex could be a tail (when it is connected to the complement by 1 edge) or a bridge (when it is connected to the complement by 2 edges). The fact that ϕ is compatible for graph morphisms implies the same property for $\alpha(\phi)$.

Notation 3.28. We will slightly abuse the notation and, for $\phi \in V^d_{g,n}$, we will simply write $\phi \in V^d_{g,n+1}$ in place of $\alpha(\phi) \in V^d_{g,n+1}$.

We now show that a quasistable modification of the universal curve can be described as the morphism $\pi'\colon \overline{\mathcal{J}}^d_{g,n+1}(\phi,\sigma_i) \to \overline{\mathcal{J}}^d_{g,n}(\phi,\sigma_i)$ that forgets the last point and stabilizes, thus mapping each $(C', p_1, \dots, p_{n+1}, F)$ to $(C, p_1, \dots, p_n, f_*F)$, where $f\colon C' \to C$ is the stabilization of $(C, p_1, \dots, p_n)$. In order to do that, for each fixed $i = 1, \dots, n$, we define a morphism $\psi\colon \overline{\mathcal{J}}^d_{g,n+1}(\phi,\sigma_i) \to \overline{\mathcal{C}}_{g,n}(\phi,\sigma_i)$ by

$$(C', p_1, \dots, p_{n+1}, L) \mapsto (C, p_1, \dots, p_n, f_*L, f(p_{n+1}))$$

where $f\colon C' \to C$ is the stabilization of the curve $(C', p_1, \dots, p_n)$. Then we show that ψ is the stabilization of π'.

Theorem 3.29. *For each $1 \leq i \leq n$, the forgetful morphism ψ defined above is the unique positively admissible quasistable modification of the universal* *curve over $\overline{\mathcal{J}}^d_{g,n}$. A tautological line bundle on $\overline{\mathcal{J}}^d_{g,n+1}(\phi,\sigma_i)$ is*

$$\mathcal{L}_{\mathrm{tau}} := \sigma_{n+1}^*(\mathcal{F}'_{\mathrm{tau}}) \otimes \sigma_1^*(\mathcal{F}'^{-1}_{\mathrm{tau}}), \tag{3.30}$$

where $\mathcal{F}'_{\mathrm{tau}} = \mathcal{F}'_{\mathrm{tau}}(\phi)$ is a tautological sheaf on the universal curve

$$\pi\colon \overline{\mathcal{C}}_{g,n+1}(\phi,\sigma_i) \to \overline{\mathcal{J}}^d_{g,n+1}(\phi,\sigma_i).$$

Proof. We apply Proposition 2.3 to show that the morphism ψ is a quasistable modification of the universal curve.

Let us begin by proving that ψ is a quasistable modification of $\mathcal{C}_{g,n}(\phi,\sigma_i)$. Firstly, if $C = C'$, then f is an isomorphism, so ψ is an isomorphism locally around $(C', p_1, \dots, p_{n+1}, L)$. If $C \neq C'$, then we have two cases. Either p_{n+1} belongs in a rational tail and, in this case, $L = f^*f_*(L)$ so ψ is again an isomorphism locally around $(C', p_1, \dots, p_{n+1}, L)$. Or, p_{n+1} is in a bridge $E \subset C'$ such that no other marked points are in E. We will now focus on this case.

If $\deg_E(L) = 0$, then by Proposition 2.3, we have that $L = f^*f_*(L)$ and again the map ψ is an isomorphism locally around $(C', p_1, \dots, p_{n+1}, L)$. We are left with the case where $\deg_E(L) = 1$. In this case, we have that $f_*(L)$ is not locally free around $f(p_{n+1})$, which is a node. More so, we have that $\psi^{-1}(C, p_1, \dots, p_n, f_*(L), f(p_{n+1}))$ is isomorphic to $\mathbb{P}^1$. Indeed, every L' obtained from gluing $L|_{E^c}$ and $\mathcal{O}_E(1)$ will have the property that $\psi(C', p_1, \dots, p_{n+1}, L') = (C, p_1, \dots, p_n, f_*(L), f(p_{n+1}))$. The possible gluings are paremeterized by a $\mathbb{P}^1$, and we are done.

Secondly we observe that $\deg_E(L_{\mathrm{tau}}) = 1$ for each exceptional component E contracted by ψ. In order to show that, it suffices to construct a nonconstant map $\delta\colon \mathbb{P}^1 \to E$ such that $\delta^*(L_{\mathrm{tau}}) = \mathcal{O}(1)$. Let $(C, p_1, \dots, p_n, L, p)$ correspond to the point in the universal curve that is contracted by E. That means that p is a node of C and that L fails to be locally free at p. We construct a family $X/\mathbb{P}^1$ by gluing two sections on two families $X_1/\mathbb{P}^1$ and $X_2/\mathbb{P}^1$. The family X_1 is the trivial family $C^{\nu_p} \times \mathbb{P}^1$ (where ν_p denotes the normalization at p) carrying the n trivial sections $p_1, \dots, p_n$ and the gluing sections are the points q_1, q_2 such that $\nu_p(q_i) = p$. The family X_2 is the blowup of the trivial family $\mathbb{P}^1 \times \mathbb{P}^1$ at $[0:1] \times [0:1]$ and $[1:0] \times [1:0]$ with a further section p_{n+1} defined as the inverse image of a constant section (different from $[0:1]$ and $[1:0]$) and the two gluing sections are the strict transforms of the sections $[0:1]$ and $[1:0]$. Then we choose any line bundle F on X with the property that $F|_{X_1} = L$ and $F|_{X_2} = \mathcal{O}(\widetilde{\Delta})$, for $\widetilde{\Delta}$ the strict transform of the diagonal in X_2. Then F is (ϕ,σ_i)-quasistable, and so the datum of (X, F) defines a morphism δ. By construction, we have that $\delta^*(L_{\mathrm{tau}}) = \sigma_{n+1}^*(F) = \sigma_{n+1}^*(\mathcal{O}(\widetilde{\Delta}))$, which equals $\mathcal{O}(1)$ because σ_{n+1}^* intersects $\widetilde{\Delta}$ at 1 reduced point.

Then we prove that the direct image $\psi_* L_{\mathrm{tau}}$ of the line bundle defined in (3.30) equals $F_{\mathrm{tau}} \otimes \pi^*(M)$ for some line bundle M. By the previous part combined with Proposition 2.3, we conclude that $\psi_* L_{\mathrm{tau}}$ is rank 1 and torsion-free. By [KP19, Appendix 7], it is enough to prove that this equality occurs on an open subset U of $\overline{\mathcal{C}}_{g,n}(\phi)$ whose complement has codimension at least 2. It is easy to show equality over the open set U that is the universal curve over the line bundle locus in $\overline{\mathcal{J}}^d_{g,n}(\phi)$ (this follows because $\psi_{|\psi^{-1}(U)}$ is an isomorphism over U, and because the restriction of F'_{tau} to the open set $\pi^{-1}(\psi^{-1}(U)) \subset \overline{\mathcal{C}}_{g,n+1}(\phi, \sigma_i)$ is a line bundle). This concludes the proof that ψ is a positively admissible quasistable modification.

Uniqueness follows from Corollary 2.4. □

3.d. **Brill–Noether classes.** To the data of a flat family $\pi\colon \mathcal{C} \to S$ of nodal curves of arithmetic genus g over a nonsingular scheme S and a rank 1 torsion-free $\mathcal{F}$ on $\mathcal{C}$ of fiberwise degree d, we can associate the Brill–Noether (or Thom–Porteous) class

$$\mathsf{w}_d(\mathcal{C}/S, \mathcal{F}) := c_{g-d}(-R^\bullet \pi_* \mathcal{F}). \tag{3.31}$$

This class is supported on the subscheme

$$\mathsf{W}_d(\mathcal{C}/S, \mathcal{F}) = \{s \in S : \; h^0(\mathcal{C}_s, \mathcal{F}_s) > 0\} \subset S$$

and when the latter is of the expected codimension $g-d$, it coincides with its fundamental class (with a suitably defined scheme structure– see [Ful98, Chapter 14]). If $h^0(\mathcal{C}_s, \mathcal{F}_s) = 0$ for all $s \in S$, then the complex $-R^\bullet\pi_*\mathcal{F} = R^1\pi_*\mathcal{F}$ is a vector bundle of rank $g-1-d$, so the Chern class $\mathsf{w}_d(\mathcal{C}/S, \mathcal{F})$ equals zero.

Here are a couple of further basic remarks on these classes.

Remark 3.32. If I is a line bundle on S, then we have

$$\mathsf{w}_d(\mathcal{C}/S, \mathcal{F}) = \mathsf{w}_d(\mathcal{C}/S, \mathcal{F} \otimes \pi^* I).$$

Indeed, by [Ful98, Example 3.2.2] we have

$$c_j(-R^\bullet \pi_* \mathcal{F} \otimes I) = \sum_{i=0}^{j} \binom{g-d-1-i}{j-i} c_i(-R^\bullet \pi_* \mathcal{F}) \cdot c_1(I)^{j-i} \tag{3.33}$$

for all $j \geq 0$. The result follows because, for $j = g-d$, the binomial coefficient vanishes unless when $i = j$.

Remark 3.34. Let $f\colon \mathcal{C}' \to \mathcal{C}$ be a semistable modification of the family of nodal curves $\pi\colon \mathcal{C} \to S$, and let $\mathcal{L}$ be a positively admissible line bundle on $\mathcal{C}$ (see Section 2.c). By Proposition 2.3, we deduce

$$R^\bullet(\pi \circ f)_* \mathcal{L} = R^\bullet \pi_*(f_* \mathcal{L}). \tag{3.35}$$

Conversely, if $\mathcal{F}$ is a rank 1 torsion free simple sheaf on a family of stable curves $\mathcal{C}/S$, there exists a quasistable modification $\mathcal{C}'$ of $\mathcal{C}$ and a line bundle $\mathcal{L}$ on $\mathcal{C}$ such that $R^\bullet f_*(\mathcal{L}) = f_*\mathcal{L} = \mathcal{F}$ and thus (3.35) occurs.

The same construction and remarks apply to the case of the semistable modification $\pi\colon \overline{\mathcal{C}}'_{g,n}(\phi) \to \overline{\mathcal{J}}^d_{g,n}(\phi)$ of the universal family, and its tautological line bundle $\mathcal{L} = \mathcal{L}_{\mathrm{tau}}$ (see [PRvZ20]). We will denote by $\mathsf{w}_d(\phi)$ the corresponding universal class in $A^d(\overline{\mathcal{J}}^d_{g,n}(\phi))$ and with $\mathsf{W}_d(\phi)$ the subscheme over which it is supported.

Remark 3.36. When restricted to smooth curves, the scheme $\mathsf{W}_d(\phi)_{|\mathcal{J}^d_{g,n}}$ is reduced, irreducible, and of relative codimension $g-d$ (it is the image of the d-th symmetric product via the Abel map). The closure of $\mathsf{W}_d(\phi)_{|\mathcal{J}^d_{g,n}}$ in $\overline{\mathcal{J}}^d_{g,n}(\phi)$ is contained in $\mathsf{W}_d(\phi)$ and when the two coincide, we have $\mathsf{w}_d(\phi) = [\mathsf{W}_d(\phi)]$.

Remark 3.37. In general, the scheme $\mathsf{W}_d(\phi)$ fails to be irreducible and of the expected dimension. For example, if ϕ is a stability condition such that the line bundles of bidegree (d_1, d_2) are ϕ-stable on curves in the boundary divisor $\Delta_{i,S}$, and either $d_1 > i$ or $d_2 > g-i$, then $\mathsf{W}_d(\phi)$ contains the pullback of $\Delta_{i,S}$ in $\overline{\mathcal{J}}^d_{g,n}(\phi)$.

We will discuss more on this matter in Proposition 3.38 and Remark 4.18.

We conclude this section by providing sufficient conditions for the Brill–Noether class 3.31 defined by the Thom–Porteous Formula to coincide with the class of the Brill–Noether locus. Some parts of the proof of the next two propositions will require to employ the fact that $\mathfrak{C}_{g,n}(\phi)$ is a stratification of $\overline{\mathcal{J}}^d_{g,n}(\phi)$, which we will discuss in the next section. For this reason, we postpone the proof of the next result to Section 4.b.1.

As in Section 3.b, we fix coordinates for $V^d_{g,n} \cong C^d_{g,n} \times D^d_{g,n}$, and let $V^d_{g,n} \ni \phi = ((x_{i,1,S})_{(i,S)}, (x_1, \ldots, x_n))$ where $x_j = x_{0,2,j}$ for each $1 \leq j \leq n$.

Proposition 3.38. *We have that $\mathsf{W}_d(\phi)$ is the closure of $\mathsf{W}_d(\phi)_{|\mathcal{J}^d_{g,n}}$ (in particular it is reduced, irreducible and of the expected codimension, and $\mathsf{w}_d(\phi) = \mathsf{W}_d(\phi)$) if and only if $\phi = ((x_{i,1,S})_{(i,S)}, (x_1, \ldots, x_n))$ is as follows*

(1) *If $d = g-1$, for $i - 3/2 < x_{i,1,S} < i + 1/2$ for all (i,S).*
(2) *If $d = g-2$, for $i - 3/2 < x_{i,1,S} < i - 1/2$ for all $(i \geq 1, S)$ and $-3/2 < x_{0,1,S} < 1/2$ for all S, and*
$$i - 2 < \frac{2g-2i-t}{2g-2} \cdot \sum_{j \in S} x_j + \left(d - \sum_{j \notin S} x_j\right) \cdot \frac{2i-2+t}{2g-2} < i+1$$
for all vine curve triples (i,t,S) with $t \geq 2$.
(3) *Never if $0 < d \leq g-3$.*
(4) *If $d < 0$, for $d - 1/2 < x_{i,1,S} < 1/2$ for all (i,S), and the coordinates $x_1, \ldots, x_n$ satisfy*
$$d - 1 < \frac{2g-2i-t}{2g-2} \cdot \sum_{j \in S} x_j + \left(d - \sum_{j \notin S} x_j\right) \cdot \frac{2i-2+t}{2g-2} < 1 \tag{3.39}$$
for all vine curve triples (i,t,S) with $t \geq 2$. (3.39).
(5) *If $d = 0$, for $-1/2 < x_{i,1,S} < 1/2$ for all $(i \geq 1, S)$, for $-3/2 < x_{0,1,S} < 1/2$ for all S, and when the coordinates $x_1, \ldots, x_n$ satisfy* (3.39) *for all vine curve triples (i,t,S) with $t \geq 2$.*

We now explicitly define a stability condition that, in each of the cases (1),(2),(4),(5) listed above, belongs to the ranges that we have identified for $\mathsf{W}_d(\phi)$ to equal the closure of its restriction to the open part. (This shows, in particular, that these ranges are not empty).

Definition 3.40. For $G \in G_{g,n}$, define the stabilized-canonical divisor K^s_G to equal zero at every vertex contained in some rational tail[2], and for every other v to equal $K^s(v) = 2g(v) - 2 + \mathrm{val}'(v)$, where $\mathrm{val}'(v)$ is the number of edges at v (counting each loop twice), except the edges that are contained in some rational tail.

Then define the stabilized canonical element $\phi^d_{\mathrm{scan}}(G) = \frac{d}{2g-2} \cdot K^s_G \in V^d_{g,n}$.

Note that the above is different from the canonical stability $\phi^d_{\mathrm{can}} \in V^d_{g,n}$ chosen as the origin in [KP17].

3.d.1. *Pull-back via Abel–Jacobi sections.* Fix integers $\mathbf{d} = (k; d_1, \ldots, d_n)$ such that $d = d_1 + \ldots + d_n$ and integers $\mathbf{f} = (f_{i,S})_{i,S}$ for every boundary divisor $G(g-i, 1, S) \in G_{g,n}$. Define the universal line bundle

$$\mathcal{L} = \mathcal{L}_{\mathbf{d},\mathbf{f}} = \omega^k_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}} \left(\sum_{j=1}^n d_j x_j + \sum_{i,S} f_{i,S^c} \cdot C_{i,S^c} \right), \tag{3.41}$$

where $C_{i,S^c} \subset \overline{\mathcal{C}}_{g,n}$ is the component[3] over the boundary divisor $\Delta_{g-i,S} = \overline{\mathcal{M}}_{G(g-i,1,S)} \subset \overline{\mathcal{M}}_{g,n}$ that contains the sections in S^c. Then define $\phi = \phi_{\mathbf{d},\mathbf{f}} \in V^d_{g,n}$ to be the multidegree of $\mathcal{L}$.

If $\phi^+ = \phi^+_{\mathbf{d},\mathbf{f}}$ is a nondegenerate small perturbation of ϕ, then we have that the universal line bundle $\mathcal{L}$ is ϕ^+-stable, and it defines a (Abel–Jacobi) section $\sigma = \sigma^+_{\mathbf{d},\mathbf{f}} \colon \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{J}}^d_{g,n}(\phi^+)$.

Remark 3.42. Assume $d < 0$, $k = 0$, and $\mathbf{d}$ satisfies $d_i \leq 1$ for all i, and at most one of the d_i's equals 1. Assume that $\mathbf{f}$ satisfies $d \leq f_{i,S^c} + \sum_{j \in S} d_j \leq 0$ for all (i, S). Then $\phi_{\mathbf{d},\mathbf{f}}$ satisfies the conditions of Item (4) in Proposition 3.38.

By Proposition 3.38 the pullback

$$\sigma^*_{\mathbf{d},\mathbf{f}}(\mathsf{w}_d) = 0 \quad \in \mathrm{A}^d(\overline{\mathcal{M}}_{g,n}), \tag{3.43}$$

gives a relation. The LHS of (3.43) can be explicitly written as a linear combination of standard generators of the tautological ring of $\overline{\mathcal{M}}_{g,n}$ by means of [PRvZ20, Theorem 1] (see also Corollary 3.7 and Equation 3.9 of *loc.cit*).

[2] a rational tail is a complete subgraph whose genus is 0 and that is connected to its complement by exactly 1 edge

[3] these unnatural conventions will simplify the formulas in Remark 7.39 and Example 7.40

4. Normal crossing stratification categories and blowups

In this section we define the axioms needed for a category of (resolved) strata of a space stratified by normal crossing divisors that are not necessarily simple normal crossing. We use this formalism to write some intersection theoretic formulas (the excess intersection formula and the GRR formula for the total Chern class) that we will use to derive our main result Theorem 7.4. Then we define the blow-up category at a stratum with transversal self-intersection. A construction of such strata categories starting from a normal crossing divisor, and more generally from a toroidal embedding, is given in [MMUV22, Definition 3.5].

The main examples we are generalizing are (a) the poset obtained by intersecting the components of a simple normal crossing divisor and (b) the stratification of $\overline{\mathcal{M}}_{g,n}$ by topological type, induced by the boundary divisors $\Delta = \Delta_{\mathrm{irr}} \cup \bigcup_{i,S} \Delta_{i,S}$ (see Section 2.d). In the latter case, the relevant category is the category $G_{g,n}$ of stable n-pointed graphs of genus g with morphisms given by graph contractions.

4.a. Categories of resolved strata for a normal crossing stratification.

Let $\mathfrak{C}$ be a finite skeletal category with a terminal object $\bullet$ such that every morphism is an epimorphism.

Remark 4.1. In $\mathfrak{C}$ we have that $\mathrm{Mor}(\alpha,\alpha) = \mathrm{Aut}(\alpha)$. Indeed, if $f \in \mathrm{Mor}(\alpha,\alpha)$, then there exist natural numbers $a > b$ such that $f^a = f^b$, and since f is an epimorphism we have that $f^{a-b} = \mathrm{Id}_A$, which proves that f is an isomorphism.

If α and β are distinct elements, we also have that if $\mathrm{Mor}(\alpha,\beta) \neq \varnothing$ then $\mathrm{Mor}(\beta,\alpha) = \varnothing$. Indeed, assume that there exists $f\colon \alpha \to \beta$ and $g\colon \beta \to \alpha$. By the observation we would have that both $f \circ g$ and $g \circ f$ are automorphisms, which implies that both f and g are isomorphisms, this contradicts the fact that $\mathfrak{C}$ is skeletal. This means that the set $\mathrm{Obj}(\mathfrak{C})$ has a natural poset structure given by $\alpha \geq \beta$ if $\mathrm{Mor}(\alpha,\beta) \neq \varnothing$.

We say that such $\mathfrak{C}$ is a *(normal crossing) stratification category* if its underlying poset is ranked with rank function cd with minimum element the terminal object, having $\mathrm{cd}(\bullet) = 0$ and it satisfies the following axiom:

Axiom 1. For each $f\colon \alpha \to \beta$ there exist exactly $\mathrm{cd}(f) := \mathrm{cd}(\alpha) - \mathrm{cd}(\beta)$ pairs

$$(\beta', \mathrm{Aut}(\beta')g) \;\in\; \mathrm{Obj}(\mathfrak{C}) \times (\mathrm{Aut}(\beta')\backslash \mathrm{Mor}(\alpha,\beta'))$$

such that, for each such pair, there exists $i\colon \beta' \to \beta$ with $\mathrm{cd}(i) = 1$ and $f = i \circ g$. (Note that (a) the existence of such i is independent of the choice of the representative g in the left coset $\overline{g} := \mathrm{Aut}(\beta')g$); and (b) since g is an epimorphism, the morphism i is necessarily unique).

From now on we will also fix some notation on $\mathfrak{C}$.

(1) We write f_α for the unique element of $\mathrm{Mor}(\alpha, \bullet)$.

(2) If $f_i\colon \alpha \to \beta_i$ are morphisms for $i = 1, \dots, m$, we define

$$\mathrm{Aut}(f_1, \dots, f_m) := \{\tau \in \mathrm{Aut}(\alpha); f_i \circ \tau = f_i \text{ for every } i = 1, \dots, m\}.$$

Note that $\mathrm{Aut}(f_\alpha) = \mathrm{Aut}(\alpha)$.

(3) For each morphism $f\colon \beta \to \gamma$ and object $\alpha \in \mathrm{Obj}(\mathfrak{C})$, we define $\overline{\mathrm{Mor}}(\alpha, f) := \mathrm{Aut}(f)\backslash \mathrm{Mor}(\alpha, \beta)$. When $f = f_\beta$, we simply write $\overline{\mathrm{Mor}}(\alpha, \beta) := \overline{\mathrm{Mor}}(\alpha, f_\beta) = \mathrm{Aut}(\beta)\backslash \mathrm{Mor}(\alpha, \beta)$.

(4) For each morphism $f\colon \alpha \to \beta$, we let S_f denote the set of all pairs $(\beta', \overline{g})$ satisfying the condition in Axiom (1). Moreover, for each $(\beta', \overline{g}) \in S_f$ we denote by $i_{\overline{g},f} := i\colon \beta' \to \beta$ the morphism defined in Axiom 1. We define $S_\alpha := S_{f_\alpha}$.

Here are the most relevant examples in this paper.

Example 4.2. (Simple normal crossing). Let X be a nonsingular variety and $D = D_1 + \ldots + D_k$ be a simple normal crossing divisor. To this we can associate a category $\mathfrak{C}$ whose objects are the strata and morphisms are the inclusions. This category $\mathfrak{C}$ is finite, skeletal, has a terminal element, every morphism is an epimorphism, it is ranked by codimension, and it satisfies Axiom 1.

The category $\mathfrak{C}$ is *simple normal crossing* if in addition to Axiom 1, it satisfies:

Axiom 2. For every $\alpha, \beta \in \mathrm{Obj}(\mathfrak{C})$ the set $\overline{\mathrm{Mor}}(\alpha, \beta)$ has at most one element.

Example 4.3. The second example is $\mathfrak{C} = G_{g,n}$ introduced in Section 2.d. The terminal object here is the trivial graph with 1 vertex of genus g carrying all the markings, and no edges. The rank function is the number of edges. The set S_f of a morphism $f\colon G \to G'$ is naturally identified with the set of edges of G that are contracted by f. In particular, S_G equals the edge set $E(G)$.

The rank 1 objects, the boundary divisors, are either graphs with two vertices connected by one edge (corresponding to the divisors $\Delta_{i,S}$, see 2.d), or the graph consisting of 1 vertex of genus $g-1$ with 1 loop.

Example 4.4. The main example in this paper is the category $\mathfrak{C} = \mathfrak{C}_{g,n}(\phi)$ that we introduced in Definition 3.23, an enhancement of the category $G_{g,n}$ discussed above. The terminal object is the trivial graph endowed with the unique function that maps its unique vertex to the integer d. The rank of an object $(G, (E_G, D_G))$ equals $|\,\mathrm{Edges}(G)| + |E_G|$. For $f\colon (G', (E'_G, D_{G'})) \to (G, (E_G, D_G))$, the set S_f is naturally identified with the set of edges contracted by f.

The rank 1 objects are $(G, (E_G, D_G))$ with G a rank 1 object of $G_{g,n}$ and (E_G, D_G) a ϕ-stable pseudodivisor (which implies $E_G = \varnothing$).

Example 4.5. To a nonsingular variety X (or DM stack) endowed with a normal crossing divisor D, [MMUV22, Definition 3.5] associates a stratification category that respects Axiom 1 above.

Note that the construction of *loc.cit* in the case $X = \overline{\mathcal{M}}_{g,n}$ and $D = \Delta_{irr} + \sum_{(i,S)} \Delta_{i,S}$, which we discussed in Example 4.3, produces the quotient of the category $G_{g,n}$ of stable graphs where 2 morphisms are identified whenever they are the same on the corresponding edge sets. (See [MMUV22, Figure 2] for examples of automorphisms that are identified to the identity). A similar phenomenon happens for the category of Example 4.4.

In this paper we prefer instead to work with the usual category of stable graphs (and its enhancements).

The next proposition is the analogue of the fact that the set of strata that contain a given stratum is in natural bijection with the subsets of the divisors that define that stratum.

Proposition 4.6. *Given a morphism $f\colon \alpha \to \beta$ and $1 \le k \le \mathrm{cd}(\alpha) - \mathrm{cd}(\beta)$, there is a natural bijection between the set of pairs*

$$\{(\gamma, \overline{j}) \in \mathrm{Obj}(\mathfrak{C}) \times \overline{\mathrm{Mor}}(\alpha,\gamma) : \ \mathrm{cd}(\gamma) - \mathrm{cd}(\beta) = k \text{ and } \exists h\colon \gamma \to \beta, f = h \circ j\}$$

(note that h above is unique) and the set $\mathcal{P}(k, S_f)$ of subsets of S_f containing k elements.

We start by observing the following:

Remark 4.7. Given a factorization $f\colon \alpha \xrightarrow{j} \gamma \xrightarrow{h} \beta$, there is a natural inclusion $j^*\colon S_h \hookrightarrow S_f$ given by $(\beta', \overline{g'}) \mapsto (\beta', \overline{g' \circ j})$.

Moreover, we claim that for each $\overline{j} \in \overline{\mathrm{Mor}}(\alpha, h)$ we have a well-defined $\overline{j}^*(S_h)$. Indeed, the map j^* is the same as $(j \circ \tau)^*$ for every $\tau \in \mathrm{Aut}(\gamma \to \beta)$.

Proof. We first observe that by Remark 4.7, there is a natural map $\lambda_{k,f}$ from the set of pairs, call it $X_{k,f}$, to the set $\mathcal{P}(k, S_f)$ of k-elements subsets of S_f, obtained by $\lambda_{k,f}((\gamma, \overline{j})) := j^*(S_h)$.

Then we prove that the cardinality of $X_{k,f}$ equals $\binom{\mathrm{cd}(\alpha)-\mathrm{cd}(\beta)}{k}$, which is also the cardinality of $\mathcal{P}(k, S_f)$. This is achieved by induction on $\mathrm{cd}(\alpha) - \mathrm{cd}(\beta)$ and double counting. For each $c \in S_f$, let $X_{k,f,c}$ be the subset of $X_{k,f}$ of elements whose image via λ_f contains c. By induction hypothesis, we have that $|X_{k,f,c}| = \binom{\mathrm{cd}(\alpha)-\mathrm{cd}(\beta)-1}{k-1}$. By Axiom 1 the number of elements of $\{(a,b) : \ a \in X_{k,f}, b \in \lambda_{k,f}(a)\}$ equals $k \cdot |X_{k,f}|$ and, it also equals $(\mathrm{cd}(\alpha) - \mathrm{cd}(\beta)) \cdot \binom{\mathrm{cd}(\alpha)-\mathrm{cd}(\beta)-1}{k-1}$. These two equalities prove that $|X_{k,f}| = \binom{\mathrm{cd}(\alpha)-\mathrm{cd}(\beta)}{k}$.

Finally we prove that each $\lambda_{k,f}$ is surjective. First we prove that this is the case for $k = \mathrm{cd}(\alpha) - \mathrm{cd}(\beta) - 1$ (or equivalently when $\mathrm{cd}(\gamma) = \mathrm{cd}(\alpha) - 1$). By the previous paragraph, $\lambda_{k,f}$ is, for this k, a function between sets of the same cardinality, so it is equivalent to prove that it is injective. Let $a_1, a_2 \in X_{k,f}$ be such that $\lambda_{k,f}(a_1) = \lambda_{k,f}(a_2)$ and let c be the only element of $S_f \setminus \lambda_{k,f}(a_1)$. If $a_1 \neq a_2$, then $\lambda_{k,f}^{-1}(\{c\})$ contains at most $\mathrm{cd}(\alpha) - \mathrm{cd}(\beta) - 2$ elements, but in the previous paragraph we have established that $|X_{k,f,c}| = \mathrm{cd}(\alpha) - \mathrm{cd}(\beta) - 1$; this contradicts the assumption $a_1 \neq a_2$.

To prove surjectivity of each $\lambda_{k,f}$ we argue by induction on $\mathrm{cd}(\alpha)-\mathrm{cd}(\beta)$. Let $S \in \mathcal{P}(k, S_f)$. Choose $T \supset S$ with $T \in \mathcal{P}(\mathrm{cd}(\alpha)-\mathrm{cd}(\beta)-1, S_f)$. By the previous paragraph, there exists $d = (\delta, \overline{h})$ such that $\lambda_{k,f}(d) = T$ and a factorization of $f = h \circ g$ through δ, so $T = g^*(T')$ and $S = g^*(S')$ for some $S' \subset T'$ subsets of S_h. We have $\mathrm{cd}(\delta)-\mathrm{cd}(\beta) = \mathrm{cd}(\alpha) - \mathrm{cd}(\beta) - 1$. By applying the induction hypothesis to $\lambda_{k,h}$, we find $c \in X_{k,h}$ such that $\lambda_{k,h}(c) = S'$, and so $\lambda_{k,f}(g^*c) = S$. This concludes the proof of surjectivity. $\square$

We will now define some important geometric notions in the stratification category.

Definition 4.8. Fix $f_i\colon \alpha_i \to \beta$ for $i = 1, \ldots, m$, and $f\colon \gamma \to \beta$. Let $g_i\colon \gamma \to \alpha_i$ for $i = 1, \ldots, m$ be a collection of morphisms such that $f_i \circ g_i = f$.

We say that the collection (g_i) is *generic* with respect to the tuple $(f, (f_i))$ if $S_f = \bigcup_{i=1}^m g_i^*(S_{f_i})$.

We say that the collection $(f, (f_i))$ is *transversal* at (g_i) if $g_i^*(S_{f_i}) \cap g_j^*(S_{f_j}) = f^*(S_\beta)$ for very $i \neq j$.

Following the above definition, we write $\mathrm{Int}(f_1, \ldots, f_m)_f$ to denote the set of all generic tuples $(g_1, \ldots, g_m)$.

Remark 4.9. Fix the same data of the above definition. Let $(\tau_1, \ldots, \tau_m)$ be a tuple in $\prod \mathrm{Aut}(f_i)$ and let $(g_1, \ldots, g_m)$ be a generic collection, then $(\tau_1 \circ g_1, \ldots, \tau_m \circ g_m)$ is also generic. A similar result holds for an automorphism $\tau \in \mathrm{Aut}(f)$. That is: (g_i) is generic if and only if $(g_i \circ \tau)$ is generic. This gives a natural left action of $\prod \mathrm{Aut}(f_i)$ and right action of $\mathrm{Aut}(f)$ on $\mathrm{Int}((f_i))$.

Following the remark, we define

$$\overline{\mathrm{Int}}(f_1, \ldots, f_m)_f := \prod \mathrm{Aut}(f_i) \backslash \mathrm{Int}(f_1, \ldots, f_m)_f$$

and

$$\widetilde{\mathrm{Int}}(f_1, \ldots, f_m)_f := \mathrm{Int}(f_1, \ldots, f_m)_f / \mathrm{Aut}(f).$$

Elements of $\overline{\mathrm{Int}}(f_1, \ldots, f_m)$ will be denoted by $(\overline{g}_1, \ldots, \overline{g}_m)$, while the elements of $\widetilde{\mathrm{Int}}(f_1, \ldots, f_m)$ will be denoted by $(g_1, \ldots, g_m)\,\mathrm{Aut}(f)$.

When $f_1 = \ldots = f_m = f'$, we write $\mathrm{SInt}((f')^m)_f$ to denote the set of sets (not tuples) $\{\overline{g}_1, \ldots, \overline{g}_m\}$ (here the g_i must be pairwise distinct) such that $(g_1, \ldots, g_m) \in \overline{\mathrm{Int}}((f')^m)_f$.

4.b. **Normal crossing stratifications.** We say that a category $\mathfrak{C}$ as in the previous section is the category of strata of a nonsingular DM-stack $X_\bullet$ if there exists a functor

$$\begin{aligned} \mathfrak{C} &\to \text{nonsingular DM-stacks} \\ \alpha &\mapsto X_\alpha \\ f\colon \alpha \to \beta &\mapsto X_f\colon X_\alpha \to X_\beta \end{aligned}$$

such that

(1) The morphisms $X_f\colon X_\alpha \to X_\beta$ are proper and local complete intersection of codimension $\mathrm{cd}(f)$.
(2) The quotient stack $\left[\frac{X_\alpha}{\mathrm{Aut}(f)}\right]$ is the normalization of the image of X_f.
(3) The normal bundle N_f of X_f can be written as $N_f = \bigoplus_{e\in S_f} \mathbb{L}_e$, where, for a pair $e = (\beta', \overline{g}) \in S_f$, we define $\mathbb{L}_e := g^*(N_{i_{g,f}})$.
(4) If $f_i\colon \alpha_i \to \beta$ for $i = 1, 2$ are two morphisms, then the following diagram

$$\begin{tikzcd}
\displaystyle\bigsqcup_{\substack{f\colon \gamma\to\beta \\ (g_1,g_2)\,\mathrm{Aut}(f)\in\widetilde{\mathrm{Int}}(f_1,f_2)_f}} [X_G/\operatorname{Aut}(g_1,g_2)] \arrow[r, "X_{g_1}"] \arrow[d, "X_{g_2}"] & X_{\alpha_1} \arrow[d, "X_{f_1}"] \\
X_{\alpha_2} \arrow[r, "X_{f_2}"] & X_\beta
\end{tikzcd}$$

is a fiber diagram.

From now on we will abuse the notation and, for $f \in \mathrm{Mor}(\alpha, \beta)$, we simply write $f\colon X_\alpha \to X_\beta$ in place of $X_f\colon X_\alpha \to X_\beta$.

Notation 4.10. We will use a prime to denote the image of a morphism $f\colon X_\alpha \to X_\beta$. In other words, $X'_f := \mathrm{Im}(f) \subseteq X_\beta$ and, in particular, $X'_\alpha := \mathrm{Im}(f_\alpha) \subseteq X_\bullet$.

We also say that the objects X'_α are the embedded strata and the objects X_α are the (resolved) strata.

The two main examples in this paper are that of $\overline{\mathcal{M}}_{g,n}$ and that of $\overline{\mathcal{J}}^d_{g,n}(\phi)$:

Example 4.11. The category $G_{g,n}$ is a category of strata of the nonsingular DM-stack $\overline{\mathcal{M}}_{g,n}$. If $G \in G_{g,n}$, we write $\overline{\mathcal{M}}_G$ for the corresponding stratum and $\overline{\mathcal{M}}'_G$ for its image in $\overline{\mathcal{M}}_{g,n}$ ([ACG11, Chapter XII, Section 10]).

Example 4.12. The category $\mathfrak{C}_{g,n}(\phi)$ is a category of strata of the nonsingular DM-stack $\overline{\mathcal{J}}^d_{g,n}(\phi)$ ([MMUV22, Section 3]). If $(G, (E, D)) \in \mathfrak{C}_{g,n}(\phi)$, we write $\mathcal{J}_{G,(E,D)}$ for[4] the corresponding stratum and $\mathcal{J}'_{G,(E,D)}$ for its image in $\overline{\mathcal{J}}^d_{g,n}(\phi)$.

The point made in Example 4.12 allows us to complete the proof of Proposition 3.38. The next section is devoted to completing that proof.

4.b.1. *Proof of Proposition 3.38.*

Proof. (1) If $d = g - 1$, the result of 3.38 follows from [KP17, Theorem 4.1].

[4]each such stratum also depends on ϕ, but we do include this dependency to ease the notation

(2) Assume that $d = g-2$. In order to reach our conclusion, we prove that ϕ is in the claimed range if and only if the intersection of $\mathsf{W}_d(\phi)$ with the boundary of $\overline{\mathcal{J}}^d_{g,n}(\phi)$ has codimension larger than the expected codimension 2. Also, the strata that generically parameterize curves whose irreducible components are singular can be excluded, because the existence of a nonzero global section is equivalent if those components are smoothened.

Firstly, we analyse the boundary divisors, which have the form $\mathcal{J}_{(G(i,1,S),D)}$. The range of ϕ in the claim is equivalent to constraining the divisor D to equal $(i-1, g-i-1)$. It is straightforward to verify that the locus cut out in $\mathcal{J}_{(G(i,1,S),D)}$ by the condition of admitting a global section has codimension at least 2 in $\mathcal{J}_{(G(i,1,S),D)}$, hence it has codimension at least 3 in $\overline{\mathcal{J}}^d_{g,n}(\phi)$.

If, on the other hand, the divisor D is of the form $(k-1, g-k-1)$ for $k \neq i$, then either $\mathcal{J}_{(G(i,1,S),D)}$ is contained in $\mathsf{W}_d(\phi)$, or their intersection has codimension 1 in $\mathcal{J}_{(G(i,1,S),D)}$. In both cases, their intersection has codimension smaller than or equal to 2 in $\overline{\mathcal{J}}^d_{g,n}(\phi)$.

Then we analyse the codimension 2 strata $\mathcal{J}_{G,D}$. If G is a tree, the stability condition is uniquely determined by the stability condition on the boundary divisors, and so is the stable degree D – the problem has been resolved in the previous paragraph. We assume therefore that $G = G(i,2,S)$ is a vine curve with 2 nodes. Using the change of coordinates (3.11), the range identified in our statement is equivalent to requesting that the stable divisor D on $G(i,2,S)$ equals one of $(i-2, g-i)$, $(i-1, g-i-1)$, $(i, g-i-2)$ or $(i+1, g-i-3)$. In all these cases, one can check that the generic element of $\mathcal{J}_{G,D}$ does not admit a global section. Conversely, if D is not one of those 4 cases, the stratum $\mathcal{J}_{G,D}$ is contained in $\mathsf{W}_d(\phi)$. This concludes our proof.

(3) Assume that $1 \leq d \leq g-3$. In order to reach our conclusion, it is enough to prove that for every ϕ, the intersection of $\mathsf{W}_d(\phi)$ with some boundary divisor has codimension smaller than or equal to the expected codimension $g-d$.

We take $i = \lfloor \frac{g}{2} \rfloor$ and pick any $S \subseteq [n]$, and show that the intersection of $\mathsf{W}_d(\phi)$ with the preimage of $\Delta_{i,S}$ in $\overline{\mathcal{J}}^d_{g,n}(\phi)$ contains a locus of codimension smaller than or equal to $g-d$ in $\overline{\mathcal{J}}^d_{g,n}(\phi)$.

The stable bidegree D such that the intersection $\mathcal{J}_{G(i,1,S),D} \cap \mathsf{W}_d(\phi)$ has largest codimension is $D = (\frac{d}{2}, \frac{d}{2})$ for d even (resp $D = (\frac{d-1}{2}, \frac{d+1}{2})$ for d odd). The intersection has codimension $\lceil \frac{g-d-1}{2} \rceil + 2$ in $\overline{\mathcal{J}}^d_{g,n}(\phi)$ and, for $d \leq g-3$, this number is smaller than or equal to $g-d$.

(4-5) Assume that $d \leq 0$.

Assume first that ϕ is not in the given range. Then, arguing as in the case $d = g - 2$ above, one can check that in some boundary divisor or in some codimension 2 vine curve locus (depending on which inequality ϕ fails to satisfy) the intersection with $\mathsf{W}_d(\phi)$ has codimension larger than the expected one (which is g for $d = 0$, and by which we mean that the locus is not empty when $d < 0$).

Assume now that ϕ is in the given range. We will use the following result:

Proposition 4.13. *If $\phi \in V^d_{g,n}$ is nondegenerate and such that the inequality*

$$\phi_{C_0} \leq \frac{\left|C_0 \cap \overline{C_0^c}\right|}{2}. \tag{4.14}$$

holds for all $(C, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{g,n}$ and for all subcurves $C_0 \subseteq C$, then

(a) if $d = 0$, then $F \in \mathsf{W}_d(\phi)$ if and only F is the trivial line bundle.
(b) if $d < 0$, then $\mathsf{W}_d(\phi) = \varnothing$.

Proof. Part (a) is [HKP18, Lemma 8, Lemma 9]. Part (b) follows from Lemma 4.15 below.

□

By applying Proposition 4.13, and observing that both ϕ and the multidegree D of line bundles are stable for graph morphisms, and arguing as in the proof of Proposition 3.5, we conclude that Inequality (4.14) is satisfied for all curves C and subcurves C_0 if and only if it is satisfied for all vine curves C (and taking C_0 to be one of its irreducible components). After applying the change of coordinates (3.11), this is equivalent to the given range.

The only remaining case to consider is when $d = 0$ and $-3/2 < x_{0,1,S} < -1/2$ for some S. In that case, the intersection of the component $\mathcal{J}_{G(0,1,S),(-1,1)}$ with $\mathsf{W}_d(\phi)$ has codimension $g+1$ in $\overline{\mathcal{J}}^d_{g,n}(\phi)$, hence the intersection is in the closure of the restriction of $\mathsf{W}_d(\phi)$ to the open part.

□

In the proof of Proposition 4.13 we used the following.

Lemma 4.15. *Assume $d < 0$. Let C be a nodal curve, and let $\phi \in V^d_{\mathrm{stab}}(C)$ be such that Inequality (4.14) holds for all subcurves $C_0 \subseteq C$. Then every ϕ-stable rank-1 torsion-free sheaf F on C satisfies $H^0(C, F) = 0$.*

The lemma generalizes to the case $d < 0$ the argument given in in [Dud18, Lemma 3.1] and [HKP18, Lemma 8].

Proof. Let F be one such sheaf. If F is ϕ-stable, the inequality

$$\deg_{C_0}(F) < \frac{\left|C_0 \cap \overline{C_0^c}\right|}{2} - \delta_{C_0}(F) + \phi_{C_0} \tag{4.16}$$

holds for all subcurves $\varnothing \neq C_0 \subsetneq C$. The latter, combined with (4.14), implies the inequality

$$\deg_{C_0}(F) < \left|C_0 \cap \overline{C_0^c}\right| - \delta_{C_0}(F) \tag{4.17}$$

for all subcurves $\varnothing \neq C_0 \subsetneq C$.

The fact that the latter inequality holds on all subcurves C_0 implies that F admits no nonzero global sections. Indeed, if such a section s existed denote by C' its support. Note that $C' \neq C$ because the degree of F is negative. Hence, $C' \neq C$ and we have the inequality

$$\deg_{C'}(F) \geq \left|C' \cap \overline{C'^c}\right| - \delta_{C'}(F),$$

contradicting (4.17). □

We conclude this interlude by observing that for all degrees "in the middle", the Brill–Noether cycle cannot be of the expected codimension.

Remark 4.18. Assume that $1 \leq d \leq g-5$. We claim that there exist no ϕ such that $\mathsf{W}_d(\phi)$ has the expected codimension $g-d$. To show this, we argue in a very similar way to the case $1 \leq d \leq g-3$ of the proof of Proposition 3.38. We let $i = \lfloor \frac{g}{2} \rfloor$ and pick any $S \subseteq [n]$. In the same way as discussed in loc.cit., for $1 \leq d \leq g-5$, the intersection of $\mathsf{W}_d(\phi)$ with the preimage of $\Delta_{i,S}$ in $\overline{\mathcal{J}}^d_{g,n}(\phi)$ contains a locus of codimension strictly smaller than $g-d$ in $\overline{\mathcal{J}}^d_{g,n}(\phi)$, and this proves our claim.

4.c. **Intersection theory formulas.** In this section we enunciate and prove some results concerning the intersection theory of this stratification. From now on in this section we fix $X_\bullet$ and its stratification functor.

Since most of our computations are done using Chern classes, we will abuse the notation as we now explain. Let p_1, p_2, q be polynomials in variables $x_{i,j}$ such that $p_1 = qp_2$. Assume that L_i are elements in the K-theory of X, and that $A = p_1(c_j(L_i))$ and $B = p_2(c_j(L_i))$ are formal polynomials in the Chern classes of L_i. We will write $\frac{A}{B}$ to mean the class $q(c_j(L_i)) \cap [X]$ in the Chow group of X.

More generally, we will write $\frac{A}{B} \in A^*(X)$ to mean that there exists polynomials p_1, p_2, q and K-theory elements L_i satisfying the conditions in the previous paragraph.

The main motivation for this is [Ful98,], which states that, for a vector bundle N

$$\frac{c(L \otimes \wedge^\bullet N)}{c_{\mathrm{rk}\, N}(N)}$$

is a polynomial in the Chern classes of L and of N.

In this language, we have the following excess intersection formula.

Proposition 4.19. *Let $f_i\colon \alpha_i \to \beta$ for $i = 1, 2$ be two morphisms in $\mathfrak{C}$ and fix classes $A_i/c_{\mathrm{cd}(f_i)}(N_{f_i}) \in A^*(X_{\alpha_i})$, then*

$$f_{1*}\Big(\frac{A_1}{c_{\mathrm{cd}(f_1)}(N_{f_1})}\Big) f_{2*}\Big(\frac{A_2}{c_{\mathrm{cd}(f_2)}(N_{f_2})}\Big) = \sum_{\substack{f\colon \gamma\to\beta \\ (g_1,g_2)\operatorname{Aut}(f)\in \\ \widetilde{\operatorname{Int}}(f_1,f_2)_f}} \frac{f_*}{|\operatorname{Aut}(g_1,g_2)|}\Big(\frac{g_1^*A_1 g_2^*A_2}{c_{\mathrm{cd}(f)}(N_f)}\Big)$$

where $g_i = 1, 2$ are the base change morphisms (as in Section 4.b, Item (4)).

Proof. This follows directly from Item (4) in Section 4.b and from the excess intersection formula (see [Ful98, Proposition 17.4.1]). □

We will also be using the following corollary of the above formula.

Corollary 4.20. *Let $f_i\colon \alpha_i \to \beta$ be two morphisms in $\mathfrak{C}$ and let*

$$\frac{A_i}{c_{\mathrm{cd}(f_i)}(N_{f_i})} \in A^*(X_{\alpha_i})$$

be such that A_i is invariant under $\operatorname{Aut}(f_i)$, then the following holds

$$\begin{aligned}\frac{f_{1*}(\frac{A_1}{c_{\mathrm{cd}(f_1)}(N_{f_1})})}{|\operatorname{Aut}(f_1)|}\frac{f_{2*}(\frac{A_2}{c_{\mathrm{cd}(f_2)}(N_{f_2})})}{|\operatorname{Aut}(f_2)|} &= \\ &= \sum_{f\colon\gamma\to\beta} \frac{f_*}{|\operatorname{Aut}(f)|}\Big(\sum_{(\overline{g}_1,\overline{g}_2)\in\overline{\operatorname{Int}}(f_1,f_2)_f} \frac{g_1^*A_1 g_2^*A_2}{c_{\mathrm{cd}(f)}(N_f)}\Big)\end{aligned}$$

Proof. We expand the formula in Proposition 4.19 to obtain

$$f_{1*}\Big(\frac{A_1}{c_{\mathrm{cd}(f_1)}(N_{f_1})}\Big) f_{2*}\Big(\frac{A_2}{c_{\mathrm{cd}(f_2)}(N_{f_2})}\Big) = \sum_{\substack{f\colon \gamma\to\beta \\ (g_1,g_2)\in\operatorname{Int}(f_1,f_2)_f}} \frac{f_*}{|\operatorname{Aut}(f)|}\Big(\frac{g_1^*A_1 g_2^*A_2}{c_{\mathrm{cd}(f)}(N_f)}\Big)$$

because $|(f\colon \gamma\to\beta, (g_1,g_2)\operatorname{Aut}(f))| = |\operatorname{Aut}(f)|/|\operatorname{Aut}(g_1,g_2)|$. From there, we have that

$$\begin{aligned} f_{1*}(\frac{A_1}{c_{\mathrm{cd}(f_1)}(N_{f_1})}) f_{2*}(\frac{A_2}{c_{\mathrm{cd}(f_2)}(N_{f_2})}) &= \\ &= \sum_{\substack{f\colon \gamma\to\beta \\ (\overline{g}_1,\overline{g}_2)\in\overline{\operatorname{Int}}(f_1,f_2)_f}} |\operatorname{Aut}(f_1)||\operatorname{Aut}(f_2)| \frac{f_*}{|\operatorname{Aut}(f)|}(\frac{g_1^*A_1 g_2^*A_2}{c_{\mathrm{cd}(f)}(N_f)})\end{aligned}$$

and the result follows. □

Next, we apply the above to obtain a self-intersection formula.

Corollary 4.21. *Let $f\colon \alpha\to\beta$, then*

$$\Big(\frac{f_*}{|\operatorname{Aut}(f)|}\Big(\frac{A}{c_{\mathrm{cd}(f)}(N_f)}\Big)\Big)^k = \sum_{f'\colon\gamma\to\beta} \frac{f'_*}{|\operatorname{Aut}(f')|}\Bigg(\sum_{\substack{(\overline{g}_1,\ldots,\overline{g}_k)\in \\ \overline{\operatorname{Int}}(f,\ldots,f)_f}} \frac{\prod_{j=1}^k g_i^*(A)}{c_{\mathrm{cd}(f')}(N_{f'})}\Bigg)$$

The latter will be used to prove the following GRR formula for the total Chern class (deduced from the usual one, involving the Chern character).

Proposition 4.22 (GRR for the total Chern class)**.** *Let $f\colon \alpha \to \beta$ be a morphism and let $\mathcal{F}$ be an element in the K-theory of X_α with rational coefficients. Then*

$$c\Big(\frac{f_*(\mathcal{F})}{|\operatorname{Aut}(f)|}\Big) = 1 + \sum_{\substack{m\geq 1\\ f'\colon \gamma\to\beta}} \frac{f'_*}{|\operatorname{Aut}(f')|}\Bigg(\sum_{\substack{\{\overline{g}_1,\dots,\overline{g}_m\}\in\\ \operatorname{SInt}((f)^m)_{f'}}} \frac{\prod_{j=1}^m g_i^*(c(\bigwedge^\bullet N_f^\vee\otimes\mathcal{F})-1)}{c_{\operatorname{cd}(f')}(N_{f'})}\Bigg)$$

(This is inspired by [Ful98, Theorem 15.3]).

Proof. We begin with the usual GRR formula

$$\operatorname{ch}\Big(\frac{f_*(\mathcal{F})}{|\operatorname{Aut}(f)|}\Big) = f_*\Big(\frac{\operatorname{ch}(\mathcal{F})}{|\operatorname{Aut}(f)|}\operatorname{td}(N_f)^{-1}\Big),$$

which, combining with the formula for the Todd class, implies

$$\operatorname{ch}_n\Big(\frac{f_*(\mathcal{F})}{|\operatorname{Aut}(f)|}\Big) = \frac{f_*}{|\operatorname{Aut}(f)|}\Bigg(\frac{\operatorname{ch}_n(\bigwedge^\bullet N_f^\vee\otimes\mathcal{F})}{c_{\operatorname{cd}(f)}(N_f)}\Bigg).$$

By the inversion formula to express the total Chern class in terms of the Chern character (see e.g. [PRvZ20, Equation 3.9]), we deduce

$$c\Big(\frac{f_*(\mathcal{F})}{|\operatorname{Aut}(f)|}\Big) = \exp\Bigg(\frac{f_*}{|\operatorname{Aut}(f)|}\Bigg(\frac{\sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\operatorname{ch}_n(\bigwedge^\bullet N_f^\vee\otimes\mathcal{F})}{c_{\operatorname{cd}(f)}(N_f)}\Bigg)\Bigg).$$

Setting $A := \sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\operatorname{ch}_n(\bigwedge^\bullet N_f^\vee\otimes\mathcal{F})$, we will then compute

$$\star := \exp\Bigg(\frac{f_*}{|\operatorname{Aut}(f)|}\Big(\frac{A}{c_{\operatorname{cd}(f)}(N_f)}\Big)\Bigg)$$

$$\begin{aligned}
\star &= 1 + \sum_{k\geq 1}\Bigg(\frac{f_*}{|\operatorname{Aut}(f)|}\Big(\frac{A}{c_{\operatorname{cd}(f)}(N_f)}\Big)\Bigg)^k\frac{1}{k!}\\
&= 1 + \sum_{k\geq 1}\sum_{f'\colon\gamma\to\beta}\frac{f'_*}{|\operatorname{Aut}(f')|}\Bigg(\sum_{(\overline{g}_1,\dots,\overline{g}_k)\in\overline{\operatorname{Int}}((f)^k)_{f'}}\frac{\prod_{i=1}^k g_i^*(A)}{c_{\operatorname{cd}(f')}(N_{f'})}\cdot\frac{1}{k!}\Bigg)\\
&= 1 + \sum_{f'\colon\gamma\to\beta}\frac{f'_*}{|\operatorname{Aut}(f')|}\Bigg(\sum_{\substack{m\geq 1\\ \{\overline{g}_1,\dots,\overline{g}_m\}\in\operatorname{SInt}((f)^m)_{f'}\\ k_1,\dots,k_m\geq 1}}\frac{\prod_{i=1}^m g_i^*(A)^{k_i}}{c_{\operatorname{cd}(f')}(N_{f'})}\cdot\frac{1}{(\sum_{i=1}^m k_i)!}\Bigg)\\
&= 1 + \sum_{f'\colon\gamma\to\beta}\frac{f'_*}{|\operatorname{Aut}(f')|}\Bigg(\sum_{\substack{m\geq 1\\ \{\overline{g}_1,\dots,\overline{g}_m\}\in\operatorname{SInt}((f)^m)_{f'}}}\frac{\prod_{i=1}^m(\exp(g_i^*(A))-1)}{c_{\operatorname{cd}(f')}(N_{f'})}\Bigg)
\end{aligned}$$

The claim is then obtained by applying again the inversion formula in the form

$$\exp(A) = c(\bigwedge^{\bullet} N_f \otimes F).$$

□

4.d. **Blow-up.** Starting from a category $\mathfrak{C}$ as in 4.a, here we define the blow-up category at a stratum with transversal self-intersection. Then for a fixed stratification functor X, we interpret the blow-up category as the stratification of the blow-up of the nonsingular DM-stack X at that stratum.

Definition 4.23. We say that an object $\delta \in \mathrm{Obj}(\mathfrak{C})$ *has transversal self-intersection* if for every pair $g_1, g_2 \colon \gamma \to \delta$, the sets $g_1^*(S_\delta)$, $g_2^*(S_\delta)$ are either equal or disjoint.

Remark 4.24. If $g_1^*(S_\delta) = g_2^*(S_\delta)$, then $\overline{g}_1 = \overline{g}_2 \in \overline{\mathrm{Mor}}(\gamma, \delta)$. See Proposition 4.6.

Example 4.25. Let $\mathfrak{C} = G_{2k+1,1}$ for some $k \geq 1$. We claim that the vine curve graph $G = G(k, 2, \{1\})$ does not have transversal self-intersection.

Indeed, let G' be the "triangle" graph with 2 vertices of genus k and a third vertex of genus 0 carrying the marking. There are two different morphisms $g_1, g_2 \colon G' \to G$ and $g_1^*(S_G) \cap g_2^*(S_G)$ consists of 1 edge.

Definition 4.26. Let δ be an object of $\mathfrak{C}$ with transversal self-intersection. We define *the blowup category* $\mathrm{Bl}_\delta \mathfrak{C}$ *of* $\mathfrak{C}$ *at* δ as follows.

First consider the following category. Its set of objects consists of pairs $(\gamma, \mathbf{m})$ where γ is an object of $\mathfrak{C}$ and $\mathbf{m}$ is a function $\overline{\mathrm{Mor}}(\gamma, \delta) \to \mathcal{P}(S_\gamma)$ such that

$$\varnothing \neq \mathbf{m}(\overline{g}) \subseteq g^*(S_\delta) \quad \text{for every} \quad \overline{g} \in \overline{\mathrm{Mor}}(\gamma, \delta). \tag{4.27}$$

Its morphisms $(\gamma_1, \mathbf{m}_1) \to (\gamma_2, \mathbf{m}_2)$ are morphisms $f \colon \gamma_1 \to \gamma_2$ such that for every $\overline{g}_1 \in \overline{\mathrm{Mor}}(\gamma_1, \delta)$ we have that one of the conditions hold

(1) there exists $\overline{g}_2 \in \overline{\mathrm{Mor}}(\gamma_2, \delta)$ such that $\overline{g}_1 = \overline{g_2 \circ f}$ and $\mathbf{m}_1(\overline{g}_1) \subseteq f^*(\mathbf{m}_2(\overline{g}_2))$,
(2) or $\mathbf{m}_1(\overline{g}_1) \cap f^*(S_{\gamma_2}) = \varnothing$.

We then define $\mathrm{Bl}_\delta \mathfrak{C}$ as a skeleton of the above category.

Proposition 4.28. *The category* $\mathrm{Bl}_\delta \mathfrak{C}$ *is naturally ranked, and it satisfies Axiom (1) from Section 4.a.*

Proof. Straightforward. □

Remark 4.29. The rank of $(\gamma, \mathbf{m})$ is

$$\mathrm{rk}(\gamma) - \sum_{\overline{g} \in \overline{\mathrm{Mor}}(\gamma, \delta)} |\mathbf{m}(\overline{g})|.$$

Moreover, the set $S_{(\gamma, \mathbf{m})}$ (the codimension 1 strata that contain a fixed stratum $(\gamma, \mathbf{m})$) is naturally identified with $S_\gamma \backslash (\bigcup_{\overline{g} \in \overline{\mathrm{Mor}}(\gamma,\delta)} \mathbf{m}(\overline{g})) \cup \overline{\mathrm{Mor}}(\gamma, \delta)$.

Recall Notation 4.10. We define $h\colon \widetilde{X}_\beta \to X_\beta$ to be the blow up of X_β at the union of the images $X'_{g_1} \subseteq X_\beta$ for every $g_1\colon \gamma \to \beta$ such that there exists $g_2\colon \gamma \to \delta$ satisfying $(g_1, g_2) \in \mathrm{Int}(f_\beta, f_\delta)_{f_\gamma}$. We define $X_{\beta,\mathbf{m}}$ to be

$$\prod_{\overline{g}\in\overline{\mathrm{Mor}}(\gamma,\delta)} \mathbb{P}\Big(\bigoplus_{e\in\mathbf{m}(\overline{g})} \mathbb{L}_e\Big).$$

Proposition 4.30. *The functor*

$$\begin{aligned}\mathrm{Bl}_\delta \mathfrak{C} &\to \text{nonsingular DM stacks}\\ (\gamma, \mathbf{m}) &\mapsto X_{\gamma,\mathbf{m}}\end{aligned}$$

is a stratification of $\mathrm{Bl}_{X'_\delta} X_\bullet$.

Proof. This follows from [MPS23, Section 4.5] (see also [KKMSD73, Theorem 6, p.90]) where the nonsingular DM stack is constructed as the star subdivision of the cone stack associated to the stratification. $\square$

Remark 4.31. When there exists no morphism $\gamma \to \delta$, there exists a unique $\mathbf{m}$ such that the pair $(\gamma, \mathbf{m}) \in \mathrm{Bl}_\delta \mathfrak{C}$. The latter is the stratum that corresponds to the strict transform of the image $X'_\gamma \subset X_\bullet$.

Remark 4.32. Suppose that δ is a stratum with transversal self intersection and $f\colon \gamma \to \beta$ is a morphism such that $\mathrm{Mor}(\beta, \delta) = \varnothing$. Let $(\gamma, \mathbf{m})$ be an object in $\mathrm{Bl}_\delta \mathfrak{C}$. Then the morphism f lifts to a morphism $(\gamma, \mathbf{m}) \to (\beta, \varnothing)$ in $\mathrm{Bl}_\delta \mathfrak{C}$ if and only if $f^* S_\beta \cap \bigcup_{\overline{g}\in\overline{\mathrm{Mor}}(\gamma,\delta)} \mathbf{m}(g) = \varnothing$. (That is, when $X'_{\gamma,\mathbf{m}}$ is contained in the strict transform of X'_β in $\mathrm{Bl}_{X'_\delta} X_\bullet$).

In this paper, the main example of the above construction is going to be the case where $\mathfrak{C}$ is the category $\mathfrak{C}_{g,n}(\phi)$ of Example 4.12, or a blowup of the latter. We now describe the example of 1 blowup of $\mathfrak{C}_{g,n}(\phi)$ at one of the centers that will be relevant for our main result.

Example 4.33. Let $\phi \in V^d_{g,n}$ and $(G, D) \in \mathfrak{C}_{g,n}(\phi)$ be the lift of a vine curve $G = G(i, t, S)$ by some ϕ-stable divisor D.

Morphisms $f\colon (G', (E', D')) \to (G, D)$ correspond to subsets $T_f \subset V(G'^{E'})$ such that the complete subgraphs $G(T_f), G(T_f^c)$ in $G'^{E'}$ are connected and of genus $i, g-i-t+1$, the markings S are on $G(T_f)$ and the markings S^c are on $G(T^c_f)$, and $D'(G(T_f)) = D(v_1)$ and $D'(G(T^c_f)) = D(v_2)$ (for v_1, v_2 the two vertices of G). We let $E(T_f) \subseteq E(G'^{E'})$ be the subset of t edges that separate $G(T_f)$ from $G(T^c_f)$.

Assume that (G, D) is a stratum[5] with transversal self-intersection. The category $\mathrm{Bl}_{(G,D)} \mathfrak{C}_{g,n}(\phi)$ defined above stratifies the blowup $\mathrm{Bl}_{\mathcal{J}'(G,D)} \overline{\mathcal{J}}^d_{g,n}(\phi)$. We can describe more explicitly its objects as tuples (G', E', D', α) such that $(G', (E', D')) \in \mathrm{Obj}(\mathfrak{C}_{g,n}(\phi))$ and α is a choice, for each morphism

[5]because of Lemma 5.27, in this paper we will never need to blowup any strata of the form (G, E, D) with $E \neq \varnothing$

$f\colon (G',D') \to (G,D)$ (up to automorphisms of (G,D)), of a subset $\varnothing \neq \alpha(\mathrm{Aut}(G,D)f) \subseteq E(T_f)$.

We now define the psi-classes associated to a given stratum $\gamma \in \mathfrak{C}$. Recall that each $e \in S_\gamma$ corresponds to a morphism $j_e\colon \gamma \to \beta_e$ where the latter has rank 1. Then define the psi-classes

$$\Psi_{\beta_e} := -c_1(\mathbb{L}_e) = -c_1(N_{X_{\beta_e}} X_\bullet), \quad \psi_{\gamma,e} := j_e^* \Psi_{\beta_e} \tag{4.34}$$

(see Item (3) in the beginning of Section 4.b) for $\mathbb{L}_e$).

We will now state and prove a pushforward formula for monomials in psi-classes under the blowdown morphism. We begin by introducing some notation.

Recall Remark 4.29. For an object $(\gamma, \mathbf{n})$ in $\mathrm{Bl}_\delta \mathfrak{C}$ we define the sets

$$\begin{aligned} S_{\gamma\setminus\delta} &:= S_\gamma \setminus \bigcup_{j\colon \gamma\to\delta} j^*(S_\delta), \\ \mathrm{FU}_\delta(\gamma,\mathbf{n}) &:= \bigcup_{j\colon \gamma\to\delta} j^*(S_\delta) \setminus \mathbf{n}(\overline{j}), \\ \mathrm{CU}_\delta(\gamma,\mathbf{n}) &:= \bigcup_{j\colon \gamma\to\delta} \mathbf{n}(\overline{j}). \end{aligned}$$

(The symbols FU and CU will acquire some meaning in Section 7 as certain collection of edges, see Equation (7.19).) Note that the unions can equivalently be taken over $\overline{\mathrm{Mor}}(\gamma,\delta)$ instead of over all morphisms. (See Remark 4.7).

We define $H^\delta_{\gamma,\mathbf{n}}((g'_{e'})_{e'\in S_{\gamma,\mathbf{n}}})$ as the set of tuples $((a_e)_{e\in S_\gamma}, (g_e)_{e\in S_\gamma})$ of non-negative integers satisfying $a_e = 0$ for every $e \notin \mathrm{FU}_\delta(\gamma,\mathbf{n})$,

$$\sum_{e\in\mathbf{n}(\overline{j})} (g_e+1) = g'_{\overline{j}} + 1 + \sum_{e\in j^*(S_\delta)\setminus\mathbf{n}(\overline{j})} a_e$$

for every $\overline{j} \in \overline{\mathrm{Mor}}(\gamma,\delta) \subseteq S_{\gamma,\mathbf{n}}$, and $g_e = g'_e$ for every $e \in S_\gamma \setminus \mathrm{CU}_\delta(\gamma,\mathbf{n}) \subseteq S_{\gamma,\mathbf{n}}$.

For a morphism $h\colon (\gamma,\mathbf{n}) \to (\beta,\mathbf{m})$ and a tuple $(g'_{e'})_{e'\in S_{\beta,\mathbf{m}}}$ we define $h^*(g'_{e'})$ as the tuple

$$(h^*(g'_{e'}))_{\widetilde{e}} := \begin{cases} g'_{e'} & \text{if } \widetilde{e} = h^*(e') \text{ for some } e' \\ -1 & \text{if } \widetilde{e} \in S_{\gamma,\mathbf{n}} \setminus h^*(S_{\beta,\mathbf{m}}). \end{cases}$$

We define $M_\delta(\gamma)$ to be the set of all function $\mathbf{m}\colon \overline{\mathrm{Mor}}(\gamma,\delta) \to \mathcal{P}(S_\gamma)$ satisfying Equation (4.27).

Corollary 4.35. *Let $p\colon \mathrm{Bl}_{X_{\delta'}} X_\bullet \to X_\bullet$ be the blowdown morphism, and fix integers $(g'_{e'} \geq 0)_{e'\in S_{\beta,\mathbf{m}}}$. Then the pushforward*

$$p_* \frac{f_{(\beta,\mathbf{m})*}}{|\,\mathrm{Aut}(\beta,\mathbf{m})|} \Big(\prod_{e'\in S_{\beta,\mathbf{m}}} \Psi_{e'}^{g'_e} \Big)$$

equals

$$\sum_{\gamma\in\mathfrak{C}} \frac{f_{\gamma *}}{|\operatorname{Aut}(\gamma)|}\left(\sum_{\substack{\mathbf{n}\in M_\delta(\gamma)\\ h\in\overline{\operatorname{Mor}}((\gamma,\mathbf{n}),(\beta,\mathbf{m}))}} \sum_{\substack{(a_e,g_e)\in\\ H^\delta_{\gamma,\mathbf{n}}(h^*(g'_{e'}))}} (-1)^{a_e}\binom{g_e}{a_e}\Psi_e^{g_e-a_e}\right).$$

Proof. Follows from [BL05, Theorem 4.8] (or [Alu10, Theorem 4.2]). □

5. Combinatorial aspects of Wall-Crossing

In this section we fix two stability conditions ϕ^+ and ϕ^- "on opposite sides of a stability hyperplane H" (Definition 5.1), and give a description of what will turn out to be the stratification category $\widetilde{\mathfrak{C}} = \widetilde{\mathfrak{C}}(\phi^+,\phi^-)$ of the resolution $\widetilde{\mathcal{J}}^d_{g,n}(\phi^+,\phi^-)$, which we will formally construct in the next section.

Objects of this category are defined in Definition 5.28 as triples (G, D, α) where (G, D) is an object of $\overline{\mathcal{J}}^d_{g,n}(\phi^+)$ and α is a certain "vine function" to be introduced in Definition 5.15. This generalizes the case of 1 blowup, described in Example 4.33. Then we study the subcategory $\widetilde{\mathfrak{C}}_E = \widetilde{\mathfrak{C}}(\phi^+,\phi^-)$ of the strata that are in the intersection of the exceptional divisors — this is the category appearing in our main result Theorem 7.4. The condition that singles out objects of $\widetilde{\mathfrak{C}}_E \subset \widetilde{\mathfrak{C}}$ is that the function α should be *full*, as defined in Definition 5.15. We then see in Proposition 5.21 that the datum of a full vine function is equivalent to that of a full forest, and this gives a simpler description of the objects of $\widetilde{\mathfrak{C}}_E$, which is the one that we will use in Theorem 7.4.

Recall from 3.b that there are 3 types of hyperplanes. If H is as in (3.12), then no blowup is required and $\mathfrak{C}_{g,n}(\phi) = \widetilde{\mathfrak{C}}$. The second case is when H is as in (3.13) and $S \neq [n]$. That case will be discussed in 5.c. The most difficult case is when H is as in (3.13) and $S = [n]$.

Definition 5.1. Let $\phi^+,\phi^- \in V^d_{g,n}$ be nondegenerate, and let H be a stability hyperplane (see Section 3.b). We say that *the polarizations $\phi^\pm$ are on opposite sides of the hyperplane H* if $\phi^0 = \frac{\phi^-+\phi^+}{2} \in H$ is the only degenerate point of the segment $[\phi^+,\phi^-] \subset V^d_{g,n}$.

In other words, $H \ni \phi_0$-semistability implies ϕ^+ or ϕ^- stability, and $\phi^\pm$ are small perturbations of ϕ_0. Throughout we fix $H, \phi^\pm$ and ϕ_0 as in the above definition.

5.a. **Extremal sets, vine functions and full forests.** In this section we prove some of our bulk combinatorial results that have to do with wall-crossing, focusing on the "extremal" multidegree, i.e. those multidegrees that are ϕ^+-stable but are not ϕ^--stable. *We shall fix a n pointed graph G of genus g and a divisor D on G throughout.* (We are not imposing stability conditions here, but the cases we are interested in are either $G = G'^{E'}$ for some $E' \subseteq E(G)$ where G' is stable, or G is obtained by forgetting the last marking on a stable $(n+1)$ pointed graph.)

For a subset $V \subseteq V(G)$ we define[6]

$$\beta^\star(V) := -\deg(D|_V) + \phi^\star(V) + \frac{|E(V,V^c)|}{2}, \qquad \text{for } \star = +,-,0.$$

Note that D is $\phi^\star$-semistable (Definition 3.17) if and only if $\beta^\star(V) \geq 0$ for every $V \subset V(G)$. Moreover, we have the following relation for $\beta^\star$ (see [AP20, Lemma 4.1])

$$\beta^\star(V) + \beta^\star(W) - |E(V \setminus W, W \setminus V)| = \beta^\star(V \cap W) + \beta^\star(V \cup W). \tag{5.2}$$

From now on in this chapter we will assume that D is ϕ^+ semistable on G. A subset $V \subsetneq V(G)$ is called *extremal* (with respect to ϕ^+, ϕ^- and D) if

$$\beta^+(V) > 0 \text{ and } \beta^-(V) < 0 \tag{5.3}$$

In particular, this implies

$$\phi^+(V) > \phi^-(V).$$

Note that if V is extremal, then $\beta^0(V) = 0$.

We are now ready to define the main object of study in this section.

Definition 5.4. We define the poset $\mathrm{Ext}(G,D) = \mathrm{Ext}_{\phi^+,\phi^-}(G,D)$ as

$$\{V \subseteq V(G); V \text{ is extremal, connected and with connected complement}\}$$

with the ordering given by inclusion.

Remark 5.5. If $\iota\colon (G,D) \to (G',D')$ is a specialization and $V' \in \mathrm{Ext}(G',D')$, we have that $\iota^{-1}(V') \in \mathrm{Ext}(G,D)$.

Remark 5.6. Each element V of $\mathrm{Ext}(G,D)$ corresponds to a morphism from G to an extremal vine curve stratum (G',D') (up to automorphisms of (G',D')) obtained by contracting $E(V,V)$ and $E(V^c,V^c)$.

We have the following results for extremal subsets.

Proposition 5.7. *Let V_1 and V_2 be two extremal subsets. Then $V_1 \cap V_2$ and $V_1 \cup V_2$ are either empty, extremal or equal to $V(G)$. Moreover, we have that $E(V_1 \setminus V_2, V_2 \setminus V_1) = \varnothing$ in all cases.*

Proof. Write $H_0 = V_1 \cap V_2$, $H_1 = V_1 \setminus H_0$, $H_2 = V_2 \setminus H_0$ and $H_3 = V_1^c \cap V_2^c$ (see Figure 1). Define $\alpha = |E(H_1,H_2)| = |E(V_1 \setminus V_2, V_2 \setminus V_1)|$.

Since $\beta^-(H_0 \cup H_1), \beta^-(H_0 \cup H_2) < 0$ we have that $\beta^0(H_0 \cup H_1) = \beta^0(H_0 \cup H_2) = 0$. By (5.2) we have

$$\beta^0(H_0 \cup H_1) + \beta^0(H_0 \cup H_2) - \alpha = \beta^0(H_0) + \beta^0(H_0 \cup H_1 \cup H_2),$$

and, because $\beta^0(H) \geq 0$ for every H, we deduce that $\alpha = 0$, and $\beta^0(H_0) = \beta^0(H_0 \cup H_1 \cup H_2) = 0$.

[6]We note that the existing definition of $\beta(V)$ in [Est01] and [AP20] are based on a different sign convention, however all relevant properties remain the same.

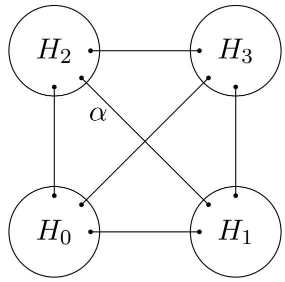

FIGURE 1

If $H_0 \neq \varnothing$, then $\beta^+(H_0) > 0$ and if $H_0 \cup H_1 \cup H_2 \neq V(G)$, then $\beta^+(H_0 \cup H_1 \cup H_2) > 0$. Since $\beta^0 = \frac{\beta^+ + \beta^-}{2}$, we have that $\beta^-(H_0) < 0$ (respectively, $\beta^-(H_0 \cup H_1 \cup H_2) < 0$) if $H_0 \neq \varnothing$ (respectively, if $H_0 \cup H_1 \cup H_2 \neq V(G)$). This finishes the proof. □

Proposition 5.8. *Let V be an extremal set.*

If $V = V_1 \sqcup V_2$ with $E(V_1, V_2) = \varnothing$ and $V_1, V_2 \neq \varnothing$, then V_1 and V_2 are extremal. Similarly, if $V^c = W_1 \sqcup W_2$ with $E(W_1, W_2) = \varnothing$ and $W_1, W_2 \neq \varnothing$, then W_1^c and W_2^c are extremal.

Proof. For the first part, we have that $\beta^0(V) = \beta^0(V_1) + \beta^0(V_2)$, since $\beta^0(V) = 0$, then $\beta^0(V_1) = \beta^0(V_2) = 0$ as well. So, $\beta^-(V_i), \beta^-(V_2) < 0$. The second part is proven similarly. □

In what follows we will also need an additional hypothesis.

Hypothesis 1. If $V \subseteq V(G)$ is extremal, then $\text{leg}(1) \in V$.

In particular, by Proposition 5.8, we have that if $V \subseteq V(G)$ is extremal, then $G(V)$ is connected. From now on in this chapter we will always assume Hypothesis 1.

Hypothesis (1) fixes the following convention on ϕ^+, ϕ^-:

Remark 5.9. If V is an extremal set in $\text{Ext}(G, D)$ then $\phi^+(V) > \phi^-(V)$. In particular, if we set $S := \text{leg}^{-1}(V)$ and $i := g(V)$ and $t = |E(V, V^c)|$, then

$$x^+_{i,t,S} > x^-_{i,t,S}.$$

for fixed coordinates $\phi^\pm = (x^\pm_{i,t,S})_{(i,t,S)} \in V^d_{g,n}$ as discussed in Section 3.b. Also, since ϕ^+ and ϕ^- are on opposite sides of a hyperplane $H = H(i_0, t_0, S_0)$, Hypothesis 1 is always satisfied upon possibly switching ϕ^+ and ϕ^-.

Remark 5.10. The results in this section hold more generally for when ϕ^+ and ϕ^- lie on opposite sides of a higher codimension stability plane (not necessarily a hyperplane, as in Definition 5.1), in which case Hypothesis 1 becomes restrictive.

Here are some important properties of $\text{Ext}(G, D)$ that follow from Hypothesis 1:

Corollary 5.11. *Let $V_1, V_2 \in \mathrm{Ext}(G,D)$, then either $V_1 \cup V_2 = V(G)$ or there exists $V \in \mathrm{Ext}(G,D)$ such that $V_1 \cup V_2 \subseteq V$.*

Proof. Assume that $V_1 \cup V_2 \neq V(G)$. Then, by Proposition 5.7, we have that $V_1 \cup V_2$ is extremal. By Hypothesis 1, we have that $V_1 \cup V_2$ is also connected. Let W be a connected component of $(V_1 \cup V_2)^c$. By Proposition 5.8, we have that W^c is extremal. Moreover, since $V_1 \cup V_2$ is connected (and G is connected), so is W^c. This proves that $W^c \in \mathrm{Ext}(G,D)$ and $V_1 \cup V_2 \subseteq W^c$. □

Corollary 5.12. *Let V_1, V_2 be elements of* $\mathrm{Ext}(G,D)$ *such that $V_1, V_2 \subseteq V$ for some $V \in \mathrm{Ext}(G,D)$. Then $V_1 \cap V_2 \in \mathrm{Ext}(G,D)$.*

Proof. By Proposition 5.7, we have that $V_1 \cap V_2$ is extremal and by Hypothesis 1 we have that $V_1 \cap V_2$ is nonempty and connected. All that is left is to prove that $(V_1 \cap V_2)^c = V^c \cup (V \setminus V_1) \cup (V \setminus V_2)$ is connected. But this follows from the fact that V^c, $V_1^c = V^c \cup (V \setminus V_1)$ and $V_2^c = V^c \cup (V \setminus V_2)$ are connected. □

We are now ready to introduce a key notion to describe the blowup category of $\mathfrak{C}_{g,n}(\phi)$ at some vine curve strata.

Definition 5.13. For each (G,D) and lower set $L \subseteq \mathrm{Ext}(G,D)$, we say that a function $\alpha\colon L \to \mathcal{P}(E(G))$ is a *vine function* if the following conditions hold

(1) $\alpha(V) \subseteq E(V, V^c)$ for every $V \in L$.
(2) For all $V \in L$ we have $\alpha(V) = \varnothing$ if and only if there exists $V' \subsetneq V$ with $V' \in \mathrm{Ext}(G,D)$ such that $\alpha(V') \cap E(V,V^c) \neq \varnothing$.

We usually think that L is part of the datum of α, and write L_α for the domain of a vine function α. We also define $|\alpha| = \bigcup_{V \in L_\alpha} \alpha(V) \subseteq E(G)$.

Definition 5.14. Given a specialization $\iota\colon (G,D) \to (G',D')$ we say that the vine functions α and α' are *compatible with* ι if

(1) $\iota^{-1}(V') \in L_\alpha$ for every $V' \in L_{\alpha'}$.
(2) if $\alpha'(V') \neq \varnothing$, then $\alpha(\iota^{-1}(V')) \neq \varnothing$.
(3) if $e' \notin |\alpha'|$ then $\iota_E(e) \notin |\alpha|$.

In that case we write $\iota\colon (G,D,\alpha) \to (G',D',\alpha')$ and say that the first triple specializes to the second.

We will also need to introduce some subcategories of the stratification category of $\mathfrak{C}_{g,n}(\phi)$, which will use only some vine functions which we call "full". We now introduce those, and then discuss how this notion is equivalent to the combinatorial notion of a full forest.

Definition 5.15. We say that α is *full* if $L_\alpha = \mathrm{Ext}(G,D)$, and $|\alpha| = E(G)$.

We will show in Proposition 5.21 how full vine functions are equivalent to the following notion.

Definition 5.16. A forest $V_\bullet \subseteq \mathrm{Ext}(G,D)$ is a *full forest* in $\mathrm{Ext}(G,D)$ if

(1) it contains all maximal elements of $\mathrm{Ext}(G,D)$, and
(2) the edge set satisfies $E(G) = \bigcup_{V\in V_\bullet} E(V,V^c)$.

We first prove some intermediate results in that direction. For a forest $V_\bullet \subseteq \mathrm{Ext}(G,D)$, and for each $V' \in \mathrm{Ext}(G,D)$, we define

$$\mathrm{next}(V') = \mathrm{next}_{V_\bullet}(V') := \bigcap_{V' \subsetneq V \in V_\bullet} V \tag{5.17}$$

(with the usual convention that the intersection over the empty set equals $V(G)$).

Lemma 5.18. *Let $V_\bullet \subseteq \mathrm{Ext}(G,D)$ be a full forest and let V_1 and V_2 be two incomparable elements in $V_\bullet$. Then $V_1 \cup V_2 = V(G)$ and $E(V_1^c, V_2^c) = \varnothing$.*

Proof. By Corollary 5.11 we have that either $V_1 \cup V_2 = V(G)$, or there exists $V \in \mathrm{Ext}(G,D)$ such that $V_1, V_2 \subseteq V$. If the latter holds, Part (1) of Definition 5.16 implies that V_1 and V_2 are comparable, a contradiction. So $V_1 \cup V_2 = V(G)$. The fact that $E(V_1^c, V_2^c) = \varnothing$ follows from Proposition 5.7. □

Proposition 5.19. *Let $V_\bullet \subseteq \mathrm{Ext}(G,D)$ be a full forest. Let $V' \in \mathrm{Ext}(G,D)$ be a nonmaximal element and let $V_1, \dots, V_m$ be the elements of $V_\bullet$ that are minimal among those containing V'. Then:*

(1) For every $i = 1, \dots, m$,
$$E(\mathrm{next}(V') \setminus V', V_i^c) \neq \varnothing.$$
(2) $\mathrm{next}(V') \neq V'$.
(3) $G(\mathrm{next}(V'))$ is connected.
(4) If $V' \in V_\bullet$, then $E(\mathrm{next}(V') \setminus V', \mathrm{next}(V') \setminus V') = \varnothing$
(5) If $V \in \mathrm{Ext}(G,D)$ is such that $\mathrm{next}(V') \subseteq V$, then there exists $i \in \{1, \dots, m\}$ such that $V_i \subseteq V$.
(6) all minimal elements of $\mathrm{Ext}(G,D)$ belong to $V_\bullet$.
(7) We have $E(G) = \bigsqcup_{V \in V_\bullet} E(V, \mathrm{next}(V) \setminus V)$.

Proof. For (1), we first notice that V_i, V_j are inconparable if $i \neq j$, because of the minimality condition. By Lemma 5.18, we have that $E(V_i^c, V_j^c) = \varnothing$ for all $i \neq j$. Since $V'^c = (\mathrm{next}(V') \setminus V') \cup \bigcup_{i=1}^m V_i^c$ and V'^c is connected, we must have that $E(\mathrm{next}(V') \setminus V', V_i^c) \neq \varnothing$. Item (2) follows immediately from (1).

For (3), just notice that $\mathrm{next}(V')$ is an extremal element by Proposition 5.7, then it is connected by Hypothesis 1.

The existence of an edge between vertices of $\mathrm{next}(V') \setminus V'$ would witness the failure for $V_\bullet$ to be full. Indeed, let $V \in V_\bullet$ be an element. If V is incomparable with V', by Lemma 5.18, we have that $V \cup V' = V(G)$, and that implies that $\mathrm{next}(V') \setminus V' \subseteq V$ and hence $E(\mathrm{next}(V') \setminus V', \mathrm{next}(V') \setminus V')) \cap E(V, V^c) = \varnothing$. If $V' \subseteq V$, then $V_i \subseteq V$ for some i, and hence $\mathrm{next}(V') \subseteq V$ which implies that $E(\mathrm{next}(V') \setminus V', \mathrm{next}(V') \setminus V')) \cap E(V, V^c) = \varnothing$. If $V \subset V'$, then it is clear that $E(\mathrm{next}(V') \setminus V', \mathrm{next}(V') \setminus V')) \cap E(V, V^c) = \varnothing$. By the condition that $V_\bullet$ is a full forest, we have that $E(\mathrm{next}(V')\setminus V', \mathrm{next}(V')\setminus V'))\cap E(G)=\varnothing$ and this completes the proof of Item (4).

Item (5). This follows from the fact that $V(G)=\mathrm{next}(V')\cup\bigcup_{j=1}^m V_j^c$ and hence $V^c=\bigcup_{j=1}^m V^c\cap V_j^c$. Since $G(V^c)$ is connected and $E(V_i^c,V_j^c)=\varnothing$ for $j\neq i$ (Lemma 5.18), so we have that there exists $i\in\{1,\dots,m\}$ such that $V^c\cap V_j^c=\varnothing$ for every $j\neq i$. This means that $V_i\subset V$.

Item (6). Let V_0 be a minimal element of $\mathrm{Ext}(G,D)$. If V_0 is also maximal, there is nothing to do. So we can assume that V_0 is nonmaximal. Assume by contradiction that $V_0\notin V_\bullet$. From Items (2) and (3) we deduce that $E(V_0,\mathrm{next}(V_0)\setminus V_0)\neq\varnothing$. Let $V\in V_\bullet$ we will prove that $E(V_0,\mathrm{next}(V_0)\setminus V_0)\cap E(V,V^c)=\varnothing$ and get a contradiction. If $V_0\subseteq V$, then $\mathrm{next}(V_0)\subset V$ (recall that $V_0\neq V$ because $V_0\notin V_\bullet$), then $E(V_0,\mathrm{next}(V_0)\setminus V_0)\cap E(V,V^c)=\varnothing$. If $V_0\not\subseteq V$, we have that $V_0\cap V\subsetneqq V_0$ is extremal by Proposition 5.7; by the minimality of V_0 in $\mathrm{Ext}(G,D)$ we have that $(V_0\cap V)^c=V_0^c\cup V^c$ is not connected. Since V_0^c,V^c are connected, we have that $V_0^c\cap V^c=\varnothing$ and hence $V_0\cup V=V(G)$. By Proposition 5.7 we have that $E(V_0^c,V^c)=\varnothing$. Hence $E(V_0,V_0^c)\cap E(V,V^c)=\varnothing$, indeed $E(V_0,V_0^c)=E(V_0\cap V,V_0^c)$ and $E(V,V^c)=E(V_0\cap V,V^c)$. In particular $E(V_0,\mathrm{next}(V_0)\setminus V_0)\cap E(V,V^c)=\varnothing$.

Item (7). Let e be an edge of G and let V' be an element in $V_\bullet$ that is maximal among those with the property that $e\in E(V',V'^c)$. For each $V'\subsetneqq V\in V_\bullet$, we must have that $e\in E(V,V)$, so we have that $e\in E(V',\mathrm{next}(V')\setminus V')$.

□

Proposition 5.20. *Let α be a vine function and define*

$$V_\bullet:=\{V\in\mathrm{Ext}(G,D):\ \alpha(V)\neq\varnothing\}\subseteq\mathrm{Ext}(G,D).$$

Then the following hold.

1. *For every $V_0\in\mathrm{Ext}(G,D)$, we have that $V_\bullet\cap\mathrm{Ext}(G,D)_{\subseteq V_0}$ is a chain.*
2. *The poset $V_\bullet$ is a forest.*
3. *$\alpha(V)\subseteq E(V,\mathrm{next}(V)\setminus V)$ for every $V\in V_\bullet$.*

Proof. Let $V_1,V_2\in V_\bullet$ be such that $V_1,V_2\subseteq V_0$ and V_1 and V_2 are incomparable. By Corollary 5.12, we have that $V_1\cap V_2\in\mathrm{Ext}(G,D)$. Moreover, we have that

$$E(V_1\cap V_2,(V_1\cap V_2)^c)\subseteq E(V_1,V_1^c)\cup E(V_2,V_2^c).$$

By the fact that α is a vine function, we have that there exists $V'\subseteq V_1\cap V_2$ such that $\alpha(V')\cap E(V_1\cap V_2,(V_1\cap V_2)^c)\neq\varnothing$. This means that either $\alpha(V')\cap E(V_1,V_1^c)\neq\varnothing$, or $\alpha(V')\cap E(V_2,V_2^c)\neq\varnothing$, which contradicts the fact that α is a vine function and $\alpha(V_1),\alpha(V_2)\neq\varnothing$. This concludes the proof of Item (1). Item (2) follows directly.

We now prove (3). If $\mathrm{next}(V)=V(G)$ there is nothing to prove. Otherwise, we have

$$E(V,V^c)\setminus E(V,\mathrm{next}(V)\setminus V)\subseteq\bigcup_{V\subsetneqq V'\in V_\bullet}E(V',V'^c),$$

so $\alpha(V)\cap(E(V,V^c)\backslash E(V,\operatorname{next}(V)\backslash V))\neq\varnothing$ would imply $\alpha(V)\cap E(V',V'^c)\neq\varnothing$ for some $V'\in V_\bullet$ with $V\subsetneq V'$, thus contradicting the assumption that α is a vine function.

□

We are now ready to prove the equivalence of full vine functions and full forests.

Proposition 5.21. *For each (G,D), the mapping defined in Proposition 5.20 induces a natural bijection between full vine functions and full forests in* $\operatorname{Ext}(G,D)$.

Proof. Assume that α is a full vine function and define $V_\bullet = V_\bullet^\alpha$ as in Proposition 5.20. By loc.cit we have that $V_\bullet$ is a forest and that $\alpha(V)\subseteq E(V,\operatorname{next}(V)\setminus V)$ for every V. The condition

$$\bigcup_{V\in V_\bullet}\alpha(V)=\bigcup_{V\in\operatorname{Ext}(G,D)}\alpha(V)=E(G)$$

implies that $\bigcup E(V,\operatorname{next}(V)\backslash V)=E(G)$, which implies $\bigcup E(V,V^c)=E(G)$. This proves that $V_\bullet$ is a full forest.

We now study the inverse mapping. For a full forest $V_\bullet$, define

$$\alpha:=\alpha_{V_\bullet}(V_0)=\begin{cases}\varnothing & \text{if } V_0\notin V_\bullet;\\ E(V_0,\operatorname{next}(V_0)\setminus V_0) & \text{if } V_0\in V_\bullet.\end{cases}$$

We claim that α is a vine function. Condition (1) in Definition 5.13 follows from the fact that $E(V_0,\operatorname{next}(V_0)\setminus V_0)\subseteq E(V_0,V_0^c)$ for every $V\in V_\bullet$.

Let us prove Condition (2). First, we see that $\alpha_{V_\bullet}(V_0)=\varnothing$ if and only if $V_0\notin V_\bullet$. By the definition of α it is clear that if $V_0\notin V_\bullet$, then $\alpha(V_0)=\varnothing$. On the other hand, if $\alpha(V_0)=\varnothing$ and $V_0\in V_\bullet$, then $E(V_0,\operatorname{next}(V_0)\backslash V_0)=\varnothing$, but this is a contradiction with the fact that $\operatorname{next}(V_0)$ induces a connected subgraph of G and $\operatorname{next}(V_0)\setminus V_0\neq\varnothing$ (see Proposition 5.19).

Now we show that if $\alpha(V_0)=\varnothing$, then we can find $V'\in\operatorname{Ext}(G,D)$ with $V'\subseteq V_0$, such that $\alpha(V')\cap E(V_0,V_0^c)\neq\varnothing$. By the previous paragraph, we have that $V_0\notin V_\bullet$. Since $V_\bullet$ contains all minimal elements of $\operatorname{Ext}(G,D)$ (by Proposition 5.19), we have that there exists $V\in V_\bullet$ contained in V_0. Let V' be the maximum such element. This maximum exists because $V_\bullet$ is a forest that contains all maximal elements of $\operatorname{Ext}(G,D)$. By Item (5) of Proposition 5.19 and the maximality of V', we have that $\operatorname{next}(V')\nsubseteq V_0$. Moreover, by Item (4) of Proposition 5.19 we have that $E(\operatorname{next}(V')\setminus V_0,\operatorname{next}(V')\cap V_0\setminus V')=\varnothing$. Since $\operatorname{next}(V')$ induces a connected subgraph (this is Item (3) of Proposition 5.19), we have that $E(V',\operatorname{next}(V')\backslash V_0)\neq\varnothing$, and since

$$E(V',\operatorname{next}(V')\setminus V_0)\subseteq E(V',\operatorname{next}(V')\setminus V')\cap E(V_0,V_0^c)=\alpha(V')\cap E(V_0,V_0')$$

we have that $\alpha(V')\cap E(V_0,V_0')\neq\varnothing$ as needed.

We now prove that if there exists a $V' \in \mathrm{Ext}(G,D)$ with $V' \subsetneq V_0$ such that $\alpha(V') \cap E(V_0, V_0^c) \neq \varnothing$, then $\alpha(V_0) = \varnothing$. Assume by contradiction that there exist $V_0, V' \in V_\bullet$ such that $\alpha(V_0) \neq \varnothing$, and that $V' \subsetneq V_0$ and $\alpha(V') \cap E(V_0, V_0^c) \neq \varnothing$. Since $\alpha(V_0), \alpha(V') \neq \varnothing$, we have that $V_0, V' \in V_\bullet$. Since $V' \subsetneq V_0$, we have that $\mathrm{next}(V') \subseteq V_0$ as well, so this means that $E(V', \mathrm{next}(V') \setminus V') \cap E(V_0, V_0^c) = \varnothing$, a contradiction (recall that $E(V', \mathrm{next}(V') \setminus V') = \alpha(V')$). This concludes the proof that α is a vine function.

The fact that α is full follows from Item (7) of Proposition 5.19.

□

Definition 5.22. A morphism $\iota\colon (G, D, V_\bullet) \to (G', D', V'_\bullet)$ is a morphism $\iota\colon (G,D) \to (G',D')$ such that $\iota^{-1}(V') \in V_\bullet$ for every $V' \in V'_\bullet$. This is equivalent of saying that ι is compatible with $\alpha_{V_\bullet}$ and $\alpha_{V'_\bullet}$.

Given morphisms $\iota_i\colon (G,D,\alpha) \to (G_i, D_i, \alpha_i)$ for $i = 1, \ldots, k$, we say that the collection $(\iota_1, \ldots, \iota_k)$ is *generic* if

(1) for every edge $e \in E(G) \setminus |\alpha|$ there exists some $i \in \{1, \ldots, k\}$ and some $e' \in E(G_i) \setminus |\alpha_i|$ such that $e = \iota_{i,E}(e')$;
(2) for every $V \in L_\alpha$ such that $\alpha(V) \neq \varnothing$, there exists i and $V' \in L_{\alpha_i}$ with $\alpha_i(V') \neq \varnothing$ such that $\iota^{-1}(V') = V$.

Remark 5.23. We will see in the next section how this definition of "generic" matches the one given in Definition 4.8.

Proposition 5.24. *Let $\iota_i\colon (G,D,\alpha) \to (G_i, D_i, \alpha_i)$ be a generic collection. If all α_i are full, then α is full as well.*

Proof. This follows directly. □

The next result will imply that all the strata that we blow-up have transversal self-intersection.

Proposition 5.25. *Assume (G,D,α) is such that G is a vine curve and $L_\alpha = \varnothing$. Let $f_i\colon (G', D', \alpha') \to (G, D, \alpha)$ for $i = 1, 2$ be generic. Assume that for all $V \in \mathrm{Ext}(G', D')$ such that $V \subseteq f_1^*(v_0) \cap f_2^*(v_0)$, we have $V \in L_{\alpha'}$. Then (f_1, f_2) is transversal.*

Proof. Denote by $V_i = f_i^*(\{v_0\})$. Assume that $V_1 \cup V_2 \neq V(G)$. This means that $V_1 \cap V_2 \in \mathrm{Ext}(G,D)$ and hence in $L_{\alpha'}$. Since $E(V_1 \cap V_2, (V_1 \cap V_2)^c) \subseteq E(V_1, V_1^c) \cup E(V_2, V_2^c)$, that would mean that there exists $V' \in L_{\alpha'}$ such that $\alpha(V') \cap E(V_1 \cap V_2, (V_1 \cap V_2)^c) \neq \varnothing$, and in turn, if $e \in \alpha(V') \subseteq E(V_1, V_1^c)$ we would have a contradiction with the fact that if $e \notin |\alpha|$ but $f_i^*(e) \in |\alpha'|$.

So $V_1 \cup V_2 = V(G)$, which means that $E(V_1, V_1^c) \cap E(V_2, V_2^c) = \varnothing$ □

The next proposition will be used to prove that the category $\widetilde{\mathfrak{C}}_Y$, defined later, is a simple normal crossing stratification (as in Example 4.2).

Proposition 5.26. *Let (G,D) be a stable, $(n+1)$-marked vine curve such that $v_{n+1} \notin V$ for every $V \in \mathrm{Ext}(G,D)$. Let $V_\bullet$ be a full forest, and $f_1, f_2\colon (G', D', V'_\bullet) \to (G, D, V_\bullet)$ be morphisms. Then $f_1 \in \mathrm{Aut}(G, D, V_\bullet) f_2$.*

Proof. Upon further contraction of $(G', D', V'_\bullet)$, we can assume that f_1, f_2 are generic. This means that either $(G', D', V'_\bullet) = (G, D, V_\bullet)$ or $V'_\bullet = \{V'_1, V'_2\}$.

We have two cases, either $V'_1 \subseteq V'_2$ (up to swapping V'_1 and V'_2) or V'_1 and V'_2 are incomparable. In the latter case, we have that $V'_1 \cup V'_2 = V(G')$ by Corollary 5.11, but that contradicts the fact that $v_{n+1} \notin V'_1 \cup V'_2$.

In the former case, we have that $g(V'_1) = g(V'_2)$, $|E(V'_1, V'^c_1)| = |E(V'_2, V'^c_2)|$, $\mathrm{leg}^{-1}(V_1) = \mathrm{leg}^{-1}(V_2)$ and $D(V_1) = D(V_2)$. But that means that all the vertices in $V'_2 \setminus V'_1$ have genus 0, there are no marked points and the degree of D equals 0. This is a contradiction with the fact that (G', D') is ϕ^+-stable. $\square$

We conclude this section by observing that the existence of a full function/forest rules out the presence of exceptional vertices.

Lemma 5.27. *If* $\mathrm{Ext}(G, D)$ *admits a full forest, then* G *is stable.*

Proof. Because D is ϕ^+ semistable, if G fails to be stable, it contains an exceptional vertex v (meaning that v has genus 0, no marked points, it has valence 2 and $D(v) = 1$). Let e_1, e_2 be the two edges of G that contain v.

If (G, D) admits a full forest $V_\bullet$, then there are V_1, V_2 such that $e_i \in E(V_i, V_i^c)$ for $i = 1, 2$. If $v \notin V_i$ for some i, then $\beta^\star_D(V_i) = \beta^\star(V_i \cup \{v\}) + 1$, in particular $\beta^\star_D(V_i) > 0$, a contradiction. This means that $v \in V_1 \cap V_2$. On the other hand, we have that $e_1, e_2 \in E(V_1 \cap V_2, (V_1 \cap V_2)^c)$, that $V_1 \cap V_2$ is extremal (by 5.7), hence connected, and $\mathrm{leg}(1) \in (V_1 \cap V_2)$. This is a contradiction.

$\square$

Recall that, as stipulated in Section 2.c, when E is empty, we simply write (G, D) in place of $(G, (\varnothing, D))$.

5.b. **The stratification categories.** In light of the results of the previous section, we are now ready to define the stratification category $\widetilde{\mathfrak{C}}$, and some other categories $\widetilde{\mathfrak{C}}_E$ and $\widetilde{\mathfrak{C}}_Y$ that will play an important role in our proof of Theorem 7.4. In the next chapter we will interpret these categories as stratification categories of the resolution of the identity map $\mathsf{Id}\colon \overline{\mathcal{J}}^d_{g,n}(\phi^+) \dashrightarrow \overline{\mathcal{J}}^d_{g,n}(\phi^-)$.

Definition 5.28. We define $\widetilde{\mathfrak{C}} = \widetilde{\mathfrak{C}}(\phi^+, \phi^-)$ as a skeleton of the category whose objects are triples (G, D, α) such that (G, D) is an object of $\mathfrak{C}_{g,n}(\phi)$ and α is a vine function with the property that $L_\alpha = \mathrm{Ext}(G, D)$. Morphisms are given as in Definition 5.14.

We define $\widetilde{\mathfrak{C}}_E$ as the full subcategory of $\widetilde{\mathfrak{C}}$ whose objects (G, D, α) with α *full*. Equivalently (Proposition 5.20) objects are triples $(G, D, V_\bullet)$ with $V_\bullet$ a full forest in $\mathrm{Ext}(G, D)$.

We then define the category $\widetilde{\mathfrak{C}}_Y$ as a skeleton of the category whose objects are triples $(G, D, V_\bullet)$ where G is a $n+1$ pointed stable graph of genus g, the divisor D is $(\phi^+, \mathrm{leg}(1))$-quasistable, and $V_\bullet$ is a full forest such that $\mathrm{leg}(n+1) \notin V$ for every $V \in V_\bullet$. (By Lemma 5.18, this implies that $V_\bullet$ is a chain). Morphisms are specializations as in Definition 5.14.

These categories will be interpreted geometrically in Remark 6.5. Note that the rank 1 objects (the divisors) of $\widetilde{\mathfrak{C}}_E$ are the triples $(G, D, V_\bullet)$ such that G is a vine curve and $V_\bullet$ contains a single element (by Hypothesis 1, the vertex containing the first marking).

5.c. **The case of "good" hyperplanes.** In this section, we fix a hyperplane $H = H(i, t, S; k)$ that satisfies $S^c \neq \varnothing$. In this case, we prove that the corresponding exceptional vine curves loci in the compactified universal Jacobian have pairwise empty intersections. The main result here is:

Proposition 5.29. *The objects of $\widetilde{\mathfrak{C}}_E$ are triples $(G, D, V_\bullet)$ satisfying either*

(1) G has no edges and $V_\bullet$ is empty. This is the terminal object.
(2) G is a vine curve and $V_\bullet$ has a single element $V = \{\operatorname{leg}(1)\}$.

By Proposition 3.16, each vine curve as in (2) above is necessarily of the form $G(i-j, t+2j, S)$, for all j satisfying $-t/2 < j \leq \min(i, g+1-t-i)$.

Proposition 5.30. *Let $V \in \operatorname{Ext}(G, D)$, then $\operatorname{leg}^{-1}(V) = S$.*

Proof. Let $G' := G/(E(V, V) \cup E(V^c, V^c))$ be the vine curve associated to V. If $\operatorname{leg}^{-1}(V) \neq S$, then $(\phi_0)_{G'}$ is nondegenerate by Proposition 3.14. This contradicts the assumption that V is extremal. □

For our next result, recall that the canonical divisor $K_G^{\log}$ of a graph G is defined by $K_G^{\log}(v) = 2g(v) - 2 + |E(v)| + |\operatorname{leg}^{-1}(v)|$ for all $v \in V(G)$.

Lemma 5.31. *If $V_1, V_2 \in \operatorname{Ext}(G, D)$, then $K_G^{\log}(V_1) = K_G^{\log}(V_2)$.*

Proof. We have that $K_G^{\log}(V) = 2g(V) - 2 + |E(V, V^c)| + |\operatorname{leg}^{-1}(V)|$. By Proposition 5.30, we have that $|\operatorname{leg}^{-1}(V_1)| = |\operatorname{leg}^{-1}(V_2)|$. By Proposition 3.16 we conclude that $2g(V_1)-2+|E(V_1, V_1^c)| = 2g(V_2)-2+|E(V_2, V_2^c)|$. □

Proposition 5.32. *If (G, D) is a ϕ^+-stable pair, then $|\operatorname{Ext}(G, D)| \leq 1$.*

Proof. Assume that we have $V_1 \neq V_2$ elements of $\operatorname{Ext}(G, D)$. By Proposition 5.30, we have that $\operatorname{leg}^{-1}(V_1) = \operatorname{leg}^{-1}(V_2) \neq \varnothing$, so $V_1 \cap V_2 \neq \varnothing$. Moreover, $\operatorname{leg}^{-1}(V_1^{-c}) = \operatorname{leg}^{-1}(V_2^c) \neq \varnothing$, so $V_1 \cup V_2 \neq V(G)$. By Propositions 5.7, 5.8 and 5.30 we have that $V_1 \cap V_2 \in \operatorname{Ext}(G, D)$.

This means that we can assume $V_1 \subseteq V_2$. By Lemma 5.31 we have that $K_G^{\log}(V_1) = K_G^{\log}(V_2)$, which implies that $K_G^{\log}(V_2 \setminus V_1) = 0$, which is a contradiction with the fact that G is stable. □

Corollary 5.33. *We have that (G, D) has a full forest $V_\bullet$ if and only if G is a vine curve and $\beta_D^-(\{\operatorname{leg}(1)\}) < 0$.*

Proof. By Proposition 5.32 we have that $\operatorname{Ext}(G, D)$ has at most one element. Since $V_\bullet$ must be nonempty, we have that $V_\bullet = \operatorname{Ext}(G, D)$. Since $V_\bullet = \{V\}$ is a full forest, we must have that $E(G) = E(V, V^c)$, and that $G(V)$ and $G(V^c)$ are connected. This means that both V and V^c are singletons and hence that G is a vine curve. □

Proof of Proposition 5.29. Let (G', D') be a pair with different specializations

$$\iota_1\colon (G,D) \to (G_1,D_1) \text{ and } \iota_2\colon (G,D)\to (G_2,D_2)$$

to extremal pairs. By Remark 5.5 we have that

$$\operatorname{Ext}(G',D') \supseteq \iota_1^{-1}(\operatorname{Ext}(G_1,D_1)) \cup \iota_2^{-1}(\operatorname{Ext}(G_2,D_2)),$$

which means that $\operatorname{Ext}(G',D')$ has at least 2 elements, contradicting Proposition 5.32. □

6. Nonsingular resolution of the identity

Let $\phi^-, \phi^+ \in V^d_{g,n}$ be on opposite sides of a stability hyperplane H (Definition 5.1). In this section we construct a nonsingular resolution $\widetilde{\mathcal{J}}^d_{g,n}(\phi^+,\phi^-)$ of the identity map $\mathsf{Id}\colon \overline{\mathcal{J}}^d_{g,n}(\phi^+) \dashrightarrow \overline{\mathcal{J}}^d_{g,n}(\phi^-)$. We construct $\widetilde{\mathcal{J}}^d_{g,n}(\phi^+,\phi^-)$ as an iterated blow up of $\overline{\mathcal{J}}^d_{g,n}(\phi^+)$ at certain strata (G^i, D^i) of vine curves with extremal bidegrees.

To define the order in which we blow up the vine curves, we first introduce a partial order. Let (G_i, D_i), for $i = 1, 2$, be a pair where G_i is a vine curve and D_i is an extremal bidegree, we also set v_i to be the vertex of G_i such that $\beta^-_{D_i}(\{v_i\}) < 0$ (i.e., it is the vertex with the first marked point, in particular ϕ^+ ans ϕ^- satisfy Hypothesis 1). We say that $(G_1, D_1) \leq (G_2, D_2)$ if there exists $(G, D) \in \mathfrak{C}_{g,n}(\phi^+)$ and morphisms $f_i\colon (G, D) \to (G_i, D_i)$ such that $f_1^{-1}(v_1) \subseteq f_2^{-1}(v_2)$. Note that, in particular, $f_i^{-1}(v_i) \in \operatorname{Ext}(G, D)$ (see Remark 5.5).

The next proposition guarantees that this preorder is indeed a partial order.

Proposition 6.1. *Assume $(G_1, D_1), (G_2, D_2)$ are vine curve strata with extremal bidegrees for ϕ^+, ϕ^-. Then $(G_1, D_1) \leq (G_2, D_2)$ if and only if $\operatorname{leg}^{-1}_{G_1}(v_1) \subseteq \operatorname{leg}^{-1}_{G_2}(v_2)$ and $g_{G_1}(v_1) \leq g_{G_2}(v_2)$ and $g_{G_1}(v_1) + |E(G_1)| \leq g_{G_2}(v_2) + |E(G_2)|$.*

Proof. The "only if" part follows immediately from the existence of a common degeneration (G, D), and the fact that for $V_1 \subseteq V_2$ then if $V_1, V_2 \in \operatorname{Ext}(G, D)$ or if $V_1^c, V_2^c \in \operatorname{Ext}(G, D)$, then $g(V_1) \leq g(V_2)$. (Because elements of $\operatorname{Ext}(G, D)$ and complements of elements of $\operatorname{Ext}(G, D)$ are connected).

For the "if" part, consider the graph G with 3 vertices w_1, w_2, w_3 with $|E(w_1, w_3)| = \lambda$ and $|E(w_1, w_2)| = |E(G_1)| - \lambda$ and $|E(w_2, w_3)| = |E(G_2)| - \lambda$. Set $g_G(w_1) = g_{G_1}(v_1)$ and $g_G(w_2) = g_{G_2}(v_2) - g_{G_1}(v_1) + \lambda - |E(G_1)| + 1$, and $g_G(w_3)$ so that $g(G) = g$. The numerical assumptions in the claim guarantee the existence of λ such that $E(w_i, w_j) \geq 1$ for all $i \neq j$ and $g(w_2), g(w_3) \geq 0$. It is then straightforward to check that the given graph G admits a morphism to G_1 (by contracting $E(w_2, w_3)$) and to G_2 (by contracting $E(w_1, w_2)$).

□

We are now ready to construct our resolution of the identity map.

Construction 6.2. *Take any extension to a total order of the partial order defined above on the set of pairs of vine curves with an extremal bidegree, and denote this extension by* $(G^1, D^1) < (G^2, D^2) < \ldots < (G^m, D^m)$.

Define J_i *inductively as follows:* $J_0 = \overline{\mathcal{J}}(\phi^+)$ *and*

$$J_i = \mathrm{Bl}_{J_{G^i,D^i,\alpha^i}}(J_{i-1})$$

where α^i *is the only vine function on* (G^i, D^i) *with* $L_{\alpha^i} = \varnothing$, *and* J_{G,D,α^i} *is the strict transform of* $J_{G^i,D^i} \subset \overline{\mathcal{J}}^d_{g,n}(\phi^+)$. *Following this, let* $\widetilde{\mathcal{J}}^d_{g,n}(\phi^+, \phi^-) := J_m$.

Similarly, let $G^i(P)$ *be the same vine curve* G^i *with an additional marked point* P *on the vertex that is not* $\mathrm{leg}_{G_i}(1)$, *and denote by* $D^i(P)$ *and* $\alpha^i(P)$ *the obvious lifts. Then define* $J_i(P)$ *inductively, starting from* $J_0(P) = \overline{\mathcal{J}}_{g,n+1}(\phi^+, P)$, *and then*

$$J_i(P) = \mathrm{Bl}_{J_{G^i(P),D^i(P),\alpha^i(P)}}(J_{i-1}),$$

and finally $\widetilde{\mathcal{J}}^d_{g,n+1}(\phi^+, \phi^-; P) := J_m(P)$.

The first point to observe is that this blowup does not depend upon the chosen extension to a total order.

Proposition 6.3. *Let* (G_i, D_i), *for* $i = 1, 2$, *be two pairs where* G_i *is a vine curve and* D_i *is an extremal bidegree. Set* α_i *to be the unique vine function on* (G_i, D_i) *such that* $L_{\alpha_i} = \varnothing$. *If* f_1, f_2 *are morphisms* $f_i \colon (G, D, \alpha) \to (G_i, D_i, \alpha_i)$ *where* L_α *contains*

$$\{V \in \mathrm{Ext}(G, D); V \subsetneq f_i^{-1}(v_i)\},$$

then $f_1^*(E(G_1)) \cap f_2^*(E(G_2)) = \varnothing$.

Proof. This follows from Proposition 6.1 and from a straightforward analysis of the possible common degenerations of two pairs (G_1, D_1) and (G_2, D_2) that satisfy the two inequalities given in *loc.cit.* □

We immediately deduce:

Corollary 6.4. *The blowup* $\widetilde{\mathcal{J}}^d_{g,n}(\phi^+, \phi^-) \to \overline{\mathcal{J}}^d_{g,n}(\phi^+)$ *is independent of the chosen extension to a total order. (It only depends on the partial order). The same is true of* $\widetilde{\mathcal{J}}^d_{g,n+1}(\phi^+, \phi^-; P)$.

Proof. By Proposition 6.3, if two vine curves are incomparable under the partial order, their intersection is transversal, hence swapping the order of the two blowups does not change the result. □

We let $\widetilde{\mathfrak{C}}$ be the category whose objects are (G, D, α), where α is a vine function with $L_\alpha = \mathrm{Ext}(G, D)$. The morphisms of $\widetilde{\mathfrak{C}}$ are given in Definition 5.14.

Remark 6.5. The category $\widetilde{\mathfrak{C}}$ defined in 5.28 is the category obtained by blowing up $\mathfrak{C}_{g,n}(\phi^+)$ (as in Definition 4.26) at (G^1, D^1), then at (G^2, D^2), $\dots$, and finally at (G^m, D^m). The case $m = 1$ was discussed in Example 4.33, and the general case follows in the same way. The category $\widetilde{\mathfrak{C}}_E$ is the subcategory of $\widetilde{\mathfrak{C}}$ generated by the exceptional divisors only.

The category $\widetilde{\mathfrak{C}}_Y$ is the subcategory of the stratification category of the stack $\widetilde{\mathcal{J}}^d_{g,n+1}(\phi^+, \phi^-; P)$ (same as $\widetilde{\mathfrak{C}}$, but with an extra marking), whose elements are the intersection of the components over the exceptional divisors that do not contain the first marking.

Our main result in this section is then

Theorem 6.6. *The stack $\widetilde{\mathcal{J}}^d_{g,n}(\phi^+, \phi^-)$ is nonsingular and the category $\widetilde{\mathfrak{C}} = \widetilde{\mathfrak{C}}(\phi^+, \phi^-)$ is its blowup stratification from $\overline{\mathcal{J}}^d_{g,n}(\phi^+)$. The same result holds for $\widetilde{\mathcal{J}}^d_{g,n+1}(\phi^+, \phi^-; P)$. Moreover, the forgetful morphism $\widetilde{\mathcal{J}}^d_{g,n+1}(\phi^+, \phi^-; P) \to \widetilde{\mathcal{J}}^d_{g,n}(\phi^+, \phi^-)$ is the quasistable modification of the universal curve.*

This follows as a combination of results in Section 4.d and Section 5.a.

Proof. To prove the first statement we apply Proposition 4.30 to the category $\mathfrak{C}_{g,n}(\phi)$ of Example 4.12. The fact that each stratum (G^i, D^i, α^i) has transversal self intersection in J_{i-1} is Proposition 5.25 (see also Remark 6.5).

The second part follows from Theorem 3.29, and the fact that blowup commutes with flat base change. □

Remark 6.7. Note that the vine curve strata G^i that are part of the datum of our blowup, do not necessarily have themselves transversal self-intersection (see Example 4.25) in $\overline{\mathcal{M}}_{g,n}$, so the procedure of Section 4.d cannot be applied to blow up the strata $G^1, \dots, G^m$ in $\overline{\mathcal{M}}_{g,n}$ to produce a nonsingular DM stack with a stratification.

Moreover, similarly to Example 4.25, one can also see that the strata (G^i, D^i) do not themselves have transversal self-intersection in $\overline{\mathcal{J}}^d_{g,n}(\phi^+)$. In our construction each stratum (G^i, D^i) only acquires a transversal self-intersection once lifted to a stratum of J_{i-1} by means of the function α^i.

Let Y' be the Cartier divisor in $\widetilde{\mathcal{J}}^d_{g,n+1}(\phi^+, \phi^-; P)$ given by the sum of all strata that correspond to (G, D, α), where (G, D) is a simple vine curve:

$$Y' = \sum_{i=1}^{m} J'_{G^i, D^i, \alpha^i}$$

Let $\mathcal{L}$ be the sheaf in $\widetilde{\mathcal{J}}^d_{g,n+1}(\phi^+, \phi^-; P)$ obtained by pulling back a tautological sheaf in $\overline{\mathcal{J}}^d_{g,n+1}(\phi^+, P)$ (see Theorem 3.29). We have the following result.

Theorem 6.8. *The line bundle $\mathcal{L}(-Y')$ is ϕ^--stable. In particular, the stack $\widetilde{\mathcal{J}}^d_{g,n}(\phi^+, \phi^-)$ comes with two morphisms that resolve the identity map*

$\overline{\mathcal{J}}^d_{g,n}(\phi^+) \dashrightarrow \overline{\mathcal{J}}^d_{g,n}(\phi^-)$. *The first is the blow-down morphism (also defined by $\mathcal{L}$), and the second is the morphism defined by $\mathcal{L}(-Y')$.*

Proof. By Proposition 3.10 and Remark 3.20, it is enough to check that $\mathcal{L}(-Y')$ is ϕ^- stable on all vine curves.

This follows from Construction 6.2. The divisor $Y' \subset \widetilde{\mathcal{J}}^d_{g,n+1}(\phi^+,\phi^-;P)$ is supported on the strata (G^i, D^i, α^i) of $\overline{\mathcal{J}}^d_{g,n}(\phi^+)$, which are exactly the vine curves where $\mathcal{L}$ fails to be ϕ^+-stable. Moreover, over each J'_{G^i,D^i}, the divisor Y' fiberwise intersects its complement at $t_i = |E(G^i)|$ points. Thus tensoring by $\mathcal{O}(-Y')$ has the effect of modifying the bidegree of $\mathcal{L}$ on each stratum (G^i, D^i, α^i) by $(-t_i, +t_i)$ (where the first element of the pair is the degree on the component of the vine curve that contains the first marking). Thus, because the bidegree D^i is extremal on G^i, the line bundle $\mathcal{L}(-Y')$ is ϕ^--stable on (G^i, D^i, α^i) for all $i = 1, \dots, m$. □

We conclude with the following observation.

Corollary 6.9. *The Cartier divisor $Y' \subset \widetilde{\mathcal{J}}^d_{g,n+1}(\phi^+,\phi^-;P)$ is simple normal crossing, and the stratification category it generates (as in Example 4.2) is $\widetilde{\mathfrak{C}}_Y$.*

Proof. The first part follows from Proposition 5.26. The second part follows directly from the definition of Y'. □

7. Wall–Crossing Formulas

Let $\phi^+, \phi^- \in V^d_{g,n}$ be on opposite sides of a stability hyperplane H (Definition 5.1). In this section we find a formula for the wall-crossing along H of Brill-Noether classes in terms of push-forward of boundary strata classes. We first give a formula in Theorem 7.4 on the nonsingular resolution $\widetilde{\mathcal{J}}^d_{g,n}(\phi^+,\phi^-)$ of the identity map

$$\mathsf{Id}\colon \overline{\mathcal{J}}^d_{g,n}(\phi^+) \dashrightarrow \overline{\mathcal{J}}^d_{g,n}(\phi^-)$$

that we defined in Section 6. Then we write a second formula in Corollary 7.24 by taking the push-forward of that difference along the blow-down morphism $p\colon \widetilde{\mathcal{J}}^d_{g,n}(\phi^+,\phi^-) \to \overline{\mathcal{J}}^d_{g,n}(\phi^+)$.

The universal quasistable family $\widetilde{\mathcal{J}}^d_{g,n+1}(\phi^+,\phi^-;P) \to \widetilde{\mathcal{J}}^d_{g,n}(\phi^+,\phi^-)$ carries two line bundles: the pullback $\mathcal{L}$ of a tautological line bundle on $\overline{\mathcal{J}}^d_{g,n}(\phi^+)$ and its modification $\mathcal{L}(-Y')$, the pullback of a tautological line bundle on $\overline{\mathcal{J}}^d_{g,n}(\phi^-)$ (see Theorem 6.8).

Our main result in Theorem 7.4 is a formula for the difference of the total Chern classes of the derived pushforward of $-\mathcal{L}(-Y')$ and that of $-\mathcal{L}$ as an explicit pushforward of classes supported on the boundary. Because on the (unprimed) "resolved" strata the normal bundles split as a direct sum of line bundles, our formula is better written on the "resolved" strata instead of the embedded ones.

Before stating the main results, let us fix some notation, for Theorem 7.4 and for Corollary 7.24.

For each pair $(G, D) \in \mathfrak{C}_{g,n}(\phi^+)$, denote by $\pi_{G,D}\colon \mathcal{C}_{G,D} \to \mathcal{J}_{G,D}$ the pullback to $\mathcal{J}_{G,D}$ of the universal quasistable family $\overline{\mathcal{J}}^d_{g,n+1}(\phi^+; P) \to \overline{\mathcal{J}}^d_{g,n}(\phi^+)$. The total space $\mathcal{C}_{\mathcal{J}_{G,D}}$ has one irreducilbe component $\mathcal{C}^+_v := \mathcal{C}_{G,D,v}$ for each vertex v of G. We denote by $\pi^+_v := \pi_{G,D,v}\colon \mathcal{C}^+_v \to \mathcal{J}_{G,D}$ the induced map. Also, for each $V \subset V(G)$, we denote by $\pi^+_V\colon \bigcup_{v\in V} \mathcal{C}^+_v \to \mathcal{J}_{G,D}$ the induced map on the union. We write $X^+ = X^+_{G,D} := \mathcal{C}^+_{\mathrm{leg}(1)}$ and $\Sigma^+ = X^+ \cap \mathcal{C}^+_{\{\mathrm{leg}(1)\}^c}$. We also write $Y^+_V = \mathcal{C}^+_{V^c}$ for every $V \subset V(G)$.

We can extend these notations to $\widetilde{\mathcal{J}}^d_{g,n}(\phi^+, \phi^-)$. Let us recall the geometry of $\widetilde{\mathcal{J}}^d_{g,n}(\phi^+, \phi^-)$ from Section 6. For each $1 \leq i \leq m$ set $\beta_i := (G_i, D_i)$ to be the vine curves strata from Construction 6.2, so $\mathcal{J}_{\beta_i}$ are the strata of $\overline{\mathcal{J}}^d_{g,n}(\phi^+)$ whose strict transforms of the images are blown-up, in the given order, in $\overline{\mathcal{J}}^d_{g,n}(\phi^+)$, to obtain $\widetilde{\mathcal{J}}^d_{g,n}(\phi^+, \phi^-)$. Recall that $\widetilde{\mathfrak{C}}_E$ is the category of the (resolutions of the closed) strata of $\widetilde{\mathcal{J}}^d_{g,n}(\phi^+, \phi^-)$ that are in the intersection of the exceptional divisors E'_i.

Objects are triples $(G, D, V_\bullet)$ for $(G, D) \in \mathfrak{C}_{g,n}(\phi)$. Each stratum $\widetilde{\mathcal{J}}_{G,D,V_\bullet}$ admits a forgetful morphism $p_{G,D,V_\bullet}$ to $\mathcal{J}_{G,D}$, and we define π_v, π_V, X, Σ and Y_V as the pullbacks via $p_{G,D,V_\bullet}$ of the corresponding items defined in the previous paragraph for (G, D). Also, we set $Y_{G,D,V_\bullet} := \bigcap_{V\in V_\bullet} Y_V$, note that, by Proposition 5.7, $Y_{G,D,V_\bullet}$ is nonempty if and only if $V_\bullet$ is a chain (as in the definition of $\widetilde{\mathfrak{C}}_Y$ in Definition 5.28), and in that case, we have that $Y_{G,D,V_\bullet} = Y_{\max(V_\bullet)}$.

Also, for some triple $(G, D, V_\bullet)$, we define

(7.1)
$$F^X_+ := -R^\bullet(\pi^+_X)_*\mathcal{L}^+(-\Sigma^+)_{|X^+};\ F^+_V := -R^\bullet(\pi_{V^c})_*\mathcal{L}^+_{|Y^+_V};\ H^+_V := F^+_V - \sum_{\substack{V'\in V_\bullet,\\ V'\gtrdot V}} F^+_{V'}$$

and

(7.2)
$$F^X := -R^\bullet(\pi_X)_*\mathcal{L}(-\widetilde{\Sigma})_{|X};\ F_V := -R^\bullet(\pi_{V^c})_*\mathcal{L}_{|Y_V};\ \text{and}\quad H_V := F_V - \sum_{\substack{V'\in V_\bullet,\\ V'\gtrdot V}} F_{V'}.$$

Note also that $\mathcal{L}$ is the pullback of $\mathcal{L}^+$, hence F^X, F_V and H_V are the pullback via $p_{G,D,V}$ of F^X_+, F^+_V and H^+_V respectively.

Let E_i be the exceptional stratum of the blowup morphism $\widetilde{\mathcal{J}}^d_{g,n}(\phi^+, \phi^-) \to \overline{\mathcal{J}}^d_{g,n}(\phi^+)$, so that $E_i \to E'_i \subset \widetilde{\mathcal{J}}^d_{g,n}(\phi^+, \phi^-)$ is the exceptional divisor, and each E'_i is contracted to $\mathcal{J}'_{\beta_i}$. Following the notation in the previous paragraph, we let then $X'_i \cup Y'_i$ denote the two irreducible components of the restriction to E'_i of the universal quasistable family $\widetilde{\mathcal{J}}^d_{g,n+1}(\phi^+, \phi^-; P) \to \widetilde{\mathcal{J}}^d_{g,n}(\phi^+, \phi^-)$, where X'_i is the component containing the first marked point and Y'_i is the other component, and denote by X_i and Y_i the base change to E_i of X'_i and Y'_i. Recall that the divisor Y' in Theorem 6.8 is precisely $\sum_i Y'_i$.

We now define psi-classes following (4.34). Each edge $e \in E(G)$ defines a morphism $f_e \colon \mathcal{J}_{G,D} \to \mathcal{J}_{G',D'}$ to some codimension one stratum (G', D'), and we set

$$\Psi_{G',D'} := -c_1\left(N_{\mathcal{J}_{G',D'}}\overline{\mathcal{J}}^d_{g,n}(\phi^+)\right), \quad \Psi_{(G,D,e)} := f_e^*(\Psi_{G',D'}). \tag{7.3}$$

In Remark 7.30 we will discuss how these compare to the usual psi-classes on $\overline{\mathcal{M}}_{g,n}$.

Similarly, for a triple $(G, D, V_\bullet)$ in $\widetilde{\mathfrak{C}}_E$, we have that $S_{G,D,V_\bullet} = V_\bullet$ (recall the definition of $S_{G,D,V_\bullet}$ in Section 4.a Item (4)) and each $V \in V_\bullet$ defines a morphism $f_{G,D,V} \colon \widetilde{\mathcal{J}}_{G,D,V_\bullet} \to E_i$ for some $i = 1, \ldots, m$. As in Section 4.b Item (3), we set $\mathbb{L}_V := f^*_{G,D,V} N_{E_i}(\widetilde{\mathcal{J}}^d_{g,n}(\phi^+, \phi^-)$ and define the psi-classes

$$\Psi_i := -c_1\left(N_{E_i}(\widetilde{\mathcal{J}}^d_{g,n}(\phi^+, \phi^-)\right), \quad \Psi_{G,D,V} := f^*_{G,D,V}(\Psi_i) = -c_1(\mathbb{L}_V)$$

on E_i and on $\widetilde{\mathcal{J}}_{G,D,V_\bullet}$ respectively.

Finally, define the coefficient

$$b_{G,D,V}((j_V)_{V\in V_\bullet}; (k_V)_{V\in V_\bullet}) := -\binom{k_V + g_V - d_V - \sum_{V'\geq V} j_{V'} + (k_{V'}+1)}{k_V + 1}$$

for each vectors $(j_V \geq 0)_{V\in V_\bullet}$ and $(k_V \geq 0)_{V\in V_\bullet}$ of nonnegative integers.

Theorem 7.4. *The difference of total Chern classes*

$$c_t(-R^\bullet\pi_*\mathcal{L}) - c_t(-R^\bullet\pi_*\mathcal{L}(-Y')) \tag{7.5}$$

in $A^*\left(\widetilde{\mathcal{J}}^d_{g,n}(\phi^+, \phi^-)\right)$ *equals the boundary class*

(7.6)
$$\sum_{\substack{\Gamma=(G,D,V_\bullet)\in \\ \widetilde{\mathfrak{C}}_E\setminus\{\bullet\}}} \frac{-f_{\Gamma *}}{|\mathrm{Aut}(\Gamma)|}\left(\sum_{\substack{(j_V\geq 0)_{V\in V_\bullet}, \\ (k_V\geq 0)_{V\in V_\bullet} \\ s\geq 0}} c_s(\widetilde{F}^X)\cdot \prod_{V\in V_\bullet} b_{G,D,V}((j_V)_V, (k_V)_V)\cdot c_{j_V}(\widetilde{H}_V)\cdot \Psi_V^{k_V}\right),$$

where the sum runs over all resolved strata $\Gamma \in \widetilde{\mathfrak{C}}_E$ *(intersection of exceptional divisors) of* $\widetilde{\mathcal{J}}^d_{g,n}(\phi^+, \phi^-)$ *(see Section 6), except the terminal object (the open stratum).*

Note that Formula (7.6) gives a total Chern class from which one can immediately deduce the difference of the Brill–Noether classes on $\widetilde{\mathcal{J}}^d_{g,n}(\phi^+, \phi^-)$. This is a more direct formula than e.g. the main result of [PRvZ20], where an explicit formula is given for the Chern character, which then requires inversion to obtain the desired Cher class.

Proof. We start our calculation by making use of the short exact sequence

$$0 \to \mathcal{L}(-Y') \to \mathcal{L} \to \mathcal{L}|_{Y'} \to 0. \tag{7.7}$$

on the quasistable family $\widetilde{\mathcal{J}}^d_{g,n+1}(\phi^+,\phi^-;P) \to \widetilde{\mathcal{J}}^d_{g,n}(\phi^+,\phi^-)$. For convenience, define

$$F := -R^\bullet\pi_*(\mathcal{L}), \quad \widetilde{F} := R^\bullet\pi_*(\mathcal{L}_{|Y'}).$$

We then apply Whitney's formula

$$c_t(-R^\bullet\pi_*(\mathcal{L}(-Y'))) - c_t(-R^\bullet\pi_*(\mathcal{L})) = (c_t(\widetilde{F}) - 1)\cdot c_t(F) \tag{7.8}$$

for the total Chern class of the three terms in (7.7). This computes the opposite of (7.5). From now on, we will mostly work on the term $c_t(\widetilde{F}) - 1$.

For each $\Gamma' \in \widetilde{\mathfrak{C}}_E$ we let

$$F^Y_{\Gamma'} := -R^\bullet\pi_*\mathcal{L}_{|Y_{\Gamma'}}.$$

We now apply the following

Lemma 7.9. *(Inclusion-exclusion principle for a simple normal crossing stratification.) Let $\mathcal{D}$ be a simple normal crossing divisor in X, and let $\mathfrak{C}$ be its category of strata. Then the following equality holds in the rational K-theory of X:*

$$\mathcal{L}|_{\mathcal{D}} = \sum_{\alpha\in\mathfrak{C}} (-1)^{\mathrm{cd}(\alpha)-1}\mathcal{L}|_{\mathcal{D}_\alpha}$$

By combining Lemma 7.9 with Lemma 6.9 (the fact that the Y_i' are indeed simple normal crossing), and the multiplicativity of the total Chern class, together with the fact that $Y_\Gamma \to Y'_\Gamma$ is étale of degree $|\operatorname{Aut}(\Gamma)|$ (by Corollary 6.9), we obtain

(7.10)

$$c_t(F)\cdot(-1 + c_t(\widetilde{F})) = c_t(F)\cdot\left(-1 + \prod_{\Gamma'\in\widetilde{\mathfrak{C}}_Y\setminus\{\bullet\}} c_t\left((-1)^{\mathrm{cd}\,\Gamma'}\frac{f_{\Gamma'*}F^Y_{\Gamma'}}{|\operatorname{Aut}(\Gamma')|}\right)\right)$$

where $\widetilde{\mathfrak{C}}_Y \subseteq \widetilde{\mathfrak{C}}_E$ is the image in $\widetilde{\mathcal{J}}^d_{g,n}(\phi^+,\phi^-)$ of the stratification induced by $Y_1',\ldots,Y_m'$ (and in the product we have removed its terminal object), and

$$f_{\Gamma'}\colon \widetilde{\mathcal{J}}_{\Gamma'} \to \widetilde{\mathcal{J}}^d_{g,n}(\phi^+,\phi^-) =: \widetilde{\mathcal{J}}$$

is the (resolution of the closed) stratum $\widetilde{\mathcal{J}}'_{\Gamma'}$.

For a fixed Γ', we now aim to write each factor of the product in the RHS of (7.10) as a pushforward via the corresponding stratum. We apply Formula (4.22) (GRR for the total Chern class) to obtain that each factor

$$c_t\left((-1)^{\mathrm{cd}\,\Gamma'}\frac{f_{\Gamma'*}F^Y_{\Gamma'}}{|\operatorname{Aut}(\Gamma')|}\right)$$

equals
(7.11)

$$1+\sum_{\substack{\Gamma\in\widetilde{\mathfrak{C}}_E,\\ k\geq 1}}\frac{f_{\Gamma *}}{|\mathrm{Aut}(\Gamma)|}\left(\sum_{\substack{\{f_1,\dots,f_k\}\in\\ \mathrm{SInt}((f_\Gamma)^k)_{f_{\Gamma'}}}}\frac{\prod_{j=1}^k\left(f_j^*c_t(\left(\bigwedge^\bullet N^\vee_{\widetilde{\mathcal{J}}_\Gamma}\widetilde{\mathcal{J}}\otimes(-1)^{\mathrm{cd}(\Gamma')}F^Y_{\Gamma'}\right)-1\right)}{c_{\mathrm{top}}N_{\widetilde{\mathcal{J}}_\Gamma}\widetilde{\mathcal{J}}}\right).$$

Note that intersections in $\widetilde{\mathfrak{C}}_Y$ are not necessarily objects of the latter category, so the product is taken over $\widetilde{\mathfrak{C}}_E\supseteq\widetilde{\mathfrak{C}}_Y$.

After that, to continue our derivation from Formula (7.10), we aim to calculate the product of the terms in (7.11) for varying Γ'. We apply the excess intersection formula Proposition 4.19 to the product (7.10), to deduce that it equals

$$\sum_{\Gamma\in\widetilde{\mathfrak{C}}_E\setminus\{\bullet\}}\frac{1}{|\mathrm{Aut}(\Gamma)|}f_{\Gamma *}\left(\sum_{\substack{\Gamma_1,\dots,\Gamma_k\in\widetilde{\mathfrak{C}}_Y\\ Q_t\subseteq\overline{\mathrm{Mor}}(\Gamma,\Gamma_t)\text{ for all }t=1,\dots,k\\ \text{such that }\cup_t Q_t\text{ is generic}}}\frac{c_t(F_\Gamma)\cdot\prod_{\substack{j=1,\dots,k,\\ f\in Q_j}}\left(f^*c_t(\left(\bigwedge^\bullet N^\vee_{\widetilde{\mathcal{J}}_{\Gamma_j}}\widetilde{\mathcal{J}}\otimes(-1)^{\mathrm{cd}(\Gamma_j)}F^Y_{\Gamma_j}\right)-1\right)}{c_{\mathrm{top}}N_{\widetilde{\mathcal{J}}_\Gamma}\widetilde{\mathcal{J}}}\right).\tag{7.12}$$

Now we focus on simplifying the term inside the pushforward $f_{\Gamma *}$. Following Proposition 4.6.for fixed $\Gamma=(G,D,V_\bullet)$, there is a natural bijection between the set of morphisms

$$\Big\{\{Q_t\subseteq\mathrm{Aut}(\Gamma)\backslash\,\mathrm{Mor}(\Gamma,\Gamma_t)\}_{t=1,\dots,k}\text{ for }\Gamma_1,\dots,\Gamma_k\in\widetilde{\mathfrak{C}}_Y\text{ s.t. }\cup_t Q_t\text{ is generic}\Big\}$$

and the set

$$\big\{\{\ell_1,\dots,\ell_M\}\subseteq\mathrm{chains}(V_\bullet)\text{ such that }V_\bullet=\cup_{i=1}^M\ell_i\big\}$$

with $M=|Q_1|+\ldots+|Q_k|$, given by

$$\{Q_1,\dots,Q_k\}\mapsto\bigcup_{t=1}^k\{f^*(V_{\Gamma_j,\bullet})\}_{f\in Q_j}.$$

Moreover, if $f_\ell\colon\Gamma\to\Gamma_t$ is a contraction that corresponds to the chain $\ell\subseteq V_\bullet$, then $f_\ell^*(F^Y_{\Gamma_t})=F_{\Gamma,\max(\ell)}$ (in particular, the latter only depends on $\max(\ell)\in V_\bullet$, and not on the whole chain). Furthermore, the pullback $f_\ell^*(N_{\widetilde{\mathcal{J}}_{\Gamma_t}}\widetilde{\mathcal{J}})$ equals a direct sum of line bundles, which allows us to expand the wedge product

$$\bigwedge^\bullet f_\ell^*(N_{\widetilde{\mathcal{J}}_{\Gamma_t}}\widetilde{\mathcal{J}})=\bigwedge^\bullet\bigoplus_{V\in\ell}\mathbb{L}_V=\sum_{S\subseteq\ell}(-1)^{|S|}\bigotimes_{V\in S}\mathbb{L}_V.$$

In light of this, we rewrite the numerator inside the pushforward via f_Γ in (7.12) as

$$\sum_{\substack{\Gamma_1,\ldots,\Gamma_k\in\widetilde{\mathfrak{C}}_Y\\ Q_t\subseteq\mathrm{Mor}(\Gamma,\Gamma_t),t=1,\ldots,k\\ \text{s. t. } \cup_t Q_t \text{ is generic}}} c_t(F_\Gamma)\cdot\prod_{\substack{j=1,\ldots,k,\\ f\in Q_j}}\left(f^*c_t\left(\bigwedge^{\bullet} N^\vee_{\widetilde{\mathcal{J}}_{\Gamma_j}}\widetilde{\mathcal{J}}\otimes(-1)^{\mathrm{cd}\,\Gamma_j}F^Y_{\Gamma_j}\right)-1\right),$$

which equals

$$c_t(F_\Gamma)\cdot\sum_{\substack{\{\ell_1,\ldots,\ell_M\}\subseteq\\ \mathrm{chains}(V_\bullet)\text{ s.t.}\\ \ell_1\cup\ldots\cup\ell_M=V_\bullet}}\prod_{i=1}^{M}\prod_{S\subseteq\ell_i}\left(c_t\left((-1)^{|S|}\bigotimes_{V\in\ell_i}\mathbb{L}_V^\vee\otimes(-1)^{|\ell_i|}F_{\Gamma,\max(\ell_i)}\right)-1\right).\tag{7.13}$$

Next, we apply the inclusion-exclusion principle in the form

$$\sum_{\{\ell_1,\ldots,\ell_M\}\subseteq\mathrm{chains}(V_\bullet)}\varphi(\ell_1,\ldots,\ell_M)=\sum_{K\subseteq V_\bullet}\sum_{\substack{\{\ell_1,\ldots,\ell_M\}\subseteq\\ \mathrm{chains}(V_\bullet)\text{ s.t.}\\ \ell_1\cup\ldots\cup\ell_M=V_\bullet\setminus K}}(-1)^{|K|}\varphi(\ell_1,\ldots,\ell_M)$$

for any function $\varphi\colon$ chains $V_\bullet\to\mathbb{Z}$, to eliminate the condition that $\bigcup_{i=1}^M\ell_i=V_\bullet$ in the last set of indices of (7.13). We thus obtain that (7.13) equals

$$\sum_{K\subseteq V_\bullet}(-1)^{|K|}c_t(F_\Gamma)\cdot\prod_{S\in\mathrm{chains}(V_\bullet\setminus K)}c_t\left(\bigotimes_{V\in S}\mathbb{L}_V^\vee\otimes\sum_{\substack{\ell\in\mathrm{chains}(V_\bullet\setminus K)\\ \text{such that }S\subseteq\ell}}(-1)^{|S|+|\ell|}F_{\Gamma,\max(\ell)}\right)\tag{7.14}$$

We now apply Lemma 7.18 to simplify (7.14), so it becomes

$$\sum_{K\subseteq V_\bullet}(-1)^{|K|}c_t\left(F_X\right)\cdot\prod_{V_0\in V_\bullet\setminus K}c_t\left(\bigotimes_{\substack{V\in V_\bullet\setminus K\\ V\leq V_0}}\mathbb{L}_V^\vee\otimes H_{K,V_0}\right).\tag{7.15}$$

After all these simplifications, we now go back and replace (7.15) as the numerator of the term in (7.12) that is pushed forward via f_Γ, to obtain that (7.12) equals

$$\sum_{\Gamma\in\widetilde{\mathfrak{C}}_E\setminus\{\bullet\}}\frac{f_{\Gamma*}}{|\mathrm{Aut}(\Gamma)|}\left(\frac{\sum_{K\subseteq V_\bullet}(-1)^{|K|}c_t\left(F_X\right)\cdot\prod_{V_0\in V_\bullet\setminus K}c_t\left(\bigotimes_{V\leq V_0}\mathbb{L}_V^\vee\otimes H_{K,V_0}\right)}{c_{\mathrm{top}}N_{\widetilde{\mathcal{J}}_\Gamma}\widetilde{\mathcal{J}}}\right).\tag{7.16}$$

Our final step to conclude repeatedly uses Formula (3.33) for the total Chern class of the tensor product of a K-theory element times a line bundle, and then divide by

$$c_{\mathrm{top}} N_{\widetilde{\mathcal{J}}_\Gamma} \widetilde{\mathcal{J}} = \prod_{V \in V_\bullet} -\Psi_V. \tag{7.17}$$

After combining the binomial coefficients by means of Vandermonde's identity, we obtain that Formula (7.16) equals the final formula (7.6). (One way to obtain the formula is to consider only the case $K = \varnothing$ in (7.15), then expanding as a polynomial in $\{\Psi_V\}_{V \in V_\bullet}$, and considering only the monomial containing $\prod_{V \in V_\bullet} \Psi_V^{a_V}$ for all $a_V \geq 1$, and then lowering the exponents a_V by one because of the division by the term in (7.17)). □

We now prove the ancillary results used in the proof of Theorem 7.4.

Lemma 7.18. *Let $V_\bullet$ be a rooted forest, and let $(x_V)_{V \in V_\bullet}$ be formal variables. Let $S \subseteq V_\bullet$ be a chain in $V_\bullet$. Then*

$$\sum_{\substack{\ell \in \mathrm{chains}(V_\bullet) \\ \text{such that } S \subseteq \ell}} (-1)^{|S|+|\ell|} x_{\max(\ell)}$$

equals

(1) $-\sum_{V \in \min(V_\bullet)} x_V$, *if $S = \varnothing$*
(2) 0, *if there is $V \in V_\bullet \setminus S$ such that $V < \max(S)$ and $S \cup \{V\}$ is a chain, and*
(3) $x_{\max(S)} - \sum_{\substack{V \in V_\bullet \\ V > \max(S)}} x_V$, *in all other cases.*

Proof. Assume that S is nonempty and that there exists $V \in V_\bullet \setminus S$ such that $V < \max(S)$ and $S \cup \{V\}$ still is a chain. Then we can write

$$\sum_{\substack{C \in \mathrm{chains}(V_\bullet) \\ S \subset C}} (-1)^{|S|+|C|-1} x_{\max(C)} =$$
$$= \sum_{\substack{C \in \mathrm{chains}(V_\bullet) \\ S \subset C, V \notin C}} (-1)^{|S|+|C|-1} x_{\max(C)} + (-1)^{|S|+|C \cup \{V\}|-1} x_{\max(C \cup \{V\})}$$

since $\max(C) = \max(C \cup \{V\})$, we have that the sum is 0.

Assume that S is nonempty and denote by $\max(S) = V_0$. Also assume that $S = \{V \in V_\bullet : V \leq V_0\}$. Then we can write

$$\sum_{\substack{C \in \mathrm{chains}(V_\bullet) \\ S \subseteq C}} (-1)^{|S|+|C|-1} x_{\max(C)} = \sum_{V \in V_\bullet} x_V \sum_{\substack{C \in \mathrm{chains}(V_\bullet) \\ S \subseteq C, \max(C) = V}} (-1)^{|S|+|C|-1}$$

If $V = V_0$, then the condition $S \subseteq C$ and $\max(C) = V_0$ is equivalent to $C = S$, so

$$\sum_{\substack{C \in \mathrm{chains}(V_\bullet) \\ S \subseteq C, \max(C) = V}} (-1)^{|S|+|C|-1} = -1.$$

If $V \gtrdot V_0$, then the condition $S \subset C$ and $\max(C) = V_0$ is equivalent to $C = S \cup \{V\}$, so

$$\sum_{\substack{C \in \mathrm{chains}(V_\bullet)\\ S \subseteq C, \max(C)=V}} (-1)^{|S|+|C|-1} = 1.$$

If $V < V_0$, then the sum is empty, and so it is 0.

If $V \gg V_0$, choose V' such that $V_0 < V' < V$, and then

$$\sum_{\substack{C \in \mathrm{chains}(V_\bullet)\\ S \subseteq C, \max(C)=V}} (-1)^{|S|+|C|-1} = \sum_{\substack{C \in \mathrm{chains}(V_\bullet),\\ S \subseteq C, \max(C)=V\\ V' \notin C}} (-1)^{|S|+|C|-1} + (-1)^{|S|+|C \cup \{V'\}|-1}$$

which equals 0.

The case $S = \varnothing$ is similar. □

And now the inclusion-exclusion principle:

Proof. (of Lemma 7.9). It is enough to prove the statement for the case of the structure sheaf $\mathcal{L} = \mathcal{O}$. Assuming that $\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2$, we have the short exact sequences

$$\begin{gathered} 0 \to \mathcal{O}(-\mathcal{D}_1 - \mathcal{D}_2) \to \mathcal{O} \to \mathcal{O}_{\mathcal{D}_1+\mathcal{D}_2} \to 0 \\ 0 \to \mathcal{O}(-\mathcal{D}_1 - \mathcal{D}_2) \to \mathcal{O}(-\mathcal{D}_2) \to \mathcal{O}_{\mathcal{D}_1}(-\mathcal{D}_2) \to 0 \\ 0 \to \mathcal{O}(-\mathcal{D}_2) \to \mathcal{O} \to \mathcal{O}_{\mathcal{D}_2} \to 0 \\ 0 \to \mathcal{O}_{\mathcal{D}_1}(-\mathcal{D}_2) \to \mathcal{O}_{\mathcal{D}_1} \to \mathcal{O}_{\mathcal{D}_1 \cap \mathcal{D}_2} \to 0. \end{gathered}$$

By combining these, we obtain the equality

$$\mathcal{O}_{\mathcal{D}_1+\mathcal{D}_2} = \mathcal{O}_{\mathcal{D}_1} + \mathcal{O}_{\mathcal{D}_2} - \mathcal{O}_{\mathcal{D}_1 \cap \mathcal{D}_2}.$$

The statement is then obtained by repeatedly decomposing $\mathcal{D}$ until all summands are irreducible. □

Our next task is to take the push-forward of Formula (7.6) via the blow-down morphism $p\colon \widetilde{\mathcal{J}}^d_{g,n}(\phi^+, \phi^-) \to \overline{\mathcal{J}}^d_{g,n}(\phi^+)$, to produce an explicit graph formula for the difference of the Brill–Noether classes. Recall the notation for the strata $\mathcal{J}_{G,D}$ that was set in the beginning of this section.

For $V \in V_\bullet$, we define the "close upper edges" and "far upper edges" as

$$\mathrm{CU}(V) := E(V, \mathrm{next}(V) \setminus V) \text{ and } \mathrm{FU}(V) := E(V, \mathrm{next}(V)^c) \tag{7.19}$$

so there is a decomposition

$$E(V, V^c) = \mathrm{CU}(V) \sqcup \mathrm{FU}(V).$$

For every collection $(g_V)_{V \in V_\bullet}$ of nonnegative integers, define the class

$$c_{(G,D,V_\bullet)}((g_V)_{V \in V_\bullet}) \in A^\bullet(\overline{\mathcal{J}}_{(G,D)})$$

to equal

$$\sum_{\substack{(a_{e,V})_{V\in V_\bullet, e\in \mathrm{FU}(V)}\\ (g_{e,V})_{V\in V_\bullet, e\in \mathrm{CU}(V)}\\ \text{such that, for all } V\in V_\bullet,\\ \sum(g_{e,V}+1)-\sum a_{e,V}=g_V+1}} \prod_{e\in E(G)} \Psi_{(G,D,e)}^{g_{e,k(e)}-\sum_{V\in S(e)} a_{e,V}} \cdot \\ \cdot \prod_{V\in S(e)} (-1)^{a_{e,V}} \binom{g_{e,k(e)}-\sum_{\substack{V'\in S(e),\\ V\subsetneq V'}} a_{e,V'}}{a_{e,V}} \tag{7.20}$$

where, the $a_{e,V}$ and $g_{e,V}$ vary over the nonnegative integers, and for $e \in E(G)$, we let $k(e) \in V_\bullet$ be the unique (by Proposition 5.19) element such that $e \in \mathrm{CU}(k(e))$, and we let $S(e) := \{V \in V_\bullet : e \in \mathrm{FU}(V)\}$.

For a given specialization $h\colon (G', D', V'_\bullet) \to (G, D, V_\bullet)$, we define the pull-back $h^*((g_V)) = h^*((g_V)_{V\in V_\bullet})_{V'\in V'_\bullet}$ by:

$$h^*((g_V)_{V\in V_\bullet})_{V'} := \begin{cases} g_V & \text{if } V' = h^{-1}(V) \text{ for some } V \in V_\bullet; \\ -1 & \text{otherwise.} \end{cases} \tag{7.21}$$

We have then the following pushforward result.

Proposition 7.22. *The following pushforward formula holds*

$$p_*\left(\frac{f_{(G,D,V_\bullet)*}}{|\mathrm{Aut}(G,D,V_\bullet)|}\left(\prod_{V\in V_\bullet} \Psi_V^{g_V}\right)\right) = \\ \sum_{(G',D')\in \mathfrak{C}_{g,n}(\phi)} \frac{f_{(G',D')*}}{|\mathrm{Aut}(G',D')|}\left(\sum_{\substack{V'_\bullet \text{ a full forest in } \mathrm{Ext}(G',D'),\\ h\in \overline{\mathrm{Mor}}((G',D',V'),(G,D,V))}} c_{(G',D',V'_\bullet)}(h^*((g_V)))\right) \tag{7.23}$$

Proof. Follows from Corollary 4.35. □

Our final step is to take the pushforward of our formula in Theorem 7.4 via the blowdown morphism $p\colon \widetilde{\mathcal{J}}^d_{g,n}(\phi^+,\phi^-) \to \overline{\mathcal{J}}^d_{g,n}(\phi^+)$. Note first that the K-theory elements defined in (7.1) are pull-backs via p of similar classes, which we will denote with the same name in the next result. The pushforward via p is then obtained by combining Theorem 7.4 and Proposition 7.22.

Corollary 7.24. *The difference* $\mathsf{w}_d(\phi^+) - \mathsf{Id}^*\mathsf{w}_d(\phi^-)$ *in* $A^{g-d}\big(\overline{\mathcal{J}}^d_{g,n}(\phi^+)\big)$ *equals*

$$- \sum_{(G,D)\in \mathfrak{C}_{g,n}(\phi)} \frac{1}{|\operatorname{Aut}(G,D)|} f_{(G,D)*} \\ \left(\sum_{\substack{V_\bullet \text{ a full forest in } \mathrm{Ext}(G,D),\\ s+\sum_V j_V+\sum g_e = g-d-|E(G)|}} \alpha(s,(j_V),(g_e))\cdot c_s(F_+^X)\cdot \prod_{V\in V_\bullet} c_{j_V}(H_V^+) \prod_e \Psi_{(G,D,e)}^{g_e}\right), \tag{7.25}$$

where each coefficient $\alpha(s,(j_V)_{V\in V_\bullet},(g_e)_{e\in E(G)})$ *is defined by*

$$\sum_{(a_{e,V})_{(e,V)}} (-1)^{|V_\bullet|} \prod_{e\in E(G)} \prod_{V\in S(e)} (-1)^{a_{e,V}} \binom{g_e+\sum_{\substack{V'\in S(e),\\ V'\subseteq V}} a_{e,V'}}{a_{e,V}} \cdot \tag{7.26}$$

$$\prod_{V\in V_\bullet} \binom{\left(\sum_{V\subseteq V'}(\operatorname{rk} H^+_{V'}-j_{V'})\right)-\left(\sum_{e\in \mathrm{CN}(V)}(g_e+1+\sum_{\substack{V'\subseteq V,\\ e\in \mathrm{FU}(V')}} a_{e,V'})\right)}{\sum_{e\in \mathrm{CU}(V)} g_e+1+\sum_{\substack{e\in\mathrm{CU}(V)\\ V'\in S(e)}} a_{e,V'}-\sum_{e\in\mathrm{FU}(V)} a_{e,V}},$$

where $\mathrm{CN}(V) := E(\mathrm{next}(V),\mathrm{next}(V)^c)$*, each* $a_{(e,V)}$ *ranges over the integers, and the indices* (e,V) *range over all* $V\in V_\bullet$ *and over all* $e\in E(V,\mathrm{FU}(V))$*.*

(Note that, because of the last binomial, the summand is zero except for finitely many natural number values of $a_{e,V}$, Also, note that H^+_V depends on $V_\bullet$).

Proof. First observe that the result amounts to taking the degree $g-d$ part of the pushforward via p of Formula (7.6).

The calculation that we are attempting has the form

$$p_*\left(\sum_{(G,D,V_\bullet)\in\widetilde{\mathfrak{C}}_E} \frac{f_{(G,D,V_\bullet)*}}{|\mathrm{Aut}(G,D,V_\bullet)|}\left(\sum_{(g_V)_{V\in V_\bullet}} p^*_{(G,D)}(\beta_{(g_V)_{V\in V_\bullet}})\prod_{V\in V_\bullet}\Psi_V^{g_V}\right)\right). \tag{7.27}$$

for suitable classes $\beta_{(g_V)_{V\in V_\bullet}}\in A^*(\overline{\mathcal{J}}_{(G,D)})$ as in (7.6). Since F^X and H_V are pullback of F^X_+ and H^+_V, by the push-pull formula, and by Proposition 7.22, we obtain that (7.27) equals

$$\sum_{(G',D')\in\mathfrak{C}_{g,n}(\phi)} \frac{f_{(G',D')*}}{|\mathrm{Aut}(G',D')|} \left(\sum_{\substack{V'_\bullet \text{ a full forest in } \mathrm{Ext}(G',D');\\ h\in\overline{\mathrm{Mor}}((G',D',V'_\bullet),(G,D,V_\bullet))\\ (g_V\geq 0)_{V\in V_\bullet}}} h^*\beta_{(g_V)_{V\in V_\bullet}}\cdot c_{(G',D',V'_\bullet)}(h^*((g_V)))\right). \tag{7.28}$$

For a tuple $(g_{V'}\geq -1)_{V'\in V'_\bullet}$ we define $h\colon (G',D',V'_\bullet)\to(G,D,V_\bullet)$ as the unique contraction with the property that $g_{V'}\geq 0$ if and only $V'=h^{-1}(V)$ for some $V\in V_\bullet$. That is, we contract each collection of vertices V' such that $g_{V'}=-1$ (see 7.21). We then define $\beta_{(g_{V'})_{V'\in V'_\bullet}} := h^*(\beta_{(g_V)_{V\in V_\bullet}})$. Formula (7.28) can then be simplified to

$$\sum_{(G',D')\in\mathfrak{C}_{g,n}(\phi)} \frac{f_{(G',D')*}}{|\mathrm{Aut}(G',D')|}\left(\sum_{\substack{V' \text{ a full forest}\\ \text{in } \mathrm{Ext}(G',D');\\ (g_{V'}\geq -1)_{V'\in V'_\bullet}}} \beta_{(g_{V'})_{V'\in V'_\bullet}}\cdot c_{(G',D',V'_\bullet)}((g_{V'}))\right). \tag{7.29}$$

Now to obtain the final result, we rename (G', D') and $V'_\bullet$ into (G, D) and $V_\bullet$. Then we eliminate the indices (g_V) by means of the equality

$$g_V = -1 + \sum (g_{e,V} + 1) - \sum a_{e,V},$$

and we replace the indices $(g_{e,V})$ with indices (g_e) defined by $g_e := g_{e,k(e)} - \sum_{V \in S(e)} a_{e,V}$. □

As promised earlier, here we compare the ψ classes on Jacobians with the classical ones on moduli of curves.

Remark 7.30. Denote by $f \colon \mathcal{J}_{G,D} \to \overline{\mathcal{M}}_G$ the forgetful morphism. For every $e \in E(G)$, we have:

$$\Psi_{(G,D,e)} = f^* \Psi_{G,e} + \Delta_{G,D,e}$$

Where $\Psi_{G,e} = -c_1(\mathbb{L}_e)$ is the first Chern class of the normal line bundle corresponding to the node e on the stratum $\overline{\mathcal{M}}_G \to \overline{\mathcal{M}}_{g,n}$, and $\Delta_{G,D,e}$ is the divisor in $\mathcal{J}_{G,D}$ corresponding to pointsrepresent sheaves that fail to be locally free at the edge e.

7.a. **The case of disjoint blowups.** Our main results, Theorem 7.4 and Corollary 7.24 massively simplify in the case when the m vine curves $\beta_1, \ldots, \beta_m$ are disjoint. This is for example the case for all hyperplanes on divisorial (or compact type) vine curves (3.12) (Proposition 3.15) –in this case m equals 1 and no blowup is required–, and for all hyperplanes of the form (3.13) with $S \neq [n]$ (Proposition 5.29).

In each of these cases, the category $\widetilde{\mathfrak{C}}_E$ only contains the terminal object and the resolved strata $(\beta_i, V^i_\bullet)$ where $V^i_\bullet = \{V_i\}$ contains only the one vertex set $V_i = \{\mathrm{leg}_{\beta_i}(1)\}$ for all $i = 1, \ldots, m$ (Proposition 5.29).

Recall the notation from the previous section. We set X_i^+, Y_i^+ (respectively, X_i, Y_i) as the two components over β_i (respectively, over $(\beta_i, V^i_\bullet)$). We denote by g_{Y_i} the genus of the fiber of Y_i and by d_{Y_i} the degree of the universal line bundle on Y_i. We also set $F^{Y_i}_+ = F^+_{V_i}$ and $F^{Y_i} = F_{V_i}$. Let t_i be the number of nodes of a general curve in β_i, so $|\operatorname{Aut}(\beta_i)| = t_i!$.

Then we have:

Corollary 7.31. *When the hyperplane $H = H(\phi^+, \phi^-)$ is such that the vine curves $\beta_1, \ldots, \beta_m$ are pairwise disjoint, Formula (7.6) simplifies to*

$$\sum_{i=1}^{m} \sum_{s_i, j_i, k_i \geq 0} \frac{1}{t_i!} \binom{g_{Y_i} - d_{Y_i} - j - 1}{k_i + 1} \cdot f_{E_i *}\big(c_{s_i}(F^{X_i}) \cdot c_{j_i}(F^{Y_i}) \cdot \Psi_i^{k_i}\big) \tag{7.32}$$

Proof. Follows immediately from Theorem 7.4. Note the simplification of the minus sign in the definition of $b_{G,D,V}((j_V);(k_V))$ and the minus sign before $f_{\Gamma *}$ in Equation (7.6). □

We can also recast the main result of 7.24. by simply taking the pushforward along each $\mathbb{P}^{t_i - 1}$ bundle $p_i \colon E_i \to \beta_i$. Note that the K-theory elements F^{X_i} and F^{Y_i} are pull-backs of corresponding elements $F^{X_i}_+$ and $F^{Y_i}_+$ on β_i.

For each i and $1 \leq r_i \leq t_i$, let Ψ_{i,r_i} be the first Chern class of the conormal bundle to the r_i-th gluing on the resolved stratum β_i.

Corollary 7.33. *When the hyperplane $H = H(\phi^+, \phi^-)$ is such that the vine curves $\beta_1, \dots, \beta_m$ are pairwise disjoint, the formula in Corollary 7.24 equals*

$$\sum_{\substack{s_i+j_i+\lambda_i=g-d-t_i \\ \text{for all } i=1,\dots,m}} \frac{1}{t_i!}\binom{g_{Y_i}-d_{Y_i}-j_i-1}{g-d-j_i-s_i} \cdot f_{\beta_i *}\big(c_{s_i}(F_+^{X_i}) \cdot c_{j_i}(F_+^{Y_i}) \cdot h_{\lambda_i}(\Psi_{i,1}, \dots, \Psi_{i,t_i})\big) \tag{7.34}$$

where h_{λ_i} is the complete homogeneous polynomial of degree λ_i in its entries.

Proof. This follows from Corollary 7.24 or, more directly, by applying the push-pull formula to Corollary 7.31 combined with the fact that the push-forward of $\Psi_i^{k_i}$ along $E_i \to \beta_i$ equals $h_{k_i-t_i+1}(\Psi_{i,1}, \dots, \Psi_{i,t_i})$. □

We now analyse some even more special cases of this, already special, formula.

Remark 7.35. (The "compact type" hyperplanes). A special case of Corollaries 7.31 and 7.33 occurs when $H(\phi^+, \phi^-)$ is a hyperplane of the form (3.12). In this case the generic locus where ϕ^+ differs from ϕ^- is a compact type boundary divisor. In particular, m equals 1 and $\widetilde{\mathcal{J}}^d_{g,n}(\phi^+, \phi^-) \to \overline{\mathcal{J}}^d_{g,n}(\phi^+)$ is the identity. The unique vine curve $\beta = \beta_1$ consists of a boundary divisor $\Delta_{g-g_Y,S} \subset \overline{\mathcal{M}}_{g,n}$ decorated with a unique pair of ϕ^+-stable bidegrees, say $(d - d_Y, d_Y)$. In this case the degree $g - d$ part of the formula in 7.31 coincides with the formula in 7.33, and they equal to

$$\sum_{s+j+\lambda=g-d-1} \binom{g_Y-d_Y-j-1}{g-d-j-s} \cdot f_{\beta *}\big(c_s(F_+^X) \cdot c_j(F_+^Y) \cdot \Psi^\lambda\big). \tag{7.36}$$

7.b. **Wall-crossing in low codimension.** We now analyse the first few cases of our main result, ordered by codimension.

7.b.1. *Codimension* 1. Let $d = g - 1$. In this case the classes $\mathsf{w}_{g-1}(\phi)$ are divisors, also known under the name of *theta divisors*. This case was the main result of [KP17].

Because each $\mathsf{w}_{g-1}(\phi)$ is a divisor class and $\overline{\mathcal{J}}^d_{g,n}(\phi^+)$ is nonsingular, the wall-crossing term equals zero across any hyperplane not of the form (3.12).

Assume that the hyperplane crossed is $H = H(g - g_Y, 1, S; d - d_Y + \frac{1}{2})$. Then Formula (7.36) collapses and it gives

$$\mathsf{w}_{g-1}(\phi^+) - \mathsf{Id}^* \mathsf{w}_{g-1}(\phi^-) = (g_Y - d_Y - 1) \cdot [\mathcal{J}_\beta]$$

for $\beta = (G(g - g_Y, 1, S), (d - d_Y, d_Y))$. This recovers [KP17, Theorem 4.1] after observing that the divisor $\mathcal{J}_\beta \subset \overline{\mathcal{J}}^d_{g,n}(\phi^+)$ is the pullback of $\Delta_{g-g_Y,S} \subset \overline{\mathcal{M}}_{g,n}$.

7.b.2. *Codimension* 2. When $d = g-2$, the classes $\mathsf{w}_{g-2}(\phi)$ have codimension 2. There are 2 types of hyperplanes where the wall-crossing term is not zero.

If the hyperplane has the form (3.12), the vine curve β is a boundary divisor. Assuming that H and β are as in the previous paragraph, then Formula 7.36 reads

$$f_{\beta*}\left(\binom{g_Y-d_Y-1}{g-d-1}c_1(F_+^X)+\binom{g_Y-d_Y-2}{g-d-1}c_1(F_+^Y)+\binom{g_Y-d_Y-1}{g-d}\Psi\right).$$

If the hyperplane is of type (3.13), then the only cases when the formula is nontrivial is for $H = H(g-g_Y-1, 2, S, d-d_Y-1)$. While this hyperplane might witness a change in stability on more than 1 vine curve, the intersection of any 2 would occur in codimension > 2 and hence not be relevant. We can read the wall-crossing term off Formula (7.33):

$$\sum_{i=1}^{m}\binom{g_{Y_i}-d_{Y_i}-1}{g-d}[\mathcal{J}_{\beta_i}]$$

Here β_i for $i = 1,\dots,m$ is a vine curve of the form $(G(g-g_{Y_i}-1, 2, S), (d-d_{Y_i}, d_{Y_i}))$ where the stability condition changes. (If S^c is not empty, then $m = 1$).

7.c. **Pullbacks via Abel–Jacobi sections.** Fix integers $\mathbf{d} = (k; d_1,\dots,d_n)$, $\mathbf{f} = (\mathbf{f_{i,S}})_{\mathbf{i,S}}$, and let $\mathcal{L} = \mathcal{L}_{\mathbf{d},\mathbf{f}}$ be the line bundle on the universal curve $\overline{\mathcal{C}}_{g,n}$ defined in Section 3.d.1. Let then ϕ^+ and ϕ^- be on opposite sides of a hyperplane H (Definition 5.1), and such that $\mathcal{L}$ is ϕ^+ stable. This defines an Abel–Jacobi section $\sigma = \sigma_{\mathbf{d},\mathbf{f}}\colon \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{J}}^d_{g,n}(\phi^+)$. We now compute the pullback of Formula 7.25 via σ.

For every $G \in G_{g,n}$, define the divisor $D = D_{\mathbf{d},\mathbf{f}}$ on G as the multidegree of $\mathcal{L}$ on any curve whose dual graph equals G. We then have a poset $\operatorname{Ext}(G) = \operatorname{Ext}(G, D)$, depending on ϕ^+, ϕ^-, defined in Section 5.

The total space $\mathcal{C}_G \to \overline{\mathcal{M}}_G$ has one irreducilbe component $\mathcal{C}_v := \mathcal{C}_{G,v}$ for each vertex v of G. We redefine $\pi_v = \pi_{G,v}\colon \mathcal{C}_v \to \overline{\mathcal{M}}_G$. Also, for each $V \subset V(G)$, we denote by $\pi_V\colon \bigcup_{v\in V}\mathcal{C}_v \to \overline{\mathcal{M}}_G$ the induced map on the union. We write $X = X_G = \mathcal{C}_{\operatorname{leg}(1)}$ and $\Sigma = X \cap \mathcal{C}_{\{\operatorname{leg}(1)\}^c}$. We also write $Y_V = \mathcal{C}_{V^c}$ for every $V \subset V(G)$.

We then define the following K-theory elements in $\overline{\mathcal{M}}_G$

$$F^X_{\mathbf{d},\mathbf{f}} := -R^\bullet(\pi_X)_*\mathcal{L}(-\Sigma)_{|X};\ \ F^{\mathbf{d},\mathbf{f}}_V := -R^\bullet(\pi_{V^c})_*(\mathcal{L}_{|Y_V});\ \ H^{\mathbf{d},\mathbf{f}}_V := F_V - \sum_{\substack{V'\in V_\bullet,\\ V'\gtrdot V}} F_{V'}.$$

The line bundle $\mathcal{L}$ also defines a section, possibly rational, $\sigma^-\colon \overline{\mathcal{M}}_{g,n} \dashrightarrow \overline{\mathcal{J}}^d_{g,n}(\phi^-)$.

Corollary 7.37. *The difference*

$$\sigma^*(\mathsf{w}_d(\phi^+)) - \sigma^*_-(\mathsf{w}_d(\phi^-))$$

equals

$$(7.38)\quad -\sum_{G\in G_{g,n}} \frac{1}{|\operatorname{Aut}(G)|} f_{G*}$$

$$\left(\sum_{\substack{V_\bullet \text{ a full forest in } \operatorname{Ext}(G),\\ s+\sum_V j_V+\sum g_e=g-d-|E(G)|}} \alpha(s,(j_V),(g_e))\cdot c_s(F^X_{\mathbf{d},\mathbf{f}})\cdot \prod_{V\in V_\bullet} c_{j_V}(H_V^{\mathbf{d},\mathbf{f}}) \prod_e \Psi^{g_e}_{(G,e)}\right)$$

where the coefficient α is defined in Equation (7.26)*.*

Proof. Follows directly by pulling back (7.25) via σ. □

When the hyperplane H is such that the vine curves that fail ϕ^--stability are disjoint, the latter can be simplified, as for Formula (7.34).

Remark 7.39. The pull-back of Formula 7.25 via the Abel–Jacobi section $\sigma_{\mathbf{d},\mathbf{f}}$ can be explicitly computed via [PRvZ20, Theorem 1].

For a full forest $V_\bullet$ in $\operatorname{Ext}(G)$, recall the definition of the next element from (5.17). Defining $Z_V := \mathcal{C}_{\operatorname{next}(V)\setminus V}$ and set $\Sigma_V = Z_V \cap \bigcup_{V'>V} \mathcal{C}_{V'^c}$ and $\Sigma'_V = Z_V \cap \mathcal{C}_V$, we have

$$H_V^{\mathbf{d},\mathbf{f}} = -R^\bullet(\pi_V)_*((\mathcal{L}_{\mathbf{d},\mathbf{f}})_{|Z_V}(-\Sigma_V)),$$

and the line bundle $(\mathcal{L}_{\mathbf{d},\mathbf{f}})_{|Z_V}(-\Sigma_V)$ equals

$$\omega^k_{Z_V/\overline{\mathcal{M}}_G}\left(k\Sigma'_V+(k-1)\Sigma_V+\sum_{\substack{\operatorname{leg}(j)\in\\ \operatorname{next}(V)\setminus V}} d_jP_j+\sum_{\operatorname{leg}(S^c)\subseteq \operatorname{next}(V)\setminus V} f_{i,S^c}C_{i,S^c}|_{Z_V}\right).$$

We note that Z_V is the disjoint union $\mathcal{C}_v$ for $v\in \operatorname{next}(V)\setminus V$ (see Proposition 5.19). Moreover, the line bundle above, restricted to each one of these components, is precisely the pullback via the projection $\overline{\mathcal{M}}_G \to \overline{\mathcal{M}}_{g(v),\operatorname{val}(v)}$ of a line bundle as in [PRvZ20, Formula 0.1].

Example 7.40. As an illustration of Remark 7.39, we write the simpler case where H corresponds to changing the stability condition on a single vine curve β with t nodes, components of genus g_X and g_Y, with markings S_β and S^c_β respectively. Then $\overline{\mathcal{M}}_\beta = \overline{\mathcal{M}}_{g_X,|S_\beta|+t}\times \overline{\mathcal{M}}_{g_Y,|S^c_\beta|+t}$ and we denote by p_X and p_Y the projections. In this case we have that

$$F^X_{\mathcal{L}} = p_X^*\left(-R^\bullet(\pi^X_*)\left(\omega^k_X((k-1)\Sigma+\sum_{j\in S_\beta} d_jP_j+\sum_{\substack{i\le g_X\\ 1\in S\subseteq S_\beta}} f_{i,S_\beta\setminus S}\cdot C^X_{i,S_\beta\setminus S})\right)\right)$$

$$F^Y_{\mathcal{L}} = p_Y^*\left(-R^\bullet(\pi^Y_*)\left(\omega^k_Y(k\Sigma+\sum_{j\in S^c_\beta} d_jP_j+\sum_{\substack{i\le g_Y\\ S\subseteq S^c_\beta}} f_{i,S^c_\beta\setminus S}\cdot C^Y_{i,S^c_\beta\setminus S})\right)\right)$$

Formula (7.38) for the difference $\sigma^*(\mathsf{w}_d(\phi^+)) - \sigma^*_-(\mathsf{w}_d(\phi^-))$ in this case becomes

$$\sum_{\substack{s+j+\lambda\\=g-d-t}} \binom{g_Y - d_Y - j - 1}{g - d - j - s} \frac{f_{\beta *}}{t!}\left(c_s(F^X_{\mathcal{L}}) \cdot c_j(F^Y_{\mathcal{L}}) \cdot h_\lambda(\Psi_1, \dots, \Psi_t)\right).$$

The Chern classes above are computed in [PRvZ20, Theorem 1].

References

[ACG11] Enrico Arbarello, Maurizio Cornalba, and Phillip A. Griffiths. *Geometry of algebraic curves. Volume II*, volume 268 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011. With a contribution by Joseph Daniel Harris.

[ACGH85] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. *Geometry of algebraic curves. Vol. I*, volume 267 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.

[Alu10] Paolo Aluffi. Chern classes of blow-ups. *Math. Proc. Cambridge Philos. Soc.*, 148(2):227–242, 2010.

[AP20] Alex Abreu and Marco Pacini. The universal tropical Jacobian and the skeleton of the Esteves' universal Jacobian. *Proc. Lond. Math. Soc. (3)*, 120(3):328–369, 2020.

[AP21] Alex Abreu and Marco Pacini. The resolution of the universal Abel map via tropical geometry and applications. *Adv. Math.*, 378:Paper No. 107520, 62, 2021.

[BHP$^+$] Younghan Bae, David Holmes, Rahul Pandharipande, Johannes Schmitt, and Rosa Schwarz. Pixton's formula and Abel–Jacobi theory on the Picard stack. arXiv:2004.08676.

[BL05] Lev Borisov and Anatoly Libgober. McKay correspondence for elliptic genera. *Ann. of Math. (2)*, 161(3):1521–1569, 2005.

[Cap94] Lucia Caporaso. A compactification of the universal Picard variety over the moduli space of stable curves. *J. Amer. Math. Soc.*, 7(3):589–660, 1994.

[CCUW20] Renzo Cavalieri, Melody Chan, Martin Ulirsch, and Jonathan Wise. A moduli stack of tropical curves. *Forum Math. Sigma*, 8:Paper No. e23, 93, 2020.

[CGH$^+$] Dawei Chen, Samuel Grushevsky, David Holmes, Martin Möller, and Johannes Schmitt. A tale of two moduli spaces: logarithmic and multi-scale differentials. arXiv:2212.04704.

[CJ18] Emily Clader and Felix Janda. Pixton's double ramification cycle relations. *Geom. Topol.*, 22(2):1069–1108, 2018.

[CMKV15] Sebastian Casalaina-Martin, Jesse Leo Kass, and Filippo Viviani. The local structure of compactified Jacobians. *Proc. Lond. Math. Soc. (3)*, 110(2):510–542, 2015.

[Dud18] Bashar Dudin. Compactified universal Jacobian and the double ramification cycle. *Int. Math. Res. Not. IMRN*, (8):2416–2446, 2018.

[EP16] Eduardo Esteves and Marco Pacini. Semistable modifications of families of curves and compactified Jacobians. *Ark. Mat.*, 54(1):55–83, 2016.

[Est01] Eduardo Esteves. Compactifying the relative Jacobian over families of reduced curves. *Trans. Amer. Math. Soc.*, 353(8):3045–3095, 2001.

[Ful98] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*

[Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, second edition, 1998.

[GZ14] Samuel Grushevsky and Dmitry Zakharov. The zero section of the universal semiabelian variety and the double ramification cycle. *Duke Math. J.*, 163(5):953–982, 2014.

[HKP18] David Holmes, Jesse Leo Kass, and Nicola Pagani. Extending the double ramification cycle using Jacobians. *Eur. J. Math.*, 4(3):1087–1099, 2018.

[HMP$^+$] David Holmes, Samouil Molcho, Rahul Pandharipande, Aaron Pixton, and Johannes Schmitt. Logarithmic double ramification cycles. arXiv:2207.06778.

[Hol21] David Holmes. Extending the double ramification cycle by resolving the Abel-Jacobi map. *J. Inst. Math. Jussieu*, 20(1):331–359, 2021.

[KKMSD73] G. Kempf, Finn Faye Knudsen, D. Mumford, and B. Saint-Donat. *Toroidal embeddings. I.* Lecture Notes in Mathematics, Vol. 339. Springer-Verlag, Berlin-New York, 1973.

[KP17] Jesse Leo Kass and Nicola Pagani. Extensions of the universal theta divisor. *Adv. Math.*, 321:221–268, 2017.

[KP19] Jesse Leo Kass and Nicola Pagani. The stability space of compactified universal Jacobians. *Trans. Amer. Math. Soc.*, 372(7):4851–4887, 2019.

[Mel19] Margarida Melo. Universal compactified Jacobians. *Port. Math.*, 76(2):101–122, 2019.

[MMUV22] Margarida Melo, Samouil Molcho, Martin Ulirsch, and Filippo Viviani. Tropicalization of the universal Jacobian. *Épijournal Géom. Algébrique*, 6:Art. 15, 51, 2022.

[Mol22a] Sam Molcho. Pullbacks of Brill-Noether classes under Abel-Jacobi sections, 2022.

[Mol22b] Sam Molcho. Smooth compactifications of the Abel-Jacobi section, 2022.

[MPS23] Samouil Molcho, Rahul Pandharipande, and Johannes Schmitt. The hodge bundle, the universal 0-section, and the log Chow ring of the moduli space of curves. *Compositio Mathematica*, 159(2):306–354, 2023.

[MW20] Steffen Marcus and Jonathan Wise. Logarithmic compactification of the Abel-Jacobi section. *Proc. Lond. Math. Soc. (3)*, 121(5):1207–1250, 2020.

[OS79] Tadao Oda and C. S. Seshadri. Compactifications of the generalized Jacobian variety. *Trans. Amer. Math. Soc.*, 253:1–90, 1979.

[Pan18] Rahul Pandharipande. A calculus for the moduli space of curves. In *Algebraic geometry: Salt Lake City 2015*, volume 97 of *Proc. Sympos. Pure Math.*, pages 459–487. Amer. Math. Soc., Providence, RI, 2018.

[PRvZ20] Nicola Pagani, Andrea T. Ricolfi, and Jason van Zelm. Pullbacks of universal Brill-Noether classes via Abel-Jacobi morphisms. *Math. Nachr.*, 293(11):2187–2207, 2020.

[Yin16] Qizheng Yin. Cycles on curves and Jacobians: a tale of two tautological rings. *Algebr. Geom.*, 3(2):179–210, 2016.

Universidade Federal Fluminense, Instituto de Matemática
Email address: alexbra1@gmail.com

Department of Mathematical Sciences, University of Liverpool
Email address: pagani@liverpool.ac.uk