Mixed Nash Equilibria in Discrete Tullock Contests

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Abstract

Through the lens of *Tullock contests,* we take a fresh look at the *contest game for crowdsourcing reviews* [15] as a *Tullock contest game with discrete strategies:* Each of n *players,* endowed with *skills,* strategically invests an *effort* for writing a *review* of some *quality* she chooses from a finite set $[Q]$; she is awarded a *payment,* out of a budget β, in proportion to her effort, and pays a player-specific *cost,* which is the product of effort and skill. So both the payment and the utility functions for a player are strictly concave in the player's effort. Players are *anonymous* if skills are identical, and so is the game. We study the *mixed Nash equilibria* of the game, where no player could deviate to increase her *expected utility*; her *support* in a Nash equilibrium is the set of qualities she chooses with strictly positive probability. By strict concavity, mixed Nash equilibria have the *Small&Consecutive-Supports* property: their supports have size at most 2. We present upper bound results and structural and (ir)rationality results about mixed Nash equilibria and their *Price of Anarchy*:

- For the anonymous Tullock contest game, there is an algorithm to compute all symmetric Nash equilibria up to a fixed precision ϵ and determine how many of them are rational; the algorithm resorts to an optimal algorithm for root-finding in univariate polynomials and takes time $\Theta\left(Q \cdot \text{poly}(n) \cdot \log 1/\epsilon\right)$. The algorithm extends naturally to games with an arbitrary strictly concave utility function.
- For two natural definitions of *Social Cost,* the corresponding *Mixed Price of Anarchy,* studied here for the first time, is upper-bounded by 1 and 2, respectively, plus an additive constant; a lower bound we provide establishes almost-tightness for the first upper bound. Corresponding bounds on the *Asymptotic Mixed Price of Anarchy* follow.
- For two families of instances of anonymous Tullock contest games with mandatory participation and $\beta = 1$, we provide *(i)* very simple, necessary and sufficient numerical conditions on the efforts for the existence of an (ir)rational Nash equilibrium and *(ii)* a *single* Nash equilibrium which is irrational, respectively. Specifically, the first family has $n = 3$, while the second has $n = 4$ and $Q = 2$. Deriving the irrational Nash equilibrium for the second instance draws heavily from connections to the theory of *symmetric polynomials*; a significant implication is that computing a Nash equilibrium for a Tullock contest game is outside PPAD [53]. Nonetheless, we observe that FIXP is an upper bound for such computations. Envisioning the Tullock contest game as a discrete *Lipschitz game* [2], we prove that, in contrast, there is some constant $\varepsilon > 0$ such that computing an additive ε-approximate pure Nash equilibrium for it is in PPAD.
- We propose a framework grounded on Algebraic Geometry as a plausible host for proving structural results disentangling rational and irrational Nash equilibria in the Tullock contest game. Towards this end and envisioning a *reduction* among contests as an isomorphism of $\mathbb{Q}$-algebras, we present an isomorphism that maps rational (resp., irrational) fully mixed Nash equilibria to rational (resp., irrational) ones. As a by-product, we leverage the Algebraic Geometry framework to completely characterize the set of *all* Tullock contest games with $n = 2$ and $Q = 2$ as an algebraic variety.

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1 The Contest Game as a Discrete Tullock Contest

We continue the study of the *contest game for crowdsourcing reviews,* or *contest game* for short, which was considered in [15], with good blockchain-related motivation, assuming a discrete strategy space, a *proportional payments* scheme and a simple product-form function for the *skill-effort* cost. Results on the existence and Price of Anarchy of pure Nash equilibria in the contest game were claimed in [15]. The contest game was extended in [46] to incorporate more general assumptions on the payments and the skill-effort function and results on the existence, dynamics and computation of pure Nash equilibria were derived. Here we revisit the contest game in its form studied in [15] and focus on its mixed Nash equilibria, connecting and integrating it into well-studied game classes such as *concave games* [56] *aggregative games* [58] and *Lipschitz games* [2], and, most prominently, to *Tullock contest games* [63].

In the *contest game,* there are n *players,* or *contestants,* with *skills* $0 < s_1 \leq s_2 \leq \ldots \leq s_n$, respectively, and a set $[Q]$ of Q *qualities*; we think of skill as an intrinsic characteristic of a contestant. Players are *anonymous* if $s_1 = s_2 = \ldots = s_n = 1$, giving rise to *anonymous contest games*. Each player chooses a quality $q \in [Q]$ and pays an *effort* f_q. (For an integer $k \geq 1$, $[k]$ denotes $\{1, \ldots, k\}$.) Efforts are distinct and monotonically strictly increasing; *voluntary participation* and *mandatory participation* require $\mathsf{f}_1 = 0$ and $\mathsf{f}_1 > 0$, respectively. $\mathcal{G}_{vp}$ is the class of contest games with voluntary participation. Skills and efforts are rational numbers of constant size (independent of n and Q). Assuming that each number other than n and Q is represented with a constant number of bits, the contest game requires an amount of $\Theta(n+Q)$ memory. For the case of anonymous players, this amount reduces to $\Theta(\log n + Q)$.

A *pure profile,* or *profile* for short, q is a vector $\langle q_1, q_2, \ldots, q_n \rangle$, where q_i is the quality chosen by player i, called her *(pure) strategy*; thus, $[Q]^n$ is the set of all pure profiles. By abuse of notation, we extend the definition of the effort f to pure profiles by setting $\mathsf{f}(\mathsf{q}) := \sum_{i \in [n]} \mathsf{f}_{q_i}$, called the *total effort* in q. Efforts are paid for out of a *reward* $\beta > 0$. We adopt *proportional payments,* where player i receives a payment equal to $\frac{\mathsf{f}_{q_i}}{\mathsf{f}(\mathsf{q})}\,\beta$ if $\mathsf{f}(\mathsf{q}) > 0$, or 0 otherwise.[1] Note that $\mathsf{f}(\mathsf{q}) = 0$ may only occur under voluntary participation. The *utility* of player i in a profile q is a rational number determined by the *utility function* as $\mathsf{U}_i(\mathsf{q}) = \frac{\mathsf{f}_{q_i}}{\mathsf{f}(\mathsf{q})}\,\beta - s_i \mathsf{f}_{q_i}$, where $s_i \mathsf{f}_{q_i}$ is the *skill-effort cost*, or *cost* for short, of player i in q; as the utility function allows representing the contest game in size much smaller than the normal form representation, the contest game is a *succinctly representable* game (cf. [54]).

The contest game is an example of an *aggregative game*, where the utility of each player is only affected by her own choice and an aggregate of the others'. Formally, a finite n-players *aggregative game* [58] is a tuple $\left\langle n, (S_i)_{i \in [n]}, (\mathsf{U}_i)_{i \in [n]}, (\mathsf{u}_i)_{i \in [n]}, \varphi \right\rangle$, where for each player i, S_i is the strategy set, with $S = \times_{i \in [n]} S_i$, $\varphi : S \to \mathsf{Y}$ is the *aggregator function* and y is an *aggregate* in the *aggregate space* Y, $\mathsf{u}_i : S_i \times \mathsf{Y} \to \mathbb{R}$ is the *aggregate utility function*, and $\mathsf{U}_i : S \to \mathbb{R}$ is the utility function, with $\mathsf{U}_i(\mathsf{s}) = \mathsf{u}_i(s_i, \varphi(\mathsf{s}))$ for each profile $\mathsf{s} \in S$. For the contest game, $\varphi(\mathsf{q}) = \mathsf{f}(\mathsf{q})$ and $\mathsf{U}_i(\mathsf{q}) = \mathsf{u}_i(\mathsf{f}_{q_i}, \varphi(\mathsf{q})) = \frac{\mathsf{f}_{q_i}}{\mathsf{f}(\mathsf{q})}\,\beta - s_i \mathsf{f}_{q_i}$; so $\mathsf{U}_i(\mathsf{q})$ is strictly decreasing in $\varphi(\mathsf{q})$ *when* q_i *is fixed*. (This is not necessary for *any* aggregative game.)

Proportional payments coincide with the *Ratio-Form Contest Success Function* [63], or *Ratio-Form* CSF for short, expressed as the player's effort over the total effort of contestants.

[1] A normative convention for Tullock contests is to set the ratio $\frac{\mathsf{f}_{q_i}}{\mathsf{f}(\mathsf{q})}$ equal to $\frac{1}{n}$ when $\mathsf{f}(\mathsf{q}) = 0$; however, there are works adopting 0 as the ratio value. We choose the latter convention as it better fits into the crowdsourcing reviews framework, where it is not reasonable to pay someone for producing no review.

So the contest game from [15] can be considered as a *Tullock contest game* [63] with *discrete* strategies, linear costs and *elasticity* 1 (i.e., efforts are raised to power 1); call it a *discrete Tullock contest (game)*. Henceforth, contest games and discrete Tullock contest games shall be used interchangeably. In contrast, strategy spaces are typically *continuous* and compact in a *continuous Tullock contest (game)*; discretizing them yields the discrete Tullock contest. As a continuous function, the *Ratio-Form* CSF is strictly concave in a player's effort, with other players' efforts fixed. Adopting linear (hence, convex) costs in a player's effort, the *conditional* utility function, conditioned on fixed efforts of the other players, is concave in a player's effort in a continuous Tullock contest with Ratio-Form CSF. In general, games with such concave utility functions are called *concave* [56]. A *best-response* of player i to the (fixed) partial profile q_{-i} is a strategy q that maximizes $\mathsf{U}_i(q, \mathsf{q}_{-i})$. For strictly concave games with continuous and compact strategy spaces, best-responses exist uniquely [56]; by strict concavity, a player's utility goes down as we move away from the best-response. Hence, in the discrete Tullock contest, a contestant has *at most two* and consecutive best-responses, which then lie on different sides of her best-response in the continuous Tullock contest.

A *mixed profile* $\boldsymbol{\sigma} = \langle \sigma_1, \ldots, \sigma_n \rangle$ is a vector of probability distributions, called *mixed strategies* and represented as vectors in $[0,1]^Q$, with σ_i^q denoting the probability of player i choosing quality q. A pure profile q is *realizable* by $\boldsymbol{\sigma}$ if $\mathsf{Pr}_{\boldsymbol{\sigma}}(\mathsf{q}) := \prod_{k \in [n]} \sigma_k^{q_k} > 0$. The *expected utility* player i gets in $\boldsymbol{\sigma}$ is $\mathsf{U}_i(\boldsymbol{\sigma}) := \sum_{\mathsf{q} \in [Q]^n} \mathsf{U}_i(\mathsf{q}) \mathsf{Pr}_{\boldsymbol{\sigma}}(\mathsf{q})$. (To save notation, we use U_i to denote the utility that player i gets in either a mixed or a pure profile.) For the next definitions, we shall need some notation: For a k-tuple $\mathsf{t} = \langle t_1, \ldots t_k \rangle$ and an integer $i \in [k]$, denote $\mathsf{t}_{-i} := \langle t_1, \ldots, t_{i-1}, t_{i+1} \ldots t_k \rangle$. The *conditional expected utility* player i gets in $\boldsymbol{\sigma}$ had she chosen quality q is a function $\mathsf{U}_i(q, \boldsymbol{\sigma}_{-i}) : [Q] \times \{\boldsymbol{\sigma}_{-i}\} \mapsto \mathbb{R}$; $\{\boldsymbol{\sigma}\}$ and $\{\boldsymbol{\sigma}_{-i}\}$ denote the sets of profiles and *partial mixed profiles* excluding player i, respectively; clearly, $\mathsf{U}_i(q, \boldsymbol{\sigma}_i) = \sum_{\mathsf{q}_{-i} \in [Q]^{n-1}} \mathsf{U}_i(q, \mathsf{q}_i) \mathsf{Pr}_{\boldsymbol{\sigma}_i}(\mathsf{q}_i)$. Thus, the conditional expected utility is a convex combination of utilities a player choosing a *fixed* pure strategy gets. Since the utility function is strictly concave in a player's effort (with other players' choices fixed), so also is conditional expected utility on the continuous strategy space $[0,1]^Q$. The *support* of player i in the mixed profile $\boldsymbol{\sigma}$, is the set of qualities q with $\sigma_i^q > 0$; the *expected total effort* is $\mathsf{f}(\boldsymbol{\sigma}) = \sum_{\mathsf{q} \in [Q]^n} \mathsf{f}(\mathsf{q}) \mathsf{Pr}_{\boldsymbol{\sigma}}(\mathsf{q})$.

A mixed (resp., pure) profile $\boldsymbol{\sigma}$ (resp., q) is a *mixed* (resp., *pure*) *Nash equilibrium* if no player could unilaterally increase her expected utility (resp., utility) by switching to a different mixed strategy (resp., strategy); formally, for each player i and mixed strategy τ (resp., strategy q), $\mathsf{U}_i(\boldsymbol{\sigma}) \geq \mathsf{U}_i(\tau, \boldsymbol{\sigma}_{-i})$ (resp., $\mathsf{U}_i(\mathsf{q}) \geq \mathsf{U}_i(q, \mathsf{q}_{-i})$); so *all* players play best-responses in a Nash equilibrium. It is known [30, Theorem 4.1] that computing a pure equilibrium for a concave game is FIXP-complete. Since a mixed Nash equilibrium may be viewed as a pure Nash equilibrium on $[0,1]^Q$, the strict concavity property of the expected utility function implies that computing a mixed Nash equilibrium for the discrete Tullock contest game is in FIXP. (FIXP is the complexity class capturing the exact computation of fixed points [25]). Since a player has at most two and consecutive best-responses in a discrete Tullock contest, all supports in a mixed Nash equilibrium have size at most 2; this is called as the *Small&Consecutive-Supports* property. Henceforth, we shall usually refer to mixed Nash equilibria simply as Nash equilibria. $\mathsf{NE}(\mathsf{CG})$ denotes the set of Nash equilibria for the discrete Tullock contest game CG. Here is a characterization of mixed Nash equilibria.

▶ **Lemma 1** ([49]). *A mixed profile $\boldsymbol{\sigma}$ is a Nash equilibrium if and only if for each player $i \in [n]$, the following conditions hold:* (1) *For any pair of qualities $q, q' \in \mathsf{supp}_i(\boldsymbol{\sigma})$, $\mathsf{U}_i(q, \boldsymbol{\sigma}_{-i}) = \mathsf{U}_i(q', \boldsymbol{\sigma}_{-i})$ (Indifference Principle, or IP).* (2) *For any pair of qualities $q \in \mathsf{supp}_i(\boldsymbol{\sigma})$ and $q' \notin \mathsf{supp}_i(\boldsymbol{\sigma})$, $\mathsf{U}_i(q, \boldsymbol{\sigma}_{-i}) \geq \mathsf{U}_i(q', \boldsymbol{\sigma}_{-i})$ (Weak Dominance Principle, or WDP).*

By Lemma 1 and the *Small&Consecutive-Supports* property, $\mathsf{U}_i(q^*, \boldsymbol{\sigma}_{-i}) = \mathsf{U}_i(q^*+1, \boldsymbol{\sigma}_{-i})$ if and only if q^* and q^*+1 are all and only global maxima of $\mathsf{U}_i(q, \boldsymbol{\sigma}_{-i})$ as a function in q. Hence, for each quality $q \in [Q] \setminus \{q^*, q^*+1\}$, $\mathsf{U}_i(q^*, \boldsymbol{\sigma}_{-i}) > \mathsf{U}_i(q, \boldsymbol{\sigma}_{-i})$. Thus, the IP implies the WDP in the contest game, and more generally in any strictly concave game; this is called the *Indifference Principle Implies Weak Dominance* property. The IP (resp., the WDP) is expressed as a system of polynomial equations (resp., inequalities). A Nash equilibrium is *rational* if all probabilities are rational numbers; else, it is *irrational*. A Nash equilibrium is *fully mixed* if all probabilities are strictly between 0 and 1. By Nash's celebrated result [49], the set of Nash equilibria for a finite game is non-empty; but it might not have a fully mixed Nash equilibrium. For a fully mixed Nash equilibrium, the WDP is vacuous, the IP is both necessary and sufficient and its solutions are the positive real points of an *algebraic variety* (see Appendix, Section B). By the *Small&Consecutive-Supports* property, a fully mixed Nash equilibrium for the contest game may exist only if $Q = 2$. A Nash equilibrium is *symmetric* if all players play the same strategy. Nash [49] showed that every finite *symmetric* game, where the utility of a player depends on her chosen strategy and the multiset of strategies chosen by others, such as the anonymous contest game, has a symmetric Nash equilibrium. By the *Small&Consecutive-Supports* property, a symmetric pure Nash equilibrium for the discrete Tullock contest game is computable in time $\Theta(Q)$.

As a measure of the quality of a profile q (resp., a mixed profile $\boldsymbol{\sigma}$), usually called *Social Value* (cf. [42]), we adopt the total effort $\mathsf{f}(\mathsf{q})$ (resp., the expected total effort $\mathsf{f}(\boldsymbol{\sigma})$). A *Social Optimum* is a profile q^* maximizing total effort. Capturing the degradation in performance due to both selfish users and the lack of central regulation, the *Mixed Price of Anarchy of a game* CG [42] is defined as $\mathsf{MPoA}(\mathsf{CG}) = \max_{\boldsymbol{\sigma} \in \mathsf{NE}(\mathsf{CG})} \frac{\mathsf{f}(\mathsf{q}^*)}{\mathsf{f}(\boldsymbol{\sigma})}$. The *Mixed Price of Anarchy of class* $\mathcal{G}$ is $\mathsf{MPoA}(\mathcal{G}) = \sup_{\mathsf{CG} \in \mathcal{G}} \mathsf{PoA}(\mathsf{CG})$. The *Pure Price of Anarchy* PPoA restricts to the set of pure Nash equilibria, provided this is non-empty. Clearly, $\mathsf{PPoA}(\mathsf{CG}) \leq \mathsf{MPoA}(\mathsf{CG})$ for any game $\mathsf{CG} \in \mathcal{G}$, implying that $\mathsf{PPoA}(\mathcal{G}) \leq \mathsf{MPoA}(\mathcal{G})$.

When the Social Value of a Social Optimum increases with the number of players, it makes sense to evaluate the Price of Anarchy *asymptotically* by considering the *Asymptotic Price of Anarchy* [24]. Formally, consider the subclass $\mathcal{G}_k := \{\mathsf{CG} \in \mathcal{G} : \mathsf{f}(\mathsf{q}^*) \geq k\}$. The *Asymptotic Mixed Price of Anarchy* (resp., the *Asymptotic Pure Price of Anarchy*) of the class $\mathcal{G}$ is $\mathsf{AMPoA}(\mathcal{G}) = \lim_{k \to \infty} \mathsf{MPoA}(\mathcal{G}_k)$ (resp., $\mathsf{APPoA}(\mathcal{G}) = \lim_{k \to \infty} \mathsf{PPoA}(\mathcal{G}_k)$). Note that an upper bound on MPoA applies also to AMPoA, while a lower bound on AMPoA applies to MPoA as well.

The *Non-Proportional Social Optimum* for a contest game was defined in [15] as the maximum possible total effort in a *feasible* profile where a player i choosing quality q gets a payment of $s_i \mathsf{f}_q$, which is the smallest payment securing a non-negative utility for her. Thus, the resulting *Price of Anarchy against a Non-Proportional Social Optimum* did not restrict the Social Optimum to proportional payments. We consider that such a comparison is unfair and contrasts the intended meaning of the Price of Anarchy [42]; We shall instead evaluate it for the contest game *by restricting Social Optima to both feasible profiles and proportional payments.*[2] Formally, the *Proportional Social Optimum* is a profile $\mathsf{q}^* \in \arg\max_{\mathsf{q} \in [Q]^n | \min_{i \in [n]} \mathsf{U}_i(\mathsf{q}) \geq 0} \mathsf{f}(\mathsf{q})$. The MPoA *(resp.* PPoA*) against a Proportional Social Optimum* is defined in a corresponding way.

[2] Had we instead considered *all* profiles and chosen a sufficiently small β, all players would choose quality Q in the Social Optimum and quality 1 in a unique Nash equilibrium, implying $\mathsf{AMPoA} = \mathsf{f}_Q/\mathsf{f}_1$, which can be made arbitrarily large.

2 Results and Significance

We embark on four major directions related to the mixed Nash equilibria: *(i)* the computation of symmetric Nash equilibria in anonymous Tullock contest games, *(ii)* their *Mixed Price of Anarchy, (iii)* their rationality and irrationality properties, and *(iv)* the structure of rational and irrational Nash equilibria through the lens of Algebraic Geometry. We derive these results:

2.1 Symmetric Nash Equilibria for an Anonymous Contest Game

We prove a significant structural property of symmetric mixed Nash equilibria: they simply do not exist for a non-anonymous discrete Tullock contest game (Proposition 2). The property naturally extends to all concave and aggregative games with non-anonymous players (Corollary 3). Proposition 2 implies that *all* symmetric Nash equilibria for a *non-anonymous* discrete Tullock contest game can be computed in $\Theta(Q)$ time — we simply have to check all pure ones, of which, by the *Small&Consecutive-Supports* property, there can be at most $Q-1$. In turn, this property is instrumental for devising an efficient algorithm with time complexity $\Theta\left(Q \cdot \text{poly}(n) \cdot \log 1/\epsilon\right)$ for computing *all* symmetric Nash equilibria of an anonymous contest game to an arbitrary precision ϵ (Theorem 4). The necessity of computing to some finite precision, versus computing exactly, is due to an algebraic obstruction: roots of a polynomial equation may be irrational or (when the degree is at least 5) even non-expressible by radicals. The algorithm is an instance of a *general* algorithm for computing all symmetric Nash equilibria for an arbitrary finite game with strictly concave utility functions that have size polynomial in n and the number of pure strategies (Theorem 5).

2.2 Bounds on the Mixed Price of Anarchy

We prove that for any contest game $\mathsf{CG} \in \mathcal{G}_{vp}$ with a Nash equilibrium $\boldsymbol{\sigma}$ and Proportional Social Optimum q^*_{pr}, it holds that $\mathsf{f}(\mathsf{q}^*_{pr}) \leq \mathsf{f}(\boldsymbol{\sigma}) + \phi(\mathsf{CG})$, where ϕ satisfies $\phi(\mathsf{CG}) \leq 2\mathsf{f}_Q$ (Theorem 7). Hence, $\mathsf{AMPoA}(\mathcal{G}_{vp}) = 1$ against a Proportional Social Optimum. We complement this upper bound with an almost-matching lower bound on the additive factor (Theorem 8). To the best of our knowledge, this result represents a rare case of a class of games where the Asymptotic Mixed Price of Anarchy is 1.

Assume that f_q divides f_{q+1} for each $q \in \{2, \ldots, Q-1\}$; this assumption, although restrictive, captures the interesting scenario in which the effort increases at a geometric pace with the achieved quality. Under this assumption, we prove that the total effort of a Non-Proportional Social Optimum is no more than twice that of a Proportional Social Optimum, plus an additive factor of f_Q (Proposition 10). This allows leveraging the upper bound on total effort against a Proportional Social Optimum and immediately deriving that, under these premises, for any Non-Proportional Social Optimum q^*_{np} and Nash equilibrium $\boldsymbol{\sigma}$, it holds that $\mathsf{f}(\mathsf{q}^*_{np}) \leq 2\mathsf{f}(\boldsymbol{\sigma}) + 5\mathsf{f}_Q$ (Theorem 11). Hence, the Asymptotic Mixed Price of Anarchy against a Non-Proportional Social Optimum is at most 2.

2.3 Contest Games with Rational and Irrational Nash Equilibria

For two families of anonymous contest games with $n = 3$ and $n = 4$, respectively, $Q = 2$, $\beta = 1$ and mandatory participation, we identify simple, necessary and sufficient numerical conditions on efforts for the non-existence of an irrational Nash equilibrium (Theorem 12) and the existence of a single and irrational Nash equilibrium (Theorem 14), respectively. The proof of Theorem 14 uses, as a combinatorial tool, the *Elementary Symmetric Polynomials*

Indifference Equation, expressing the IP in Tullock contest games (Proposition 13) and named after the involved *elementary symmetric polynomials* [45]; On top of Proposition 13, the proof of Theorem 14 employs the *Girard-Newton* formulae for elementary symmetric polynomials. (A similar involvement of elementary symmetric polynomials in a study on mixed Nash equilibria in *continuous* contest games is established, perhaps not coincidentally, in [28].)

By irrationality and uniqueness, as established in Theorem 14, computing a Nash equilibrium is outside Linear-FIXP = PPAD [25]. It is natural to ask whether there is any fragment of the general problem in which computing a Nash equilibrium for a contest game is in PPAD. Indeed there is: We establish that the discrete Tullock contest game is a discrete *Lipschitz game* [2] with *Lipschitz constant* [2] $o(1/n)$; in turn, this implies existence of an ε-approximate pure Nash equilibrium for any $\varepsilon > 0$ (Theorem 15). This allows leveraging a result showing membership in PPAD [35, Theorem 3.2] to conclude that *there is some constant* $\varepsilon > 0$ such that computing an ε-approximate pure Nash equilibrium is in PPAD (Theorem 17).

Resolving the existence of irrational Nash equilibria, even without *characterizing* the conditions under which a discrete Tullock contest has a unique and irrational Nash equilibrium, is a necessary prerequisite for directing efforts towards the appropriate complexity class. Besides excluding PPAD membership, the existence of contest games with only irrational Nash equilibria is an important pivotal step towards proving appropriate PPAD- or FIXP-hardness results for (some kind of) approximate Nash equilibria.

2.4 Irrationality Vs. Rationality Through the Algebraic Geometry Lens

The anonymous contest game with a unique and irrational Nash equilibrium in Theorem 14 self-poses the decision and search problems about the existence and computation, respectively, of an irrational Nash equilibrium in a multi-player contest game. The classical way to address it is by developing suitable reductions, as in, e.g., [9, 12, 14, 32], within the landscapes of $\exists\mathbb{R}$ [57] and FIXP [25], the two prominent complexity classes for hosting (among others) decision and search problems regarding the computation of fixed points, respectively, when these could be irrational. For the decision problem, it might be tempting to wonder whether the irrational contest game in Theorem 14 can be used as a *gadget* in an $\exists\mathbb{R}$-hardness proof of deciding the existence of an irrational Nash equilibrium in a multi-player contest game, as in [9, 12]; this would point to searching for a reduction of a *general* game to a game in the restricted (measure-zero perhaps) class (as opposed to a general one) that preserves the existence of an irrational Nash equilibrium. To the best of our knowledge, all previous reductions in this vein reduce to a game either in the whole class of strategic games as in [9, 14], or in some wide class of games, such as that of symmetric games [14, 32] or that of *win-lose* games [12].

Nevertheless it remains challenging to investigate reductions among *general* games that preserve the existence of irrational Nash equilibria. We propose to explore the possibility and the potential of Algebraic Geometry as a host for such investigations; Algebraic Geometry has long been employed in algorithms for computing mixed Nash equilibria — see the book by Sturmfels [61] and references therein. In Algebraic Geometry terms we envision a complexity-theoretic reduction as an isomorphism of $\mathbb{Q}$-algebras. (Section B in the Appendix serves as a Primer on Algebraic Geometry and includes formal definitions for all such terms we shall need to use.) We derive two modest yet interesting results; one is general and the other pertains to contest games:

- We present a bijective mapping between general games with n players and 2 strategies that preserve the existence of irrational Nash equilibria (Theorem 19).

- We take advantage of the proposed framework of Algebraic Geometry by completely characterizing the set of *all* contest games with $n = 2$ and $Q = 2$ as algebraic varieties (Theorem 20). The proof is assisted by importing results from Algebraic Geometry and the software package (*Macauley2*). So is the proof of a complete characterization of Nash equilibria for *anonymous* contest games with $n = 2$ and $Q = 2$ (Theorem 21).

We believe that investing into establishing connections with Algebraic Geometry is well worth the effort, as the approach is exemplifying a novel pathway (through polynomial maps between algebraic varieties) for addressing a wealth of complexity questions about (and not only) the contest game. Setting a precedent for this type of exploration, we hope to inspire a multitude of further research. Planning to exclusively focus on such complexity questions in a future work, we aspire to achieve impactful outcomes, possibly through appropriate complexity results in Algebraic Geometry.

3 Related Work and Comparison

The observation that the contest game is an instance of a Tullock contest with discrete strategies, linear costs and a *Contest Success Function* [59] in ratio-form, implying the cornerstone property that the corresponding utility function is strictly concave, remained elusive in both [15] and the (the conference version of) [46]. Remarkably, the *first* work in the crowdsourcing literature that adopted Tullock contests (exactly as [15] later did) as a tool to design incentive mechanisms had actually come in 2015 [44] for a continuous Tullock contest. It has been shown in [6] that pure Nash equilibria might not not exist in a continuous Tullock contest where elasticities take a common value exceeding 2. Complexity results about the existence of pure Nash equilibria in *general* discrete aggregative games appear in [60]. Mixed Nash equilibria for a continuous Tullock contest were considered in [7, 26, 29].

The *utility gradient* of a game with compact strategies is defined as the vector each component of which is the partial derivative of each player's utility function with respect to the player's own strategy; the game is *strictly monotone* if the inner product of the difference between two arbitrary profiles and the difference between the corresponding utility gradients is strictly negative. An important property of a strictly monotone game is that it has strictly concave utility functions; using this property, it was shown in [56, Theorem 2] that a strictly monotone game has a unique pure Nash equilibrium. Another subclass of strictly concave games known to have a pure Nash equilibrium are the so called *socially concave* games [22, Theorem 3.1]. Linear Cournot oligopoly games, well-known to be strategically equivalent to Tullock contest games (see, e.g., [62]), have been shown to be socially concave [22, Lemma 4.2]; hence, so is the Tullock contest game.

Ewerhart [27] proves that a Tullock contest with linear costs and anonymous players belongs to the class of *best-response* potential games [65], containing the class of exact potential games and related to the classes of weighted, ordinal and generalized ordinal potential games — see [65, Figure 1] for their relations. This yields that the contest game with anonymous players and linear costs is a best-response potential game. Best-response dynamics in Tullock contests have been recently studied in [23, 33, 34] for continuous Tullock contests. A short conference paper [17] outlines a general model for a Tullock contest game, in which, effectively, also payments are *player-specific* [47]: there is no common β; instead, effectively each player i has her own β_i. [17] claims (with no proofs) the existence of pure Nash equilibria and an upper bound on the Pure Price of Anarchy.

Related to Theorem 4 is [54, Theorem 7.1], proving that a symmetric Nash equilibrium in a symmetric game with n players and Q strategies can be computed to precision ϵ in time

polynomial in n^Q, the number of bits required for the representation of the utility functions and $\log 1/\epsilon$. For anonymous contest games, where the utility functions can be described with $\Theta(\log n + Q)$ bits, the algorithm in [54, Theorem 7.1] specializes to an algorithm running in time $\Theta\left(\text{poly}(n^Q) \cdot \log 1/\epsilon\right)$. The algorithm in Theorem 4 significantly improves on running time, while computing *all* (as opposed to an arbitrary one) Nash equilibria, thanks to the small-supports property, which is a consequence of the strict concavity property.

The first strategic game with a single irrational Nash equilibrium is the famous *Three-Man Poker Game,* discussed in the celebrated work of Nash [49] and originally solved by Nash and Shapley [48]. Other examples of strategic games with a (single) irrational Nash equilibrium come from certain classes of *congestion games* [30, 31], *financial networks games* [39, 40] or from the class of 3-player games [9, 12, 13]. Less well-known in the Algorithmic Game Theory community are some examples of Bayesian games defined from Tullock contests with only irrational Bayes-Nash equilibria; see, e.g., [1, Example 2]. A preprint by Ewerhart [28] presents the *first* (to our knowledge) example of a 5-players Tullock contest game with a single Nash equilibrium that is even not expressible by radicals, by virtue of *Galois Theory* (see, e.g. [28, Section 2]). In the spirit of Theorem 14, we have no reason to doubt the existence of a corresponding 5-players Tullock contest game with $Q = 2$.

Birmpas *et al.* [15, Theorem 2] analysed the Pure Price of Anarchy against a Non-Proportional Social Optimum for contest games with voluntary participation, restricting their analysis to contest games with $Q = 3$ and corresponding efforts $\mathsf{f}_2 = 1$ and $\mathsf{f}_3 = \alpha$, for some $\alpha > 1$; they prove that for any pure Nash equilibrium q and Non-Proportional Social Optimum q^*_{np}, $\mathsf{f}(\mathsf{q}^*_{np}) \leq 4\mathsf{f}(\mathsf{q}) + 6\alpha$, implying that $\mathsf{APPoA}(\mathcal{G}_{vp}) \leq 4$ against a Non-Proportional Social Optimum. They provide no lower bound on the corresponding Pure Price on Anarchy; instead, they prove that an absolute additive factor of $\alpha - 1$ is unavoidable. The upper bound we prove on the Mixed Price of Anarchy against a Non-Proportional Social Optimum significantly improves and extends the one claimed by Birmpas *et al.* [15, Theorem 2] for their restricted setting with $Q = 3$, where the assumption that f_q divides f_{q+1} for each $q \in \{2, \ldots, Q-1\}$ is satisfied by design. Specifically, our upper bound concerns the Mixed Price of Anarchy and an arbitrary number of qualities, while it more than halves the upper bound in [15, Theorem 2]. It is significant that we upper-bound the Mixed Price of Anarchy as pure Nash equilibria may fail to exist for the contest games considered in [15]: although Birmpas *et al.* [15, Theorem 1] claim the existence of pure Nash equilibria under voluntary participation for their contest games with $Q = 3$, we provide counterexamples to existence under voluntary participation in the Appendix (Section C). Hence, the Pure Price of Anarchy is not even well-defined for the contest games considered in [15]. The Asymptotic Price of Anarchy was introduced in [24] in the context of selfish bin packing. Subsequently, both attracted interest — see, e.g., [11] and references therein.

The *Logit-Form* CSF $\frac{\varphi\left(\mathsf{f}_{q_i}\right)}{\sum_{k\in[n]}\varphi\left(\mathsf{f}_{q_k}\right)}$ [19] generalizes the *Ratio-Form* CSF (cf. [10, Section 2.4]). The particular case where $\varphi\left(\mathsf{f}_{q_i}\right) = \alpha_i\left(\mathsf{f}_{q_i}\right)^{r_i}$, where $r_i, \alpha_i > 0$ are the effort's *elasticity* or *decisiveness* [26] and the *efficiency* or *weight,* respectively, gives rise to a *generalized Tullock contest* [63] with $\widehat{\mathsf{U}}_i^{\mathsf{r},\alpha}(\mathsf{q}) = \frac{\alpha_i(\mathsf{f}_{q_i})^{r_i}}{\mathsf{f}^{\mathsf{r},\alpha}(\mathsf{q})}\,\beta - s_i\,\alpha_i(\mathsf{f}_{q_i})^{r_i}$; r and α are the *elasticity* and the *weight* vectors, respectively, and $\mathsf{f}^{\mathsf{r},\alpha}(\mathsf{q}) = \sum_{k=1}^{n}\alpha_k\left(\mathsf{f}_{q_k}\right)^{r_k}$. When $r_i \leq 1$ for all players i, the *Generalized-Form* CSF remains strictly concave and the shown results on the Tullock contest for the case $\mathsf{r} = \mathsf{1}$ shown in Section 4 carry over. He *et al.* [38] consider the complexity of computing or deciding the existence of pure Nash equilibria for continuous Tullock contests. Although *existence* results for that model may carry over to the discrete Tullock contest game, where possible efforts are constrained to specific discrete values, computational algorithms

might not – the computational methods outlined in [38] perform, e.g., binary search over continuous strategy spaces (such as their claimed FPTAS for computing a ε-pure Nash equilibrium, if one exists [38, Algorithm 1]), or *mirror decent* based on the exact gradient of their utility function (such as their claimed method for finding a pure Nash equilibrium when $r_i \leq 1$ [38, Section 4.1], employing the MMD *with perfect gradient algorithm* from [16], under additional regularity assumptions on the utility functions).

4 Symmetric Nash Equilibria for an Anonymous Contest Game

We start with a property of symmetric mixed Nash equilibria in Tullock contest games:

▶ **Proposition 2.** *In the discrete Tullock contest game, players with different skills cannot play the same mixed strategy in a Nash equilibrium. Hence, there can be no symmetric non-pure Nash equilibrium unless the game is anonymous.*

Proof. Given a partial profile q_{-k} and a partial mixed profile $\boldsymbol{\sigma}_{-k}$ for any player $k \in [n]$, denote $\mathsf{f}(\mathsf{q}_{-k}) := \sum_{l\in[n]\setminus\{k\}} \mathsf{f}_{q_l}$ and $\mathsf{Pr}_{\boldsymbol{\sigma}_{-k}}(\mathsf{q}_{-k}) := \prod_{l\in[n]\setminus\{k\}} \sigma_l(q_l)$. For a quality $q \in [Q]$ and a partial mixed profile $\boldsymbol{\sigma}_{-k}$, $\mathsf{U}_k(q, \boldsymbol{\sigma}_{-k}) = \beta \sum_{\mathsf{q}\in[Q]^n} \frac{\mathsf{Pr}_{\boldsymbol{\sigma}_{-k}}(\mathsf{q}_{-k})\,\mathsf{f}_q}{\mathsf{f}_q + \mathsf{f}(\mathsf{q}_{-k})} - s_k\mathsf{f}_q$ if $\mathsf{f}(\mathsf{q}) = \mathsf{f}_q + \mathsf{f}(\mathsf{q}_k) > 0$, or 0 if $\mathsf{f}(\mathsf{q}) = 0$. Define the function $\mathsf{g}(k) = \frac{\mathsf{Pr}_{\boldsymbol{\sigma}_{-k}}(\mathsf{q}_{-k})\,\mathsf{f}_q}{\mathsf{f}_q + \mathsf{f}(\mathsf{q}_{-k})}$ for $k \in [n]$. Assume, by way of contradiction, that there is a mixed Nash equilibrium $\boldsymbol{\sigma}$ in which players $i, j \in [n]$, with $\Delta_{i,j} := s_j - s_i > 0$, play the same mixed strategy. The assumption implies that $\mathsf{g}(i) = \mathsf{g}(j)$, or $\frac{\mathsf{Pr}_{\boldsymbol{\sigma}_{-i}}(\mathsf{q}_{-i})\,\mathsf{f}_q}{\mathsf{f}_q + \mathsf{f}(\mathsf{q}_{-i})} = \frac{\mathsf{Pr}_{\boldsymbol{\sigma}_j}(\mathsf{q}_{-j})\,\mathsf{f}_q}{\mathsf{f}_q + \mathsf{f}(\mathsf{q}_{-j})}$. It follows that $\sum_{\mathsf{q}\in[Q]^n} \frac{\mathsf{Pr}_{\boldsymbol{\sigma}_{-i}}(\mathsf{q}_{-i})\,\mathsf{f}_q)}{\mathsf{f}_q + \mathsf{f}(\mathsf{q}_{-i})} = \sum_{\mathsf{q}\in[Q]^n} \frac{\mathsf{Pr}_{\boldsymbol{\sigma}_{-j}}(\mathsf{q}_{-j})\,\mathsf{f}_q}{\mathsf{f}_q + \mathsf{f}(\mathsf{q}_{-j})}$. Hence, $\mathsf{U}_j(q, \boldsymbol{\sigma}_{-j}) = \mathsf{U}_i(q, \boldsymbol{\sigma}_{-i}) - \Delta_{i,j}\mathsf{f}_q$. By the *Small&Consecutive-Supports* property, $\mathsf{U}_j(q^*, \boldsymbol{\sigma}_{-j}) = \mathsf{U}_i(q^*, \boldsymbol{\sigma}_{-i}) - \Delta_{i,j}\mathsf{f}_{q^*}$ and $\mathsf{U}_j(q^*+1, \boldsymbol{\sigma}_{-j}) = \mathsf{U}_i(q^*+1, \boldsymbol{\sigma}_{-i}) - \Delta_{i,j}\mathsf{f}_{q^*+1}$, where q^* and q^*+1 are the two consecutive qualities. By Lemma 1, we have $\mathsf{U}_i(q^*, \boldsymbol{\sigma}_{-i}) = \mathsf{U}_i(q^*+1, \boldsymbol{\sigma}_{-i})$ and $\mathsf{U}_j(q^*, \boldsymbol{\sigma}_{-j}) = \mathsf{U}_j(q^*+1, \boldsymbol{\sigma}_{-j})$. It follows that $\mathsf{U}_i(q^*, \boldsymbol{\sigma}_{-i}) - \Delta_{i,j}\mathsf{f}_{q^*} = \mathsf{U}_i(q^*+1, \boldsymbol{\sigma}_{-i}) - \Delta_{i,j}\mathsf{f}_{q^*+1}$; that is, $\mathsf{f}_{q^*} = \mathsf{f}_{q^*+1}$, contradicting the fact that f is strictly increasing. The claims follow. □

The careful reader may have observed that the proof of Proposition 2 does not use the particular definition of the utility function in the discrete Tullock contest game, which happens to be aggregative, but only uses the *Small&Consecutive-Supports* property, which is a consequence of its strict concavity. Hence, Proposition 2 readily extends to all (finite) discrete concave games, even to those that are not aggregative.

▶ **Corollary 3.** *In a concave game with non-anonymous players, players cannot play the same mixed strategy in a Nash equilibrium. Hence, there can be no symmetric non-pure Nash equilibrium unless the game is anonymous.*

We now use Proposition 2 to establish that symmetric mixed Nash equilibria for an anonymous and discrete Tullock contest game are computable in pseudo-polynomial (resp, polynomial) time with respect to n (resp., to Q):

▶ **Theorem 4.** *There is an algorithm to compute, given an anonymous and discrete Tullock contest game, all symmetric Nash equilibria to a fixed precision ϵ and count the number of rational Nash equilibria in time $\Theta\left(Q \cdot \mathrm{poly}(n) \cdot \log 1/\epsilon\right)$.*

Proof. By Proposition 2, we have to search for both pure and mixed symmetric Nash equilibria. Here is an algorithm to compute them all in the anonymous game:

(1) Check all pure strategies. Each can be checked in $\Theta(1)$ time, for a total time $\Theta(Q)$.
(2) We now consider mixed strategies, aiming to construct and solve a univariate polynomial equation expressing the IP, which is both necessary and sufficient for a Nash equilibrium in a concave game. Specifically, for each mixed support $\{q^*, q^*+1\}$, where $1 \leq q^* < Q$ (thanks to the *Small&Consecutive-Supports Property*) and players choose qualities q^* and q^*+1 with probabilities x and $1-x$, respectively:
(2.a) For any i, formulate the univariate (in x) IP equation $\mathsf{U}_i(q^*, \boldsymbol{\sigma}_{-i}) = \mathsf{U}_i(q^*+1, \boldsymbol{\sigma}_{-i})$; this is an equation of degree at most $n-1$ with coefficients of size polynomial in n and Q.
(2.b) Solve the equation in (2.a) using an *optimal* algorithm by Pan [50, 51, 52] for root-finding in univariate polynomials, which either returns a solution (to precision ϵ) in the interval $(0,1)$ or replies that the equation has no solution. (Note that the returned set of solutions to precision ϵ includes all rational, irrational and non-expressible by radicals solutions as well; by Galois Theory, the latter might exist for $n \geq 5$.)

Since there are $Q-1$ supports to check and Pan's algorithm has (optimal) $\Theta(\text{poly}(n) \cdot \log 1/\epsilon)$ time complexity, the total time complexity is $\Theta\left(Q \cdot \text{poly}(n) \cdot \log 1/\epsilon\right)$. We remark that the algorithm in [54] relies on importing the approach in the seminal paper of Renagar [55], which also leads to time complexity $\Theta\left(\text{poly}(n) \cdot \log 1/\epsilon\right)$ but resorts to *Quantifier Elimination* algorithms (see, e.g., [5, Chapter 14]), which are not always practical.

For counting the number of rational Nash equilibria, we use the *Rational Root Theorem*: *For each rational solution $\frac{p}{q}$, where p and q are relatively prime, of a univariate polynomial equation with integer coefficients, p divides the constant term and q divides the coefficient of the leading term.* The *Rational Roots Theorem* gives a finite number of possible fractions and checking if they are roots yields the rational Nash equilibria. □

Although the time complexity of the algorithm from Theorem 4 is polynomial in both n and Q, the algorithm does not count as a polynomial one since its input size is $\Theta(\log n + Q)$ bits; however, it becomes a polynomial algorithm when n is polynomial in Q. This algorithm is an instance of a general algorithm to compute all Nash equilibria of a discrete and anonymous, concave game whose size is polynomial in n and the number $\widetilde{Q}$ of pure strategies:

▶ **Theorem 5.** *There is an algorithm to compute, given an arbitrary, discrete, anonymous and concave game whose utility functions have size polynomial in n and the number $\widetilde{Q}$ of pure strategies, all symmetric Nash equilibria to a fixed precision ϵ and count the number of rational Nash equilibria in time $\Theta\left(\widetilde{Q} \cdot \text{poly}(n) \cdot \log 1/\epsilon\right)$.*

5 The Mixed Price of Anarchy

We establish almost-matching bounds on MPoA for contest games with voluntary participation against a Proportional Social Optimum and a Non-Proportional Social Optimum, respectively. Let $\mathcal{P}_{\mathsf{q}}(q)$ denote the set of players choosing quality q in q. Say that player i *necessarily contributes in the profile* q if $i \in \mathcal{P}_{\mathsf{q}}(q)$ for some quality $q > 1$. Generalizing to mixed profiles, player i *necessarily contributes in the mixed profile* $\boldsymbol{\sigma}$ if $\sigma_i^1 = 0$; so, player i necessarily contributes in any profile that can be realized in $\boldsymbol{\sigma}$.

5.1 MPoA Against a Proportional Social Optimum

We start with an upper bound on Proportional Social Optimum. A simple technical fact, which we prove using *Schwarz's Inequality*, will be instrumental for the proof:

▶ **Lemma 6.** *Consider two sequences $a_1, \dots a_k$ and $x_1, \dots, x_k$ of $k \geq 1$ real numbers such that $a_j > 0$ and $x_j \geq 0$ for each $j \in [k]$. Then, $\sum_{j=1}^{k} a_j x_j \cdot \sum_{j=1}^{k} \frac{x_j}{a_j} \geq \left(\sum_{j=1}^{k} x_j\right)^2$.*

▶ **Theorem 7.** *Fix a Nash equilibrium $\boldsymbol{\sigma}$ and a Proportional Social Optimum q^*_{pr} for the contest game CG with voluntary participation. Then, $\mathsf{f}(\mathsf{q}^*_{pr}) \leq \mathsf{f}(\boldsymbol{\sigma}) + \phi(\mathsf{CG})$, where $\phi(\mathsf{CG}) = \mathsf{f}_2$ if $Q = 2$ and $\phi(\mathsf{CG}) = \max_{c \geq 2} \min\left\{c(\mathsf{f}_Q - \mathsf{f}_2), \frac{c}{c-1}\mathsf{f}_Q\right\}$ if $Q > 2$. Hence, $\mathsf{AMPoA}(\mathcal{G}_{vp}) = \mathsf{APPoA}(\mathcal{G}_{vp}) = 1$ against a Proportional Social Optimum.*

Proof. If $\mathsf{f}(\mathsf{q}^*_{pr}) = 0$, the claim trivially holds. So assume that $\mathsf{f}(\mathsf{q}^*_{pr}) > 0$ and consider a player $i \in [n]$ who always contributes in q^*_{pr}. Consider any quality $q \geq 2$ such that $i \in \mathcal{P}_{\mathsf{q}^*_{pr}}(q)$. Since q^*_{pr} is feasible, $\frac{\beta\,\mathsf{f}_q}{\mathsf{f}(\mathsf{q}^*_{pr})} - s_i\mathsf{f}_q \geq 0$. Since $\mathsf{f}_q > 0$, $s_i \leq \frac{\beta}{\mathsf{f}(\mathsf{q}^*_{pr})}$. Assume first that $\sigma_i^1 > 0$, i.e., i does not always contribute in $\boldsymbol{\sigma}$. By definition of conditional expected utility, $\mathsf{U}_i(2, \boldsymbol{\sigma}_{-i}) = \sum_{\mathsf{q}_{-i} \in [Q]^{n-1}} \left(\mathsf{Pr}_{\boldsymbol{\sigma}_{-i}}(\mathsf{q}_{-i}) \left(\frac{\beta\,\mathsf{f}_2}{\mathsf{f}(\mathsf{q}_{-i}) + \mathsf{f}_2} - s_i\mathsf{f}_2\right)\right) = \beta\,\mathsf{f}_2 \sum_{\mathsf{q}_{-i} \in [Q]^{n-1}} \left(\frac{\mathsf{Pr}_{\boldsymbol{\sigma}_{-i}}(\mathsf{q}_{-i})}{\mathsf{f}(\mathsf{q}_{-i}) + \mathsf{f}_2}\right) - s_i\mathsf{f}_2$. Since $\boldsymbol{\sigma}$ is a Nash equilibrium and $1 \in \mathsf{supp}_i(\boldsymbol{\sigma})$, Lemma 1 implies that $\mathsf{U}_i(2, \boldsymbol{\sigma}_{-i}) \leq \mathsf{U}_i(1, \boldsymbol{\sigma}_{-i}) = 0$. Since $\mathsf{f}_2 > 0$, it follows that $s_i \geq \beta \sum_{\mathsf{q}_{-i} \in [Q]^{n-1}} \frac{\mathsf{Pr}_{\boldsymbol{\sigma}_{-i}}(\mathsf{q}_{-i})}{\mathsf{f}(\mathsf{q}_{-i}) + \mathsf{f}_2} \geq \beta \sum_{\mathsf{q}_{-i} \in [Q]^{n-1}} \frac{\mathsf{Pr}_{\boldsymbol{\sigma}_{-i}}(\mathsf{q}_{-i})}{\mathsf{f}(\mathsf{q}) + \mathsf{f}_2}$. Set $k := |[Q]^{n-1}| = Q^{n-1}$ and map every partial profile $\mathsf{q}_{-i} \in [Q]^{n-1}$ to a distinct index $j \in [k]$. Defining $x_j := \mathsf{Pr}_{\boldsymbol{\sigma}_{-i}}(\mathsf{q}_{-i}) \geq 0$ and $a_j := \mathsf{f}(\mathsf{q}) + \mathsf{f}_2 > 0$ for each $j \in [k]$, by Lemma 6 and observing that $\sum_{j \in [k]} x_j = \sum_{\mathsf{q}_{-i} \in [Q]^{n-1}} \mathsf{Pr}_{\boldsymbol{\sigma}_{-i}}(\mathsf{q}_{-i}) = 1$, it follows that $s_i \geq \frac{\beta}{\sum_{\mathsf{q}_{-i} \in [Q]^{n-1}} (\mathsf{Pr}_{\boldsymbol{\sigma}_{-i}(\mathsf{q}_{-i})}(\mathsf{f}(\mathsf{q}) + \mathsf{f}_2))} = \frac{\beta}{\mathsf{f}(\boldsymbol{\sigma}) + \mathsf{f}_2}$. Since $s_i \leq \frac{\beta}{\mathsf{f}(\mathsf{q}^*_{pr})}$, it follows that $\mathsf{f}(\mathsf{q}^*_{pr}) \leq \mathsf{f}(\boldsymbol{\sigma}) + \mathsf{f}_2$. For $Q = 2$, the proven inequality is the claim. For $Q > 2$, taking a sufficiently large c yields $\min\left\{c(\mathsf{f}_Q - \mathsf{f}_2), \frac{c}{c-1}\mathsf{f}_Q\right\} \geq \mathsf{f}_Q$, so that $\phi(\mathsf{CG}) \geq \mathsf{f}_Q > \mathsf{f}_2$ and the claim holds.

So far the claim has been proved assuming that there exists a player that is always contributing in q^*_{pr} but not always contributing in $\boldsymbol{\sigma}$. It remains to consider the case where every always contributing player in q^*_{pr} always contributes in $\boldsymbol{\sigma}$. Then, the number $c \geq 1$ of always contributing players in q^*_{pr} is at most the number of always contributing players in $\boldsymbol{\sigma}$. It follows that $\mathsf{f}(\mathsf{q}^*_{pr}) \leq c\,\mathsf{f}_Q$ and $\mathsf{f}(\boldsymbol{\sigma}) \geq c\,\mathsf{f}_2$, so that $\Delta := \mathsf{f}(\mathsf{q}^*_{pr}) - \mathsf{f}(\boldsymbol{\sigma}) \leq c\,(\mathsf{f}_Q - \mathsf{f}_2)$. For $c \geq 2$, we prove an alternative upper bound on Δ. Let i be an always contributing player in q^*_{pr} who is playing with positive probability the lowest quality in $\boldsymbol{\sigma}$. Formally, there are qualities $k, h \in [Q] \setminus \{1\}$ such that *(i)* $i \in \mathcal{P}_{\mathsf{q}^*_{pr}}(k)$, $\sigma_i^h > 0$ and $\sigma_i^q = 0$ for each quality $q < h$, and *(ii)* there exists no other always contributing player i' in q^*_{pr} such that $\sigma_{i'}^q > 0$ for some quality $q < h$. Clearly, if $h = Q$, then $\mathsf{f}(\mathsf{q}^*_{pr}) \leq \mathsf{f}(\boldsymbol{\sigma})$ immediately holds and we are done; so assume that $h < Q$, implying that $h + 1 \leq Q$. By definition of conditional expected utility, $\mathsf{U}_i(h+1, \boldsymbol{\sigma}_{-i}) = \sum_{\mathsf{q}_{-i} \in [Q]^{n-1}} \left(\mathsf{Pr}_{\boldsymbol{\sigma}_{-i}}(\mathsf{q}_{-i}) \left(\frac{\beta\,\mathsf{f}_{h+1}}{\mathsf{f}(\mathsf{q}_{-i}) + \mathsf{f}_{h+1}} - s_i\mathsf{f}_{h+1}\right)\right) = \beta\,\mathsf{f}_{h+1} \sum_{\mathsf{q}_{-i} \in [Q]^{n-1}} \left(\frac{\mathsf{Pr}_{\boldsymbol{\sigma}_{-i}}(\mathsf{q}_{-i})}{\mathsf{f}(\mathsf{q}_{-i}) + \mathsf{f}_{h+1}}\right) - s_i\mathsf{f}_{h+1}$ and $\mathsf{U}_i(h, \boldsymbol{\sigma}_{-i}) = \sum_{\mathsf{q}_{-i} \in [Q]^{n-1}} \left(\mathsf{Pr}_{\boldsymbol{\sigma}_{-i}}(\mathsf{q}_{-i}) \left(\frac{\beta\,\mathsf{f}_h}{\mathsf{f}(\mathsf{q}_{-i}) + \mathsf{f}_h} - s_i\mathsf{f}_h\right)\right) = \beta\,\mathsf{f}_h \sum_{\mathsf{q}_{-i} \in [Q]^{n-1}} \left(\frac{\mathsf{Pr}_{\boldsymbol{\sigma}_{-i}}(\mathsf{q}_{-i})}{\mathsf{f}(\mathsf{q}_{-i}) + \mathsf{f}_h}\right) - s_i\mathsf{f}_h$. As $\boldsymbol{\sigma}$ is a Nash equilibrium and quality $h \in \mathsf{supp}_i(\boldsymbol{\sigma})$, by Lemma 1, $\mathsf{U}_i(h+1, \boldsymbol{\sigma}_{-i}) \leq \mathsf{U}_i(h, \boldsymbol{\sigma}_{-i})$. Plugging in the above expressions for $\mathsf{U}_i(h+1, \boldsymbol{\sigma}_{-i})$ and $\mathsf{U}_i(h, \boldsymbol{\sigma}_{-i})$, and dividing through by $\mathsf{f}_{h+1} - \mathsf{f}_h > 0$ yields $s_i \geq \beta \sum_{\mathsf{q}_{-i} \in [Q]^{n-1}} \frac{\mathsf{Pr}_{\boldsymbol{\sigma}_{-i}}(\mathsf{q}_{-i})\,\mathsf{f}(\mathsf{q}_{-i})}{(\mathsf{f}(\mathsf{q}_{-i}) + \mathsf{f}_h)(\mathsf{f}(\mathsf{q}_{-i}) + \mathsf{f}_{h+1})}$. As done before, defining $k := |[Q]^{n-1}| = Q^{n-1}$, $x_j := \mathsf{Pr}_{\boldsymbol{\sigma}_{-i}}(\mathsf{q}_{-i}) \geq 0$ (implying that $\sum_{j \in [k]} x_j = 1$) and $a_j := \frac{\mathsf{f}(\mathsf{q}_{-i})}{(\mathsf{f}(\mathsf{q}_{-i}) + \mathsf{f}_h)(\mathsf{f}(\mathsf{q}_{-i}) + \mathsf{f}_{h+1})} > 0$ by Lemma 6, we obtain $s_i \geq$

$\beta \sum_{\mathsf{q}_{-i}\in[Q]^{n-1}} \frac{\mathsf{Pr}_{\boldsymbol{\sigma}_{-i}}(\mathsf{q}_{-i})\,\mathsf{f}(\mathsf{q}_{-i})}{(\mathsf{f}(\mathsf{q}_{-i})+\mathsf{f}_h)(\mathsf{f}(\mathsf{q}_{-i})+\mathsf{f}_{h+1})} \geq \frac{\beta}{\sum_{\mathsf{q}_{-i}\in[Q]^{n-1}}\left(\mathsf{Pr}_{\boldsymbol{\sigma}_{-i}}(\mathsf{q}_{-i})\frac{(\mathsf{f}(\mathsf{q}_{-i})+\mathsf{f}_h)(\mathsf{f}(\mathsf{q}_{-i})+\mathsf{f}_{h+1})}{\mathsf{f}(\mathsf{q}_{-i})}\right)} =$
$\frac{\beta}{\sum_{\mathsf{q}_{-i}\in[Q]^{n-1}}\left(\mathsf{Pr}_{\boldsymbol{\sigma}_{-i}}(\mathsf{q}_{-i})\left(\mathsf{f}(\mathsf{q}_{-i})+\mathsf{f}_{h+1}+\mathsf{f}_h+\frac{\mathsf{f}_h\,\mathsf{f}_{h+1})}{\mathsf{f}(\mathsf{q}_{-i})}\right)\right)}$. For any profile q realized in $\boldsymbol{\sigma}$, since i is choosing qualities of at least h, we have $\mathsf{f}(\boldsymbol{\sigma}) \geq \sum_{\mathsf{q}_{-i}\in[Q]^{n-1}}\left(\mathsf{Pr}_{\boldsymbol{\sigma}_{-i}}(\mathsf{q}_{-i})\,(\mathsf{f}(\mathsf{q}_{-i})+\mathsf{f}_h)\right)$. Moreover, by assumption, in each partial profile q_{-i} realized in $\boldsymbol{\sigma}_{-i}$ there are at least $c-1$ players who are not playing qualities smaller than h. It follows that $\mathsf{f}(\mathsf{q}_{-i}) \geq (c-1)\mathsf{f}_h$. Hence, we conclude that $s_i \geq \frac{\beta}{\sum_{\mathsf{q}_{-i}\in[Q]^{n-1}}\left(\mathsf{Pr}_{\boldsymbol{\sigma}_{-i}}(\mathsf{q}_{-i})\left(\mathsf{f}(\mathsf{q}_{-i})+\mathsf{f}_{h+1}+\mathsf{f}_h+\frac{\mathsf{f}_h\,\mathsf{f}_{h+1})}{\mathsf{f}(\mathsf{q}_{-i})}\right)\right)} \geq$
$\frac{\beta}{\mathsf{f}(\boldsymbol{\sigma})+\frac{c}{c-1}\mathsf{f}_{h+1}}$. Since $s_i \leq \frac{\beta}{\mathsf{f}(\mathsf{q}^*_{pr})}$, this implies that $\mathsf{f}(\mathsf{q}^*_{pr}) \leq \mathsf{f}(\boldsymbol{\sigma}) + \frac{c}{c-1}\mathsf{f}_{h+1} \leq$ $\mathsf{f}(\boldsymbol{\sigma})+\frac{c}{c-1}\mathsf{f}_Q$. It follows that $\Delta \leq \frac{c}{c-1}\,\mathsf{f}_Q$ when $c \geq 2$. Together with the proved bound $\Delta \leq c(\mathsf{f}_Q-\mathsf{f}_2)$ for $c \geq 1$, it follows that $\Delta \leq \max\left\{\mathsf{f}_Q-\mathsf{f}_2, \max_{c\geq 2}\min\left\{c(\mathsf{f}_Q-\mathsf{f}_2), \frac{c}{c-1}\,\mathsf{f}_Q\right\}\right\}$. Observe that for $c=2$ the attained minimum is by $2(\mathsf{f}_Q-\mathsf{f}_2) > \mathsf{f}_Q-\mathsf{f}_2$. Hence, $\Delta \leq$ $\max_{c\geq 2}\min\left\{c(\mathsf{f}_Q-\mathsf{f}_2), \frac{c}{c-1}\,\mathsf{f}_Q\right\} = \phi(\mathsf{CG})$, as needed. Since the difference between the total effort of a Proportional Social Optimum and that of any Nash equilibrium is bounded by $\phi(\mathsf{CG}) \leq 2\mathsf{f}_Q$, the tight bound on the Asymptotic Mixed Price of Anarchy follows. □

An almost-matching lower bound holds even when restricting to the Pure Price of Anarchy and to anonymous contest games:

▶ **Theorem 8.** *There exists an anonymous contest game with voluntary participation, which has, for any integer $c \geq 2$, a pure Nash equilibrium q and a Proportional Social Optimum q^*_{pr} such that $\mathsf{f}(\mathsf{q}^*_{pr}) \geq \mathsf{f}(\mathsf{q}) + c(\mathsf{f}_Q-\mathsf{f}_2)$.*

Proof. Fix an integer $c \geq 2$. Consider the anonymous contest game with $n=c$ anonymous players, $\beta = c\cdot\mathsf{f}_Q$, and $Q=3$, with $\mathsf{f}_Q = \mathsf{f}_3 \leq \frac{(c-1)}{c-2}\,\mathsf{f}_2$ if $c \geq 3$, and $\mathsf{f}_Q > \mathsf{f}_2$ if $c=2$. The profile q in which all players choose quality Q is feasible as $\frac{\beta\,\mathsf{f}_Q}{c\,\mathsf{f}_Q} - \mathsf{f}_Q = 0$. So since q maximizes total effort, it is a Proportional Social Optimum. Consider now the profile q in which all players choose quality 2. In q each player's utility is $\frac{\beta\cdot\mathsf{f}_2}{c\cdot\mathsf{f}_2} - \mathsf{f}_2 = \mathsf{f}_Q - \mathsf{f}_2 > 0$. Hence, no player has an incentive to deviate to quality 1. Now if a player chooses quality Q, she gets utility $\frac{\beta\,\mathsf{f}_Q}{\mathsf{f}_Q+(c-1)\mathsf{f}_2} - \mathsf{f}_Q = \frac{c\,\mathsf{f}_Q^2}{\mathsf{f}_Q+(c-1)\mathsf{f}_2} - \mathsf{f}_Q = \frac{(c-1)\mathsf{f}_Q(\mathsf{f}_Q-\mathsf{f}_2)}{\mathsf{f}_Q+(c-1)\mathsf{f}_2}$. The last quantity never exceeds $\mathsf{f}_Q-\mathsf{f}_2$ since, by assumption, $c=2$ or $c \geq 3$ and $\mathsf{f}_Q \leq \frac{(c-1)\mathsf{f}_2}{c-2}$. So q is a pure Nash equilibrium, and the claim follows. □

5.2 MPoA Against a Non-Proportional Social Optimum

We now show how to extend the upper bound in Theorem 7 to the case of Non-Proportional Social Optimum. To do so, we need an assumption on the effort function, namely that for each $q \in \{2,\ldots,Q-1\}$, f_q divides f_{q+1}. For every quality q, we define a set A_q and a number n^*_q which are constructed by backward induction as follows. A_Q is the set of all players with lower indices that can choose quality Q and receive a non-negative utility; n^*_Q denotes the largest index of a player in A_Q. A_{Q-1} is the set of all players with lower indices other than those in A_Q that can choose quality $Q-1$ and receive a non-negative utility, assuming that the players in A_Q are playing quality Q; n^*_{Q-1} denotes the largest index of a player in A_{Q-1}. In general, A_q is the set of all players with lower indices other than those in $\bigcup_{h>q} A_h$ that can choose

quality q and receive a non-negative utility, assuming that, for each $h > q$, the players in A_h are playing quality h; n_q^* denotes the largest index of a player in A_q. Formally, set $n_{Q+1}^* := 0$; for every $q = Q, Q-1, \ldots, 2$, define $n_q^* := \max\left\{i \in [n] \cup \{0\} : i \in A_q \cup \{n_{q+1}^*\}\right\}$, where A_q is defined as the set of all $i \in [n] \setminus [n_{q+1}^*]$ such that $\sum_{h=q+1}^{Q}(n_h^* - n_{h+1}^*)\mathsf{f}_h + (i - n_{q+1}^*)\mathsf{f}_q \leq \frac{\beta}{s_i}$. We start with proving the existence of a Proportional Social Optimum with a certain property.

▶ **Proposition 9.** *For a contest game with voluntary participation, assume that for each $q \in \{2, \ldots, Q-1\}$, f_q divides f_{q+1}. Then, there is a Proportional Social Optimum q_{pr}^* with $\mathcal{P}_{\mathsf{q}_{pr}^*}(q) = [n_q^*] \setminus [n_{q+1}^*]$ for every $q \in [Q] \setminus \{1\}$.*

Proof. We define two transformations which, given a feasible Proportional Social Optimum, construct a second Proportional Social Optimum with a required property.

- Transformation T_1 constructs a *contiguous* [46, Section 4] Proportional Social Optimum: one in which the quality chosen by any player i is at least as large as any quality chosen by a player j with $j > i$. Towards this end, consider a feasible, non-contiguous Proportional Social Optimum q' such that $q_i' < q_j'$ for players i and j with $i < j$. By the feasibility of q', $\frac{\beta \cdot \mathsf{f}_{q_i'}}{\mathsf{f}(\mathsf{q}')} - s_i\mathsf{f}_{q_i'} \geq 0$ and $\frac{\beta \cdot \mathsf{f}_{q_j'}}{\mathsf{f}(\mathsf{q}')} - s_j\mathsf{f}_{q_j'} \geq 0$. Let q'' be the profile in which i and j swap their chosen qualities; so $q_i'' = q_j'$ and $q_j'' = q_i'$; Clearly, $\mathsf{f}(\mathsf{q}'') = \mathsf{f}(\mathsf{q}')$; hence, q'' is also a Proportional Social Optimum. For player i, $\frac{\beta \cdot \mathsf{f}_{q_i''}}{\mathsf{f}(\mathsf{q}'')} - s_i\mathsf{f}_{q_i''} = \frac{\beta \cdot \mathsf{f}_{q_j'}}{\mathsf{f}(\mathsf{q}')} - s_i\mathsf{f}_{q_j'} \geq \frac{\beta \cdot \mathsf{f}_{q_j'}}{\mathsf{f}(\mathsf{q}')} - s_j\mathsf{f}_{q_j'} \geq 0$. Since $\mathsf{f}_{q_j'} > 0$, $\frac{\beta \cdot \mathsf{f}_{q_j'}}{\mathsf{f}(\mathsf{q}')} - s_j\mathsf{f}_{q_j'} \geq 0$ implies that $\frac{\beta}{\mathsf{f}(\mathsf{q}')} - s_j \geq 0$. It follows that $\frac{\beta \cdot \mathsf{f}_{q_j''}}{\mathsf{f}(\mathsf{q}'')} - s_j\mathsf{f}_{q_j''} \geq 0$, so that the Proportional Social Optimum q'' is feasible.
  Observe that T_1 preserves the number of always contributing players. So a finite number of applications of T_1 is sufficient to obtain a contiguous Proportional Social Optimum starting with a non-contiguous one.
- Transformation T_2 constructs a Proportional Social Optimum where, for every quality $q \in \{2, \ldots, Q-1\}$, the number of players contributing a review of quality q is strictly smaller than $\frac{\mathsf{f}_{q+1}}{\mathsf{f}_q}$. Towards this end, consider a Proportional Social Optimum q such that $|\mathcal{P}_{\mathsf{q}}(q)| \geq \frac{\mathsf{f}_{q+1}}{\mathsf{f}_q}$ for some quality $q \in \{2, \ldots, Q-1\}$. Consider the smallest index $i \in \mathcal{P}_{\mathsf{q}}(q)$. Consider a subset $\mathcal{R} \subseteq \mathcal{P}_{\mathsf{q}}(q) \setminus \{i\}$ with $|\mathcal{R}| = \frac{\mathsf{f}_{q+1}}{\mathsf{f}_q} - 1$. Let q' be the profile obtained by setting $q_i' = q+1$, $q_j' = 0$ for each $j \in \mathcal{R}$ and $q_j' = q_j$ for all $j \in [n] \setminus (\mathcal{R} \cup \{i\})$. Clearly, $\mathsf{f}(\mathsf{q}') = \mathsf{f}(\mathsf{q})$; hence, q' is also a Proportional Social Optimum. Since $\frac{\beta \cdot \mathsf{f}_q}{\mathsf{f}(\mathsf{q})} - s_j\mathsf{f}_q \geq 0$ for each $j \in \mathcal{R} \cup \{i\}$, it follows that $0 \leq \sum_{j \in \mathcal{R} \cup \{i\}} \left(\frac{\beta\,\mathsf{f}_q}{\mathsf{f}(\mathsf{q})} - s_j\mathsf{f}_q\right) = \frac{\mathsf{f}_{q+1}}{\mathsf{f}_q} \cdot \frac{\beta\,\mathsf{f}_q}{\mathsf{f}(\mathsf{q})} - \mathsf{f}_q \sum_{j \in \mathcal{R} \cup \{i\}} s_j \leq \frac{\beta\,\mathsf{f}_{q+1}}{\mathsf{f}(\mathsf{q})} - \frac{\mathsf{f}_{q+1}}{\mathsf{f}_q}\,\mathsf{f}_q\,s_i = \frac{\beta\,\mathsf{f}_{q+1}}{\mathsf{f}(\mathsf{q})} - s_i\mathsf{f}_{q+1}$. Clearly, T_2 strictly decreases the number of always contributing players.

The properties of the transformations T_1 and T_2 established above imply that a finite sequence of their applications yields a contiguous Proportional Social Optimum q with $|\mathcal{P}_{\mathsf{q}}(q)| < \frac{\mathsf{f}_{q+1}}{\mathsf{f}_q}$ for each $q \in \{2, \ldots, Q-1\}$. If $\mathsf{q} = \mathsf{q}_{pr}^*$ we are done; so assume $\mathsf{q} \neq \mathsf{q}_{pr}^*$. Define q as the largest quality such that $\mathcal{P}_{\mathsf{q}}(q) \neq \mathcal{P}_{\mathsf{q}_{pr}^*}(q)$.

Assume that $|\mathcal{P}_{\mathsf{q}}(q)| > |\mathcal{P}_{\mathsf{q}_{pr}^*}(q)|$. Define $i^* := \min\left(\mathcal{P}_{\mathsf{q}}(q) \setminus \mathcal{P}_{\mathsf{q}_{pr}^*}(q)\right)$. As *(i)* both q and q_{pr}^* are contiguous, *(ii)* $\mathcal{P}_{\mathsf{q}}(h) = \mathcal{P}_{\mathsf{q}_{pr}^*}(h)$ for each $h > q$, *(iii)* $|\mathcal{P}_{\mathsf{q}}(q)| > |P_{\mathsf{q}_{pr}^*}(q)|$ and *(iv)*

q is feasible, we derive $s_{i^*}\mathsf{f}_q \leq \frac{\beta \cdot \mathsf{f}_q}{\mathsf{f}(\mathsf{q})} \leq \frac{\beta \cdot \mathsf{f}_q}{\sum_{j\in[i^*]} \mathsf{f}_{q_j}} = \frac{\beta \cdot \mathsf{f}_q}{\sum_{h=q+1}^{Q} (n^*_h - n^*_{h+1})\,\mathsf{f}_{q_h} + (i^* - n^*_{q+1})\mathsf{f}_q}$, which contradicts the definition of n^*_q. Hence, $|\mathcal{P}_{\mathsf{q}}(q)| < |\mathcal{P}_{\mathsf{q}^*_{pr}}(q)|$. As both q and q^*_{pr} are contiguous, $\mathcal{P}_{\mathsf{q}}(h) = \mathcal{P}_{\mathsf{q}^*_{pr}}(q)$ for each quality $h > q$ and $|\mathcal{P}_{\mathsf{q}}(q)| < |\mathcal{P}_{\mathsf{q}^*_{pr}}(q)|$, it follows that $\mathsf{f}_{\mathsf{q}^*_{pr}} \geq \sum_{h=q+1}^{Q}(n^*_h - n^*_{h+1})\mathsf{f}_{q_h} + |\mathcal{P}_{\mathsf{q}^*_{pr}}(q)|\mathsf{f}_q$ and $\mathsf{f}(\mathsf{q}) \leq \sum_{h=q+1}^{Q}(n^*_h - n^*_{h+1})\mathsf{f}_{q_h} + |\mathcal{P}_{\mathsf{q}}(q)|\,\mathsf{f}_q + \sum_{h=1}^{q-1}|\mathcal{P}_{\mathsf{q}}(h)|\mathsf{f}_{q_h}$, which yields $\mathsf{f}(\mathsf{q}^*_{pr}) - \mathsf{f}(\mathsf{q}) \geq \mathsf{f}_q - \sum_{h=1}^{q-1}|\mathcal{P}_{\mathsf{q}})(h)|\mathsf{f}_{q_h}$. Since $|\mathcal{P}_{\mathsf{q}}(h)| < \frac{\mathsf{f}_{h+1}}{\mathsf{f}_h}$ for each $h \in \{2,\ldots,Q-1\}$, it follows that $\sum_{h=1}^{q-1}|\mathcal{P}_{\mathsf{q}}(h)|\mathsf{f}_{q_h} < \mathsf{f}_q$. Hence, $\mathsf{f}(\mathsf{q}^*_{pr}) > \mathsf{f}(\mathsf{q})$. A contradiction to the fact that q is a Proportional Social Optimum. □

Removing the assumption that f_q divides f_{q+1} for all $q \in [Q-1]$ from Proposition 9, or proving that the assumption is necessary, remains a very interesting open problem. We now establish a connection between Proportional and Non-Proportional Social Optima.

▶ **Proposition 10.** *For a contest game with voluntary participation, assume that for each $q \in \{2,\ldots,Q-1\}$, f_q divides f_{q+1}, and consider a Proportional Social Optimum q^*_{pr} and a Non-Proportional Social Optimum q^*_{np}. Then, $\mathsf{f}(\mathsf{q}^*_{np}) \leq 2\,\mathsf{f}(\mathsf{q}^*_{pr}) + \mathsf{f}_Q$.*

Proof. [15, Lemma 2] implies that there exists a Non-proportional Social Optimum q^*_{np} with $\mathsf{f}(\mathsf{q}^*_{np}) \leq \mathsf{f}_Q\,(i^*+1)$, where $i^* := \max\{i \in [n] \mid \mathsf{f}_Q \cdot \sum_{i\in[i^*]} s_i \leq \beta\}$. Using $q = Q$ in Proposition 9 and recalling that $n^*_{Q+1} := 0$, we derive that there exists a Proportional Social Optimum q^*_{pr} in which n^*_Q players choose quality Q, so that $\mathsf{f}(\mathsf{q}^*_{pr}) \geq \mathsf{f}_Q\,n^*_Q$. By the definition of n^*_Q, we have $\frac{\beta}{n^*_Q+1} - \mathsf{f}_Q\,s_{n^*_Q+1} < 0$. Assume that $n^*_Q < \frac{i^*}{2}$. Then, $i^* - n^*_Q \geq \frac{i^*+1}{2}$ and $n^*_Q + 1 \leq \frac{i^*+1}{2}$, by $s_{n^*_Q+1} > \frac{\beta}{\mathsf{f}_Q\,(n^*_Q+1)} \geq \frac{\beta}{\mathsf{f}_Q\,\frac{i^*+1}{2}}$. So $\sum_{i=n^*_Q+1}^{i^*}\mathsf{f}_Q s_i \geq \frac{i^*+1}{2}s_{n^*_Q+1} > \frac{i^*+1}{2}\,\frac{\mathsf{f}_Q\beta}{\mathsf{f}_Q\,\frac{i^*+1}{2}} = \beta$, contradicting the definition of i^*. Hence, $n^*_Q \geq \frac{i^*}{2}$. Then, $\mathsf{f}(\mathsf{q}^*_{np}) \leq \mathsf{f}_Q\,i^* + \mathsf{f}_Q \leq 2\,\mathsf{f}_Q\,n^*_Q + \mathsf{f}_Q \leq 2\mathsf{f}(\mathsf{q}^*_{pr}) + \mathsf{f}_Q$, as needed. □

We are now ready to show almost tight bounds on the Mixed Price of Anarchy and the Asymptotic Mixed Price of Anarchy. Combining Theorem 7 and Proposition 10, we obtain:

▶ **Theorem 11.** *For a contest game with voluntary participation, assume that for each $q \in \{2,\ldots,Q-1\}$, f_q divides f_{q+1} and fix a Nash equilibrium $\boldsymbol{\sigma}$ and a Non-Proportional Social Optimum q^*_{np}. Then, it holds that $\mathsf{f}(\mathsf{q}^*_{np}) \leq 2\mathsf{f}(\boldsymbol{\sigma}) + 2\phi(\mathsf{CG}) + \mathsf{f}_Q$. Hence, $\mathsf{AMPoA}(\mathcal{G}_{vp}) \leq 2$ against a Non-Proportional Social Optimum.*

Proof. Fix a Proportional Social Optimum q^*_{pr}. By Theorem 7 and Lemma 10, $2\mathsf{f}(\boldsymbol{\sigma}) + 2\phi(\mathsf{CG}) + \mathsf{f}_Q \geq 2\mathsf{f}(\mathsf{q}^*_{pr}) + \mathsf{f}_Q \geq \mathsf{f}(\mathsf{q}^*_{np})$, which implies that $\mathsf{AMPoA}(\mathcal{G}_{vp}) \leq 2$ against a Non-Proportional Social Optimum. □

Proving a lower bound to complement Theorem 11 remains an interesting open problem.

6 Contest Games with Rational and Irrational Nash Equilibria

We show via an exhaustive case analysis (the complete proof is in the Appendix (Section F)):

▶ **Theorem 12.** *Consider the anonymous contest game CG with $n = 3$, $\beta = 1$ and mandatory participation. Assume that $\mathsf{f}_1 > \frac{1}{3}$ and $\min_{\ell\in[Q-1]}(\mathsf{f}_{\ell+1} - \mathsf{f}_\ell)$ is sufficiently large. Then:*

(1) *CG has no irrational Nash equilibrium.*

(2) CG *has a unique, non-pure and rational Nash equilibrium, which is symmetric, if and only if for each quality* $\ell \in [Q-1]$, $0 < \dfrac{(2\mathsf{f}_{\ell+1}+\mathsf{f}_\ell)(2-3(2\mathsf{f}_\ell+\mathsf{f}_{\ell+1}))}{\mathsf{f}_{\ell+1}-\mathsf{f}_\ell} < 1$.

We now turn to a class of contest games with a unique Nash equilibrium that is irrational. As a tool, we derive the *Elementary Symmetric Polynomials Indifference Equation,* or IE for short, which expresses the IP. It owes its name to the *elementary symmetric polynomials* from the theory of symmetric polynomials [45] it involves. We show (the complete proof is in the Appendix (Lemma G)):

▶ **Proposition 13** (Elementary Symmetric Polynomials Indifference Equation)**.** *Consider an anonymous contest game under mandatory participation. Then, in a Nash equilibrium* $\boldsymbol{\sigma}$*, for a mixed player* $i \in [n]$ *with* $\mathsf{supp}_i(\boldsymbol{\sigma}) = \{\ell, \ell+1\}$*,*

$$\sum_{1\le t\le n-1}\left(\frac{t\left(\mathsf{f}_{\ell+1}^2-\mathsf{f}_\ell^2\right)}{((n-t)\,\mathsf{f}_{\ell+1}+t\,\mathsf{f}_\ell)\,((n-t)\,\mathsf{f}_\ell+t\,\mathsf{f}_{\ell+1})}\cdot\right.$$

$$\left.\left(\sum_{S\subseteq[n]\setminus\{i\}||S|=t}\left(e_{n-1-|S|}(\boldsymbol{\sigma}_{-\{i\}\cup S})\,\left(1+\sum_{k=1}^{|S|}(-1)^k e_k(\boldsymbol{\sigma}\mid S)\right)\right)\right)\right)$$

$$-\frac{1}{n}\left(\sum_{k=1}^{n-1}(-1)^k e_k(\boldsymbol{\sigma}_{-i})-e_{n-1}(\boldsymbol{\sigma}_{-i})\right) \;=\; \frac{1}{n}+\mathsf{f}_{\ell+1}-\mathsf{f}_\ell\,.$$

We now use Proposition 13 to show (the complete proof is in the Appendix (Section H)):

▶ **Theorem 14.** *Every anonymous contest game with* $n=4$*,* $Q=2$ *and a sufficiently large ratio* $\alpha := \mathsf{f}_2/\mathsf{f}_1$ *has an irrational Nash equilibrium, which is unique, fully mixed and symmetric.*

In view of Theorem 14, it is natural to investigate the existence of special cases of the general problem of computing a Nash equilibrium that lie *in* PPAD. Towards this end, we consider the computation of an *ε-approximate pure Nash equilibrium*, with $\varepsilon > 0$: a profile q such that for any player i and quality $\widehat{q} \in [Q]$, $\mathsf{U}_i(\widehat{q}, \mathsf{q}_{-i}) \le \mathsf{U}_i(\mathsf{q}) + \varepsilon$. The *Lipschitz constant* of a game G [2] is given by $\delta = \delta(\mathsf{G}) := \max\{|\mathsf{U}_i(s_i, \mathsf{s}'_{-i}) - \mathsf{U}_i(s_i, \mathsf{s}'_{-i})|\}$, where the maximum ranges over all players i, all strategies s_i in her strategy set and all pairs s'_{-i} and s''_{-i} that differ in the strategy chosen by one player. Thus, for a game with Lipschitz constant δ, the utility of a player does not change by more than δ when a single opponent changes her strategy. A *λ-Lipschitz game* G, or just *Lipschitz game,* has $\delta(\mathsf{G}) = \lambda$; for any $\varepsilon > 0$, an $\frac{\varepsilon}{2n}$-Lipschitz game with n players has an ε-pure Nash equilibrium [2, Proposition 2.3]. If the Lipschitz constant of a game is $o\left(\frac{1}{n}\right)$, then, when the number n of players grows, any game with sufficiently many players has an additive ε-pure Nash equilibrium, since in this case a player's utility is essentially independent of the strategies chosen by her opponents. Formally, for a function $\delta : \mathbb{N} \to [0,1]$ such that $\delta(n) = o\left(\frac{1}{n}\right)$. for every $\varepsilon > 0$, there is an integer $N = N(\varepsilon)$ such that every game with $n \ge \max\{Q, N(\varepsilon)\}$ players and Lipschitz constant smaller than $\delta(n)$ has an ε-pure Nash equilibrium [2, Corollary 2.4]. We use it to show:

▶ **Theorem 15.** *For any* $\varepsilon > 0$*, there is an integer* $N = N(\varepsilon)$ *such that every contest game with* $n \ge \max\{N(\varepsilon), \Theta(Q)\}$ *players has an* ϵ*-approximate pure Nash equilibrium.*

Proof. Fix a contest game G and consider a player i, a quality q_i and partial quality profiles q'_{-i} and q''_{-i} in which player j chooses qualities q'_j and q''_j, respectively, and all other players choose the same qualities in the two partial quality profiles. Denote as $(q_i, q'_j, \mathsf{q}_{-\{i,j\}})$ (resp.,

$(q_i, q''_j, \mathsf{q}_{-\{i,j\}})$) the profiles in which player i chooses quality q_i, player j chooses quality q'_j (resp., quality q''_j) and all other players choose qualities as in q_{-i}; so all other players choose the same qualities in the profiles $(q_i, q'_j, \mathsf{q}_{-\{i,j\}})$ and $(q_i, q''_j, \mathsf{q}_{-\{i,j\}})$. Denote the number of players choosing quality q in a profile q as $\mathsf{N}_{\mathsf{q}}(q)$. Clearly, $\mathsf{N}_{(q_i,q''_j,\mathsf{q}_{-\{i,j\}})}(q'_j) = \mathsf{N}_{(q_i,q'_j,\mathsf{q}_{-\{i,j\}})}(q'_j) - 1$ and $\mathsf{N}_{(q_i,q''_j,\mathsf{q}_{-\{i,j\}})}(q'') = \mathsf{N}_{(q_i,q'_j,\mathsf{q}_{-\{i,j\}})}(q''_j) + 1$. Define the function $\delta(n) := \delta(\mathsf{G})$. We prove a simple fact (the complete proof is in the Appendix (Section I)):

▶ **Lemma 16.** $\delta(n) = o(1/n)$.

Define $\mathsf{f}_{\min} := \mathsf{f}_2$ if $\mathsf{f}_1 = 0$ or f_1 if $\mathsf{f}_1 > 0$; so $\mathsf{f}_{\min}$ is the smallest non-zero effort. (Using $\mathsf{f}_{\min}$ allows proving the result for either mandatory or voluntary participation.) Note that $\left(\sum_{q\in[Q]} \mathsf{f}_q \mathsf{N}_{(q_i,q'_j,\mathsf{q}_{-\{i,j\}})}(q)\right)\left(\sum_{q\in[Q]} \mathsf{f}_q \mathsf{N}_{(q_i,q''_j,\mathsf{q}_{-\{i,j\}})}(q)\right) \geq \mathsf{f}^2_{\min}\left(\sum_{q\in[Q]} \mathsf{N}_{(q_i,q'_j,\mathsf{q}_{-\{i,j\}})}(q)\right)\left(\sum_{q\in[Q]} \mathsf{N}_{(q_i,q''_j,\mathsf{q}_{-\{i,j\}})}(q)\right) = \mathsf{f}^2_{\min}\cdot n^2 = \Theta(n^2)$. Hence, $\min\left\{\left|\left(\sum_{q\in[Q]} \mathsf{f}_q \mathsf{N}_{(q_i,q'_j,\mathsf{q}_{-\{i,j\}})}(q)\right)\left(\sum_{q\in[Q]} \mathsf{f}_q\, \mathsf{N}_{(q_i,q''_j,\mathsf{q}_{-\{i,j\}})}(q)\right)\right|\right\} = \Omega(n^2)$. Since $\Omega(n^2) \subset \omega(n)$, this implies that $\delta(n) = o\left(\frac{1}{n}\right)$. Hence, by [2, Corollary 2.4], the claim follows. □

We recall from [35, Theorem 3.2] (see also [36, Theorem 7] for a simplified statement): *There is* some constant $\varepsilon > 0$ such that for all functions $\lambda(n) = \Theta(n^{-\frac{3}{4}})$, computing an ε-approximate pure Nash equilibrium of a λ-Lipschitz game is PPAD-complete. Since the contest game is a λ-Lipschitz game for some function $\lambda(n)$, the PPAD upper bound from [35, Theorem 3.2] is inherited to the class of contest games. For establishing PPAD-hardness, a reduction has been used in [35, Theorem 3.5] from the PPAD-complete problem of computing a (mixed) Nash equilibrium for a suitable polymatrix game. We do not know how to adjust the reduction in [35, Theorem 3.5] to the special case of contest games; doing so (or otherwise proving) remains a very interesting open problem. Hence, we are only able to claim:

▶ **Proposition 17.** *There is some constant $\varepsilon > 0$ such that computing an ε-approximate pure Nash equilibrium of a contest game is in* PPAD.

We finally show:

▶ **Theorem 18.** *Consider a contest game* CG *with proportional payments, anonymous players, mandatory participation, $n = 3$, $Q = 2$ and $\beta = 1$. Assume that* (1) $0 < \mathsf{f}_1 < \frac{1}{27}$ *and* (2) $4\mathsf{f}_1 < \mathsf{f}_2 \leq \frac{1-3\mathsf{f}_1}{6}$. *Then, the mixed profile $(0, x, y)$ is a Nash equilibrium if and only if $x = y = \frac{2\mathsf{f}_1+\mathsf{f}_2}{2(\mathsf{f}_2-\mathsf{f}_1)}$.*

Proof. Represent a mixed profile as $(\sigma_1, \sigma_2, \sigma_3)$, where σ_i is the probability that player i chooses quality 1, $i \in [3]$. First assume that $(0, x, y)$ is a Nash equilibrium. Since CG is symmetric, Theorem 12 implies that there is no irrational Nash equilibrium. We use a formulation of Lemma 1 from [61, Theorem 6.4] reproduced in the Appendix (Section A); these have the parameters i (for the player) and k (for the choice of pure strategy of player i). If $x = 0$ (and analogously if $x = 1$), the equations for $i = 2$ reduce to $\mathsf{f}_1(\mathsf{f}_1 - \mathsf{f}_2)y(y-1) = 0$. By mandatory participation, this implies $y(y-1) = 1$, which gives a pure strategy, a contradiction. So, we may assume that $0 < x, y < 1$. A computation using *Macaulay2* gives the following system of equations and inequality for x and y: *(i)* $(x-y)(x+y-1)\mathsf{f}_2(\mathsf{f}_1-\mathsf{f}_2) = 0$, *(ii)* $y(y-1)(x-y)\mathsf{f}_2(\mathsf{f}_1-\mathsf{f}_2) = 0$, *(iii)* $y(y-1)(2\mathsf{f}_1 y - 2\mathsf{f}_2 y - 2\mathsf{f}_1 - \mathsf{f}_2)\mathsf{f}_2(\mathsf{f}_1-\mathsf{f}_2) = 0$ and *(iv)* $(\mathsf{f}_1-\mathsf{f}_2)x + (\mathsf{f}_1-\mathsf{f}_2)y + 6\mathsf{f}_1^2 + 15\mathsf{f}_1\mathsf{f}_2 + 6\mathsf{f}_2^2 - 4\mathsf{f}_1 - 2\mathsf{f}_2 \geq 0$. Since $0 < y < 1$ and participation is mandatory, (ii) implies $x = y$, which makes *(i)* superfluous. Combining these proves the claim. Now, combining (1), $0 < y < 1$ and the inequality above gives a system of two inequalities in f_1 and f_2. Simple calculations yield the ranges described in (1) and (2). □

7 Irrationality Vs. Rationality via Algebraic Geometry

Our original motivation has been to import techniques from Algebraic Geometry (see the Primer in the Appendix (Section B)) into constructing reductions in hardness proofs for deciding the existence of irrational Nash equilibria in the contest game. Consider an n-players game with (expected) utility function U. Abusing notation, denote the game by U. We aim to construct the utility function $\widehat{\mathsf{U}}$ for another game from a targeted class of games and, in particular, for the class of contest games with $Q = 2$. To reach this goal, it must hold that $\widehat{\mathsf{U}}$ has an irrational Nash equilibrium if and only if U has an irrational Nash equilibrium. Towards this end, we construct a suitable isomorphism between algebraic varieties, defined over $\mathbb{Q}$ and preserving positive real points between the two varieties $\widehat{V}$ and V representing the Nash equilibria of $\widehat{\mathsf{U}}$ and U, respectively. A promising approach towards such a construction is grounded on rational positive dilation of the coordinates, as done below for n-player, 2-strategies games. Given that a computational hardness reveals a kind of *universality*, 2-strategies games have had a central role also in recent works proving universality results about Nash equilibria [3, 4, 18, 64], perhaps for some yet uncovered reason – see [20, Open Problem 4.4].) We show:

▶ **Theorem 19.** *Assume that the utility functions U and $\widehat{\mathsf{U}}$ of the n-players, 2-strategies games SG and $\widehat{\mathsf{SG}}$, respectively, satisfy $\widehat{\mathsf{U}}_i(j_1,\ldots,j_n) = a_{j_1}^{(1)}\cdots a_{j_{i-1}}^{(i-1)}a_{j_{i+1}}^{(i+1)}\cdots a_{j_n}^{(n)}\,\mathsf{U}_i(j_1,\ldots,j_n)$, for a vector $\boldsymbol{\alpha} = \left(a_{j_i}^{(i)}\right)_{i\in[n],j_1,\ldots,j_n\in[2]}$ of positive rationals, for each $i\in[n]$. Then, fully mixed Nash equilibria of SG are in bijective correspondence with fully mixed Nash equilibria of $\widehat{\mathsf{SG}}$ via the map given by coordinate-wise multiplication with $\boldsymbol{\alpha}$. Hence, SG has an irrational (resp., rational) fully mixed Nash equilibrium if and only if $\widehat{\mathsf{SG}}$ does.*

Proof. By [61, Theorem 6.6], the fully mixed Nash equilibria for U with $d_i = 2$ for each i are given by the positive real points of the variety V defined by the polynomials f_i in the polynomial ring $\mathsf{R} = \mathbb{Q}[p_{j_k}^{(k)} \mid k\in[n], j_k\in\{1,2\}]$, with

$$f_i := \sum_{j_1,\ldots,j_{i-1},j_{i+1},\ldots,j_n\in\{1,2\}} \left(\mathsf{U}_i(j_1,\ldots,j_{i-1},2,j_{i+1},\ldots,j_n) - \mathsf{U}_i(j_1,\ldots,j_{i-1},1,j_{i+1},\ldots,j_n)\right)\prod_{k\in[n]\setminus\{i\}} p_{j_k}^{(k)},$$

where $i\in[2]$ and $p_{j_k}^{(k)}$ denotes the probability that player k chooses pure strategy j_k. For each $k\in[n]$ and $j_k\in\{1,2\}$, $a_{j_k}^{(k)}$ denotes a positive rational number. Denote as $\varphi:\mathsf{R}\to\mathsf{R}$ the map that fixes the ground field $\mathbb{Q}$ and maps $p_{j_k}^{(k)}$ to $a_{j_k}^{(k)}p_{j_k}^{(k)}$. Since this is an isomorphism of $\mathbb{Q}$-algebras, it induces an isomorphism $\mathbb{A}_{\mathbb{C}}^{2k}\to\mathbb{A}_{\mathbb{C}}^{2k}$. From a geometric point of view, this isomorphism dilates the coordinate $p_{j_k}^{(k)}$ by the positive rational factor $a_{j_k}^{(k)}$. φ maps a polynomial f_k to the polynomial

$$\widehat{f_k} := \sum_{j_1,\ldots,j_{i-1},j_{i+1},\ldots,j_n\in\{1,2\}} \left(\widehat{\mathsf{U}}_i(j_1,\ldots,j_{i-1},2,j_{i+1},\ldots,j_n) - \widehat{\mathsf{U}}_i(j_1,\ldots,j_{i-1},1,j_{i+1},\ldots,j_n)\right)\prod_{k\in[n]\setminus\{i\}} p_{j_k}^{(k)},$$

where $\widehat{\mathsf{U}}_i(j_1,\ldots,j_n) := \mathsf{U}_i(j_1,\ldots,j_n)a_{j_1}^{(1)}\cdots a_{j_{i-1}}^{(i-1)}a_{j_{i+1}}^{(i+1)}\cdots a_{j_n}^{(n)}$. So, the $\widehat{f_k}$'s define a variety $\widehat{V}$ whose positive real points are the fully mixed Nash equilibria of the game $\widehat{\mathsf{U}}$. As the dilation factors are positive rational numbers, φ preserves fully mixed Nash equilibria; so an irrational (resp., rational) fully mixed Nash equilibrium of U is mapped to an irrational (resp., rational) fully mixed Nash equilibrium of $\widehat{\mathsf{U}}$ and vice versa. The dilation φ maps the defining polynomials for the fully mixed Nash equilibria of U to the defining polynomials

for the fully mixed Nash equilibria of $\widehat{\mathsf{U}}$. Since the $a_{j_i}^{(i)}$ are positive rational numbers, its corresponding map of affine spaces restricts to a self-bijection of positive rational (resp., irrational) points, as needed. □

In view of Theorem 19, a few interesting open problems pose themselves. First, is it *in general* necessary for a strategic game *with* an irrational Nash equilibrium, that a fully mixed Nash equilibrium, if existing, has to be irrational? In the same vein, the weaker question for a discrete Tullock contest game with $Q = 2$ might be more tractable. Settling this necessity could be instrumental for *verifying* the existence of an irrational Nash equilibrium in such a contest game.

▶ **Theorem 20.** *Represent the 2-players, 2-strategies strategic game with utility tables* $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ *and* $\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ *as the point* $(a_{11}, a_{12}, a_{21}, a_{22}, b_{21}, b_{12}, b_{21}, b_{22}) \in \mathbb{R}^8$*; consider the subset* $\mathcal{C} \subseteq \mathbb{R}^8$ *of contest games with* $n = 2$ *and* $Q = 2$*. Then:*

(1) *$\mathcal{C}$ is the constructible set of points satisfying the sets* (1) *and* (2)

(1/i) $a_{22} = a_{21} + a_{12} - a_{11}$	(2/i) $a_{11} \neq a_{12}$	(2/v) $b_{11} \neq b_{21}$
(1/ii) $b_{22} = b_{21} + b_{12} - b_{11}$	(2/ii) $a_{11} \neq a_{22}$	(2/vi) $b_{11} \neq b_{22}$
(1/iii) $a_{11} - a_{12} = b_{11} - b_{21}$	(2/iii) $a_{21} \neq a_{22}$	(2/vii) $b_{12} \neq b_{22}$
(1/iv) $a_{21}b_{21} = a_{12}b_{12}$	(2/iv) $a_{21} \neq a_{12}$	(2/viii) $b_{21} \neq b_{12}$
	or $a_{21} = a_{12} = 0$	*or* $b_{21} = b_{12} = 0$

of algebraic conditions. The closure of $\mathcal{C}$ is the irreducible variety of dimension 4 *and degree* 2 *defined by (1/i) through (1/iv).*

(2) *The set of contest games with $n = 2$, $Q = 2$ and efforts $\mathsf{f}_1, \mathsf{f}_2 \neq 0$, such that $\beta \neq \mathsf{f}_1 + \mathsf{f}_2$ is the set of games with $n = 2$, $Q = 2$ and utility tables* $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{21} + a_{12} - a_{11} \end{pmatrix}$ *and* $\begin{pmatrix} a_{11} - a_{12} + \frac{a_{12}b}{a_{21}} & \frac{a_{12}b}{a_{21}} \\ b & b - a_{11} + a_{12} \end{pmatrix}$*, where $a_{11}, b \in \mathbb{R}$ and $a_{21}, a_{12} \in \mathbb{R} \setminus \{0\}$.*

Proof. We start with (1). By simple transformations on the defining equations for the utilities of a contest game, we obtain $a_{ij}(\mathsf{f}_i+\mathsf{f}_j)-\beta\mathsf{f}_i+s_1\mathsf{f}_i(\mathsf{f}_i+\mathsf{f}_j) = 0$ and $b_{ij}(\mathsf{f}_i+\mathsf{f}_j)-\beta\mathsf{f}_j+s_2\mathsf{f}_j(\mathsf{f}_i+\mathsf{f}_j) = 0$ for $i, j \in \{1, 2\}$. The ideal I generated by the left-hand sides of these equations in the complex polynomial ring in the appearing variables defines an algebraic variety V in the complex affine space with coordinates a_{ij}, b_{ij}, f_i, β and s_i for $i, j \in \{1, 2\}$. To remove irreducible components from V, where the utilities may not be well-defined due to a division by zero, I is saturated at the principal ideal generated by the product of the polynomials $\mathsf{f}_i+\mathsf{f}_j$, with $i, j \in \{1, 2\}$. By *Chevalley's Theorem* (Theorem 24 in the Appendix), the projection of the resulting variety to the coordinates a_{ij} and b_{ij}, $i, j \in \{1, 2\}$, is a constructible set in its ambient affine space and its Zariski closure $\widehat{V}$ coincides with its Euclidean closure. A *Macaulay2* computation for $\widehat{V}$ yields the defining equations (1/i) - (1/iv) for $\widehat{V}$ and the values for dimension and degree. (Note that since every non-zero point on $\widehat{V}$ is smooth, the real part $\widehat{V}(\mathbb{R})$ (the closure of $\mathcal{C}$ in its real ambient space) is dense inside $\widehat{V}$ and gives the same values for the constructible set $\mathcal{C}$.) To get the set of all defining equation and non-equations for the $\mathcal{C}$, note that the condition $\mathsf{f}_1 \neq \mathsf{f}_2$ (which is the standing assumption for contest games) is equivalent to (2/i) - (2/viii). For (2), note that the assumption $\beta \neq \mathsf{f}_1 + \mathsf{f}_2$ implies that $a_{21}, a_{12} \neq 0$; hence, we get the required representation by using the conditions from (1). □

Note that conclusions (1) and (2) in Theorem 20 are *not* inverses of each other: (1) gives a characterization of contest games in the form of algebraic equations on the utilities; (2) gives an alternative characterization in the form of a parametrization of the utilities by rational

functions in the four free variables a_{11}, b, a_{21} and a_{12}. Since we showed in Theorem 20 that the variety is 4 dimensional; so at least 4 parameters are needed and the parametrization is as small as possible. (To justify this in a different way, we used the set of relations with homogeneous defining polynomials from (1) to derive the parametrization and used *Macaulay2* to check that this set is minimal.) Although, by *Chevalley's Theorem* (Theorem 24 in the Appendix), the set of *all* contest games of arbitrary size is constructible, computing the defining equations of its Zariski closure in the general case with arbitrary n and Q seems infeasible by using *Macaulay2*. Nevertheless, it would be interesting to determine its dimension and degree as a variety in the affine space of contest games. A *Macaulay2* computation shows that for $n = 3$ and $Q = 2$, there are 226 polynomial equations, which cut out a variety of dimension 12 and degree 2 in the 24-dimensional vector space of all strategic games with $n = 3$. Some preliminary computations suggest that in the case with general n and Q, the dimension is around $\frac{nQ^n}{2}$ and the degree is equal to Q, but this remains to be proven. We finally show:

▶ **Proposition 21.** *An anonymous contest game* CG *with $n = 2$, $Q = 2$ and mandatory participation has a rational, fully mixed Nash equilibrium if and only if $2(\mathsf{f}_1 + \mathsf{f}_2) = 1$. If $2(\mathsf{f}_1 + \mathsf{f}_2) = 1$, then every fully mixed profile is a Nash equilibrium.*

Proof. We shall investigate conditions for the existence of a fully mixed Nash equilibrium $\{p_i^\lambda\}_{i\in[2],\ell\in\{0,1\}}$. Clearly, $\mathbb{E}(\mathsf{U}_i(\ell) = p_i^\ell\left(\frac{1}{2} - \mathsf{f}(\ell)\right) + (1 - p_i^\ell)\left(\frac{\mathsf{f}(\ell)}{\mathsf{f}(\ell)+\mathsf{f}_{\bar\ell}} - \mathsf{f}(\ell)\right) = p_i^\ell\left(\frac{1}{2} - \frac{\mathsf{f}(\ell)}{\mathsf{f}(\ell)+\mathsf{f}_{\bar\ell}}\right) + \frac{\mathsf{f}(\ell)}{\mathsf{f}(\ell)+\mathsf{f}_{\bar\ell}} - \mathsf{f}(\ell)$; $\mathbb{E}(\mathsf{U}_i^{\bar\ell}) = (1 - p_i^\ell)\left(\frac{1}{2} - \mathsf{f}_{\bar\ell}\right) + p_i^\ell\left(\frac{\mathsf{f}_{\bar\ell}}{\mathsf{f}(\ell)+\mathsf{f}_{\bar\ell}} - \mathsf{f}_{\bar\ell}\right) = p_i^\ell\left(-\frac{1}{2} + \frac{\mathsf{f}_{\bar\ell}}{\mathsf{f}(\ell)+\mathsf{f}_{\bar\ell}}\right) + \frac{1}{2} - \mathsf{f}_{\bar\ell}$. By the *Indifference Principle*, $\mathbb{E}(\mathsf{U}_i^\ell) = \mathbb{E}(\mathsf{U}_i^{\bar\ell})$ if and only if $0 \cdot p_i^\ell = \frac{1}{2} - \mathsf{f}_{\bar\ell} - \frac{\mathsf{f}(\ell)}{\mathsf{f}(\ell)+\mathsf{f}_{\bar\ell}} + \mathsf{f}(\ell) = (\mathsf{f}(\ell) - \mathsf{f}_{\bar\ell})\left(1 - \frac{1}{2(\mathsf{f}(\ell)+\mathsf{f}_{\bar\ell})}\right)$ if and only if (since $\mathsf{f}(\ell) \neq \mathsf{f}_{\bar\ell}$) $2(\mathsf{f}(\ell) + \mathsf{f}_{\bar\ell}) = 1$; then, every fully mixed profile is a Nash equilibrium. □

The defining equations and inequalities for Nash equilibria from [61, Theorem 6.4], which restate Lemma 1, may be handy for characterizing mixed Nash equilibria other than the fully mixed one. For example, it is straightforward to derive directly:

▶ **Corollary 22.** *For an anonymous contest game with $n = 2$, $Q = 2$, mandatory participation and utility tables $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, the mixed profile $((1,0),(x,y))$ is a Nash equilibrium if and only if $a_{11} \geq a_{21}$ and $b_{11} = b_{12}$.*

8 A Few More Open Problems

1. The techniques used to show the MPoA bounds are specifically tailored to the special form of both payments and skill-effort functions in the discrete Tullock contest game. In addition, the MPoA refers to Social Optima that also obey particular restrictions. It is an intriguing open problem whether these results could be extended (and how) to more general classes of concave or aggregative games.
2. Beyond the scope of the present article lies a complexity-theoretic question that remains tantalizingly open: *What is the complexity of computing an approximate Nash equilibrium or some (strong or weak) approximation of a Nash equilibrium for the contest game?* We conjecture PPAD-hardness and FIXP-hardness, respectively.

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A Sturmfels' Form [61, Theorem 6.4] of Lemma 1

Consider a game G, where player i has d_i pure strategies and *utility matrix* $\mathsf{U}^{(i)}$, where $i \in [n]$, in the game G. A mixed profile $\boldsymbol{\sigma} = \{\sigma_i^k\}_{i\in[n],k\in[d_i]}$ is a *Nash equilibrium* for G if and only if the pair $(\{\sigma_i\}_{i\in[n]}, \{\pi_i\})$ is a zero of the polynomials

$$\sigma_i^k \left(\pi_i - \sum_{j_1=1}^{d_1} \cdots \sum_{j_{i-1}=1}^{d_{i-1}} \sum_{j_{i+1}=1}^{d_{i+1}} \cdots \sum_{j_n=1}^{d_n} \mathsf{U}^{(i)}_{j_1\ldots j_{i-1}kj_{i+1}\ldots j_n} \sigma_1^{j_1} \ldots \sigma_{i-1}^{j_{i-1}} \sigma_{i+1}^{j_{i+1}} \ldots \sigma_n^{j_n} \right),$$

with $i \in [n]$ and $k \in [d_i]$, where

$$\pi_i \quad = \quad \sum_{j_1=1}^{d_1} \sum_{j_2=1}^{d_2} \cdots \sum_{j_n=1}^{d_n} \mathsf{U}^{(i)}_{j_1 j_2 \ldots j_n} \cdot \sigma_1^{j_1} \sigma_2^{j_2} \cdots \sigma_n^{j_n}$$

and all parenthesized expressions in these polynomials is always nonnegative.

B A Primer on Algebraic Geometry

Some basic familiarity of the reader with basic concepts of Abstract Algebra (e.g., rings, fields and vector spaces) and Topology (e.g., topology, constructible sets, closures) is assumed. For a panoramic view of Algebraic Geometry, we refer the reader to [8, 37] or to [5, Chapter 4] for a concise exposition of the machinery needed for the proofs using Algebraic Geometry. The Primer attempts to distill this machinery. *Macaulay2* is a computer algebra system for computation in Commutative Algebra and Algebraic Geometry.

A *field* $\Bbbk$ is a set endowed with two operations, *addition* and *multiplication*, satisfying certain axioms. We shall restrict to the three fields $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Q}$. The *Euclidean closure* operator is an operator $\mathbf{K}_e$ that extends a subset A of the finite-dimensional vector space $\Bbbk^d$ to include all points arbitrarily close to points in A under the usual Euclidean metric d_e, called *distance,* on $\Bbbk^d$; formally, $\mathbf{K}_e\mathsf{A} := \{x \in \Bbbk^d \mid (\forall \varepsilon > 0)(\exists a \in \mathsf{A})\, \mathsf{d}(x, a) < \varepsilon\}$. For an integer e, with $1 \le e < d$, a *projection map* from $\Bbbk^d$ to $\Bbbk^e$ maps a point $(x_1, \ldots, x_d) \in \Bbbk^d$ to the point $(x_1, \ldots, x_e) \in \Bbbk^e$; that is, it projects it to its first e coordinates.

The space of polynomials in d indeterminates $x_1, \ldots, x_d$, with coefficients in $\Bbbk$, equipped with an addition, a multiplication and a scalar muliplication that make it a commutative algebra over $\Bbbk$, is denoted as $\Bbbk[x_1, \ldots, x_d]$ and called the *polynomial ring* over $\Bbbk$; when $\Bbbk = \mathbb{C}$, it is called the *complex polynomial ring.*

Henceforth, we shall assume that the field $\Bbbk$ is algebraically closed. A finite subset $S \subseteq \Bbbk[x_1, \ldots, x_d]$ defines an *affine algebraic set* $V = V(S)$ as the set of all common zeros of the polynomials in S; roughly speaking, V is the set of points defined by polynomial equations. V is a subset of $\Bbbk^d$, which is called the *ambient affine space* of V. An *affine algebraic variety* embedded in $\Bbbk^d$ is an affine algebraic set (defined by a finite subset $S \subseteq \Bbbk[x_1, \ldots, x_d]$) that is not the union of two proper affine algebraic sets; so it is *irreducible.*[3] When S is a set of linear polynomials, the induced affine algebraic variety is called *linear.* The *Zariski topology* on $\Bbbk^d$ is defined as the topology on $\Bbbk^d$ whose closed sets are affine algebraic sets. The *Zariski closure* of a set $\mathsf{A} \subseteq \Bbbk^d$ extends A to the smallest (affine) algebraic set containing it. (When A is *constructible*, the Zariski closure of A coincides with its Euclidean closure.)

[3] The currently dominating trend is to define an *affine algebraic variety* as an affine algebraic set and reserve the term of *irreducible variety,* or, in full, *irreducible affine algebraic variety* for an algebraic set that is not the union of two proper affine algebraic subsets.

The *coordinate ring of* V, denoted as $\Bbbk[V]$, is defined to be the set of all polynomial functions $\mathsf{f}\mid_V : V \to \Bbbk$, where $\mathsf{f} \in \Bbbk[x_1, \ldots, x_d]$; $\mathsf{f}\mid_V$, is the restriction of f to V. The ring $\Bbbk[V]$ is called the *ring of polynomial functions on* V. A major strength of Computational Algebraic Geometry is due to that $\Bbbk[V]$ reflects most geometric properties, such as dimension and degree, of V in pure algebraic terms; thus, many questions about varieties can be addressed by symbolic computations in $\Bbbk[V]$ using the technique of *Gröbner bases*, which is implemented in *Macaulay2*, for instance (see below).

A *reducible* algebraic variety V is the union of a finite number of affine algebraic varieties, called *subvarieties*; it can be expressed uniquely as the union of finitely many, *maximal* subvarieties, called the *irreducible components* of V. For an *irreducible* algebraic variety V, if $V = V_1 \cup V_2$, then either $V = V_1$ or $V = V_2$.

The *dimension* of an affine algebraic variety V, denoted as $\mathsf{dim} := \mathsf{dim}(V)$, is the maximal length of a chain $V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_{\mathsf{dim}}$ of distinct, nonempty, closed irreducible subvarieties of V. By way of example, a finite point set, a curve and a surface have dimension 0, 1 and 2, respectively. The *degree* of an affine algebraic variety V embedded in $\Bbbk^d$ is the maximal (finite) number of intersection points of V with a linear variety $L \subseteq \Bbbk^d$ such that $\mathsf{dim}(V) + \mathsf{dim}(L) = d$. For example, a variety given by a single polynomial of degree r has degree r. The *codimension* of a subvariety $Z \subseteq V$ is defined as $\mathsf{dim}(V) - \mathsf{dim}(Z)$.

The *correspondence* $\mathbb{I}$ maps an affine algebraic variety $V \subseteq \Bbbk^d$ to the *ideal* I *associated to* V or simply the ideal of V, as $\mathsf{I} = \mathbb{I}(V) := \{\mathsf{f} \in \Bbbk[x_1, \ldots, x_d] \mid \mathsf{f}(\alpha_1, \ldots, \alpha_d) = 0 \text{ for all } (\alpha_1, \ldots, \alpha_d) \in V\}$. A finite set of non-zero polynomials $G = \{\mathsf{g}_1, \ldots, \mathsf{g}_s\} \subseteq \Bbbk[x_1, \ldots, x_d]$ *generates* the ideal $\mathsf{I} \subseteq \Bbbk[x_1, \ldots, x_d]$ if I is the smallest ideal in $\Bbbk[x_1, \ldots, x_d]$ containing G; that is, I is the intersection of all ideals containing S. A *principal ideal* is an ideal generated by a singleton set G. If V has the irreducible decomposition $V = V_1 \cup V_2 \cup \ldots \cup V_k$, then $\mathbb{I}(V) = \mathbb{I}(V_1) \cap \mathbb{I}(V_2) \cap \ldots \cap \mathbb{I}(V_k)$; so the ideal of the Zariski closure of a set $\mathsf{A} \subseteq \Bbbk^d$ is the intersection of principal ideals. The *correspondence* $\mathbb{V}$, also called *vanishing locus,* maps an ideal I to the set $\mathbb{V}(\mathsf{I}) := \{(\alpha_1, \ldots, \alpha_d) \in \Bbbk^d \mid \mathsf{f}(\alpha_1, \ldots, \alpha_d) = 0 \text{ for all } \mathsf{f} \in \mathsf{I}\}$; so $\mathbb{V}(\mathsf{I})$ is the set of common zeros of functions $\mathsf{f} \in \Bbbk[x_1, \ldots, x_d]$ in I. The *saturation of an ideal* I_1 *at an ideal* I_2 is a new ideal associated to the affine algebraic variety consisting of those irreducible components of $\mathsf{V}(\mathsf{I}_1)$ that are not contained in $\mathsf{V}(\mathsf{I}_2)$. An ideal $\mathbb{I}(V)$ is always *radical*. *Hilbert's Nullstellensatz* links the geometry of algebraic sets to the algebra of ideals.

▶ **Theorem 23.** *[Hilbert's Nullstellensatz (or Zero-Locus-Theorem)] For an algebraically closed field* $\Bbbk$*, there is an order-reversing bijection between the set of radical ideals in* $\Bbbk[x_1, \ldots, x_d]$ *and the set of affine algebraic subsets of* $\Bbbk^d$*.*

So, *Hilbert's Nullstellensatz* links the geometry of algebraic sets to the algebra of ideals.

The *Affine Jacobi criterion* yields a necessary and sufficient condition for an affine algebraic variety $V \subset \Bbbk^d$ with ideal $\mathsf{I}(V) = \{\mathsf{f}_1, \ldots, \mathsf{f}_r\}$ to be *smooth* at a point $p \in V$: the rank of the $r \times d$ Jacobian matrix $\left[\frac{\partial f_i}{\partial x_j}(p)\right]_{1 \le i \le r, 1 \le j \le d}$ is $d - \mathsf{dim}(V)$. A set X is *Zariski-dense* in a set Y if the Zariski closure of X equals Y. A necessary and sufficient condition for the set $V(\mathbb{R})$ of points with real coordinates on an affine algebraic variety $V \subseteq \mathbb{C}^d$ to be Zariski-dense in V is that V has the *smoothness* property: it contains a real point at which it is smooth. Although the existence of a smooth real point is not checkable in *Macaulay2,* one can instead check in *Macaulay2* whether the variety is smooth, i.e., all of its points are smooth; then, in particular, every real point will be smooth.

A subset of a topological space is *locally closed* if it is the intersection of an open set and a closed set. A *constructible set* is a disjoint finite union of locally closed sets. Removing a

lower-dimensional variety from an affine algebraic variety $V \subset \Bbbk^d$ (for instance, removing a curve from a surface) results in a constructible set. We have the famous *Chevalley's Theorem*:

▶ **Theorem 24** (*Chevalley's Theorem*). *The image of a constructible subset of an algebraic variety under a polynomial map is a constructible set.*

A *morphism*, also called *regular map*, between two algebraic varieties is a polynomial map between them. In case the algebraic varieties are defined over a smaller field $\Bbbk \subset \Bbbk$ and all the coefficients of the polynomial map lie in $\Bbbk$, we say that the morphism is defined over $\Bbbk$. By *Chevalley's Theorem*, the image of the *projection* of an affine algebraic variety $V = \mathbb{V}(\mathsf{I}) \subseteq \Bbbk^d$ to the coordinates $x_1, \ldots, x_e$ of $\Bbbk^d$, with $e < d$, is a constructible set. Its Zarisky closure is the algebraic set associated with the contracted ideal $\mathsf{I} \cap \Bbbk[x_1, \ldots, x_e]$. An *isomorphism* between two algebraic varieties is a bijective morphism between them whose inverse map is also a morphism.

A finite set G of non-zero polynomials that generates the ideal I is a *Gröbner basis* of I if the leading monomial (with respect to a fixed monomial order) of every polynomial in I is a multiple of the leading monomial of some polynomial in G. Gröbner bases can be computed using *Macaulay2*. An important application of Gröbner basis is to eliminate variables; *elimination* consists of finding generators for the *i-th elimination ideal* of the ideal $\mathsf{I} \subseteq \Bbbk[x_1, \ldots, x_d]$, where $1 < i \leq n$, defined as $\mathsf{I} \cap [x_i, x_{i+1} \ldots, x_d]$.

For a fixed ideal $\mathsf{I} \subseteq \Bbbk[x_1, \ldots, x_d]$, one can define the *quotient ring* $\Bbbk[x_1, \ldots, x_d]/\mathsf{I}$, which is a set consisting of the cosets of I in $\mathbb{C}[x_1, \ldots, x_d]$. By *Hilbert's Nullstellensatz,* there is an order-reversing bijection between the varieties $V \subseteq \mathbb{C}^d$ and the set of radical ideals in $\mathbb{C}[x_1, \ldots, x_d]$. Using this bijection, we define a correspondence mapping a variety $\mathsf{V} \subseteq \mathbb{C}^d$ to the quotient ring $\mathbb{C}[x_1, \ldots, x_d]/\mathbb{I}(V)$. Under this correspondence, morphisms of algebraic varieties are in bijection with $\mathbb{C}$-algebra homomorphisms of the corresponding quotient rings. For us, this concretely means the following: To construct an isomorphism $V \to W$ between two varieties $V = \mathbb{V}(\mathsf{I})$ and $W = \mathbb{V}(\mathsf{J})$, it suffices to construct a bijective $\mathbb{C}$-algebra homomorphism $\mathbb{C}[x_1, \ldots, x_d]/\mathsf{J} \to \mathbb{C}[x_1, \ldots, x_d]/\mathsf{I}$.

C Counterexample to a Claim in [15] about the Existence of Pure Nash Equilibrium

▶ **Proposition 25.** *There is a contest game with $n = 2$, $Q = 3$ and voluntary participation, admitting no pure Nash equilibrium.*

Proof. Set $\beta := 36$, $s_1 := 5$, $s_2 := 14$ $\mathsf{f}_2 := 1$ and $\mathsf{f}_3 := 2$. We verify in Table 1 that for each of the 9 profiles, there is an improvement step for some player. □

D Proof of Lemma 2

For (1), assume, by way of contradiction, that there is a Nash equilibrium $\boldsymbol{\sigma}$ with $\mathsf{supp}_i(\boldsymbol{\sigma}) = \mathsf{supp}_j(\boldsymbol{\sigma})$, $|\mathsf{supp}_i(\boldsymbol{\sigma})| = 2$ and $s_i < s_j$. Set $\Delta := s_j - s_i > 0$. As i and j are playing the same mixed strategy in $\boldsymbol{\sigma}$, we have, for each quality $q \in [Q]$, $\sum_{\mathsf{q} \in [Q]^n} \frac{\mathsf{Pr}_{\boldsymbol{\sigma}}(\mathsf{q}_{-i})\mathsf{f}_q}{\mathsf{f}_q + \mathsf{f}(\mathsf{q}_{-i})} = \sum_{\mathsf{q} \in [Q]^n} \frac{\mathsf{Pr}_{\boldsymbol{\sigma}}(\mathsf{q}_{-j})\mathsf{f}_q}{\mathsf{f}_q + \mathsf{f}(\mathsf{q}_{-j})}$. Hence, $\mathsf{U}_j(q, \boldsymbol{\sigma}_{-j}) = \mathsf{U}_i(q, \boldsymbol{\sigma}_{-i}) - \Delta\mathsf{f}_q$. Since the common support of players i and j contains two consecutive qualities, say q^* and $q^* + 1$, and $\boldsymbol{\sigma}$ is a Nash equilibrium, Lemma 1 implies that $\mathsf{U}_i(q^*, \boldsymbol{\sigma}_{-i}) = \mathsf{U}_i(q^* + 1, \boldsymbol{\sigma}_{-i})$ and $\mathsf{U}_j(q^*, \boldsymbol{\sigma}_{-j}) = \mathsf{U}_j(q^* + 1, \boldsymbol{\sigma}_{-j})$. Using $\mathsf{U}_j(q^*, \boldsymbol{\sigma}_{-j}) = \mathsf{U}_i(q^*, \boldsymbol{\sigma}_{-i}) - \Delta\mathsf{f}_{q^*}$ and $\mathsf{U}_j(q^* + 1, \boldsymbol{\sigma}_{-j}) = \mathsf{U}_i(q^* + 1, \boldsymbol{\sigma}_{-i}) - \Delta\mathsf{f}_{q^*+1}$, we derive

In profile q:	$\mathsf{U}_1(\mathsf{q})$	$\mathsf{U}_2(\mathsf{q})$	Player	Takes an improvement step to:
$(1,1)$	0	0	2	$(1,2)$
$(1,2)$	0	22	1	$(2,2)$
$(1,3)$	0	8	1	$(2,3)$
$(2,1)$	31	0	2	$(2,2)$
$(2,2)$	13	4	1	$(3,2)$
$(2,3)$	7	-4	2	$(2,1)$
$(3,1)$	26	0	1	$(2,1)$
$(3,2)$	14	-2	2	$(3,1)$
$(3,3)$	8	-10	2	$(3,1)$

Table 1 The game CG and the improving deviations.

that $\mathsf{U}_i(q^*, \boldsymbol{\sigma}_{-i}) - \Delta\mathsf{f}_{q^*} = \mathsf{U}_i(q^*+1, \boldsymbol{\sigma}_{-i}) - \Delta\mathsf{f}_{q^*+1}$, or $\mathsf{f}_{q^*} = \mathsf{f}_{q^*+1}$. A contradiction to the assumption that f is strictly increasing. (1) rules out the existence of symmetric non-pure Nash equilibria in non-anonymous contest games, proving (2). Finally, (3) follows from strict concavity of the expected utilities.

E Proof of Lemma 6

Define $b_j = \sqrt{a_j x_j}$ and $c_j = \sqrt{x_j/a_j}$ for each $j \in [k]$. Since, for every $j \in [k]$, a_j is positive and x_j is non-negative, both b_j and c_j are real numbers. Thus, by Schwarz's Inequality,

$$\left(\sqrt{\sum_{j=1}^{k} a_j x_j \cdot \sum_{j=1}^{k} \frac{x_j}{a_j}}\right)^2 = \left(\sqrt{\sum_{j=1}^{k} b_j^2 \cdot \sum_{j=1}^{k} c_j^2}\right)^2 \geq \left(\sum_{j=1}^{k} (b_j c_j)\right)^2 = \left(\sum_{j=1}^{k} x_j\right)^2,$$

as needed.

F Proof of Theorem 12

Denote the three anonymous players as i, j and k. For ease of notation, write $\mathsf{M} := \max_{\ell \in [Q]} \mathsf{f}_\ell = \mathsf{f}_Q$ and $\mathsf{m} := \min_{\ell \in [Q]} \mathsf{f}_\ell = \mathsf{f}_1$. If CG has no mixed Nash equilibrium, then (1) holds vacuously. So assume CG has a mixed Nash equilibrium. Consider a mixed Nash equilibrium $\boldsymbol{\sigma}$. Since a player has at most two (consecutive) best-responses in a Tullock contest with discrete strategy sets and strictly concave utility functions, a mixed player i has $\mathsf{supp}_i(\boldsymbol{\sigma}) = \{\ell, \bar{\ell}\}$; where, for ease of notation, we denote $\ell+1$ as $\bar{\ell}$. The expected conditional utility of player i had she chosen quality ℓ is

$$\begin{aligned}
& \mathsf{U}_i(\ell, \boldsymbol{\sigma}_{-i}) \\
= \; & \left(\frac{1}{3} - \mathsf{f}_\ell\right) \sigma_j^\ell \, \sigma_k^\ell + \left(\frac{\mathsf{f}_\ell}{2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}}} - \mathsf{f}_\ell\right) \sigma_j^\ell \, \sigma_k^{\bar{\ell}} + \left(\frac{\mathsf{f}_\ell}{2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}}} - \mathsf{f}_\ell\right) \sigma_j^{\bar{\ell}} \, \sigma_k^\ell \\
& + \left(\frac{\mathsf{f}_\ell}{2\mathsf{f}_{\bar{\ell}} + \mathsf{f}_\ell} - \mathsf{f}_\ell\right) \sigma_j^{\bar{\ell}} \, \sigma_k^{\bar{\ell}} \\
= \; & \frac{1}{3} \sigma_j^\ell \, \sigma_k^\ell + \frac{\mathsf{f}_\ell}{2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}}} \left(\sigma_j^\ell \, \sigma_k^{\bar{\ell}} + \sigma_j^{\bar{\ell}} \, \sigma_k^\ell\right) + \frac{\mathsf{f}_\ell}{2\mathsf{f}_{\bar{\ell}} + \mathsf{f}_\ell} \, \sigma_j^{\bar{\ell}} \, \sigma_k^{\bar{\ell}} - \mathsf{f}_\ell \, .
\end{aligned}$$

By the *Indifference Principle,* $\mathsf{U}_i(\ell, \boldsymbol{\sigma}_{-i}) = \mathsf{U}_i(\bar{\ell}, \boldsymbol{\sigma}_{-i})$, or

$$\begin{aligned} & \frac{1}{3}\left(\sigma_j^\ell \sigma_k^\ell - \sigma_j^{\bar{\ell}} \sigma_k^{\bar{\ell}}\right) + \left(\frac{\mathsf{f}_\ell}{2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}}} - \frac{\mathsf{f}_{\bar{\ell}}}{2\mathsf{f}_{\bar{\ell}} + \mathsf{f}_\ell}\right)\left(\sigma_j^\ell \sigma_k^{\bar{\ell}} + \sigma_j^{\bar{\ell}} \sigma_k^\ell\right) \\ & + \frac{\mathsf{f}_\ell}{\mathsf{f}_\ell + 2\mathsf{f}_{\bar{\ell}}} \sigma_j^{\bar{\ell}} \sigma_k^{\bar{\ell}} - \frac{\mathsf{f}_{\bar{\ell}}}{2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}}} \sigma_j^\ell \sigma_k^\ell + \mathsf{f}_{\bar{\ell}} - \mathsf{f}_\ell \\ = \; & 0\,. \end{aligned}$$

We proceed by case analysis, using the anonymity of players to exclude symmetric cases:

(1) $\sigma_j^\ell = \sigma_j^{\bar{\ell}} = \sigma_k^\ell = \sigma_k^{\bar{\ell}} = 0$: (None of the other players is playing on $\{\ell, \bar{\ell}\}$.) Then, by the *Indifference Principle*, $\mathsf{f}_\ell - \mathsf{f}_{\bar{\ell}} = 0$. A contradiction. The second case is symmetric.

(2) Either $\sigma_j^\ell = \sigma_j^{\bar{\ell}} = 0$ and $\sigma_k^\ell, \sigma_k^{\bar{\ell}} > 0$ or $\sigma_k^\ell = \sigma_k^{\bar{\ell}} = 0$ and $\sigma_j^\ell, \sigma_j^{\bar{\ell}} > 0$: (Either player j or player k is not playing on $\{\ell, \bar{\ell}\}$, and the other player plays on $\{\ell, \bar{\ell}\}$ — these are two cases.) Then, by the *Indifference Principle*, $\mathsf{f}_\ell - \mathsf{f}_{\bar{\ell}} = 0$. A contradiction.

(3) Either $\sigma_j^\ell, \sigma_k^\ell > 0$ and $\sigma_j^{\bar{\ell}} = \sigma_k^{\bar{\ell}} = 0$ or $\sigma_j^\ell = \sigma_k^\ell = 0$ and $\sigma_j^{\bar{\ell}}, \sigma_k^{\bar{\ell}} > 0$: (Either (both j and k play only on ℓ) or (both j and k play only on $\bar{\ell}$) - these are two cases). In the first case, by the *Indifference Principle*, $\frac{1}{3}\sigma_j^\ell \sigma_k^\ell - \frac{\mathsf{f}_{\bar{\ell}}}{2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}}} \sigma_j^\ell \sigma_k^\ell + \mathsf{f}_{\bar{\ell}} - \mathsf{f}_\ell = 0$, or $\sigma_j^\ell \sigma_k^\ell = \frac{3(2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}})}{2} > \frac{9\mathsf{m}}{2} > 1$, since $\mathsf{m} > \frac{2}{9}$. A contradiction. In the second case, by the *Indifference Principle*, $-\frac{1}{3}\sigma_j^{\bar{\ell}} \sigma_k^{\bar{\ell}} + \frac{\mathsf{f}_\ell}{\mathsf{f}_\ell + 2\mathsf{f}_{\bar{\ell}}} \sigma_j^{\bar{\ell}} \sigma_k^{\bar{\ell}} + \mathsf{f}_{\bar{\ell}} - \mathsf{f}_\ell = 0$, or $\sigma_j^{\bar{\ell}} \sigma_k^{\bar{\ell}} = \frac{3(\mathsf{f}_\ell + 2\mathsf{f}_{\bar{\ell}})}{2} > \frac{9\mathsf{m}}{2} > \frac{3}{2}$, since $\mathsf{m} > \frac{1}{3}$. A contradiction.

(4) Either $(\sigma_j^\ell > 0$ and $\sigma_j^{\bar{\ell}} = \sigma_k^{\bar{\ell}} = \sigma_k^{\bar{\ell}} = 0)$ or $(\sigma_j^{\bar{\ell}} > 0$ and $\sigma_j^\ell = \sigma_j^{\bar{\ell}} = \sigma_k^{\bar{\ell}} = 0)$: (Only one player is playing on only one link in $\{\ell, \bar{\ell}\}$ — these are four cases.) Then, by the *Indifference Principle*, $\mathsf{f}_\ell - \mathsf{f}_{\bar{\ell}} = 0$. A contradiction.

(5) Either $(\sigma_j^\ell, \sigma_k^{\bar{\ell}} > 0$ and $\sigma_j^{\bar{\ell}} = \sigma_k^\ell = 0)$ or $(\sigma_j^{\bar{\ell}}, \sigma_k^\ell > 0$ and $\sigma_j^\ell = \sigma_k^{\bar{\ell}} = 0)$: (One player is playing on only one link in $\{\ell, \bar{\ell}\}$ and the other player is playing only on the other — these are two cases.) Then, in the first case, by the *Indifference Principle*, $\left(\frac{\mathsf{f}_\ell}{2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}}} - \frac{\mathsf{f}_{\bar{\ell}}}{2\mathsf{f}_{\bar{\ell}} + \mathsf{f}_\ell}\right) \sigma_j^\ell \sigma_k^{\bar{\ell}} + \mathsf{f}_{\bar{\ell}} - \mathsf{f}_\ell = 0$, or $\frac{(\mathsf{f}_\ell - \mathsf{f}_{\bar{\ell}})(\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}})}{(2\mathsf{f}_{\bar{\ell}} + \mathsf{f}_\ell)(2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}})} \sigma_j^\ell \sigma_k^{\bar{\ell}} + \mathsf{f}_{\bar{\ell}} - \mathsf{f}_\ell = 0$, or $\sigma_j^\ell \sigma_k^{\bar{\ell}} = \frac{(2\mathsf{f}_{\bar{\ell}} + \mathsf{f}_\ell)(2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}})}{\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}}} = \left(1 + \frac{\mathsf{f}_{\bar{\ell}}}{\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}}}\right)(2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}}) \geq \left(1 + \frac{\mathsf{m}}{2\mathsf{M}}\right) \cdot 3\mathsf{m} > 1$, since $\mathsf{m} > \frac{1}{3}$. A contradiction. The second case is treated similarly.

(6) $(\sigma_j^\ell, \sigma_j^{\bar{\ell}} > 0$ and either $(\sigma_k^\ell > 0$ and $\sigma_k^{\bar{\ell}} = 0)$ or $(\sigma_k^\ell = 0$ and $\sigma_k^{\bar{\ell}} > 0)$: (One player is playing on $\{\ell, \bar{\ell}\}$ and the other player is playing only on one link in $\{\ell, \bar{\ell}\}$ — these are four cases.) Then, in the first case where $\sigma_k^\ell > 0$ and $\sigma_k^{\bar{\ell}} = 0$, by the *Indifference Principle,*

$$\frac{1}{3}\sigma_j^\ell \sigma_k^\ell + \left(\frac{\mathsf{f}_\ell}{2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}}} - \frac{\mathsf{f}_{\bar{\ell}}}{2\mathsf{f}_{\bar{\ell}} + \mathsf{f}_\ell}\right) \sigma_j^{\bar{\ell}} \sigma_k^\ell - \frac{\mathsf{f}_{\bar{\ell}}}{2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}}} \sigma_j^\ell \sigma_k^\ell + \mathsf{f}_{\bar{\ell}} - \mathsf{f}_\ell \;=\; 0\,,$$

or

$$\sigma_k^\ell \left(\frac{1}{3}\sigma_j^\ell + \left(\frac{\mathsf{f}_\ell}{2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}}} - \frac{\mathsf{f}_{\bar{\ell}}}{2\mathsf{f}_{\bar{\ell}} + \mathsf{f}_\ell}\right) \sigma_j^{\bar{\ell}} - \frac{\mathsf{f}_{\bar{\ell}}}{2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}}} \sigma_j^\ell\right) \;=\; \mathsf{f}_\ell - \mathsf{f}_{\bar{\ell}}\,.$$

or

$$\sigma_k^\ell \left(\sigma_j^\ell \left(\frac{1}{3} - \left(\frac{\mathsf{f}_\ell}{2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}}} - \frac{\mathsf{f}_{\bar{\ell}}}{2\mathsf{f}_{\bar{\ell}} + \mathsf{f}_\ell}\right) - \frac{\mathsf{f}_{\bar{\ell}}}{2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}}}\right) + \left(\frac{\mathsf{f}_\ell}{2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}}} - \frac{\mathsf{f}_{\bar{\ell}}}{2\mathsf{f}_{\bar{\ell}} + \mathsf{f}_\ell}\right)\right) \;=\; \mathsf{f}_\ell - \mathsf{f}_{\bar{\ell}}\,.$$

Note that

$$\begin{aligned} & \frac{1}{3} - \left(\frac{\mathsf{f}_\ell}{2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}}} - \frac{\mathsf{f}_{\bar{\ell}}}{2\mathsf{f}_{\bar{\ell}} + \mathsf{f}_\ell}\right) - \frac{\mathsf{f}_{\bar{\ell}}}{2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}}} \\ = \; & \frac{1}{3} - \frac{\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}}}{2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}}} + \frac{\mathsf{f}_{\bar{\ell}}}{2\mathsf{f}_{\bar{\ell}} + \mathsf{f}_\ell} \\ = \; & \frac{(2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}})(2\mathsf{f}_{\bar{\ell}} + \mathsf{f}_\ell) - 3\,(\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}})(2\mathsf{f}_{\bar{\ell}} + \mathsf{f}_\ell) + 3\mathsf{f}_{\bar{\ell}}(2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}})}{3(2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}})(2\mathsf{f}_{\bar{\ell}} + \mathsf{f}_\ell)} \end{aligned}$$

$$= \frac{-4\mathsf{f}_\ell^2 - \mathsf{f}_{\bar\ell}^2 + 5\mathsf{f}_\ell\mathsf{f}_{\bar\ell}}{3(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell)}$$

$$= \frac{-(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})^2 + 9\mathsf{f}_\ell\mathsf{f}_{\bar\ell}}{3(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell)}\,.$$

So we have

$$\sigma_k^\ell\left(\sigma_j^\ell\,\frac{-(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})^2 + 9\mathsf{f}_\ell\mathsf{f}_{\bar\ell}}{3(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell)} + \frac{\mathsf{f}_\ell^2 - \mathsf{f}_{\bar\ell}^2}{(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell)}\right) = \mathsf{f}_\ell - \mathsf{f}_{\bar\ell}\,,$$

or

$$\sigma_k^\ell\left(\sigma_j^\ell\,\frac{-(2\mathsf{f}_\ell^2 + \mathsf{f}_{\bar\ell}^2 + 9\mathsf{f}_\ell\mathsf{f}_{\bar\ell})}{3(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell)(\mathsf{f}_\ell - \mathsf{f}_{\bar\ell})} + \frac{\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}}{(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell}\right) = 1\,.$$

Since $\sigma_j^\ell, \sigma_k^\ell < 1$, it follows that

$$\sigma_k^\ell\left(\sigma_j^\ell\,\frac{-(2\mathsf{f}_\ell^2 + \mathsf{f}_{\bar\ell}^2 + 9\mathsf{f}_\ell\mathsf{f}_{\bar\ell}}{3(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell)(\mathsf{f}_\ell - \mathsf{f}_{\bar\ell})} + \frac{\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}}{(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell)}\right)$$

$$< \frac{-(2\mathsf{f}_\ell^2 + \mathsf{f}_{\bar\ell})^2 + 9\mathsf{f}_\ell\mathsf{f}_{\bar\ell}}{3(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell)(\mathsf{f}_\ell - \mathsf{f}_{\bar\ell})} + \frac{\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}}{(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell)}$$

$$\leq \frac{9(\mathsf{M}^2 - \mathsf{m}^2)}{27\mathsf{m}^2(\mathsf{m} - \mathsf{M})} + \frac{2\mathsf{M}}{9\mathsf{m}^2}$$

$$= -\frac{9(\mathsf{M} + \mathsf{m})}{27\mathsf{m}^2} + \frac{2\mathsf{M}}{9\mathsf{m}^2}$$

$$= -\frac{3(\mathsf{M} + 9\mathsf{m})}{27\mathsf{m}^2}$$

$$< 0\,.$$

A contradiction. In the second case where $\sigma_k^\ell = 0$ and $\sigma_k^{\bar\ell} > 0$, by the *Indifference Principle,*

$$-\frac{1}{3}\sigma_j^\ell\sigma_k^{\bar\ell} + \left(\frac{\mathsf{f}_\ell}{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}} - \frac{\mathsf{f}_{\bar\ell}}{2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell}\right)\sigma_j^\ell\,\sigma_k^{\bar\ell} + \frac{\mathsf{f}_{\bar\ell}}{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}}\sigma_j^{\bar\ell}\,\sigma_k^{\bar\ell} + \mathsf{f}_{\bar\ell} - \mathsf{f}_\ell = 0\,,$$

or

$$\sigma_k^{\bar\ell}\left(-\frac{1}{3}\sigma_j^\ell + \left(\frac{\mathsf{f}_\ell}{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}} - \frac{\mathsf{f}_{\bar\ell}}{2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell}\right)\sigma_j^\ell + \frac{\mathsf{f}_{\bar\ell}}{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}}\sigma_j^{\bar\ell}\right) = \mathsf{f}_\ell - \mathsf{f}_{\bar\ell}\,.$$

or

$$\sigma_k^{\bar\ell}\left(\sigma_j^\ell\left(-\frac{1}{3} + \left(\frac{\mathsf{f}_\ell}{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}} - \frac{\mathsf{f}_{\bar\ell}}{2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell}\right) - \frac{\mathsf{f}_{\bar\ell}}{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}}\right) + \frac{\mathsf{f}_{\bar\ell}}{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}}\right) = \mathsf{f}_\ell - \mathsf{f}_{\bar\ell}\,.$$

Clearly,

$$-\frac{1}{3} + \left(\frac{\mathsf{f}_\ell}{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}} - \frac{\mathsf{f}_{\bar\ell}}{2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell}\right) - \frac{\mathsf{f}_{\bar\ell}}{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}} = -\frac{1}{3} + \frac{\mathsf{f}_\ell - \mathsf{f}_{\bar\ell}}{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}} - \frac{\mathsf{f}_{\bar\ell}}{2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell}$$

$$= \frac{-(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell) + 3\,(\mathsf{f}_\ell - \mathsf{f}_{\bar\ell})(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell) - 3\mathsf{f}_{\bar\ell}(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})}{3(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell)}$$

$$= \frac{-(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell) + 3\,\mathsf{f}_\ell(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell) - 3\mathsf{f}_{\bar\ell}\cdot 3(\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})}{3(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell)}$$

$$= \frac{5\mathsf{f}_\ell^2 - 7\mathsf{f}_{\bar\ell}^2 + 2\mathsf{f}_\ell\mathsf{f}_{\bar\ell}}{3(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell)}\,.$$

So we have

$$\sigma_k^\ell\left(\sigma_j^\ell\,\frac{-\mathsf{f}_\ell^2 - 17\mathsf{f}_{\bar\ell}^2 - 2\mathsf{f}_\ell\mathsf{f}_{\bar\ell}}{3(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell)} + \frac{\mathsf{f}_{\bar\ell}}{(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})}\right) = \mathsf{f}_\ell - \mathsf{f}_{\bar\ell}\,,$$

or

$$\sigma_k^\ell\cdot\frac{1}{\mathsf{f}_{\bar\ell} - \mathsf{f}_\ell}\cdot\left(\sigma_j^\ell\,\frac{\mathsf{f}_\ell^2 + 17\mathsf{f}_{\bar\ell}^2 + 2\mathsf{f}_\ell\mathsf{f}_{\bar\ell}}{3(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell)} - \frac{\mathsf{f}_{\bar\ell}}{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}}\right) = 1\,.$$

Since $\sigma_j^\ell < 1$, it follows that

$$\sigma_j^\ell\,\frac{\mathsf{f}_\ell^2 + 17\mathsf{f}_{\bar\ell}^2 + 2\mathsf{f}_\ell\mathsf{f}_{\bar\ell}}{3(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell)} - \frac{\mathsf{f}_{\bar\ell}}{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}} < \frac{\mathsf{f}_\ell^2 + 17\mathsf{f}_{\bar\ell}^2 + 2\mathsf{f}_\ell\mathsf{f}_{\bar\ell}}{3(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell)} - \frac{\mathsf{f}_{\bar\ell}}{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}}$$

$$= \frac{\mathsf{f}_\ell^2 + 17\mathsf{f}_{\bar\ell}^2 + 2\mathsf{f}_\ell\mathsf{f}_{\bar\ell} - 3\mathsf{f}_{\bar\ell}(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell)}{3(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell)}$$

$$= \frac{\mathsf{f}_\ell^2 + 11\mathsf{f}_{\bar\ell}^2 - \mathsf{f}_\ell\mathsf{f}_{\bar\ell}}{3(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell)}$$

$$= \frac{\mathsf{f}_\ell^2 + 11\mathsf{f}_{\bar\ell}^2 - \mathsf{f}_\ell\mathsf{f}_{\bar\ell}}{3(2\mathsf{f}_\ell^2 + 2\mathsf{f}_{\bar\ell}^2 + 5\mathsf{f}_\ell\mathsf{f}_{\bar\ell})}\,.$$

Since $\sigma_k^\ell \le 1$ and $\min_{\bar\ell,\ell\in[Q]|\mathsf{f}_{\bar\ell}>\mathsf{f}_\ell} \mathsf{f}_{\bar\ell} - \mathsf{f}_\ell$ is sufficiently large, it follows that $\sigma_k^\ell \cdot \frac{1}{\mathsf{f}_{\bar\ell} - \mathsf{f}_\ell} \cdot \frac{\mathsf{f}_\ell^2 + 11\mathsf{f}_{\bar\ell}^2 - \mathsf{f}_\ell\mathsf{f}_{\bar\ell}}{3(2\mathsf{f}_\ell^2 + 2\mathsf{f}_{\bar\ell}^2 + 5\mathsf{f}_\ell\mathsf{f}_{\bar\ell})} < 1$. A contradiction.

(7) $\underline{\sigma_j^\ell, \sigma_j^{\bar\ell}, \sigma_k^\ell, \sigma_k^{\bar\ell} > 0}$: (Each other player is playing on $\{\ell, \bar\ell\}$.) By the *Indifference Principle,*

$$\begin{aligned}
&\frac{1}{3}\left((1-\sigma_j^{\bar\ell})(1-\sigma_k^{\bar\ell}) - \sigma_j^{\bar\ell}\sigma_k^{\bar\ell}\right) + \left(\frac{\mathsf{f}_\ell}{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}} - \frac{\mathsf{f}_{\bar\ell}}{2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell}\right)\left((1-\sigma_j^{\bar\ell})\,\sigma_k^{\bar\ell} + \sigma_j^{\bar\ell}\,(1-\sigma_k^{\bar\ell})\right)\\
&+\frac{\mathsf{f}_\ell}{\mathsf{f}_\ell + 2\mathsf{f}_{\bar\ell}}\sigma_j^{\bar\ell}\,\sigma_k^{\bar\ell} - \frac{\mathsf{f}_{\bar\ell}}{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}}(1-\sigma_j^{\bar\ell})\,(1-\sigma_k^{\bar\ell}) + \mathsf{f}_{\bar\ell} - \mathsf{f}_\ell\\
=\;& 0\,,
\end{aligned}$$

or

$$\begin{aligned}
&\frac{1}{3}\left(1-\sigma_j^{\bar\ell} - \sigma_k^{\bar\ell}\right) + \left(\frac{\mathsf{f}_\ell}{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}} - \frac{\mathsf{f}_{\bar\ell}}{2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell}\right)\left(\sigma_j^{\bar\ell} + \sigma_k^{\bar\ell} - 2\,\sigma_j^{\bar\ell}\,\sigma_k^{\bar\ell}\right)\\
&+\frac{\mathsf{f}_\ell}{\mathsf{f}_\ell + 2\mathsf{f}_{\bar\ell}}\sigma_j^{\bar\ell}\,\sigma_k^{\bar\ell} - \frac{\mathsf{f}_{\bar\ell}}{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}}\left(1-\sigma_j^{\bar\ell} - \sigma_k^{\bar\ell} + \sigma_j^{\bar\ell}\,\sigma_k^{\bar\ell}\right) + \mathsf{f}_{\bar\ell} - \mathsf{f}_\ell\\
=\;& 0\,,
\end{aligned}$$

or

$$\begin{aligned}
&\left(-2\left(\frac{\mathsf{f}_\ell}{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}} - \frac{\mathsf{f}_{\bar\ell}}{2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell}\right) + \frac{\mathsf{f}_\ell}{\mathsf{f}_\ell + 2\mathsf{f}_{\bar\ell}} - \frac{\mathsf{f}_{\bar\ell}}{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}}\right)\sigma_j^{\bar\ell}\,\sigma_k^{\bar\ell}\\
&+\left(-\frac{1}{3} + \frac{\mathsf{f}_\ell}{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}} - \frac{\mathsf{f}_{\bar\ell}}{2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell} + \frac{\mathsf{f}_{\bar\ell}}{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}}\right)(\sigma_j^{\bar\ell} + \sigma_k^{\bar\ell}) + \frac{1}{3} - \frac{\mathsf{f}_{\bar\ell}}{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}} + \mathsf{f}_{\bar\ell} - \mathsf{f}_\ell\\
=\;& 0
\end{aligned}$$

The coefficient of $\sigma_j^{\bar\ell}\,\sigma_k^{\bar\ell}$ is $\frac{2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell}{2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell} - \frac{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}}{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}} = 0$. The coefficient of $\sigma_j^{\bar\ell} + \sigma_k^{\bar\ell}$ is

$$\begin{aligned}
&-\frac{1}{3} + \frac{\mathsf{f}_\ell}{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}} - \frac{\mathsf{f}_{\bar\ell}}{2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell} + \frac{\mathsf{f}_{\bar\ell}}{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}}\\
=\;& -\frac{1}{3} + \frac{\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}}{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}} - \frac{\mathsf{f}_{\bar\ell}}{2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell}\\
=\;& \frac{-(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell) + 3(\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell)(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell) - 3\mathsf{f}_{\bar\ell}(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})}{3(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell)}\\
=\;& \frac{-\left(4\mathsf{f}_\ell\mathsf{f}_{\bar\ell} + 2\mathsf{f}_\ell^2 + 2\mathsf{f}_{\bar\ell}^2 + \mathsf{f}_\ell\mathsf{f}_{\bar\ell}\right) + 3\left(2\mathsf{f}_{\bar\ell}^2 + \mathsf{f}_\ell\mathsf{f}_{\bar\ell} + 2\mathsf{f}_\ell\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell^2\right) - 6\mathsf{f}_\ell\mathsf{f}_{\bar\ell} - 3\mathsf{f}_{\bar\ell}^2}{3(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell)}\\
=\;& \frac{\mathsf{f}_\ell^2 + \mathsf{f}_{\bar\ell}^2 - 2\mathsf{f}_\ell\mathsf{f}_{\bar\ell}}{3(2\mathsf{f}_\ell + \mathsf{f}_\ell)(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell)}\\
=\;& \frac{(\mathsf{f}_{\bar\ell} - \mathsf{f}_\ell)^2}{3(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell)}\,.
\end{aligned}$$

The constant term is

$$\begin{aligned}
\frac{1}{3} - \frac{\mathsf{f}_{\bar\ell}}{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}} + \mathsf{f}_{\bar\ell} - \mathsf{f}_\ell &= \frac{2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell} - 3\mathsf{f}_{\bar\ell} + 3(\mathsf{f}_{\bar\ell} - \mathsf{f}_\ell)(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})}{3(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})}\\
&= \frac{2(\mathsf{f}_\ell - \mathsf{f}_{\bar\ell}) + 3(\mathsf{f}_{\bar\ell} - \mathsf{f}_\ell)(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})}{3(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})}\\
&= \frac{(\mathsf{f}_{\bar\ell} - \mathsf{f}_\ell)(3(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}) - 2)}{3(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})}\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{(\mathsf{f}_{\bar\ell} - \mathsf{f}_\ell)^2}{3(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell)}(\sigma_j^\ell + \sigma_k^\ell) + \frac{(\mathsf{f}_{\bar\ell} - \mathsf{f}_\ell)(3(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}) - 2)}{3(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell})} &= 0\,,
\end{aligned}$$

or

$$\begin{aligned}
\sigma_j^{\bar\ell} + \sigma_k^{\bar\ell} &= \frac{(2\mathsf{f}_{\bar\ell} + \mathsf{f}_\ell)(2 - 3(2\mathsf{f}_\ell + \mathsf{f}_{\bar\ell}))}{\mathsf{f}_{\bar\ell} - \mathsf{f}_\ell}\,.
\end{aligned}$$

Since $\mathsf{f}_{\bar{\ell}} > \mathsf{f}_\ell > 0$, we have $2\mathsf{f}_{\bar{\ell}} + \mathsf{f}_\ell, \mathsf{f}_{\bar{\ell}} - \mathsf{f}_\ell > 0$. Since $\sigma_j^{\bar{\ell}} + \sigma_k^{\bar{\ell}} > 0$, it follows that $2 - 3(2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}}) > 0$ is necessary for the existence of a mixed Nash equilibrium. Similarly, we derive:

$$\sigma_k^{\bar{\ell}} + \sigma_i^{\bar{\ell}} \;=\; \sigma_i^{\bar{\ell}} + \sigma_j^{\bar{\ell}} \;=\; \frac{(2\mathsf{f}_{\bar{\ell}} + \mathsf{f}_\ell)(2 - 3(2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}}))}{\mathsf{f}_{\bar{\ell}} - \mathsf{f}_\ell}\,.$$

By symmetry, it follows that

$$\sigma_i^{\bar{\ell}} \;=\; \sigma_j^{\bar{\ell}} \;=\; \sigma_k^{\bar{\ell}} \;=\; \frac{(2\mathsf{f}_{\bar{\ell}} + \mathsf{f}_\ell)(2 - 3(2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}}))}{2(\mathsf{f}_{\bar{\ell}} - \mathsf{f}_\ell)}\,.$$

Note that $\sigma_i^{\bar{\ell}}$, $\sigma_j^{\bar{\ell}}$ and $\sigma_k^{\bar{\ell}}$ represent probabilities if and only if $0 < \frac{(2\mathsf{f}_{\bar{\ell}} + \mathsf{f}_\ell)(2 - 3(2\mathsf{f}_\ell + \mathsf{f}_{\bar{\ell}}))}{\mathsf{f}_{\bar{\ell}} - \mathsf{f}_\ell} < 1$. By Proposition 2, no player could increase her utility by switching to a quality other than ℓ and $\bar{\ell}$.

G Proof of Proposition 13

Consider a mixed Nash equilibrium $\boldsymbol{\sigma}$. Fix a mixed player $i \in [n]$. By strict concavity, $|\mathsf{supp}_i(\boldsymbol{\sigma})| = 2$; let $\mathsf{supp}_i(\boldsymbol{\sigma}) = \{\ell, \bar{\ell}\}$, where, for ease of notation, $\bar{\ell}$ denotes $\ell + 1$. The expected conditional utility of player $i \in [n]$ had she chosen quality ℓ is

$$\begin{aligned}
\mathsf{U}_i(\ell, \boldsymbol{\sigma}_{-i}) &= \sum_{0 \le t \le n-1} \sum_{S \subseteq [n]\setminus\{i\} || S| = t} \left(\frac{\mathsf{f}_\ell}{(n-t)\,\mathsf{f}_\ell + t\,\mathsf{f}_{\bar{\ell}}} - \mathsf{f}_\ell \right) \prod_{j \in ([n]\setminus\{i\})\setminus S} \sigma_j^\ell \prod_{k \in S} \sigma_k^{\bar{\ell}} \\
&= \sum_{0 \le t \le n-1} \frac{\mathsf{f}_\ell}{(n-t)\,\mathsf{f}_\ell + t\,\mathsf{f}_{\bar{\ell}}} \sum_{S \subseteq [n]\setminus\{i\} || S| = t} \prod_{j \in ([n]\setminus\{i\})\setminus S} \sigma_j^\ell \prod_{k \in S} \sigma_k^{\bar{\ell}} \\
&\quad \underbrace{-\mathsf{f}_\ell \sum_{0 \le t \le n-1} \sum_{S \subseteq [n]\setminus\{i\} || S| = t} \prod_{j \in ([n]\setminus\{i\})\setminus S} \sigma_j^\ell \prod_{k \in S} \sigma_k^{\bar{\ell}}}_{=1} \\
&= \sum_{0 \le t \le n-1} \frac{\mathsf{f}_\ell}{(n-t)\,\mathsf{f}_\ell + t\,\mathsf{f}_{\bar{\ell}}} \sum_{S \subseteq [n]\setminus\{i\} || S| = t} \prod_{j \in ([n]\setminus\{i\})\setminus S} \sigma_j^\ell \prod_{k \in S} \sigma_k^{\bar{\ell}} - \mathsf{f}_\ell\,.
\end{aligned}$$

By the *Indifference Principle*, $\mathsf{U}_i(\ell, \boldsymbol{\sigma}_{-i}) = \mathsf{U}_i(\bar{\ell}, \boldsymbol{\sigma}_{-i})$, or

$$\begin{aligned}
&\sum_{0 \le t \le n-1} \frac{\mathsf{f}_\ell}{(n-t)\,\mathsf{f}_\ell + t\,\mathsf{f}_{\bar{\ell}}} \sum_{S \subseteq [n]\setminus\{i\} || S| = t} \prod_{j \in ([n]\setminus\{i\})\setminus S} \sigma_j^\ell \prod_{k \in S} \sigma_k^{\bar{\ell}} - \mathsf{f}_\ell \\
= &\sum_{0 \le t \le n-1} \frac{\mathsf{f}_{\bar{\ell}}}{(n-t)\,\mathsf{f}_{\bar{\ell}} + t\,\mathsf{f}_\ell} \sum_{S \subseteq [n]\setminus\{i\} || S| = t} \prod_{j \in ([n]\setminus\{i\})\setminus S} \sigma_j^{\bar{\ell}} \prod_{k \in S} \sigma_k^{\ell} - \mathsf{f}_{\bar{\ell}}\,.
\end{aligned}$$

Note that the term with $t = 0$ in the LHS is $\frac{1}{n} \prod_{j \in [n]\setminus\{i\}} \sigma_j^\ell$ and the term with $t = 0$ in the RHS is $\frac{1}{n} \prod_{j \in [n]\setminus\{i\}} \sigma_j^{\bar{\ell}}$. Hence,

$$\begin{aligned}
&\frac{1}{n} \prod_{j \in [n]\setminus\{i\}} \sigma_j^\ell + \sum_{1 \le t \le n-1} \frac{\mathsf{f}_\ell}{(n-t)\,\mathsf{f}_\ell + t\,\mathsf{f}_{\bar{\ell}}} \sum_{S \subseteq [n]\setminus\{i\} || S| = t} \prod_{j \in ([n]\setminus\{i\})\setminus S} \sigma_j^\ell \prod_{k \in S} \sigma_k^{\bar{\ell}} - \mathsf{f}_\ell \\
= &\frac{1}{n} \prod_{j \in [n]\setminus\{i\}} \sigma_j^{\bar{\ell}} + \sum_{1 \le t \le n-1} \frac{\mathsf{f}_{\bar{\ell}}}{(n-t)\,\mathsf{f}_{\bar{\ell}} + t\,\mathsf{f}_\ell} \sum_{S \subseteq [n]\setminus\{i\} || S| = t} \prod_{j \in ([n]\setminus\{i\})\setminus S} \sigma_j^{\bar{\ell}} \prod_{k \in S} \sigma_k^{\ell} - \mathsf{f}_{\bar{\ell}}\,.
\end{aligned}$$

or

$$\begin{aligned}
&\frac{1}{n} \prod_{j \in [n]\setminus\{i\}} (1 - \sigma_j^{\bar{\ell}}) + \sum_{1 \le t \le n-1} \frac{\mathsf{f}_\ell}{(n-t)\,\mathsf{f}_\ell + t\,\mathsf{f}_{\bar{\ell}}} \sum_{S \subseteq [n]\setminus\{i\} || S| = t} \prod_{j \in [n]\setminus(\{i\} \cup S)} (1 - \sigma_j^{\bar{\ell}}) \prod_{k \in S} \sigma_k^{\bar{\ell}} - \mathsf{f}_\ell \\
= &\frac{1}{n} \prod_{j \in [n]\setminus\{i\}} \sigma_j^{\bar{\ell}} + \sum_{1 \le t \le n-1} \frac{\mathsf{f}_{\bar{\ell}}}{(n-t)\,\mathsf{f}_{\bar{\ell}} + t\,\mathsf{f}_\ell} \sum_{S \subseteq [n]\setminus\{i\} || S| = t} \prod_{j \in [n]\setminus(\{i\} \setminus S)} \sigma_j^{\bar{\ell}} \prod_{k \in S} (1 - \sigma_k^{\bar{\ell}}) - \mathsf{f}_{\bar{\ell}}\,.
\end{aligned}$$

By abuse of notation, define the vectors $\boldsymbol{\sigma} = (\sigma_1^{\bar{\ell}}, \ldots, \sigma_n^{\bar{\ell}})$ and $\boldsymbol{\sigma}_{-\{i\}} = (\sigma_1^{\bar{\ell}}, \ldots, \sigma_{i-1}^{\bar{\ell}}, \sigma_{i+1}^{\bar{\ell}}, \ldots, \sigma_n^{\bar{\ell}})$. The *Indifference Equation* becomes

$$\begin{aligned} & \frac{1}{n}\sum_{k=0}^{n-1}(-1)^k e_k(\boldsymbol{\sigma}_{-i}) \\ & + \sum_{1\le t\le n-1} \frac{\mathsf{f}_\ell}{(n-t)\,\mathsf{f}_\ell + t\,\mathsf{f}_{\bar{\ell}}} \sum_{S\subseteq[n]\setminus\{i\}||S|=t} \left(\sum_{k=0}^{n-1-|S|}(-1)^k e_k\left(\boldsymbol{\sigma}_{-\{i\}\cup S}\right)\right) e_{|S|}(\boldsymbol{\sigma}\mid S) - \mathsf{f}_\ell \\ = \; & \frac{1}{n}\, e_{n-1}(\boldsymbol{\sigma}_{-i}) \\ & + \sum_{1\le t\le n-1} \frac{\mathsf{f}_{\bar{\ell}}}{(n-t)\,\mathsf{f}_{\bar{\ell}} + t\,\mathsf{f}_\ell} \sum_{S\subseteq[n]\setminus\{i\}||S|=t} e_{n-1-|S|}(\boldsymbol{\sigma}_{-\{i\}\cup S}) \left(\sum_{k=0}^{|S|}(-1)^k e_k(\boldsymbol{\sigma}\mid S)\right) - \mathsf{f}_{\bar{\ell}}\,. \end{aligned}$$

Fix any set $S \subseteq [n]\setminus\{i\}$ with $|S| = t$. Setting $T := [n]\setminus(\{i\}\cup S)$ we get

$$\begin{aligned} \left(\sum_{k=0}^{n-1-|S|}(-1)^k e_k(\boldsymbol{\sigma}_{-\{i\}\cup S})\right) e_{|S|}(\boldsymbol{\sigma}\mid S) &= \left(\sum_{k=0}^{|T|}(-1)^k e_k\left(\boldsymbol{\sigma}\mid T\right)\right) e_{n-1-|T|}(\boldsymbol{\sigma}_{-\{i\}\cup T}) \\ &= \left(\sum_{k=0}^{|S|}(-1)^k e_k(\boldsymbol{\sigma}\mid S)\right) e_{n-1-|S|}\left(\boldsymbol{\sigma}_{-\{i\}\cup S}\right)\,, \end{aligned}$$

setting $S := T$. Hence, the *Indifference Equation* becomes

$$\begin{aligned} & \frac{1}{n}\sum_{k=0}^{n-1}(-1)^k e_k(\boldsymbol{\sigma}_{-i}) \\ & + \sum_{1\le t\le n-1} \frac{\mathsf{f}_\ell}{(n-t)\,\mathsf{f}_\ell + t\,\mathsf{f}_{\bar{\ell}}} \sum_{S\subseteq[n]\setminus\{i\}||S|=t} \left(\sum_{k=0}^{|S|}(-1)^k e_k(\boldsymbol{\sigma}\mid S)\right) e_{n-1-|S|}\left(\boldsymbol{\sigma}_{-\{i\}\cup S}\right) - \mathsf{f}_\ell \\ = \; & \frac{1}{n}\, e_{n-1}(\boldsymbol{\sigma}_{-i}) \\ & + \sum_{1\le t\le n-1} \frac{\mathsf{f}_{\bar{\ell}}}{(n-t)\,\mathsf{f}_{\bar{\ell}} + t\,\mathsf{f}_\ell} \sum_{S\subseteq[n]\setminus\{i\}||S|=t} e_{n-1-|S|}(\boldsymbol{\sigma}_{-\{i\}\cup S})\,] \left(\sum_{k=0}^{|S|}(-1)^k e_k(\boldsymbol{\sigma}\mid S)\right) - \mathsf{f}_{\bar{\ell}}\,, \end{aligned}$$

or

$$\begin{aligned} & \sum_{1\le t\le n-1}\left(\frac{\mathsf{f}_{\bar{\ell}}}{(n-t)\,\mathsf{f}_{\bar{\ell}} + t\,\mathsf{f}_\ell} - \frac{\mathsf{f}_\ell}{(n-t)\,\mathsf{f}_\ell + t\,\mathsf{f}_{\bar{\ell}}}\right) \sum_{S\subseteq[n]\setminus\{i\}||S|=t} e_{n-1-|S|}(\boldsymbol{\sigma}_{-\{i\}\cup S})\,] \left(\sum_{k=0}^{|S|}(-1)^k e_k(\boldsymbol{\sigma}\mid S)\right) \\ = \; & \frac{1}{n}\left(\sum_{k=0}^{n-1}(-1)^k e_k(\boldsymbol{\sigma}_{-i}) - e_{n-1}(\boldsymbol{\sigma}_{-i})\right) + \mathsf{f}_{\bar{\ell}} - \mathsf{f}_\ell \end{aligned}$$

or

$$\begin{aligned} & \sum_{1\le t\le n-1}\frac{t\left(\mathsf{f}_{\bar{\ell}}^2 - \mathsf{f}_\ell^2\right)}{\left((n-t)\,\mathsf{f}_{\bar{\ell}} + t\,\mathsf{f}_\ell\right)\left((n-t)\,\mathsf{f}_\ell + t\,\mathsf{f}_{\bar{\ell}}\right)} \sum_{S\subseteq[n]\setminus\{i\}||S|=t} e_{n-1-|S|}(\boldsymbol{\sigma}_{-\{i\}\cup S}) \left(\sum_{k=0}^{|S|}(-1)^k e_k(\boldsymbol{\sigma}\mid S)\right) \\ = \; & \frac{1}{n}\left(\sum_{k=0}^{n-1}(-1)^k e_k(\boldsymbol{\sigma}_{-i}) - e_{n-1}(\boldsymbol{\sigma}_{-i})\right) + \mathsf{f}_{\bar{\ell}} - \mathsf{f}_\ell\,, \end{aligned}$$

or

$$\sum_{1\le t\le n-1}\frac{t\left(\mathsf{f}_{\bar{\ell}}^2 - \mathsf{f}_\ell^2\right)}{\left((n-t)\,\mathsf{f}_{\bar{\ell}} + t\,\mathsf{f}_\ell\right)\left((n-t)\,\mathsf{f}_\ell + t\,\mathsf{f}_{\bar{\ell}}\right)} \sum_{S\subseteq[n]\setminus\{i\}||S|=t} \left(e_{n-1-|S|}(\boldsymbol{\sigma}_{-\{i\}\cup S}) \left(1 + \sum_{k=1}^{|S|}(-1)^k e_k(\boldsymbol{\sigma}\mid S)\right)\right)$$

$$= \frac{1}{n}\left(1+\sum_{k=1}^{n-1}(-1)^k e_k(\boldsymbol{\sigma}_{-i}) - e_{n-1}(\boldsymbol{\sigma}_{-i})\right) + \mathsf{f}_{\bar{\ell}} - \mathsf{f}_{\ell}\,.$$

or

$$\sum_{1\le t\le n-1} \frac{t\left(\mathsf{f}_{\bar{\ell}}^2 - \mathsf{f}_{\ell}^2\right)}{\left((n-t)\,\mathsf{f}_{\bar{\ell}} + t\,\mathsf{f}_{\ell}\right)\left((n-t)\,\mathsf{f}_{\ell} + t\,\mathsf{f}_{\bar{\ell}}\right)} \sum_{S\subseteq[n]\setminus\{i\}||S|=t} \left(e_{n-1-|S|}(\boldsymbol{\sigma}_{-\{i\}\cup S})\left(1+\sum_{k=1}^{|S|}(-1)^k e_k(\boldsymbol{\sigma}\mid S)\right)\right)$$
$$-\frac{1}{n}\left(\sum_{k=1}^{n-1}(-1)^k e_k(\boldsymbol{\sigma}_{-i}) - e_{n-1}(\boldsymbol{\sigma}_{-i})\right)$$
$$= \frac{1}{n} + \mathsf{f}_{\bar{\ell}} - \mathsf{f}_{\ell}\,.$$

H Proof of Theorem 14

Consider a fully mixed Nash equilibrium $\boldsymbol{\sigma}$. For the special case where $n = 4$, the *Indifference Equation* becomes

$$\sum_{1\le t\le 3} \frac{t\left(\mathsf{f}_2^2 - \mathsf{f}_1^2\right)}{\left((4-t)\,\mathsf{f}_2 + t\,\mathsf{f}_1\right)\left((4-t)\,\mathsf{f}_1 + t\,\mathsf{f}_2\right)} \sum_{S\subseteq[4]\setminus\{i\}||S|=t} \left(e_{3-|S|}(\boldsymbol{\sigma}_{-\{i\}\cup S})\left(1+\sum_{k=1}^{|S|}(-1)^k e_k(\boldsymbol{\sigma}\mid S)\right)\right)$$
$$-\frac{1}{4}\left(\sum_{k=1}^{3}(-1)^k e_k(\boldsymbol{\sigma}_{-i}) - e_3(\boldsymbol{\sigma}_{-i})\right)$$
$$= \frac{1}{4} + \mathsf{f}_2 - \mathsf{f}_1\,.$$

Expanding yields

$$\frac{\left(\mathsf{f}_2^2 - \mathsf{f}_1^2\right)}{\left(3\mathsf{f}_2 + \mathsf{f}_1\right)\left(3\,\mathsf{f}_1 + \mathsf{f}_2\right)} \sum_{S\subseteq[4]\setminus\{i\}||S|=1} \left(e_2(\boldsymbol{\sigma}_{-\{i\}\cup S})\left(1+\sum_{k=1}^{|S|}(-1)^k e_k(\boldsymbol{\sigma}\mid S)\right)\right)$$
$$+\frac{2\left(\mathsf{f}_2^2 - \mathsf{f}_1^2\right)}{\left(2\,\mathsf{f}_2 + 2\,\mathsf{f}_1\right)\left(2\,\mathsf{f}_1 + 2\,\mathsf{f}_2\right)} \sum_{S\subseteq[4]\setminus\{i\}||S|=2} \left(e_1(\boldsymbol{\sigma}_{-\{i\}\cup S})\left(1+\sum_{k=1}^{|S|}(-1)^k e_k(\boldsymbol{\sigma}\mid S)\right)\right)$$
$$+\frac{3\left(\mathsf{f}_2^2 - \mathsf{f}_1^2\right)}{\left(\mathsf{f}_2 + 3\,\mathsf{f}_1\right)\left(\mathsf{f}_1 + 3\,\mathsf{f}_2\right)} \sum_{S\subseteq[4]\setminus\{i\}||S|=3} \left(e_0(\boldsymbol{\sigma}_{-\{i\}\cup S})\left(1+\sum_{k=1}^{|S|}(-1)^k e_k(\boldsymbol{\sigma}\mid S)\right)\right)$$
$$-\frac{1}{4}\left(\sum_{k=1}^{3}(-1)^k e_k(\boldsymbol{\sigma}_{-i}) - e_3(\boldsymbol{\sigma}_{-i})\right)$$
$$= \frac{1}{4} + \mathsf{f}_2 - \mathsf{f}_1\,,$$

or

$$\frac{\left(\mathsf{f}_2^2 - \mathsf{f}_1^2\right)}{\left(3\mathsf{f}_2 + \mathsf{f}_1\right)\left(3\,\mathsf{f}_1 + \mathsf{f}_2\right)} \left(\sum_{S\subseteq[4]\setminus\{i\}||S|=1} \left(e_2(\boldsymbol{\sigma}_{-\{i\}\cup S})\left(1+\sum_{k=1}^{|S|}(-1)^k e_k(\boldsymbol{\sigma}\mid S)\right)\right)\right.$$
$$\left.+3\sum_{S\subseteq[4]\setminus\{i\}||S|=3} \left(1+\sum_{k=1}^{|S|}(-1)^k e_k(\boldsymbol{\sigma}\mid S)\right)\right)$$
$$+\frac{\mathsf{f}_2 - \mathsf{f}_1}{2\left(\mathsf{f}_2 + \mathsf{f}_1\right)} \sum_{S\subseteq[4]\setminus\{i\}||S|=2} \left(e_1(\boldsymbol{\sigma}_{-\{i\}\cup S})\left(1+\sum_{k=1}^{|S|}(-1)^k e_k(\boldsymbol{\sigma}\mid S)\right)\right)$$

$$
\begin{aligned}
& -\frac{1}{4}\left(\sum_{k=1}^{3}(-1)^k e_k(\boldsymbol{\sigma}_{-i}) - e_3(\boldsymbol{\sigma}_{-i})\right) \\
= \quad & \frac{1}{4} + \mathsf{f}_2 - \mathsf{f}_1
\end{aligned}
$$

or

$$
\begin{aligned}
& \frac{\left(\mathsf{f}_2^2 - \mathsf{f}_1^2\right)}{(3\mathsf{f}_2 + \mathsf{f}_1)\,(3\,\mathsf{f}_1 + \mathsf{f}_2)}\ \cdot \\
& \left(\sum_{S\subseteq[4]\setminus\{i\}||S|=1}\left(e_2(\boldsymbol{\sigma}_{-\{i\}\cup S})\,\left(1+\sum_{k=1}^{|S|}(-1)^k e_k(\boldsymbol{\sigma}\mid S)\right)\right) + 3\sum_{S\subseteq[4]\setminus\{i\}||S|=3}\left(1+\sum_{k=1}^{|S|}(-1)^k e_k(\boldsymbol{\sigma}\mid S)\right)\right) \\
& + \frac{\mathsf{f}_2 - \mathsf{f}_1}{2\,(\mathsf{f}_2 + \mathsf{f}_1)}\sum_{S\subseteq[4]\setminus\{i\}||S|=2}\left(e_1(\boldsymbol{\sigma}_{-\{i\}\cup S})\,\left(1+\sum_{k=1}^{|S|}(-1)^k e_k(\boldsymbol{\sigma}\mid S)\right)\right) \\
& -\frac{1}{4}\left(\sum_{k=1}^{3}(-1)^k e_k(\boldsymbol{\sigma}_{-i}) - e_3(\boldsymbol{\sigma}_{-i})\right) \\
= \quad & \frac{1}{4} + \mathsf{f}_2 - \mathsf{f}_1
\end{aligned}
$$

We now calculate:

1.

$$
\begin{aligned}
& \sum_{S\subseteq[4]\setminus\{i\}||S|=1}\left(e_2(\boldsymbol{\sigma}_{-\{i\}\cup S})\,\left(1+\sum_{k=1}^{|S|}(-1)^k e_k(\boldsymbol{\sigma}\mid S)\right)\right) \\
= \quad & \sum_{S\subseteq[4]\setminus\{i\}||S|=1} e_2(\boldsymbol{\sigma}_{-\{i\}\cup S})\,(1 - e_1(\boldsymbol{\sigma}\mid S)) \\
= \quad & e_2(\boldsymbol{\sigma}_{-\{1,4\}})\left(1-e_1(\sigma_1^2)\right) + e_2(\boldsymbol{\sigma}_{-\{2,4\}})\left(1-e_1(\sigma_2^2)\right) + e_2(\boldsymbol{\sigma}_{-\{3,4\}})\left(1-e_1(\sigma_3^2)\right) \\
= \quad & \sigma_2^2\sigma_3^2\left(1-\sigma_1^2\right) + \sigma_1^2\sigma_3^2\left(1-\sigma_2^2\right) + \sigma_1^1\sigma_2^2\left(1-\sigma_3^2\right) \\
= \quad & \sigma_2^2\sigma_3^2 + \sigma_1^2\sigma_3^2 + \sigma_1^2\sigma_2^2 - 3\sigma_1^2\sigma_2^2\sigma_3^2 \\
= \quad & e_2(\boldsymbol{\sigma}_{-i}) - 3e_3(\boldsymbol{\sigma}_{-i})\,.
\end{aligned}
$$

2.

$$
\begin{aligned}
\sum_{S\subseteq[4]\setminus\{i\}||S|=3}\left(1+\sum_{k=1}^{|S|}(-1)^k e_k(\boldsymbol{\sigma}\mid S)\right) &= 1+\sum_{k=1}^{3}(-1)^k e_k(\boldsymbol{\sigma}\mid[3]) \\
&= 1 - e_1(\sigma_1^2,\sigma_2^2,\sigma_3^2) + e_2(\sigma_1^2,\sigma_2^2,\sigma_3^2) - e_3(\sigma_1^2,\sigma_2^2,\sigma_3^2) \\
&= 1 - \sigma_1^2 - \sigma_2^2 - \sigma_3^2 + \sigma_1^2\sigma_2^2 + \sigma_2^2\sigma_3^2 + \sigma_1^2\sigma_3^2 - \sigma_1^2\sigma_2^2\sigma_3^2 \\
&= 1 - e_1(\boldsymbol{\sigma}_{-i}) + e_2(\boldsymbol{\sigma}_{-i}) - e_3(\boldsymbol{\sigma}_{-i})\,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& \sum_{S\subseteq[4]\setminus\{i\}||S|=1} e_2(\boldsymbol{\sigma}_{-\{i\}\cup S})\,\left(1+\sum_{k=1}^{|S|}(-1)^k e_k(\boldsymbol{\sigma}\mid S)\right) + 3\sum_{S\subseteq[4]\setminus\{i\}||S|=3}\left(1+\sum_{k=1}^{|S|}(-1)^k e_k(\boldsymbol{\sigma}\mid S)\right) \\
= \quad & 3 - 3\left(\sigma_1^2+\sigma_2^2+\sigma_3^2\right) + 4\left(\sigma_1^2\sigma_2^2+\sigma_2^2\sigma_3^2+\sigma_1^2\sigma_3^2\right) - 6\sigma_1^2\sigma_2^2\sigma_3^2 \\
= \quad & 3 - 3e_1(\boldsymbol{\sigma}_{-i}) + 4e_2(\boldsymbol{\sigma}_{-i}) - 6e_3(\boldsymbol{\sigma}_{-i})\,.
\end{aligned}
$$

We continue to calculate:

3.

$$
\begin{aligned}
& \sum_{S\subseteq[4]\setminus\{i\}||S|=2} e_1(\boldsymbol{\sigma}_{-\{i\}\cup S}) \left(1+\sum_{k=1}^{|S|}(-1)^k e_k(\boldsymbol{\sigma}\mid S)\right) \\
= & \sum_{S\subseteq[4]\setminus\{i\}||S|=2} e_1(\boldsymbol{\sigma}_{-\{i\}\cup S}) \left(1+\sum_{k=1}^{2}(-1)^k e_k(\boldsymbol{\sigma}\mid S)\right) \\
= & \; e_1(\boldsymbol{\sigma}_{-\{i\}\cup\{1,2\}}) \left(1+\sum_{k=1}^{2}(-1)^k e_k(\boldsymbol{\sigma}\mid\{1,2\})\right) \\
& +e_1(\boldsymbol{\sigma}_{-\{i\}\cup\{1,3\}}) \left(1+\sum_{k=1}^{2}(-1)^k e_k(\boldsymbol{\sigma}\mid\{1,3\})\right)+e_1(\boldsymbol{\sigma}_{-\{i\}\cup\{2,3\}}) \left(1+\sum_{k=1}^{3}(-1)^k e_k(\boldsymbol{\sigma}\mid\{2,3\})\right) \\
= & \; e_1(\sigma_3^2) \left(1+\sum_{k=1}^{2}(-1)^k e_k(\sigma_1^2,\sigma_2^2)\right) \\
& +e_1(\sigma_2^2) \left(1+\sum_{k=1}^{2}(-1)^k e_k(\sigma_1^2,\sigma_3^2)\right)+e_1(\sigma_1^2) \left(1+\sum_{k=1}^{2}(-1)^k e_k(\sigma_2^2,\sigma_3^2)\right) \\
= & \; \sigma_3^2 \left(1-e_1(\sigma_1^2,\sigma_2^2)+e_2(\sigma_1^2,\sigma_2^2)\right) \\
& +\sigma_2^2 \left(1-e_1(\sigma_1^2,\sigma_3^2)+e_2(\sigma_1^2,\sigma_3^2)\right)+\sigma_1^2 \left(1-e_1(\sigma_2^2,\sigma_3^2)+e_2(\sigma_2^2,\sigma_3^2)\right) \\
= & \; \sigma_3^2 \left(1-\sigma_1^2-\sigma_2^2+\sigma_1^2\sigma_2^2\right) \\
& +\sigma_2^2 \left(1-\sigma_1^2-\sigma_3^2+\sigma_1^2\sigma_3^2\right)+\sigma_1^2 \left(1-\sigma_2^2-\sigma_3^2+\sigma_2^2\sigma_3^2\right) \\
= & \; \sigma_1^2+\sigma_2^2+\sigma_3^2-2\left(\sigma_1^2\sigma_2^2+\sigma_1^2\sigma_3^2+\sigma_2^2\sigma_3^2\right)+3\sigma_1^2\sigma_2^2\sigma_3^2 \\
= & \; e_1(\boldsymbol{\sigma}_{-i})-2e_2(\boldsymbol{\sigma}_{-i})+3e_3(\boldsymbol{\sigma}_{-i})\,.
\end{aligned}
$$

Finally:

4.

$$
\sum_{k=1}^{3}(-1)^k e_k(\boldsymbol{\sigma}_{-i})-e_3(\boldsymbol{\sigma}_{-i}) \quad = \quad -e_1(\boldsymbol{\sigma}_{-i})+e_2(\boldsymbol{\sigma}_{-i})-2e_3(\boldsymbol{\sigma}_{-i})\,.
$$

Hence, the *Indifference Equation* is

$$
\begin{aligned}
& \frac{\left(\mathsf{f}_2^2-\mathsf{f}_1^2\right)}{(3\mathsf{f}_2+\mathsf{f}_1)\,(3\,\mathsf{f}_1+\mathsf{f}_2)}\;(3-3e_1(\boldsymbol{\sigma}_{-i})+4e_2(\boldsymbol{\sigma}_{-i})-6e_3(\boldsymbol{\sigma}_{-i})) \\
& +\frac{\mathsf{f}_2-\mathsf{f}_1}{2\;(\mathsf{f}_2+\mathsf{f}_1)}\;(e_1(\boldsymbol{\sigma}_{-i})-2e_2(\boldsymbol{\sigma}_{-i})+3e_3(\boldsymbol{\sigma}_{-i})) \\
& -\frac{1}{4}\;(-e_1(\boldsymbol{\sigma}_{-i})+e_2(\boldsymbol{\sigma}_{-i})-2e_3(\boldsymbol{\sigma}_{-i})) \\
= & \; \frac{1}{4}+\mathsf{f}_2-\mathsf{f}_1\,.
\end{aligned}
$$

or

$$
\begin{aligned}
& \left(\underbrace{\frac{-3\left(\mathsf{f}_2^2-\mathsf{f}_1^2\right)}{(3\mathsf{f}_2+\mathsf{f}_1)\,(3\,\mathsf{f}_1+\mathsf{f}_2)}+\frac{\mathsf{f}_2-\mathsf{f}_1}{2\;(\mathsf{f}_2+\mathsf{f}_1)}+\frac{1}{4}}_{:=\mathsf{A}}\right) e_1(\boldsymbol{\sigma}_{-i}) \\
& \left(\underbrace{\frac{4\left(\mathsf{f}_2^2-\mathsf{f}_1^2\right)}{(3\mathsf{f}_2+\mathsf{f}_1)\,(3\,\mathsf{f}_1+\mathsf{f}_2)}-\frac{\mathsf{f}_2-\mathsf{f}_1}{\mathsf{f}_2+\mathsf{f}_1}-\frac{1}{4}}_{:=\mathsf{B}}\right) e_2(\boldsymbol{\sigma}_{-i})
\end{aligned}
$$

$$\left(\underbrace{\frac{-6\left(\mathsf{f}_2^2-\mathsf{f}_1^2\right)}{\left(3\mathsf{f}_2+\mathsf{f}_1\right)\left(3\,\mathsf{f}_1+\mathsf{f}_2\right)}+\frac{3\left(\mathsf{f}_2-\mathsf{f}_1\right)}{2\,\left(\mathsf{f}_2+\mathsf{f}_1\right)}+\frac{1}{2}}_{:=\mathsf{C}}\right)e_3(\boldsymbol{\sigma}_{-i})$$

$$= \underbrace{-\frac{3\left(\mathsf{f}_2^2-\mathsf{f}_1^2\right)}{\left(3\mathsf{f}_2+\mathsf{f}_1\right)\left(3\,\mathsf{f}_1+\mathsf{f}_2\right)}+\frac{1}{4}+\mathsf{f}_2-\mathsf{f}_1}_{:=\mathsf{D}}\,.$$

Now solve the system of Indifference Equations to compute a Nash equilibrium:

Step 1: Express each of $e_1(\boldsymbol{\sigma}_{-i})$, $e_2(\boldsymbol{\sigma}_{-i})$ and $e_3(\boldsymbol{\sigma}_{-i})$ in terms of $e_1(\boldsymbol{\sigma})$, $e_2(\boldsymbol{\sigma})$ and $e_3(\boldsymbol{\sigma})$:
For simplicity, denote $x=\sigma_1^2$, $y=\sigma_2^2$, $z=\sigma_3^2$ and $w=\sigma_4^2$. For each $k\geq 1$, denote $p_k(\boldsymbol{\sigma})=x^k+y^k+z^k+w^k$, the *k-th power sum*. Recall the *Girard-Newton formulae*

$$p_k(x_1,x_2,\ldots,x_n) \;=\; (-1)^{k-1}ke_k(x_1,x_2,\ldots,x_n)+\sum_{\iota=1}^{k-1}(-1)^{k+1-\iota}e_{k-\iota}(x_1,x_2,\ldots,x_n)p_\iota(x_1,x_2,\ldots,x_n)\,,$$

for $1\leq k\leq n$. (Recall that $0<x<1$.) Note that:
(1) $e_1(\boldsymbol{\sigma}_{-i})=e_1(\boldsymbol{\sigma})-x$.
(2)

$$\begin{aligned}
e_2(\boldsymbol{\sigma}_{-i}) &= yz+zw+yw\\
&= \frac{1}{2}\left((z+y+w)^2-(z^2+y^2+w^2)\right)\\
&= \frac{1}{2}\left((e_1(\boldsymbol{\sigma})-x)^2-(p_2(\boldsymbol{\sigma})-x^2)\right)\\
&= \frac{1}{2}\left(e_1^2(\boldsymbol{\sigma})-2e_1(\boldsymbol{\sigma})x+x^2-p_2(\boldsymbol{\sigma})+x^2\right)\\
&= \frac{1}{2}\left(2x^2-2e_1(\boldsymbol{\sigma})x+e_1^2(\boldsymbol{\sigma})-p_2(\boldsymbol{\sigma})\right)\\
&= x^2-e_1(\boldsymbol{\sigma})x+e_2(\boldsymbol{\sigma})\,,
\end{aligned}$$

using the *Girard-Newton formula* with $k=2$ and $n=4$.
(3) $e_3(\boldsymbol{\sigma}_{-i})=\dfrac{e_3(\boldsymbol{\sigma})}{x}$.

Step 2: Wrap up: Substituting (1), (2) and (3) into the *Indifference Equation*, we get

$$\mathsf{A}(e_1(\boldsymbol{\sigma})-x)+\mathsf{B}\left(x^2-e_1(\boldsymbol{\sigma})x+e_2(\boldsymbol{\sigma})\right)+\mathsf{C}\,\frac{e_3(\boldsymbol{\sigma})}{x} \;=\; \mathsf{D}\,.$$

or (since $x>0$)

$$\mathsf{B}x^3-\left(\mathsf{A}+\mathsf{B}e_1(\boldsymbol{\sigma})\right)x^2+\left(\mathsf{A}e_1(\boldsymbol{\sigma})+\mathsf{B}e_2(\boldsymbol{\sigma})-\mathsf{D}\right)x+\mathsf{C}e_3(\boldsymbol{\sigma}) \;=\; 0\,. \tag{1}$$

Denote as (x_1,x_2,x_3) the three roots of (1). Since (1) is also satisfied by y, z and w and each of $e_1(\boldsymbol{\sigma})$, $e_2(\boldsymbol{\sigma})$ and $e_3(\boldsymbol{\sigma})$ is a symmetric function in x, y, z and w, it follows that $(x_1,x_2,x_3)=(y_1,y_2,y_3)=(z_1,z_2,z_3)=(w_1,w_2,w_3)$. This implies that $e_1(\boldsymbol{\sigma})=4x$, $e_2(\boldsymbol{\sigma})=6x^2$ and $e_3(\boldsymbol{\sigma})=4x^3$. Hence, (1) becomes

$$\mathsf{B}x^3-\left(\mathsf{A}+4x\right)x^2+\left(\mathsf{A}\cdot 4x+\mathsf{B}\cdot 8x^2-\mathsf{D}\right)x+\left(\mathsf{C}\cdot 4x^3\right)]\;=\;0\,,$$

or

$$\left(-4+9\mathsf{B}+4\mathsf{C}\right)x^3+3\mathsf{A}x^2-\mathsf{D}x \;=\; 0\,. \tag{2}$$

Since $\boldsymbol{\sigma}$ is a Nash equilibrium, the root $x_1=0$ is rejected. The remaining two roots are

$$x_{2,3} \;=\; \frac{-3\mathsf{A}\pm\sqrt{9\mathsf{A}^2+4\,\left(-4+9\mathsf{B}+4\mathsf{C}\right)\mathsf{D}}}{2\left(-4+9\mathsf{B}+4\mathsf{C}\right)}\,,$$

assuming that $-4+9\mathsf{B}+4\mathsf{C}\geq 0$. We now choose f_2 and f_ℓ so that $9\mathsf{A}^2+4\,\left(-4+9\mathsf{B}+4\mathsf{C}\right)\mathsf{D}>0$ subject to (1) $-4+9\mathsf{B}+4\mathsf{C}\geq 0$, and (2) $0<x_2<1$ **or** $0<x_3<1$, yields an irrational Nash equilibrium (either (x_2,x_2,x_2,x_2) or (x_3,x_3,x_3,x_3) or both). Set $\mathsf{f}_2:=\alpha\mathsf{f}_\ell$, where $\alpha>1$ is sufficiently large. Then:

$$
\begin{aligned}
\mathsf{A} &= -3\,\frac{\alpha^2-1}{(3\alpha+1)(\alpha+3)} + \frac{\alpha-1}{2(\alpha+1)} + \frac{1}{4} \\
&= \frac{-12(\alpha^2-1)(\alpha+1) + 2(\alpha-1)(3\alpha+1)(\alpha+3) + (3\alpha+1)(\alpha+3)(\alpha+1)}{4(3\alpha+1)(\alpha+3)(\alpha+1)} \\
&= \frac{-12\alpha^3 - 12\alpha^2 + 12\alpha + 12 + (3\alpha+1)(\alpha+3)[2(\alpha-1)+(\alpha+1)]}{4(3\alpha+1)(\alpha+3)(\alpha+1)} \\
&= \frac{-12\alpha^3 - 12\alpha^2 + 12\alpha + 12 + (3\alpha+1)(\alpha+3)(3\alpha-1)}{4(3\alpha+1)(\alpha+3)(\alpha+1)} \\
&= \frac{-12\alpha^3 - 12\alpha^2 + 12\alpha + 12 + (9\alpha^2-1)(\alpha+3)}{4(3\alpha+1)(\alpha+3)(\alpha+1)} \\
&= \frac{-12\alpha^3 - 12\alpha^2 + 12\alpha + 12 + 9\alpha^3 + 27\alpha^2 - \alpha - 3}{4(3\alpha+1)(\alpha+3)(\alpha+1)} \\
&= \frac{-3\alpha^3 + 15\alpha^2 + 11\alpha + 9}{4(3\alpha+1)(\alpha+3)(\alpha+1)} \\
&= \frac{-3\alpha^3 + o(\alpha^3)}{12\alpha^3 + o(\alpha^3)}) \\
&= -\frac{1}{4} + o(1)\,.
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{B} &= 4\,\frac{\alpha^2-1}{(3\alpha+1)(\alpha+3)} - \frac{\alpha-1}{\alpha+1} - \frac{1}{4} \\
&= \frac{16(\alpha^2-1)(\alpha+1) - 4(\alpha-1)(3\alpha+1)(\alpha+3) - (3\alpha+1)(\alpha+3)(\alpha+1)}{4(3\alpha+1)(\alpha+3)(\alpha+1)} \\
&= \frac{16\alpha^3 + 16\alpha^2 - 16\alpha - 16 - (3\alpha+1)(\alpha+3)[4(\alpha-1)+(\alpha+1)]}{4(3\alpha+1)(\alpha+3)(\alpha+1)} \\
&= \frac{16\alpha^3 + 16\alpha^2 - 16\alpha - 16 - (3\alpha+1)(\alpha+3)(5\alpha-3)}{4(3\alpha+1)(\alpha+3)(\alpha+1)} \\
&= \frac{16\alpha^3 + 16\alpha^2 - 16\alpha - 16 - (\alpha^2-4\alpha-3)(\alpha+3)}{4(3\alpha+1)(\alpha+3)(\alpha+1)} \\
&= \frac{16\alpha^3 + 16\alpha^2 - 16\alpha - 16 - (\alpha^3+3\alpha^2-4\alpha^2-12\alpha-3\alpha-9)}{4(3\alpha+1)(\alpha+3)(\alpha+1)} \\
&= \frac{15\alpha^3 + 17\alpha^2 - \alpha - 7}{4(3\alpha+1)(\alpha+3)(\alpha+1)} \\
&= \frac{15\alpha^3 + o(\alpha^3)}{12\alpha^3 + o(\alpha^3)} \\
&= \frac{15}{12} + o(1)\,.
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{C} &= -6\,\frac{\alpha^2-1}{(3\alpha+1)(\alpha+3)} + \frac{3(\alpha-1)}{2(\alpha+1)} + \frac{1}{2} \\
&= \frac{-12(\alpha^2-1)(\alpha+1) + 3(\alpha-1)(3\alpha+1)(\alpha+3) + (3\alpha+1)(\alpha+3)(\alpha+1)}{2(3\alpha+1)(\alpha+3)(\alpha+1)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-12(\alpha^2-1)(\alpha+1)+(3\alpha+1)(\alpha+3)[3(\alpha-1)+\alpha+1]}{2(3\alpha+1)(\alpha+3)(\alpha+1)}\\
&= \frac{-12(\alpha^2-1)(\alpha+1)+(3\alpha+1)(\alpha+3)[2(2\alpha-1)]}{2(3\alpha+1)(\alpha+3)(\alpha+1)}\\
&= \frac{-6(\alpha^2-1)(\alpha+1)+(3\alpha+1)(\alpha+3)(2\alpha-1)}{(3\alpha+1)(\alpha+3)(\alpha+1)}\\
&= \frac{-6\alpha^3-6\alpha^2+6\alpha+6+(3\alpha+1)(\alpha+3)(2\alpha-1)}{(3\alpha+1)(\alpha+3)(\alpha+1)}\\
&= \frac{-6\alpha^3-6\alpha^2+6\alpha+6+(3\alpha^2+10\alpha+3)(2\alpha-1)}{(3\alpha+1)(\alpha+3)(\alpha+1)}\\
&= \frac{-6\alpha^3-6\alpha^2+6\alpha+6+6\alpha^3-3\alpha^2+20\alpha^2-10\alpha+6\alpha-3)}{(3\alpha+1)(\alpha+3)(\alpha+1)}\\
&= \frac{11\alpha^2+2\alpha+3}{(3\alpha+1)(\alpha+3)(\alpha+1)}\\
&= \frac{11\alpha^2+o(\alpha^2)}{3\alpha^3+o(\alpha^3)}\\
&= o(1)\,.
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{D} &= -3\,\frac{\alpha^2-1}{(3\alpha+1)(\alpha+3)}+\alpha-1+\frac{1}{4}\\
&= \frac{-12(\alpha^2-1)+4\alpha(3\alpha+1)(\alpha+3)-3(3\alpha+1)(\alpha+3)}{4\alpha(3\alpha+1)(\alpha+3)}\\
&= \frac{-12(\alpha^2-1)+(4\alpha-3)(3\alpha+1)(\alpha+3)}{4\alpha(3\alpha+1)(\alpha+3)}\\
&= \frac{-12\alpha^2+12+(12\alpha^2-5\alpha-3)(\alpha+3)}{4\alpha(3\alpha+1)(\alpha+3)}\\
&= \frac{-12\alpha^2+12+12\alpha^3+36\alpha^2-5\alpha^2-15\alpha-3\alpha-9}{4\alpha(3\alpha+1)(\alpha+3)}\\
&= \frac{12\alpha^3+19\alpha^2-18\alpha+3}{4\alpha(3\alpha+1)(\alpha+3)}\\
&= \frac{12\alpha^3+o(\alpha^3)}{12\alpha^3+o(\alpha^3)}\\
&= 1+o(1)\,.
\end{aligned}
$$

$$
\begin{aligned}
-4+9\mathsf{B}+4\mathsf{C} &= -4+9\left(\frac{15}{12}+o(1)\right)+4\,o(1)\\
&= \frac{29}{4}+o(1)\,.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
9\mathsf{A}^2+4(-4+9\mathsf{B}+4\mathsf{C})\mathsf{D} &= \frac{9}{16}+o(1)+4\left(\frac{29}{4}+o(1)\right)(1+o(1))\\
&= \frac{473}{16}+o(1)
\end{aligned}
$$

$$> \quad 0\,,$$

for sufficiently large $\alpha > 1$. Hence, x_2 is irrational while x_3 is a negative irrational, and as such it is rejected. Therefore, it is only left to prove that $0 < x_2 < 1$. Clearly,

$$\begin{aligned} x_2 &= \frac{-3\left(-\frac{1}{4}+o(1)\right)+\sqrt{\frac{473}{16}+o(1)}}{2\left(\frac{29}{4}+o(1)\right)} \\ &= \frac{\frac{3}{4}+\sqrt{\frac{473}{16}}+o(1)}{\frac{29}{2}+o(1)} \\ &= \frac{2}{29}\cdot\frac{3+\sqrt{473}}{4}+o(1) \\ &= \frac{3+\sqrt{473}}{58}+o(1) \\ &\in (0,1)\,, \end{aligned}$$

for sufficiently large α.

I Proof of Lemma 16

By the definition of $\delta(\mathsf{G})$,

$$\begin{aligned} &\delta(n) \\ &= \max\left\{\left|\mathsf{U}_i(q_i,q'_j,\mathsf{q}_{-\{i,j\}})-\mathsf{U}_i(q_i,q'_j,\mathsf{q}_{-\{i,j\}})\right|\right\} \\ &= \max\left\{\left|\frac{\mathsf{f}_{q_i}}{\sum_{q\in[Q]}\mathsf{f}_q\,\mathsf{N}_{(q_i,q'_j,\mathsf{q}_{-\{i,j\}})}(q)}-\mathsf{f}_{q_i}-\left(\frac{\mathsf{f}_{q_i}}{\sum_{q\in[Q]}\mathsf{f}_q\,\mathsf{N}_{(q_iq''_j,\mathsf{q}_{-\{i,j\}})}(q)}-\mathsf{f}_{q_i}\right)\right|\right\} \\ &= \max\left\{\left|\mathsf{f}_{q_i}\left(\frac{1}{\sum_{q\in[Q]}\mathsf{f}_q\,\mathsf{N}_{(q_i,q'_j,\mathsf{q}_{-\{i,j\}})}(q)}-\frac{1}{\sum_{q\in[Q]}\mathsf{f}_q\,\mathsf{N}_{(q_i,q''_j,\mathsf{q}_{-\{i,j\}})}(q)}\right)\right|\right\} \\ &= \max\left\{\left|\mathsf{f}_{q_i}\frac{\mathsf{f}_{q'_j}\left(\mathsf{N}_{(q_i,q''_j,\mathsf{q}_{-\{i,j\}})}(q'_j)-1\right)+\mathsf{f}_{q''_j}\left(\mathsf{N}_{(q_i,q''_j,\mathsf{q}_{-\{i,j\}})}(q''_j)+1\right)-\mathsf{f}_{q'_j}\,\mathsf{N}_{(q_i,q'_j,\mathsf{q}_{-\{i,j\}})}(q'_j)-\mathsf{f}_{q''_j}\,\mathsf{N}_{(q_i,q''_j,\mathsf{q}_{-\{i,j\}})}(q''_j)}{\left(\sum_{q\in[Q]}\mathsf{f}_q\,\mathsf{N}_{(q_i,q'_j,\mathsf{q}_{-\{i,j\}})}(q)\right)\left(\sum_{q\in[Q]}\mathsf{f}(q)\,\mathsf{N}_{(q_i,q''_j,\mathsf{q}_{-\{i,j\}})}(q)\right)}\right|\right\} \\ &= \max\left\{\left|\frac{\mathsf{f}_{q_i}(\mathsf{f}_{q''_j}-\mathsf{f}_{q'_j})}{\left(\sum_{q\in[Q]}\mathsf{f}_q\,\mathsf{N}_{(q'_j,\mathsf{q}_{-j})}(q)\right)\left(\sum_{q\in[Q]}\mathsf{f}_q\,\mathsf{N}_{(q''_j,\mathsf{q}_{-j})}(q)\right)}\right|\right\} \\ &= \frac{\Theta(1)}{\min\left\{\left|\left(\sum_{q\in[Q]}\mathsf{f}_q\,\mathsf{N}_{(q_i,q'_j,\mathsf{q}_{-\{i,j\}})}(q)\right)\left(\sum_{q\in[Q]}\mathsf{f}_q\,\mathsf{N}_{(q_i,q''_j,\mathsf{q}_{-\{i,j\}})}(q)\right)\right|\right\}} \end{aligned}$$

To prove that $\delta(n) = o\left(\frac{1}{n}\right)$, we have to prove that

$$\min\left\{\left|\left(\sum_{q\in[Q]}\mathsf{f}_q\,\mathsf{N}_{(q_i,q'_j,\mathsf{q}_{-\{i,j\}})}(q)\right)\left(\sum_{q\in[Q]}\mathsf{f}(q)\,\mathsf{N}_{(q_i,q''_j,\mathsf{q}_{-\{i,j\}})}(q)\right)\right|\right\} = \omega(n)\,.$$

We shall prove that

$$\min\left\{\left|\left(\sum_{q\in[Q]}\mathsf{f}_q\,\mathsf{N}_{(q_i,q'_j,\mathsf{q}_{-\{i,j\}})}(q)\right)\left(\sum_{q\in[Q]}\mathsf{f}(q)\,\mathsf{N}_{(q_i,q''_j,\mathsf{q}_{-\{i,j\}})}(q)\right)\right|\right\} = \Omega(n^2)\,.$$

Define $\mathsf{f}_{\min} := \mathsf{f}_2$ if $\mathsf{f}_1 = 0$ or f_1 if $\mathsf{f}_1 > 0$; so $\mathsf{f}_{\min}$ is the smallest non-zero effort. (Using $\mathsf{f}_{\min}$ allows proving the result for either mandatory or voluntary participation.) Note that

$$\begin{aligned}
&\left(\sum_{q\in[Q]} \mathsf{f}_q \mathsf{N}_{(q_i,q'_j,\mathsf{q}_{-\{i,j\}})}(q)\right)\left(\sum_{q\in[Q]} \mathsf{f}_q \mathsf{N}_{(q_i,q''_j,\mathsf{q}_{-\{i,j\}})}(q)\right)\\
\geq\ & \mathsf{f}_{\min}^2 \left(\sum_{q\in[Q]} \mathsf{N}_{(q_i,q'_j,\mathsf{q}_{-\{i,j\}})}(q)\right)\left(\sum_{q\in[Q]} \mathsf{N}_{(q_i,q''_j,\mathsf{q}_{-\{i,j\}})}(q)\right)\\
=\ & \mathsf{f}_{\min}^2 \cdot n^2 \ =\ \Theta(n^2)\,.
\end{aligned}$$

Hence,

$$\min\left\{\left|\left(\sum_{q\in[Q]} \mathsf{f}_q \mathsf{N}_{(q_i,q'_j,\mathsf{q}_{-\{i,j\}})}(q)\right)\left(\sum_{q\in[Q]} \mathsf{f}_q \mathsf{N}_{(q_i,q''_j,\mathsf{q}_{-\{i,j\}})}(q)\right)\right|\right\} \ =\ \Omega(n^2)\,.$$

Since $\Omega(n^2) \subset \omega(n)$, this implies that $\delta(n) = o\left(\frac{1}{n}\right)$.