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Six loop critical exponent analysis for Lee-Yang and percolation theory

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Abstract. Using the recent six loop renormalization group functions for Lee-Yang and percolation theory constructed by Schnetz from a scalar cubic Lagrangian, we deduce the ϵ expansion of the critical exponents for both cases. Estimates for the exponents in three, four and five dimensions are extracted using two-sided Padé approximants and shown to be compatible with values from other approaches.

1 Introduction.

In recent years there has been a significant advance in the renormalization of core quantum field theories to high loop order due to the development of new techniques such as the Laporta algorithm, [1], the FORCER package, [2, 3], high numerical precision methods based on difference equations to determine massive tadpole graphs, [1], and graphical functions [4, 5]. These tools have opened the door to β-functions to five loops and beyond [6, 7, 8, 9, 10, 11, 12, 13, 14]. While [6, 7, 8] have extended the renormalization of Quantum Chromodynamics (QCD) to five loops and that of Gross-Neveu-Yukawa theory to the same order, [9], renormalization at orders beyond five have also been achieved with the six loop β-function of scalar ϕ^3 theory available now in [14] and the seven loop ϕ^4 theory β-function provided in [11]. Such precision for gauge theories, for instance, has refined uncertainties in observables for particle physics experiments as well as in applications to condensed matter physics based on scalar and Gross-Neveu-Yukawa theories. More specifically for the latter critical exponent estimates governing the properties of phase transitions have been refined.

Of the suite of theories mentioned earlier the six loop renormalization group functions of scalar ϕ^3 theory have been established most recently in [14] using graphical functions. As a by-product the renormalization group functions for Lee-Yang and percolation theory were deduced to the same order in the modified minimal subtraction ($\overline{\text{MS}}$) scheme, [15]. For instance the connection of the Lee-Yang singularity problem with scalar ϕ^3 theory was originally made in [16]. The physics these models relate to are important in condensed matter with critical exponents calculated via a variety of methods such as high temperature and Monte Carlo techniques in fixed dimensions between two and six where the latter is the critical dimension of a scalar theory with a cubic interaction. More modern approaches such as the conformal bootstrap programme and functional renormalization group techniques have also been used to study both problems. We note that Lee-Yang theory also has applications to particle theory due to the relation of the associated edge singularity problem with the QCD equation of state [17]. This relies on an accurate value of the exponent σ in three dimensions. Another example where σ is important arose in a recent QCD lattice analysis relating to baryon number, [18]. There the volume scaling of Lee-Yang zeroes, when there is crossover behaviour, is determined by σ. For a deeper overview of these connections see [19].

The other technique used to determine exponent estimates is that of the ϵ expansion. It is based on the cubic scalar theory renormalization group functions in $d = 6 - \epsilon$ spacetime dimensions with the critical exponents determined as a power series in ϵ at the Wilson-Fisher fixed point, [20]. For the Lee-Yang and percolation situations the exponents were produced over many years from low order up to five loops [12, 13, 21, 22, 23, 24]. Compared with results known at each respective time from other techniques exponent estimates derived from the ϵ expansion were in solid agreement for three, four and five dimensions. This may appear odd since to reach say three dimensions from six one would have to set $\epsilon = 3$ thereby raising the question of convergence. In [13, 24] exponents were deduced numerically by several approaches one of which was that of two-sided Padé approximants. While one side was the exponent value in strictly six dimensions there were lower dimensional boundary conditions for each exponent. These were the respective values of the exponents in two dimensions derived from the separate conformal field theories that govern transitions in the Lee-Yang or percolation cases. The values of critical exponents of such field theories were determined *exactly* using conformal symmetry in [25]. For the Lee-Yang model exponent values are also known exactly in one dimension. So in computing two-sided approximants in [13, 24] those for Lee-Yang used two constraints while only one was available for percolation theory.

Given this background it is therefore the purpose of this article to extend the results of [13, 22, 23, 24] to six loops given the advance made in [14, 15]. With the extra order and the lower dimension constraints more Padé approximants are available which has allowed us to provide an uncertainty measure on the estimates we present here. This is qualified of course by the fact that a rational polynomial approximation to a continuous function of d in $2 \leq d \leq 6$ may not itself be continuous. This transpired to be the case in that several of the approximants which were constructed had singularities or were not monotonic. The last criterion is derived from the behaviour of available estimates from other approaches meaning that in $2 \leq d \leq 6$ they were either increasing or decreasing monotonically. Subject to these caveats we managed to obtain exponent estimates for both Lee-Yang and percolation theory that were credible compared with other techniques. Moreover they were not significantly different from previous lower loop values thereby indicating a degree of convergence as the loop order increases.

The article is structured as follows. The focus in Section 2 is on Lee-Yang theory where we record the ϵ expansion of the key critical exponents before detailing the two-sided Padé formalism. Exponent estimates using this method and their comparison with known values complete the section. The parallel analysis for percolation theory is given in Section 3 followed by concluding remarks in Section 4.

2 Lee-Yang exponents.

For the first part of the analysis we concentrate on the Lee-Yang problem which is underpinned by a non-unitary Lagrangian with a single scalar field ϕ self-interacting cubically with coupling g obeying the Lagrangian

$$L ~=~ \frac{1}{2} (\partial_\mu \phi)^2 ~+~ \frac{ig}{6} \phi^3 \tag{2.1}$$

with the relation to the Lee-Yang edge singularity being elucidated in [16]. En route we will illustrate the approach to our analysis which will also be applied to percolation theory. As the six loop $\overline{\text{MS}}$ β-function for (2.1) as well as the ϕ field anomalous dimension were recorded in [14, 15] their associated critical exponents are straightforward to extract at the Wilson-Fisher critical point, $g^\star$, in $d = 6 - \epsilon$ as

$$\begin{aligned}
\hat{\eta} ~=~& -\frac{1}{9}\epsilon - \frac{43}{729}\epsilon^2 + \left[\frac{16}{243}\zeta_3 - \frac{8375}{236196} \right] \epsilon^3 + \left[\frac{4}{81}\zeta_4 - \frac{3883409}{76527504} - \frac{716}{19683}\zeta_3 - \frac{80}{2187}\zeta_5 \right] \epsilon^4 \\
&+ \left[\frac{136}{2187}\zeta_3^2 + \frac{37643}{59049}\zeta_3 + \frac{80524}{59049}\zeta_5 - \frac{1545362585}{24794911296} - \frac{4172}{2187}\zeta_7 - \frac{179}{6561}\zeta_4 - \frac{100}{2187}\zeta_6 \right] \epsilon^5 \\
&+ \left[\frac{68}{729}\zeta_3\zeta_4 + \frac{3584}{6561}\zeta_3^3 + \frac{4624}{3645}\zeta_{5,3} + \frac{37643}{78732}\zeta_4 + \frac{99440}{59049}\zeta_6 + \frac{129968}{177147}\zeta_3^2 + \frac{1734712}{59049}\zeta_9 \right. \\
&- \frac{152041988501}{2677850419968} - \frac{440890397}{258280326}\zeta_3 - \frac{158942329}{28697814}\zeta_5 - \frac{6992407}{354294}\zeta_7 - \frac{15616}{59049}\zeta_3\zeta_5 \\
&\left. - \frac{7781}{1215}\zeta_8 \right] \epsilon^6 ~+~ O(\epsilon^7) \\
\hat{\omega} ~=~& \epsilon - \frac{125}{162}\epsilon^2 + \left[\frac{20}{27}\zeta_3 + \frac{36755}{52488} \right] \epsilon^3 + \left[\frac{5}{9}\zeta_4 + \frac{160}{81}\zeta_5 - \frac{31725355}{17006112} - \frac{9673}{2187}\zeta_3 \right] \epsilon^4 \\
&+ \left[\frac{17088604709}{5509980288} - \frac{21980}{243}\zeta_7 - \frac{9673}{2916}\zeta_4 + \frac{200}{81}\zeta_6 + \frac{1384}{243}\zeta_3^2 + \frac{1050770}{19683}\zeta_5 + \frac{12094613}{354294}\zeta_3 \right] \epsilon^5 \\
&+ \left[\frac{692}{81}\zeta_3\zeta_4 + \frac{5968}{81}\zeta_{5,3} + \frac{23680}{729}\zeta_3^3 + \frac{1880255}{59049}\zeta_3^2 + \frac{5235625}{78732}\zeta_6 + \frac{9801800}{6561}\zeta_9 + \frac{12094613}{472392}\zeta_4 \right. \\
&- \frac{3218943170927}{595077871104} - \frac{22622670455}{114791256}\zeta_3 - \frac{677377966}{1594323}\zeta_5 - \frac{14402270}{19683}\zeta_7 - \frac{81733}{243}\zeta_8
\end{aligned}$$

$$- \frac{16640}{2187}\zeta_3\zeta_5 \Bigg] \epsilon^6 ~+~ O(\epsilon^7) \tag{2.2}$$

where $\hat{\eta} = \gamma_\phi(g^\star)$, $\hat{\omega} = \beta'(g^\star)$, ζ_n is the Riemann zeta function and $\zeta_{5,3} = \sum_{m>n\geq 1} \frac{1}{m^5 n^3}$ is a multiple zeta. For this section our notation is to place a hat on the exponents connected with Lee-Yang theory to distinguish them from similar exponents we will compute in percolation theory. In addition to $\hat{\eta}$ and $\hat{\omega}$ there are several other exponents that are important for physical problems and which can be derived from scaling laws [26]. These are, [26],

$$\begin{aligned} \hat{\sigma} &= \frac{[d-2+\hat{\eta}]}{[d+2-\hat{\eta}]} ~~,~~ \hat{\phi} ~=~ 1 ~+~ \hat{\sigma} ~~,~~ \hat{\theta} ~=~ \hat{\nu}_c \hat{\omega} \\ \hat{\nu} &= \frac{2}{[d-2+\hat{\eta}]} ~~,~~ \hat{\nu}_c ~=~ \frac{2}{[d+2-\hat{\eta}]} \end{aligned} \tag{2.3}$$

where $\hat{\theta}$ and $\hat{\omega}$ are correction to scaling exponents and $\hat{\nu}$ and $\hat{\nu}_c$ are not independent since [27]

$$\frac{1}{\hat{\nu}} ~+~ \frac{1}{\hat{\nu}_c} ~=~ d ~. \tag{2.4}$$

In addition it is known that the exponent $\hat{\beta}$ is unity in all dimensions d [22, 27]. Consequently the ϵ expansion of the exponents we will find estimates for are

$$\begin{aligned} \hat{\sigma} &= \frac{1}{2} - \frac{1}{12}\epsilon - \frac{79}{3888}\epsilon^2 + \left[\frac{1}{81}\zeta_3 - \frac{10445}{1259712} \right] \epsilon^3 + \left[\frac{1}{108}\zeta_4 - \frac{4047533}{408146688} - \frac{161}{26244}\zeta_3 - \frac{5}{729}\zeta_5 \right] \epsilon^4 \\ &+ \left[\frac{17}{1458}\zeta_3^2 + \frac{20101}{78732}\zeta_5 + \frac{112399}{944784}\zeta_3 - \frac{1601178731}{132239526912} - \frac{1043}{2916}\zeta_7 - \frac{161}{34992}\zeta_4 - \frac{25}{2916}\zeta_6 \right] \epsilon^5 \\ &+ \Bigg[\frac{17}{972}\zeta_3\zeta_4 + \frac{224}{2187}\zeta_3^3 + \frac{289}{1215}\zeta_{5,3} + \frac{32669}{236196}\zeta_3^2 + \frac{49645}{157464}\zeta_6 + \frac{112399}{1259712}\zeta_4 + \frac{216839}{39366}\zeta_9 \\ &\quad - \frac{158574097133}{14281868906496} - \frac{156752701}{153055008}\zeta_5 - \frac{107952203}{344373768}\zeta_3 - \frac{7029955}{1889568}\zeta_7 - \frac{7781}{6480}\zeta_8 \\ &\quad - \frac{976}{19683}\zeta_3\zeta_5 \Bigg] \epsilon^6 ~+~ O(\epsilon^7) \\ \hat{\nu}_c &= \frac{1}{4} + \frac{1}{36}\epsilon + \frac{29}{23328}\epsilon^2 + \left[\frac{1}{486}\zeta_3 - \frac{8879}{7558272} \right] \epsilon^3 \\ &+ \left[\frac{1}{648}\zeta_4 - \frac{4526999}{2448880128} - \frac{107}{157464}\zeta_3 - \frac{5}{4374}\zeta_5 \right] \epsilon^4 \\ &+ \Bigg[\frac{17}{8748}\zeta_3^2 + \frac{20011}{472392}\zeta_5 + \frac{111757}{5668704}\zeta_3 - \frac{1845636677}{793437161472} - \frac{1043}{17496}\zeta_7 - \frac{107}{209952}\zeta_4 \\ &\quad - \frac{25}{17496}\zeta_6 \Bigg] \epsilon^5 \\ &+ \Bigg[\frac{17}{5832}\zeta_3\zeta_4 + \frac{112}{6561}\zeta_3^3 + \frac{289}{7290}\zeta_{5,3} + \frac{4141}{177147}\zeta_3^2 + \frac{12355}{236196}\zeta_6 + \frac{111757}{7558272}\zeta_4 + \frac{216839}{236196}\zeta_9 \\ &\quad - \frac{191795557319}{85691213438976} - \frac{404651861}{8264970432}\zeta_3 - \frac{150269137}{918330048}\zeta_5 - \frac{7142599}{11337408}\zeta_7 - \frac{7781}{38880}\zeta_8 \\ &\quad - \frac{488}{59049}\zeta_3\zeta_5 \Bigg] \epsilon^6 ~+~ O(\epsilon^7) \\ \hat{\theta} &= \frac{1}{4}\epsilon - \frac{107}{648}\epsilon^2 + \left[\frac{5}{27}\zeta_3 + \frac{8129}{52488} \right] \epsilon^3 + \left[\frac{5}{36}\zeta_4 + \frac{40}{81}\zeta_5 - \frac{3818417}{8503056} - \frac{9475}{8748}\zeta_3 \right] \epsilon^4 \\ &+ \left[\frac{7972456399}{11019960576} - \frac{9475}{11664}\zeta_4 - \frac{5495}{243}\zeta_7 + \frac{50}{81}\zeta_6 + \frac{346}{243}\zeta_3^2 + \frac{263750}{19683}\zeta_5 + \frac{11918591}{1417176}\zeta_3 \right] \epsilon^5 \end{aligned}$$

$$
+ \left[\frac{173}{81}\zeta_3\zeta_4 + \frac{1492}{81}\zeta_{5,3} + \frac{5920}{729}\zeta_3^3 + \frac{959221}{118098}\zeta_3^2 + \frac{2450450}{6561}\zeta_9 + \frac{5256775}{314928}\zeta_6 + \frac{11918591}{1889568}\zeta_4 \right.
- \frac{377937484105}{297538935552} - \frac{11090116969}{229582512}\zeta_3 - \frac{1335259205}{12754584}\zeta_5 - \frac{29209567}{157464}\zeta_7 - \frac{81733}{972}\zeta_8
\left. - \frac{4160}{2187}\zeta_3\zeta_5 \right] \epsilon^6 + O(\epsilon^7) \; . \quad (2.5)
$$

To gauge how each series behaves we note that the analytic expressions translate into

$$
\begin{aligned}
\hat{\eta} &= -0.111111\epsilon - 0.058985\epsilon^2 + 0.043690\epsilon^3 - 0.078954\epsilon^4 + 0.208254\epsilon^5 - 0.667225\epsilon^6 + O(\epsilon^7) \\
\hat{\sigma} &= 0.500000 - 0.083333\epsilon - 0.020319\epsilon^2 + 0.006549\epsilon^3 - 0.014382\epsilon^4 + 0.038113\epsilon^5 - 0.122697\epsilon^6 \\
&\quad + O(\epsilon^7) \\
\hat{\nu}_c &= 0.250000 + 0.027778\epsilon + 0.001243\epsilon^2 + 0.001299\epsilon^3 - 0.002180\epsilon^4 + 0.005989\epsilon^5 - 0.019451\epsilon^6 \\
&\quad + O(\epsilon^7) \\
\hat{\theta} &= 0.250000\epsilon - 0.165123\epsilon^2 + 0.377477\epsilon^3 - 1.088632\epsilon^4 + 3.731811\epsilon^5 - 14.380734\epsilon^6 + O(\epsilon^7) \\
\hat{\omega} &= \epsilon - 0.771605\epsilon^2 + 1.590668\epsilon^3 - 4.532626\epsilon^4 + 15.435688\epsilon^5 - 59.254423\epsilon^6 + O(\epsilon^7) \quad (2.6)
\end{aligned}
$$

numerically. We note that a numerical prediction was provided for $\hat{\phi}$ to $O(\epsilon^6)$ in [26]. While its terms to $O(\epsilon^3)$ used the then known three loops results of [22, 23] the subsequent terms to $O(\epsilon^6)$ were found by constructing a constrained $[6, 0]$ Padé approximant from estimates of $\hat{\phi}$ in $d = 3$, 4 and 5 dimensions. The coefficients of the resulting higher order term in ϵ are significantly smaller than the corresponding ones of $\hat{\sigma}$ in (2.6) when the scaling relation between $\hat{\phi}$ and $\hat{\sigma}$ is employed.

In previous perturbative analyses of critical exponents in ϕ^3 theory one method of extracting estimates was to use Padé approximants and we follow the same approach here. While the Padé method uses rational polynomials to approximate a function in a parameter which is regarded as small there is no guarantee that an approximant will capture the salient features of the exact function. For instance if the function is continuous an approximant may contain discontinuities or if the function is monotonic the Padé may have stationary points. In the present situation the aim is to create approximants that are valid in $2 \leq d \leq 6$ and then read off estimates for three, four and five dimensions. The obvious concern is that while ϵ is regarded as small in relation to the establishment of the Wilson-Fisher fixed point perturbatively it would be unlikely to be applicable when $\epsilon = 3$. However in previous lower loop analyses the construction of Padé approximants for Lee-Yang and percolation theory exponents benefitted greatly from knowledge of the *exact* exponents in two dimensions in both instances. These have been established from connecting the critical theory in $d \neq 2$ to a known conformal field theory in two dimensions which were all classified in [25]. In the Lee-Yang case exact exponents are also available in one dimension. Therefore we will use such available information to construct what is termed two-sided or constrained Padé approximants for each of the above exponents as well as those for percolation.

d	$\hat{\eta}$	$\hat{\sigma}$	$\hat{\nu}_c$	$\hat{\theta}$	$\hat{\omega}$	Δ_ϕ
1	-1	$-\frac{1}{2}$	$\frac{1}{2}$	(1)	(2)	-1
2	$-\frac{4}{5}$	$-\frac{1}{6}$	$\frac{5}{12}$	$\frac{5}{6}$	2	$-\frac{2}{5}$
6	0	$\frac{1}{2}$	$\frac{1}{4}$	0	0	2

Table 1: Values of Lee-Yang exponents in one, two and six dimensions.

One advantage of using the lower dimensional constraints is that the approximants should approach a more reliable value in three dimensions. Looking at it from another point of view not using the constraints for a Padé exponent could give a value well away from the exact value in two dimensions with an indication of this provided in [24]. We have recorded the values of the exponents in low dimensions in Table 1 where knowing $\hat{\eta}$ in one and two dimensions determines $\hat{\sigma}$ and $\hat{\nu}_c$ whereas $\hat{\theta}$ for two dimensions was taken from [26] and it determines $\hat{\omega}$. In $d = 1$ the values for $\hat{\theta}$ and $\hat{\omega}$ are from [26, 28, 29] but are bracketed since a doubt was expressed in [26] as to whether the values were reliable due to possible logarithmic behaviour in these correction to scaling exponents. Though it was noted in [26] that the values we have bracketed may be valid in the limit $d \rightarrow 1^+$. Given this we have used a cautious approach in constructing Padé approximants for $\hat{\theta}$ and $\hat{\omega}$ by using the two dimensional constraint only. One comment in relation to the distinction between approximants with one and two constraints is worth noting. This is that the use of a $d = 1$ value informs the *gradient* of the approximant in the region above two dimensions rather than solely the actual boundary point. In influencing the slope it refines the direction of the approximant towards three dimensions. The final exponent in Table 1, Δ_ϕ, relates to the full dimension of the ϕ field which is primarily used in the conformal bootstrap formalism. We include it in our discussions in order to translate exponent estimates from that and other techniques to the remaining exponents in the table. For completeness we note the relations are, [30],

$$\hat{\eta} ~=~ 2 ~-~ d ~+~ 2\Delta_\phi ~~~,~~~ \hat{\sigma} ~=~ \frac{\Delta_\phi}{[d - \Delta_\phi]} ~~~,~~~ \hat{\nu}_c ~=~ \frac{1}{[d - \Delta_\phi]} \tag{2.7}$$

and

$$\hat{\omega} ~=~ -~d ~+~ \Delta_{\phi^3} \tag{2.8}$$

where Δ_{ϕ^3} is the full dimension of the spin-0 conformal primary field ϕ^3 which is present in (2.1).

In using the two-sided Padé approach for lower loops, [13, 24], the results for exponents in three dimensions were generally in line with estimates from other techniques. The aim is that with the inclusion of six loop information uncertainties can be refined. This should be possible if one considers the data available for an L loop exponent. Including the canonical dimension means there are $(L + 1)$ coefficients in the ϵ expansion of an exponent. Including two constraints from lower dimensions gives a total of $(L + 3)$ pieces of information to construct the constrained Padé approximants. Allowing for a normalization drops this to $(L + 2)$ data points to construct the set of rational polynomials at L loops. Excluding the $[p, 0]$ Padés means at L loops there are $(L + 2)$ possible approximants where the $[p/q]$ approximant, $\mathcal{P}_{[p/q]}(\epsilon)$, is defined by

$$\mathcal{P}_{[p/q]}(\epsilon) ~=~ \frac{\sum_{m=0}^{p} a_m \epsilon^m}{1 + \sum_{n=1}^{q} b_n \epsilon^n} ~. \tag{2.9}$$

So at six loops there will be eight Padés to include in any analysis. For exponents where there is only one constraint from low dimension there will be $(L+1)$ possible approximants. If we construct approximants starting at two loops this means the total available number of approximants will be $\frac{1}{2}(L-1)(L+4)$ and $\frac{1}{2}(L-1)(L+6)$ when one and two constraints are employed respectively. When the leading term of the ϵ expansion is $O(\epsilon)$ these numbers reduce by unity at each loop order. Aside from this caveat this ought to improve uncertainty estimates, especially as L increases. This needs to be qualified though by recalling the continuity and monotonicity criteria which means the available number of satisfactory approximants will be less than either total. What the specific number is for any exponent can not be determined until explicit expressions for each approximant are constructed.

Therefore we have devised a sifting algorithm to isolate bona fide exponents. First, approximants which have a discontinuity anywhere in the range $2 \leq d \leq 6$ are disregarded even if the discontinuity is located below three dimensions. While the four and five dimensional estimates from such a Padé may be in the neighbourhood of values from other methods its lack of contiguity with the two dimensional boundary condition would bring doubt into the reliabilty of the two higher dimension estimates. The second criteria is to have monotonicity in the same range so that plots of $\mathcal{P}_{[p/q]}(\epsilon)$ should be free from turning or inflection points. This sift is based on the evidence of lower order behaviour and the assumption that the six loop refinement should not lead to major deviations in the overall structure. The final condition is not unconnected with the previous one in that when examining the valid plots of all the Padés for an exponent they have a similar appearance. Though we note that locating inflection points, which this later sift effectively relates to, is also based on solving for such points analytically. Any Padé approximant that passes through the sieve is retained for the statistical analysis.

Exp	d	μ_4	μ_5	μ_6	μ	$\mu_{\rm wt}$
$\hat{\eta}$	3	− 0.582343	− 0.571817	− 0.577538	− 0.578(6)	− 0.578(5)
	4	− 0.358010	− 0.348864	− 0.353143	− 0.355(6)	− 0.354(6)
	5	− 0.152577	− 0.150706	− 0.151357	− 0.152(1)	− 0.152(1)
$\hat{\sigma}$	3	0.076041	0.079097	0.074756	0.078(9)	0.078(8)
	4	0.259275	0.261320	0.258843	0.260(6)	0.260(5)
	5	0.398278	0.398613	0.398328	0.3984(10)	0.3985(8)
$\hat{\nu}_c$	3	0.358377	0.358895	0.357898	0.3585(13)	0.3584(12)
	4	0.314629	0.314999	0.314524	0.3147(7)	0.3147(6)
	5	0.279627	0.279692	0.279649	0.2796(2)	0.27965(11)
$\hat{\theta}$	3	———	0.594581	0.569373	0.596(20)	0.590(20)
	4	———	0.385131	0.368677	0.390(16)	0.384(16)
	5	———	0.196856	0.193744	0.200(6)	0.198(6)
$\hat{\omega}$	3	1.579682	1.634611	1.602104	1.616(1)	1.615(1)
	4	1.159805	1.208799	1.181551	1.1928(10)	1.1924(10)
	5	0.686603	0.701431	0.694545	0.6965(2)	0.6970(2)

Table 2: Exponent estimates for Lee-Yang theory using constrained Padé approximants containing the four, five and six loop averages, the mean and mean weighted by loop order for 3, 4 and 5 dimensions.

From the set of approximants extracted using the above criteria we will evaluate several measures of the exponents. The main ones are the mean μ and weighted mean $\mu_{\rm wt}$ defined by

$$\mu ~=~ \frac{\sum_m \mathcal{P}_m}{\sum_n 1} ~~,~~ \mu_{\rm wt} ~=~ \frac{\sum_m w_m \mathcal{P}_m}{\sum_n w_n} \tag{2.10}$$

where the indexing set on the summation symbols corresponds to the set of valid approximants for each exponent, represented by $\mathcal{P}_m$, and w_m are non-unit weights. Clearly taking the formal limit $w_m \to 1$ in $\mu_{\rm wt}$ produces μ which also clarifies the meaning of the denominator in the definition of μ. We have chosen the w_m to be the loop order that the $\mathcal{P}_m$ originated from based on the assumption that as the loop order increases the higher order corrections should lead to a more accurate estimate. By the same token we define the uncertainty via the parallel standard deviations given by

$$\varsigma ~=~ \sqrt{\frac{\sum_m (\mu - \mathcal{P}_m)^2}{\sum_n 1}} ~~,~~ \varsigma_{\rm wt} ~=~ \sqrt{\frac{\sum_m w_m (\mu_{\rm wt} - \mathcal{P}_m)^2}{\sum_n w_n}} \tag{2.11}$$

with the same aim that $\varsigma_{\rm wt}$ should produce a refined uncertainty in relation to ς.

Ref & Method	d	$\hat{\eta}$	$\hat{\sigma}$	$\hat{\nu}_c$	$\hat{\theta}$	$\hat{\omega}$
[26] OA	1	− 0.984(24)	− 0.498(3)	0.502(3)	1	1.992(12)
	2	− 0.77(5)	− 0.161(8)	0.420(4)	$\frac{5}{6}$	1.986(19)
	3	− 0.52(13)	0.0877(25)	0.363(8)	0.622(12)	1.72(8)
	4	− 0.325(7)	0.2648(15)	0.3162(4)	0.412(8)	1.30(3)
	5	− 0.13(3)	0.402(5)	0.2804(10)	0.205(5)	0.73(2)
[33] T	3	− 0.576(10)	0.076(2)	0.359(1)	———	———
	4	− 0.36(3)	0.258(5)	0.314(2)	———	———
	5	− 0.14(5)	0.401(9)	0.280(2)	———	———
[34] CB	3	− 0.574	0.076	0.358	———	———
	4	− 0.354	0.259	0.314	———	———
[35] CB	2	− 0.798(12)	− 0.1664(20)	0.4168(10)	———	———
	3	− 0.530(5)	0.085(1)	0.3617(4)	———	———
	4	− 0.3067(5)	0.2685(1)	0.3171(3)	———	———
	5	− 0.090(3)	0.4105(5)	0.2821(1)	———	———
[27] FRG	3	− 0.586(29)	0.0742(56)	0.3581(19)	———	———
	4	− 0.316(16)	0.2667(32)	0.3167(8)	———	———
	5	− 0.126(6)	0.4033(12)	0.2807(2)	———	———
[37] FRG	4	− 0.325(3)	0.2648(6)	0.3162(2)	———	———
	5	− 0.1344(1)	0.40166(2)	0.28033(1)	———	———
[36] CB	3	− 0.651	0.062	0.354	———	———
	4	− 0.353	0.259	0.315	———	———
	5	− 0.124	0.404	0.2801	———	———
[13] ϵ	3	− 0.580(7)	0.078(2)	0.359(1)	———	———
	4	− 0.356(6)	0.2604(14)	0.3151(3)	———	———
	5	− 0.1521(13)	0.3984(2)	0.2797(1)	———	———
[30] CB	3	− 0.564	0.078	0.359	———	———
	4	− 0.346	0.261	0.315	———	———
	5	− 0.140	0.399	0.280	———	———
This work	3	− 0.578(5)	0.078(8)	0.3584(12)	0.590(20)	1.615(1)
	4	− 0.354(6)	0.260(5)	0.3147(6)	0.384(16)	1.1924(10)
	5	− 0.152(1)	0.3985(8)	0.27965(11)	0.198(6)	0.6970(2)

Table 3: Summary of exponents in various dimensions from different methods derived from scaling laws using either $\hat{\sigma}$ or Δ_ϕ as input.

We have applied this process to all the exponents in (2.6) by first constructing the Padé approximants analytically. Using the numerical representation to select the relevant approximants we arrive at the results of Table 2. For each exponent estimates are recorded in three different dimensions. There are several columns of data. Those headed with μ_L for $L = 4$, 5 and 6 are the averages of the selected approximants at L loops only. These are provided as a guide or an indication of the progression of the inclusion of higher order contributions. They are not to be regarded as bona fide estimates. Returning to an earlier point that one can never be sure which approximants will be reliable it turned out that only seven passed the sift test for $\hat{\theta}$ which was the lowest number of all five exponents. In fact no $\hat{\theta}$ four loop ones did which is the reason there are no entries for that exponent. The final two columns contain the two means calculated from all Padés that survived the sifting process where the uncertainties are deduced from the respective ς and $\varsigma_{\rm wt}$. There are several general observations. First the uncertainties are invariably smaller as d increases which can be seen across all five exponents in $d = 5$ with $\hat{\theta}$ being an exception. The reason for this is that there were only seven Padés for $\hat{\theta}$ with the absence of four loop estimates affecting the uncertainty. In places the uncertainties from the weighted measures are tighter than those from the usual mean. This is partly because more approximants were available at five and six loops and their effect can be gauged in the trends apparent in μ_L.

In order to place our six loop exponent estimates in context we have compiled a summary of previous results in Table 3 which appear in chronological order. First we have recorded values for the available dimensions and note that only two articles provide estimates of the correction to scaling exponents $\hat{\theta}$ and $\hat{\omega}$. Aside from those from [26] three dimensional estimates have been extracted from studies on regularized spheres and in particular using a recent technique based on so-called fuzzy spheres, [30, 31, 32]. The respective exponents from [30, 31] are summarized in Table 4. For Table 3 other techniques were employed such as high temperature expansions, (T), [33], conformal bootstrap method, (CB), [30, 34, 35, 36], functional renormalization group, (FRG), [27, 37], ϵ expansions (ϵ), [13, 34, 35, 36], or other approaches (OA), [26, 30]. In most of these articles the exponent that was calculated was invariably either $\hat{\sigma}$ or Δ_ϕ since it was shown in [16] that there is only one independent exponent reflecting the renormalization properties of (2.1). In [13] the ϵ expansion coupled with the scaling laws produced expressions for each exponent prior to finding estimates. The values recorded from [13] in Table 3 are those computed from the five loop constrained Padé estimates in the same way that the results of Table 2 were arrived at. They are included for comparison with our new order rather than the overall constrained values of that paper. Whichever of $\hat{\sigma}$ or Δ_ϕ was determined in an article we derived the others using the scaling laws of (2.3) or (2.7) in order to compare with the constrained Padés. In this respect methods that used a fixed dimension approach to estimate exponents were either unable to provide values for dimensions one or two, to test whether they tallied with the exact values, or could make predictions which were not in good agreement, or the predictions for these low dimensions were indeed reliable. This was the case for [26] as noted in Table 3. We included those $d = 1$ and 2 values since they illustrate, for example, one side of the debate on whether the corrections to scaling are logarithmic or not.

Ref	Method	$\hat{\eta}$	$\hat{\sigma}$	$\hat{\nu}_c$	$\hat{\theta}$	$\hat{\omega}$
[30]	E	$-$ 0.572(4)	0.0768(8)	0.3589(3)	0.579(3)	1.613(6)
	Z	$-$ 0.5790(32)	0.0774(6)	0.3591(2)	———	———
	X	$-$ 0.5698(16)	0.0772(3)	0.3591(2)	———	———
[31]	———	$-$ 0.42	0.11	0.37	0.63	1.71

Table 4: Estimates of three dimensional exponents using the value of Δ_ϕ as input to the scaling laws from [30] where E, Z and X denote eigenvalues and the matrix element of fuzzy spheres Z and X respectively, as well as the estimates from [31].

While the majority of exponent estimates arise from more modern techniques, such as the functional renormalization group and conformal bootstrap which are continuum field theory based, it is worth discussing our estimates in comparison to those. In particular we concentrate on $\hat{\eta}$, $\hat{\sigma}$ and to a lesser extent on $\hat{\omega}$ as these tend to have been directly measured. What is interesting is that the three dimensional values of $\hat{\eta}$, aside from [36], are very much in accord. In four dimensions a similar picture is apparent but perhaps with much less overlap of uncertainties. In five dimensions the deviation of $\hat{\eta}$ of perturbation theory from the two continuum techniques is more distinct. This seems peculiar in that it might have been expected that the five dimensional perturbative results should be more accurate when summing down from the critical dimension. The much earlier estimates of [26, 33] for $\hat{\eta}$ do not lean towards either side due to the large uncertainties. The position with $\hat{\sigma}$ is not dissimilar unsurprisingly since most methods derive an estimate for $\hat{\eta}$, or Δ_ϕ in the case of the conformal bootstrap, and apply the scaling relations (2.7). What is reassuring in the comparison of our results with other approaches is the consistency of the two-sided Padé construction indicating that the resummation down from six dimensions can indeed capture low dimension properties. Finally the situation with our $\hat{\omega}$ values is not as conclusive merely because there appears to be only one earlier analysis across all three discrete dimensions, [26]. In that instance our three estimates do not overlap with those of [26]. However comparing our estimate for $\hat{\omega}$ with the recent use of the fuzzy sphere method in three dimensions, [30], there is good agreement. By way of a final comment on resummation, an alternative approach to potentially improve convergence of the high order ϵ series is to use the Padé-Borel method. This involves first replacing the unconstrained series itself by a Borel transform. Its integrand is related to the original series but with the coefficients of ϵ weighted by $1/L!$ where L is the loop order. The resulting series in the Borel variable is then replaced by a Padé approximant before evaluating the transform for the three values of ϵ used here. However in following this procedure it is not possible to constrain the integrand approximant by the low dimensional exponent values. Hence the bridge to low dimensions afforded by exact exponent values in one and two dimensions, and of great benefit to the direct Padé approach, cannot be accessed in the Padé-Borel case.

3 Percolation exponents.

This section is devoted to a similar analysis to that of the previous one but for percolation theory except that we will provide a constrained Padé analysis for a larger number of exponents. Like the Lee-Yang model the underlying continuum quantum field theory governing criticality is a cubic scalar theory with a critical dimension of six. More specifically it corresponds to the replica limit of the $(N+1)$-state Potts model [38]. The renormalization group functions for percolation theory were determined to five loops in [13, 22, 23, 24, 39, 40]. More recently their extension was given to six loops in the HYPERLOG package of [15]. Unlike the previous section *two* core renormalization group functions are the key to extracting the main scaling dimensions which are the field and mass operator anomalous dimensions $\gamma_\phi(g)$ and $\gamma_{\mathcal{O}}(g)$ respectively with $\mathcal{O} = \frac{1}{2}\phi^2$. The correspondence with the critical exponents is

$$\eta \;=\; \gamma_\phi(g^\star) \quad , \quad \frac{1}{\nu} \;=\; 2 \;-\; \eta \;+\; \gamma_{\mathcal{O}}(g^\star) \quad , \quad \omega \;=\; \beta'(g^\star) \tag{3.1}$$

while the β-function connects with the correction to scaling exponent ω. Using the results of [15] we find

$$\begin{aligned}\eta \;=\;& -\frac{1}{21}\epsilon - \frac{206}{9261}\epsilon^2 + \left[\frac{256}{7203}\zeta_3 - \frac{93619}{8168202}\right]\epsilon^3 \\ &+ \left[\frac{64}{2401}\zeta_4 + \frac{189376}{9529569}\zeta_3 - \frac{320}{3087}\zeta_5 - \frac{103309103}{14408708328}\right]\epsilon^4 \\ &+ \left[\frac{47344}{3176523}\zeta_4 + \frac{2337824}{9529569}\zeta_5 + \frac{77003747}{600362847}\zeta_3 - \frac{187744}{7411887}\zeta_3^2 - \frac{664}{16807}\zeta_7 - \frac{400}{3087}\zeta_6 \right. \\ &\left. - \frac{43137745921}{3630994498656}\right]\epsilon^5\end{aligned}$$

$$+ \left[\frac{77003747}{800483796}\zeta_4 - \frac{13252084726607}{2135024765209728} - \frac{350196946003}{1059040062108}\zeta_3 - \frac{325047556}{600362847}\zeta_5 - \frac{7328344}{466948881}\zeta_3^2 - \frac{2079575}{352947}\zeta_7 - \frac{1812032}{7411887}\zeta_3\zeta_5 - \frac{1159098}{4117715}\zeta_8 - \frac{93872}{2470629}\zeta_3\zeta_4 + \frac{37376}{352947}\zeta_3^3 + \frac{361824}{4117715}\zeta_{5,3} + \frac{2782048}{453789}\zeta_9 + \frac{2816440}{9529569}\zeta_6 \right] \epsilon^6 + O(\epsilon^7)$$

$$\frac{1}{\nu} = 2 - \frac{5}{21}\epsilon - \frac{653}{18522}\epsilon^2 + \left[\frac{356}{7203}\zeta_3 - \frac{332009}{32672808} \right] \epsilon^3 + \left[\frac{110219}{9529569}\zeta_3 - \frac{3760}{21609}\zeta_5 + \frac{89}{2401}\zeta_4 - \frac{59591131}{57634833312} \right] \epsilon^4 + \left[\frac{298060003}{2401451388}\zeta_3 - \frac{119568216869}{14523977994624} - \frac{134000}{7411887}\zeta_3^2 - \frac{4700}{21609}\zeta_6 + \frac{2952}{16807}\zeta_7 + \frac{110219}{12706092}\zeta_4 + \frac{3242404}{9529569}\zeta_5 \right] \epsilon^5 + \left[\frac{298060003}{3201935184}\zeta_4 - \frac{9315646748605}{8540099060838912} - \frac{2028254681339}{4236160248432}\zeta_3 - \frac{375720260}{600362847}\zeta_5 - \frac{17931317}{2117682}\zeta_7 - \frac{12592520}{466948881}\zeta_3^2 - \frac{3902656}{7411887}\zeta_3\zeta_5 - \frac{67000}{2470629}\zeta_3\zeta_4 + \frac{46976}{352947}\zeta_3^3 + \frac{224544}{4117715}\zeta_{5,3} + \frac{723022}{4117715}\zeta_8 + \frac{15623285}{38118276}\zeta_6 + \frac{24682792}{3176523}\zeta_9 \right] \epsilon^6 + O(\epsilon^7)$$

$$\omega = \epsilon - \frac{671}{882}\epsilon^2 + \left[\frac{372}{343}\zeta_3 + \frac{40639}{57624} \right] \epsilon^3 + \left[\frac{279}{343}\zeta_4 - \frac{348539}{151263}\zeta_3 - \frac{1360}{343}\zeta_5 - \frac{317288185}{304946208} \right] \epsilon^4 + \left[\frac{11664257531}{800483796}\zeta_3 + \frac{601352852897}{691617999744} - \frac{348539}{201684}\zeta_4 - \frac{55596}{2401}\zeta_7 - \frac{1700}{343}\zeta_6 + \frac{207440}{117649}\zeta_3^2 + \frac{17305178}{453789}\zeta_5 \right] \epsilon^5 + \left[\frac{43023079}{7411887}\zeta_3^2 - \frac{17253383458933}{201721916592}\zeta_3 - \frac{59563537247}{400241898}\zeta_5 + \frac{11664257531}{1067311728}\zeta_4 + \frac{33644508241033}{406671383849472} - \frac{1388669831}{2117682}\zeta_7 - \frac{13215057}{117649}\zeta_8 - \frac{10633120}{352947}\zeta_3\zeta_5 + \frac{46720}{2401}\zeta_3^3 + \frac{311160}{117649}\zeta_3\zeta_4 + \frac{3495600}{117649}\zeta_{5,3} + \frac{85910695}{1815156}\zeta_6 + \frac{122890160}{151263}\zeta_9 \right] \epsilon^6 + O(\epsilon^7) \quad (3.2)$$

to six loops. These three exponents then appear in the definition of scaling dimensions through the scaling laws

$$\alpha = 2 - d\nu \;,\; \beta = \frac{1}{2}[d-2+\eta]\nu \;,\; \gamma = [2-\eta]\nu \;,\; \delta = \frac{[d+2-\eta]}{[d-2+\eta]} \;,$$
$$\sigma = \frac{2}{[d+2-\eta]\nu} \;,\; \tau = 1 + \frac{2d}{[d+2-\eta]} \;,\; d_f = \frac{1}{2}[d+2-\eta] \;,\; \Omega = \frac{2\omega}{[d+2-\eta]} \quad (3.3)$$

where Ω is a second correction to scaling exponent. In Lee-Yang theory it equates to $\hat{\theta}$ while the fractal dimension, d_f, is equivalent to $1/\hat{\nu}_c$. We note that the fractal dimension was denoted by D in [41].

Using these scaling laws we deduce

$$\alpha = -1 + \frac{1}{7}\epsilon - \frac{443}{12348}\epsilon^2 + \left[\frac{178}{2401}\zeta_3 - \frac{370187}{21781872} \right] \epsilon^3$$

$$+\left[\frac{267}{4802}\zeta_4+\frac{143861}{6353046}\zeta_3-\frac{1880}{7203}\zeta_5-\frac{133741081}{38423222208}\right]\epsilon^4$$
$$+\left[\frac{143861}{8470728}\zeta_4+\frac{1561982}{3176523}\zeta_5+\frac{304564789}{1600967592}\zeta_3-\frac{67000}{2470629}\zeta_3^2-\frac{2350}{7203}\zeta_6+\frac{4428}{16807}\zeta_7 -\frac{127531100903}{9682651996416}\right]\epsilon^5$$
$$+\left[\frac{23488}{117649}\zeta_3^3-\frac{1985370060917}{2824106832288}\zeta_3-\frac{1760162509991}{632599930432512}-\frac{182554231}{200120949}\zeta_5-\frac{17904749}{1411788}\zeta_7 -\frac{6882916}{155649627}\zeta_3^2-\frac{1951328}{2470629}\zeta_3\zeta_5-\frac{33500}{823543}\zeta_3\zeta_4+\frac{336816}{4117715}\zeta_{5,3}+\frac{1084533}{4117715}\zeta_8 +\frac{12341396}{1058841}\zeta_9+\frac{15031085}{25412184}\zeta_6+\frac{304564789}{2134623456}\zeta_4\right]\epsilon^6 \;+\; O(\epsilon^7)$$

$$\beta = 1-\frac{1}{7}\epsilon-\frac{61}{12348}\epsilon^2-\left[\frac{19405}{21781872}+\frac{38}{2401}\zeta_3\right]\epsilon^3$$
$$+\left[\frac{440}{7203}\zeta_5+\frac{5281}{6353046}\zeta_3-\frac{57}{4802}\zeta_4-\frac{84318803}{38423222208}\right]\epsilon^4$$
$$+\left[\frac{7457829179}{9682651996416}-\frac{46787653}{1600967592}\zeta_3-\frac{13406}{117649}\zeta_5-\frac{1642}{16807}\zeta_7+\frac{550}{7203}\zeta_6+\frac{5281}{8470728}\zeta_4 +\frac{6688}{2470629}\zeta_3^2\right]\epsilon^5$$
$$+\left[\frac{457789037921}{2824106832288}\zeta_3-\frac{2938640361995}{1897799791297536}-\frac{46787653}{2134623456}\zeta_4-\frac{7472812}{3176523}\zeta_9-\frac{1302571}{8235430}\zeta_8 -\frac{129405}{941192}\zeta_6-\frac{21816}{4117715}\zeta_{5,3}-\frac{14144}{352947}\zeta_3^3+\frac{3344}{823543}\zeta_3\zeta_4+\frac{166480}{823543}\zeta_3\zeta_5+\frac{417731}{151263}\zeta_7 +\frac{1398094}{155649627}\zeta_3^2+\frac{12603706}{66706983}\zeta_5\right]\epsilon^6 \;+\; O(\epsilon^7)$$

$$\delta = 2+\frac{2}{7}\epsilon+\frac{565}{6174}\epsilon^2+\left[\frac{371953}{10890936}-\frac{64}{2401}\zeta_3\right]\epsilon^3$$
$$+\left[\frac{300656141}{19211611104}-\frac{77584}{3176523}\zeta_3-\frac{48}{2401}\zeta_4+\frac{80}{1029}\zeta_5\right]\epsilon^4$$
$$+\left[\frac{67649411155}{4841325998208}-\frac{84030083}{800483796}\zeta_3-\frac{496256}{3176523}\zeta_5-\frac{19396}{1058841}\zeta_4+\frac{100}{1029}\zeta_6+\frac{498}{16807}\zeta_7 +\frac{46936}{2470629}\zeta_3^2\right]\epsilon^5$$
$$+\left[\frac{297068306929}{1412053416144}\zeta_3+\frac{9026360188351}{948899895648768}-\frac{84030083}{1067311728}\zeta_4-\frac{695512}{151263}\zeta_9-\frac{593860}{3176523}\zeta_6 -\frac{271368}{4117715}\zeta_{5,3}-\frac{9344}{117649}\zeta_3^3+\frac{23468}{823543}\zeta_3\zeta_4+\frac{453008}{2470629}\zeta_3\zeta_5+\frac{1738647}{8235430}\zeta_8+\frac{2084555}{470596}\zeta_7 +\frac{2925010}{155649627}\zeta_3^2+\frac{70128049}{200120949}\zeta_5\right]\epsilon^6 \;+\; O(\epsilon^7)$$

$$d_f = 4-\frac{10}{21}\epsilon+\frac{103}{9261}\epsilon^2+\left[\frac{93619}{16336404}-\frac{128}{7203}\zeta_3\right]\epsilon^3$$
$$+\left[\frac{103309103}{28817416656}-\frac{94688}{9529569}\zeta_3-\frac{32}{2401}\zeta_4+\frac{160}{3087}\zeta_5\right]\epsilon^4$$
$$+\left[\frac{43137745921}{7261988997312}-\frac{77003747}{1200725694}\zeta_3-\frac{1168912}{9529569}\zeta_5-\frac{23672}{3176523}\zeta_4+\frac{200}{3087}\zeta_6+\frac{332}{16807}\zeta_7\right.$$

$$+ \frac{93872}{7411887}\zeta_3^2 \Big] \epsilon^5$$

$$+ \Big[\frac{350196946003}{2118080124216}\zeta_3 + \frac{13252084726607}{4270049530419456} - \frac{77003747}{1600967592}\zeta_4 - \frac{1408220}{9529569}\zeta_6 - \frac{1391024}{453789}\zeta_9$$

$$- \frac{180912}{4117715}\zeta_{5,3} - \frac{18688}{352947}\zeta_3^3 + \frac{46936}{2470629}\zeta_3\zeta_4 + \frac{579549}{4117715}\zeta_8 + \frac{906016}{7411887}\zeta_3\zeta_5 + \frac{2079575}{705894}\zeta_7$$

$$+ \frac{3664172}{466948881}\zeta_3^2 + \frac{162523778}{600362847}\zeta_5 \Big] \epsilon^6 \ + \ O(\epsilon^7)$$

$$\gamma = 1 + \frac{1}{7}\epsilon + \frac{565}{12348}\epsilon^2 + \Big[\frac{408997}{21781872} - \frac{102}{2401}\zeta_3 \Big] \epsilon^3$$

$$+ \Big[\frac{302378687}{38423222208} - \frac{154423}{6353046}\zeta_3 - \frac{153}{4802}\zeta_4 + \frac{1000}{7203}\zeta_5 \Big] \epsilon^4$$

$$+ \Big[\frac{112615442545}{9682651996416} - \frac{210989483}{1600967592}\zeta_3 - \frac{838058}{3176523}\zeta_5 - \frac{154423}{8470728}\zeta_4 - \frac{1144}{16807}\zeta_7 + \frac{1250}{7203}\zeta_6$$

$$+ \frac{53624}{2470629}\zeta_3^2 \Big] \epsilon^5$$

$$+ \Big[\frac{1069791985075}{2824106832288}\zeta_3 + \frac{11157768253963}{1897799791297536} - \frac{210989483}{2134623456}\zeta_4 - \frac{22078564}{3176523}\zeta_9 - \frac{8043215}{25412184}\zeta_6$$

$$- \frac{293184}{4117715}\zeta_{5,3} - \frac{42176}{352947}\zeta_3^3 + \frac{26812}{823543}\zeta_3\zeta_4 + \frac{136064}{352947}\zeta_3\zeta_5 + \frac{218038}{4117715}\zeta_8 + \frac{4086728}{155649627}\zeta_3^2$$

$$+ \frac{30321311}{4235364}\zeta_7 + \frac{106931995}{200120949}\zeta_5 \Big] \epsilon^6 \ + \ O(\epsilon^7)$$

$$\nu = \frac{1}{2} + \frac{5}{84}\epsilon + \frac{589}{37044}\epsilon^2 + \Big[\frac{716519}{130691232} - \frac{89}{7203}\zeta_3 \Big] \epsilon^3$$

$$+ \Big[\frac{344397667}{230539333248} - \frac{222359}{38118276}\zeta_3 - \frac{89}{9604}\zeta_4 + \frac{940}{21609}\zeta_5 \Big] \epsilon^4$$

$$+ \Big[\frac{33500}{7411887}\zeta_3^2 + \frac{141995802917}{58095911978496} - \frac{313903867}{9605805552}\zeta_3 - \frac{711901}{9529569}\zeta_5 - \frac{222359}{50824368}\zeta_4 - \frac{738}{16807}\zeta_7$$

$$+ \frac{1175}{21609}\zeta_6 \Big] \epsilon^5$$

$$+ \Big[\frac{1893082324019}{16944640993728}\zeta_3 + \frac{29757051275785}{34160396243355648} - \frac{313903867}{12807740736}\zeta_4 - \frac{13649285}{152473104}\zeta_6$$

$$- \frac{6170698}{3176523}\zeta_9 - \frac{361511}{8235430}\zeta_8 - \frac{56136}{4117715}\zeta_{5,3} - \frac{11744}{352947}\zeta_3^3 + \frac{16750}{2470629}\zeta_3\zeta_4 + \frac{975664}{7411887}\zeta_3\zeta_5$$

$$+ \frac{3793208}{466948881}\zeta_3^2 + \frac{17842757}{8470728}\zeta_7 + \frac{83802155}{600362847}\zeta_5 \Big] \epsilon^6 \ + \ O(\epsilon^7)$$

$$\sigma = \frac{1}{2} - \frac{1}{98}\epsilon^2 + \Big[\frac{5}{343}\zeta_3 - \frac{773}{172872} \Big] \epsilon^3 + \Big[\frac{3551}{605052}\zeta_3 + \frac{15}{1372}\zeta_4 - \frac{120}{2401}\zeta_5 - \frac{246103}{203297472} \Big] \epsilon^4$$

$$+ \Big[\frac{1763659}{44471322}\zeta_3 - \frac{149476871}{51230962944} - \frac{718}{117649}\zeta_3^2 - \frac{150}{2401}\zeta_6 + \frac{199}{4802}\zeta_7 + \frac{3551}{806736}\zeta_4$$

$$+ \frac{300005}{3176523}\zeta_5 \Big] \epsilon^5$$

$$+ \Big[\frac{7387844}{3176523}\zeta_9 - \frac{18252766321}{134481277728}\zeta_3 - \frac{4189853447}{4236160248432} - \frac{143110153}{800483796}\zeta_5 - \frac{42017779}{16941456}\zeta_7$$

$$- \frac{362972}{2470629}\zeta_3\zeta_5 - \frac{124291}{14823774}\zeta_3^2 - \frac{1077}{117649}\zeta_3\zeta_4 + \frac{2250}{117649}\zeta_{5,3} + \frac{14080}{352947}\zeta_3^3 + \frac{24757}{941192}\zeta_8$$

$$+ \frac{1763659}{59295096}\zeta_4 + \frac{5768575}{50824368}\zeta_6 \Big] \epsilon^6 \; + \; O(\epsilon^7)$$

$$\tau \;=\; \frac{5}{2} - \frac{1}{14}\epsilon - \frac{313}{24696}\epsilon^2 + \left[\frac{16}{2401}\zeta_3 - \frac{150697}{43563744} \right] \epsilon^3$$
$$+ \left[\frac{13348}{3176523}\zeta_3 + \frac{12}{2401}\zeta_4 - \frac{20}{1029}\zeta_5 - \frac{124383401}{76846444416} \right] \epsilon^4$$
$$+ \Big[\frac{77797763}{3201935184}\zeta_3 - \frac{45091644259}{19365303992832} - \frac{11734}{2470629}\zeta_3^2 - \frac{249}{33614}\zeta_7 - \frac{25}{1029}\zeta_6 + \frac{3337}{1058841}\zeta_4$$
$$+ \frac{141704}{3176523}\zeta_5 \Big] \epsilon^5$$
$$+ \Big[\frac{173878}{151263}\zeta_9 - \frac{1662624619001}{1265199860865024} - \frac{340712273113}{5648213664576}\zeta_3 - \frac{78589417}{800483796}\zeta_5 - \frac{2080571}{1882384}\zeta_7$$
$$- \frac{1738647}{32941720}\zeta_8 - \frac{1012433}{311299254}\zeta_3^2 - \frac{113252}{2470629}\zeta_3\zeta_5 - \frac{5867}{823543}\zeta_3\zeta_4 + \frac{2336}{117649}\zeta_3^3$$
$$+ \frac{67842}{4117715}\zeta_{5,3} + \frac{170515}{3176523}\zeta_6 + \frac{77797763}{4269246912}\zeta_4 \Big] \epsilon^6 \; + \; O(\epsilon^7)$$

$$\Omega \;=\; \frac{1}{4}\epsilon - \frac{283}{1764}\epsilon^2 + \left[\frac{93}{343}\zeta_3 + \frac{324689}{2074464} \right] \epsilon^3 + \left[\frac{279}{1372}\zeta_4 - \frac{328337}{605052}\zeta_3 - \frac{340}{343}\zeta_5 - \frac{294452729}{1219784832} \right] \epsilon^4$$
$$+ \Big[\frac{11454697649}{3201935184}\zeta_3 + \frac{520662674885}{2766471998976} - \frac{328337}{806736}\zeta_4 - \frac{13899}{2401}\zeta_7 - \frac{425}{343}\zeta_6 + \frac{51860}{117649}\zeta_3^2$$
$$+ \frac{8542549}{907578}\zeta_5 \Big] \epsilon^5$$
$$+ \Big[\frac{84810295}{7260624}\zeta_6 - \frac{16905364832761}{806887666368}\zeta_3 - \frac{57749587781}{1600967592}\zeta_5 + \frac{11454697649}{4269246912}\zeta_4$$
$$+ \frac{70447924854521}{1626685535397888} - \frac{1394517869}{8470728}\zeta_7 - \frac{13215057}{470596}\zeta_8 - \frac{2658280}{352947}\zeta_3\zeta_5 + \frac{11680}{2401}\zeta_3^3$$
$$+ \frac{77790}{117649}\zeta_3\zeta_4 + \frac{873900}{117649}\zeta_{5,3} + \frac{30722540}{151263}\zeta_9 + \frac{44591123}{29647548}\zeta_3^2 \Big] \epsilon^6 \; + \; O(\epsilon^7) \qquad (3.4)$$

for the remaining exponents. We have included the ϵ expansion of ν here since we constucted approximants for both it and $1/\nu$ following the approach of [13, 24]. Numerically evaluating the expressions gives

$$\alpha \;=\; -\,1.000000 + 0.142857\epsilon - 0.035876\epsilon^2 + 0.072120\epsilon^3 - 0.186722\epsilon^4 + 0.638338\epsilon^5$$
$$-\,2.633771\epsilon^6 \; + \; O(\epsilon^7)$$
$$\beta \;=\; 1.000000 - 0.142857\epsilon - 0.004940\epsilon^2 - 0.019915\epsilon^3 + 0.049299\epsilon^4 - 0.168762\epsilon^5$$
$$+\,0.694682\epsilon^6 \; + \; O(\epsilon^7)$$
$$\delta \;=\; 2.000000 + 0.285714\epsilon + 0.091513\epsilon^2 + 0.002111\epsilon^3 + 0.045269\epsilon^4 - 0.137837\epsilon^5$$
$$+\,0.574005\epsilon^6 \; + \; O(\epsilon^7)$$
$$d_f \;=\; 4.000000 - 0.476190\epsilon + 0.011122\epsilon^2 - 0.015630\epsilon^3 + 0.030960\epsilon^4 - 0.102275\epsilon^5$$
$$+\,0.415376\epsilon^6 \; + \; O(\epsilon^7)$$
$$\eta \;=\; -\,0.047619\epsilon - 0.022244\epsilon^2 + 0.031261\epsilon^3 - 0.061921\epsilon^4 + 0.204551\epsilon^5$$
$$-\,0.830752\epsilon^6 \; + \; O(\epsilon^7)$$
$$\gamma \;=\; 1.000000 + 0.142857\epsilon + 0.045756\epsilon^2 - 0.032289\epsilon^3 + 0.088124\epsilon^4 - 0.300814\epsilon^5$$
$$+\,1.244406\epsilon^6 \; + \; O(\epsilon^7)$$
$$\nu \;=\; 0.500000 + 0.059524\epsilon + 0.015900\epsilon^2 - 0.009370\epsilon^3 + 0.029559\epsilon^4 - 0.101463\epsilon^5$$

$$
\begin{aligned}
& + 0.422051\epsilon^6 \; + \; O(\epsilon^7) \\
\frac{1}{\nu} \;&=\; 2.000000 - 0.238095\epsilon - 0.035255\epsilon^2 + 0.049249\epsilon^3 - 0.127439\epsilon^4 + 0.432873\epsilon^5 \\
& - 1.780994\epsilon^6 \; + \; O(\epsilon^7) \\
\sigma \;&=\; 0.500000 - 0.010204\epsilon^2 + 0.013051\epsilon^3 - 0.034148\epsilon^4 + 0.116861\epsilon^5 \\
& - 0.483033\epsilon^6 \; + \; O(\epsilon^7) \\
\tau \;&=\; 2.500000 - 0.071429\epsilon - 0.012674\epsilon^2 + 0.004551\epsilon^3 - 0.011312\epsilon^4 + 0.037497\epsilon^5 \\
& - 0.152981\epsilon^6 \; + \; O(\epsilon^7) \\
\omega \;&=\; \epsilon - 0.760771\epsilon^2 + 2.008933\epsilon^3 - 7.041302\epsilon^4 + 30.214779\epsilon^5 \\
& - 147.820814\epsilon^6 \; + \; O(\epsilon^7) \\
\Omega \;&=\; 0.250000\epsilon - 0.160431\epsilon^2 + 0.482439\epsilon^3 - 1.701469\epsilon^4 + 7.347235\epsilon^5 \\
& - 36.066283\epsilon^6 \; + \; O(\epsilon^7)
\end{aligned}
\tag{3.5}
$$

where we note that α, d_f, σ, ω and Ω are alternating series to six loops. Equally the general trend in the magnitude of the coefficient of the new order is similar to the Lee-Yang model since it is invariably the case that it is an order of magnitude larger than the five loop one. However in the analysis of the previous section it was apparent that despite this the constrained Padé approximants could produce exponent estimates that were in close proximity to values from other methods. Therefore the expectation is that a parallel analysis will produce percolation exponents that are commensurate with other techniques. We have followed the same process of constructing the constrained approximants for the ϵ expansion of each exponent with one difference. That is that only one constraint is available for a low dimension with the two dimensional critical exponents already determined from a minimal conformal field theory with zero central charge in [25]. We have recorded these for reference in Table 5. In previous work, [24], a different two dimensional value of ω was used to set up the constrained Padé which was 2 and not $\frac{3}{2}$ which is employed here. This is based on the arguments given in [41].

d	α	β	δ	d_f	η	γ	ν	σ	τ	ω	Ω
2	$-\frac{2}{3}$	$\frac{5}{36}$	$\frac{91}{5}$	$\frac{91}{48}$	$\frac{5}{24}$	$\frac{43}{18}$	$\frac{4}{3}$	$\frac{36}{91}$	$\frac{187}{91}$	$\frac{3}{2}$	$\frac{72}{91}$
6	-1	1	2	4	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{2}$	0	0

Table 5: Values of percolation exponents in two and six dimensions.

Recalling that to six loops there is a total of 25 potential approximants, we have constructed the two-sided Padé approximants with one constraint for each exponent. Of the 25 we have removed those d-dimensional expressions which fail the same the sifting test used for Lee-Yang theory. In the case of η no approximants survived. For α and Ω only nine Padés could be used while for ω it was eight which was the smallest number for this set of exponents. Indeed this aspect of the analysis is reflected in Table 6 which records our results in a parallel way to Table 2. For instance for these three exponents there was no five loop approximant for α and only one for each of ω and Ω which could be used for the means of the final two columns of Table 6. For α there was a degree of stability indicated by the individual four and six loop averages and reflected in the uncertainties. For the two correction to scaling exponents Ω has a similar stability over the three higher loop means. For ω this is not the case with the individual loop averages increasing at each loop order. For the remaining exponents between 12 and 18 Padé approximants survived the sieving process with δ, σ and τ being at the upper end.

Exp	d	μ_4	μ_5	μ_6	μ	$\mu_{\rm wt}$
α	3	− 0.701254	———	− 0.690435	− 0.71(2)	− 0.704(14)
	4	− 0.772721	———	− 0.763521	− 0.78(1)	− 0.775(12)
	5	− 0.873788	———	− 0.871315	− 0.877(5)	− 0.875(4)
β	3	0.428833	0.414161	0.424477	0.43(2)	0.425(15)
	4	0.659279	0.651654	0.656490	0.66(1)	0.657(8)
	5	0.845736	0.844668	0.845156	0.846(2)	0.845(2)
δ	3	5.207390	5.073557	5.187514	5.1(2)	5.1(2)
	4	3.185867	3.159635	3.180578	3.17(4)	3.17(3)
	5	2.396300	2.394183	2.395631	2.395(4)	2.395(3)
$\delta^\dagger$	3	5.207142	5.072673	5.187413	5.16(10)	5.15(9)
	4	3.185852	3.159604	3.180573	3.18(3)	3.17(2)
	5	2.396300	2.394183	2.395631	2.396(3)	2.395(2)
d_f	3	2.512775	2.500241	2.525863	2.51(2)	2.51(2)
	4	3.042408	3.036236	3.046923	3.040(9)	3.041(8)
	5	3.527550	3.526712	3.527823	3.527(2)	3.527(1)
γ	3	1.845780	1.771066	1.827194	1.81(5)	1.81(4)
	4	1.453871	1.422552	1.443809	1.44(2)	1.437(18)
	5	1.182134	1.178250	1.180328	1.180(4)	1.180(3)
ν	3	0.909753	0.886821	0.899178	0.90(2)	0.90(2)
	4	0.696981	0.687393	0.691739	0.694(9)	0.693(8)
	5	0.575161	0.573989	0.574299	0.575(1)	0.575(1)
$\nu^\dagger$	3	0.896445	0.878554	0.887482	0.89(1)	0.889(15)
	4	0.691889	0.686013	0.688240	0.690(5)	0.689(5)
	5	0.574628	0.574023	0.574080	0.5745(9)	0.5743(7)
σ	3	0.441521	0.456470	0.445084	0.444(9)	0.446(8)
	4	0.474008	0.482269	0.476428	0.475(6)	0.476(5)
	5	0.493211	0.494452	0.493727	0.4931(13)	0.4934(11)
τ	3	2.192358	2.195820	2.192879	2.195(5)	2.194(4)
	4	2.314033	2.316064	2.314442	2.315(3)	2.315(2)
	5	2.417326	2.417649	2.417428	2.4174(7)	2.4175(5)
ω	3	1.362005	1.451995*	1.521970	1.41(11)	1.44(10)
	4	1.115530	1.195616*	1.248740	1.02(16)	1.18(9)
	5	0.702260	0.727000*	0.740577	0.63(9)	0.72(3)
Ω	3	0.597626	0.596757*	0.607541	0.601(6)	0.602(7)
	4	0.403609	0.402615*	0.410698	0.406(4)	0.407(5)
	5	0.208204	0.207715*	0.209611	0.2088(11)	0.2089(11)

Table 6: Exponent estimates for percolation using constrained Padé approximants containing the four, five and six loop averages, the mean and mean weighted by loop order for 3, 4 and 5 dimensions. The estimates for $\delta^\dagger$ and $\nu^\dagger$ were produced from approximants for $1/\delta$ and $1/\nu$ respectively. Entries marked with * indicate only one valid approximant was available.

Exp	d	μ_4	μ_5	μ_6	μ	$\mu_{\rm wt}$
	3	5.207390	5.073557	5.187514	5.15(14)	5.14(12)
δ	4	3.185867	3.159635	3.180578	3.17(4)	3.17(3)
	5	2.396300	2.394183	2.395631	2.396(3)	2.395(3)
	3	0.904049	0.883522	0.894166	0.90(2)	0.895(20)
ν	4	0.694799	0.686835	0.690240	0.693(7)	0.691(8)
	5	0.574932	0.574004	0.574205	0.575(1)	0.5745(10)

Table 7: Estimates for δ and ν from combining data from all available approximants for each exponent.

For the former there is a marked difference in the estimates across the three dimensions in that the central value is less reliable as a guide to behaviour as d decreases. This lies in the fact that the difference in the endpoint boundaries at two and six dimensions, respectively 18.5 and 2, is by far the largest range of the exponent set as can be seen in Table 5. So it would be difficult to produce the level of uncertainty of the five dimensional estimate in the three dimensions. To try to address this we followed the same strategy used in [13] which was to construct two-sided Padés for $1/\delta$ in d-dimensions and deduce estimates for δ from the set of valid approximants. In this approach the magnitude of $1/\delta$ only varies by around 0.2 between the boundary dimensions in contrast to 16.5 for δ. The associated results for this are recorded against the entry denoted as $\delta^\dagger$ in Table 6. While the respective central values are similar to the direct approach for δ the uncertainties are smaller. The situation with σ and τ is cleaner in that with the endpoint differences being two orders of magnitude less than that of δ the central values are settled judging by the trend of the four, five and six loop means. This results in significantly smaller uncertainties in a dimensional comparison with δ with possibly a more accurate three dimensional estimate.

Law	$d = 3$	$d = 4$	$d = 5$
δ	− 0.02(3)	− 0.08(1)	− 0.054(3)
d_f	− 0.02(4)	− 0.082(16)	− 0.054(2)
τ	− 0.025(17)	− 0.084(10)	− 0.0547(25)

Table 8: Estimates for η derived from the scaling law of the indicated exponent.

The estimates for β, d_f and γ, compiled from 13 or 14 acceptable approximants, have similar general properties in that the weighted means have tighter uncertainties reflecting the trend of the individual loop means. In order to provide balance on the use of scaling laws to extract individual exponent estimates, for example, we note that in three dimensions $d_f \Omega = 1.51(3)$. Within uncertainties this is consistent with the bound valid for all dimensions of $d_f \Omega \leq \frac{3}{2}$ given in [41]. For four and five dimensions the bound is comfortably satisfied. Our Padé estimates for ω also satisfactorily obey the same bound and within the uncertainty in three dimensions. Finally for ν we followed the same strategy as for δ here and in [13] and constructed two-sided Padés for $1/\nu$. Estimates for ν are provided in Table 6 with those derived from $1/\nu$ marked as $\nu^\dagger$. For both means the uncertainty on ν from using the series for $1/\nu$ was smaller. Given that we have around several dozen approximants for each of δ and ν we have combined the data for both and extracted additional estimates for these exponents. In other words we evaluate the accepted approximants for $1/\delta$ and $1/\nu$ in each of the three dimensions, compute their reciprocals before feeding these into the procedures used to compute μ and $\mu_{\rm wt}$. The results of this exercise are provided in Table

7. For the case of δ the combination is similar to that for $1/\delta$ in terms of the central value and uncertainty. A similar observation is applicable for ν.

Exp	Ref & Year	$d = 3$	$d = 4$	$d = 5$
α	[13] 2021	− 0.64(4)	− 0.75(2)	− 0.870(1)
	This work	**− 0.704(14)**	**− 0.775(12)**	**− 0.875(4)**
β	[42] 1976	0.39(2)	0.52(3)	0.66(5)
	[43] 1976	0.41(1)	———	———
	[44] 1990	0.405(25)	———	———
	[45] 1990	———	0.639(20)	0.835(5)
	[46] 2014	0.4180(10)	———	———
	[24] 2015	0.4273	0.6590	0.8457
	[47] 2021	0.4053(5)	———	———
	[13] 2021	0.429(4)	0.658(1)	0.8454(2)
	This work	**0.425(15)**	**0.657(8)**	**0.845(2)**
δ	[48] 1980	5.3	3.9	3.0
	[49] 1997	———	3.198(6)	———
	[50] 1998	5.29(6)	———	———
	[13] 2021	5.16(4)	3.175(8)	2.3952(12)
	This work	**5.14(12)**	**3.17(3)**	**2.395(3)**

Table 9: Summary of estimates for α, β and δ in various dimensions.

As no η approximant passed the sieving process we have explored an alternative to estimate this exponent. Examining the scaling laws of (3.3) it is apparent that δ, d_f and τ solely depend on η. Therefore we have used their respective estimates for μ_{wt} in Table 6 to extract a measure of η. The results are provided in Table 8 where the scaling law that was used is indicated in the first column. The three dimensional values have large relative uncertainties with those associated from the τ scaling law ensuring that the estimate is fully negative. For the other two dimensions the uncertainty gives a better refinement with those in five dimensions appearing to be the most reliable. Part of the reason why the direct η approximants are unreliable is that while the boundary values from six to two dimensions go from 0 to $\frac{5}{24}$ the leading term of η is negative for $\epsilon \neq 0$. So as one approaches the lower dimensions any approximant would have to have a sharp change of slope to reach a relatively large value in two dimensions. This is not something which a rational polynomial is guaranteed to accommodate. Of the results using the three scaling laws it would appear that those from τ are the most reliable partly as the uncertainty of the three dimensional estimate ensures it is negative in three dimensions.

Having discussed our analysis in detail we now place our exponent estimates in context by examining them in relation to results from other methods. Therefore we have constructed Tables 9, 10, 11, 12 and 13. For each exponent these contain estimates in the three dimensions of interest in chronological order so that the progress in refining central values and uncertainties over half a century can be appreciated. It is not the case that the number of direct evaluations is at the same level for all the exponents nor indeed for each of the three different dimensions. So central values for one exponent may not have reached the level of commensurate accuracy as other exponents. In addition for each set of exponents we conclude with a boldface entry which are our final estimates from this study. Of great benefit in compiling previous results given in these tables was the excellent source of [51] which contains a live record of percolation exponents. We note that in Table 9 the estimate for β associated with [46] was derived from the scaling law using estimates of d_f and $1/\nu$

in Tables 10 and 11. The conformal bootstrap estimate for ν from [52] in Table 11 was deduced from a scaling law for the coupling of the energy operator in the underlying conformal field theory in that article. Also we included the constrained Padé value for ω from [13] in Table 13 in order to compare with the present results.

Exp	Ref & Year	$d = 3$	$d = 4$	$d = 5$
d_f	[53] 1985	———	3.12(2)	3.69(2)
	[49] 1997	———	3.0472(14)	———
	[50] 1998	2.523(4)	———	———
	[54] 1998	2.530(4)	———	———
	[55] 2000	2.5230(1)	———	———
	[56] 2001	———	3.046(7)	———
	[57] 2001	———	3.046(5)	———
	[58] 2005	2.5226(1)	———	———
	[46] 2014	2.52293(10)	———	———
	[24] 2015	———	3.0479	3.528
	[59] 2018	———	3.0437(11)	3.524(2)
	[52] 2018	———	3.003	———
	[60] 2021	———	3.0446(7)	3.5260(14)
	This work	**2.51(2)**	**3.041(8)**	**3.527(1)**
η	[45] 1990	– 0.07(5)	– 0.12(4)	– 0.075(20)
	[49] 1997	———	– 0.0944(28)	———
	[50] 1998	– 0.046(8)	———	———
	[54] 1998	– 0.059(9)	———	———
	[57] 2001	———	– 0.0929(9)	———
	[24] 2015	– 0.0470	– 0.0954	– 0.0565
	[13] 2021	– 0.03(1)	– 0.084(4)	– 0.0547(10)
	This work	– **0.025(17)**	– **0.084(10)**	– **0.0547(25)**

Table 10: Summary of estimates for d_f and η in various dimensions.

Taking a general overview of all the tables several themes are apparent. First there is a general trend over time of more precise values for exponents which can be associated with the improvement in computing technology. The evidence for this is the more accurate central values and tighter uncertainties. This is particularly the case across each dimension for the exponents where there has been more focus such as d_f, ν and τ. Aside from some of the earlier years for these three examples there appears to be good agreement within uncertainties to two and sometimes three decimal places. In particular for ν and τ several different techniques were used to arrive at the recorded values. For the most part the Monte Carlo or high temperature results dominate the tables with a few estimates from conformal bootstrap or the functional renormalization group approach in addition to the present perturbative analysis. In this respect the results from the six loop constrained Padé approximants are in close accord with previous four and five loop estimates. Where there is a subtle discrepancy in say the three dimensional exponents from the Padé this might be due to using the two dimensional conformal field theory values for the two-sided approach. Any fixed dimension computation will not have the freedom to bridge between discrete fixed spacetime dimensions. For other exponents clearly only five and six loop results are available for α but the slight discrepancy of perturbative estimates in three dimensions from results in the last thirty or so years is apparent in β and δ for instance. For the remaining non-correction to scaling exponents, aside from η the three, four and five dimensional estimates from this six loop exercise, and lower loop ones, are comfortably within uncertainties from other techniques which is a reassuring observation indicative that the perturbative approach is independently competitive as a technique.

Exp	Ref & Year	$d = 3$	$d = 4$	$d = 5$
γ	[42] 1976	1.80(5)	1.6(1)	1.3(1)
	[43] 1976	1.6	———	———
	[61] 1978	1.66(7)	1.48(8)	1.18(7)
	[45] 1990	1.805(20)	1.435(15)	1.185(5)
	[24] 2015	1.8357	1.4500	1.1817
	[47] 2021	1.819(3)	———	———
	[13] 2021	1.78(3)	1.430(6)	1.1792(7)
	This work	**1.81(4)**	**1.437(18)**	**1.180(3)**
ν	[62] 1976	0.80(5)	———	———
	[43] 1976	0.8(1)	———	———
	[53] 1985	———	———	0.51(5)
	[45] 1990	0.872(7)	0.6782(50)	0.571(3)
	[49] 1997	———	0.689(10)	———
	[50] 1998	0.875(1)	———	———
	[55] 2000	0.8765(18)	———	———
	[63] 2005	———	———	0.569(5)
	[64] 2013	0.8764(12)	———	———
	[65] 2014	0.8751(11)	———	———
	[46] 2014	0.8762(12)	———	———
	[24] 2015	0.8960	0.6920	0.5746
	[66] 2016	0.8774(13)	0.6852(28)	0.5723(18)
	[52] 2018	———	0.693	———
	[67] 2020	———	0.6845(6)	0.5757(7)
	[13] 2021	0.88(2)	0.686(2)	0.5739(1)
	[60] 2021	———	0.6845(23)	0.5737(33)
	[68] 2022	0.8762(7)	0.6842(16)	0.5720(43)
	This work	**0.895(20)**	**0.691(8)**	**0.5745(10)**

Table 11: Summary of estimates for γ and ν in various dimensions.

The situation with our estimates of η needs separate comment from the other exponents. It is evident from Table 10 that the four, five and six loop estimates are in the same ballpark. As noted earlier we used scaling laws to derive the present values from other exponents which was the method used at lower orders. So there the agreement is no surprise. Instead there appears to be no overlap with results from other techniques. For instance the central values from [49] and [57] are just about reached by the lower end of the uncertainty band of the result from this work. No parallel comment can be applied in the five dimensional case as only one value is available from other methods. The situation in three dimensions closely resembles that of four dimensions. Whether that could be explained by the use of the two dimensional boundary condition cannot be ascertained for what is always a difficult exponent to measure accurately given its proximity to zero. As a minor observation it is worth noting that all the earlier three dimensional estimates for η in Table 10 are negative within uncertainties which in one sense justifies our use of the τ scaling law for our η estimate.

Exp	Ref & Year	$d=3$	$d=4$	$d=5$
σ	[69] 1976	0.42(6)	———	———
	[49] 1997	0.4522(8)	———	———
	[60] 1998	0.445(10)	———	———
	[46] 2014	0.4524(6)	———	———
	[24] 2015	0.4419	0.4742	0.4933
	[13] 2021	0.452(7)	0.4789(14)	0.49396(13)
	This work	**0.446(8)**	**0.476(5)**	**0.4934(11)**
τ	[48] 1980	———	2.26	2.33
	[49] 1997	2.18906(8)	2.3127(6)	———
	[50] 1998	2.189(2)	———	———
	[54] 1998	2.186(2)	———	———
	[56] 2001	———	2.313(3)	2.412(4)
	[57] 2001	2.190(2)	2.313(2)	———
	[70] 2006	2.189(1)	———	———
	[46] 2014	2.18909(5)	———	———
	[24] 2015	2.1888	2.3124	2.4171
	[59] 2018	2.1892(1)	2.3142(5)	2.419(1)
	[13] 2021	2.1938(12)	2.3150(8)	2.4175(2)
	[71] 2023	———	———	2.4177(3)
	This work	**2.194(4)**	**2.315(2)**	**2.4175(5)**

Table 12: Summary of estimates for σ and τ in various dimensions.

Finally for the two correction to scaling exponents, ω and Ω, there does not appear to be a settled picture for the former. This is the one case where the five and six loop perturbative estimates are indeed out of line with the four loop one in three dimensions. However the reason for the discrepancy is relatively simple. While all three methods employed the two-sided Padé approximants a different two dimensional boundary condition was used in the four loop analysis of [24] which was 2 rather than the value of $\frac{3}{2}$ here and in [13]. The former value from [74] has been superseded by the latter from [41]. Curiously with the former value ω estimates were more in keeping with the central values of [50, 55] whereas the higher loop order ones are not out of line with [45]. For four and five dimensions the two dimensional constraint does not seem to play a significant role in that the estimates from all approaches have a solid overlap within uncertainties. With regard to Ω there are only a few non-loop based results for four and five dimensions and the loop estimates are generally in sink with them. In three dimensions aside from the results of [54, 59] the six loop estimate seems to be in the same company.

Exp	Ref & Year	$d = 3$	$d = 4$	$d = 5$
ω	[45] 1990	1.26(23)	0.94(15)	0.96(26)
	[49] 1997	———	1.13(10)	———
	[50] 1998	1.61(5)	———	———
	[55] 2000	1.62(13)	———	———
	[72] 2010	———	1.0(2)	———
	[24] 2015	1.6334	1.2198	0.7178
	[13] 2021	1.35(5)	1.10(6)	0.69(3)
	This work	**1.44(10)**	**1.18(9)**	**0.72(3)**
Ω	[45] 1990	0.50(9)	0.31(5)	0.27(7)
	[49] 1997	———	0.37(4)	———
	[50] 1998	0.64(2)	———	———
	[54] 1998	0.73(8)	———	———
	[55] 2000	0.64(5)	———	———
	[73] 2000	0.65(2)	———	———
	[57] 2001	0.60(8)	0.5(1)	———
	[24] 2015	———	0.4008	0.2034
	[59] 2018	0.77(3)	———	———
	[13] 2021	———	———	0.210(2)
	[71] 2023	———	———	0.27(2)
	This work	**0.602(7)**	**0.407(5)**	**0.2089(11)**

Table 13: Summary of estimates for ω and Ω in various dimensions.

4 Discussion.

It is worth offering some general comments and overview of our study. First, we have derived estimates for the critical exponents for two physics problems that are governed by a scalar field theory with a cubic self-interaction by using the recently derived six loop renormalization group functions of [14, 15]. For practical applications the relevant exponent values are those in three, four and five dimensions which necessitates a resummation of the ϵ expansion. To ensure reliability of the lower dimensional values we extended the earlier two-sided Padé calculations of [13, 24]. For both Lee-Yang and percolation theory the new estimates were not significantly dissimilar from the lower loop ones indicating a degree of convergence and in a few cases an improvement on previous uncertainties using the weighted Padé approximants. While this is reassuring what is perhaps worth noting is that the latest estimates in general are not out of line with those by other methods. Although this has to be qualified in that it is they are in accord with results from more recent years such as the conformal bootstrap programme or the functional renormalization group technique. With the improvement of computer technology and modern analytic techniques it appears evident that higher order perturbation theory can remain competitive. For instance in the Lee-Yang case the new fuzzy sphere method for three dimensions has close overlap with the loop results. The latter would not have been the case without the two-sided Padé approximants since the two dimensional boundary condition on the rational polynomials was essential in shaping the monotonic behaviour of the exponents in $2 \leq d \leq 6$. From the point of view of higher order computations the graphical function method has been applied to ϕ^4 theory to seven loops which is the current state of the art. In principle the renormalization of ϕ^3 theory could be extended to the same order with that method. However to execute such a computation would probably correspond to a significant increase in the level of difficulty for a scalar cubic theory.

Data Availability Statement. The data that support the findings of this article are openly available [75].

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