

# Renormalization properties of the mass operator $A_\mu^a A_\mu^a$ in three dimensional Yang-Mills theories in the Landau gauge

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## Abstract

Massive renormalizable Yang-Mills theories in three dimensions are analysed within the algebraic renormalization in the Landau gauge. In analogy with the four dimensional case, the renormalization of the mass operator  $A_\mu^a A_\mu^a$  turns out to be expressed in terms of the fields and coupling constant renormalization factors. We verify the relation we obtain for the operator anomalous dimension by explicit calculations in the large  $N_f$  expansion. The generalization to other gauges such as the nonlinear Curci-Ferrari gauge is briefly outlined.

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# 1 Introduction.

Recently, much work has been devoted to the study of the operator  $A_\mu^a A_\mu^a$  in four dimensional Yang-Mills theories in the Landau gauge, where a renormalizable effective potential for this operator can be consistently constructed [1, 2]. This has produced analytic evidence of a non-vanishing condensate  $\langle A_\mu^a A_\mu^a \rangle$ , resulting in a dynamical mass generation for the gluons [1, 2]. A gluon mass in the Landau gauge has been reported in lattice simulations [3] as well as in a recent investigation of the Schwinger-Dyson equations [4]. Besides being multiplicatively renormalizable to all orders of perturbation theory in the Landau gauge, the operator  $A_\mu^a A_\mu^a$  displays remarkable properties. In fact, it has been proven [5] by using BRST Ward identities that the anomalous dimension  $\gamma_{A^2}(a)$  of the operator  $A_\mu^a A_\mu^a$  in the Landau gauge is not an independent parameter, being expressed as a combination of the gauge beta function,  $\beta(a)$ , and of the anomalous dimension,  $\gamma_A(a)$ , of the gauge field, according to the relation

$$\gamma_{A^2}(a) = - \left( \frac{\beta(a)}{a} + \gamma_A(a) \right), \quad a = \frac{g^2}{16\pi^2}, \quad (1)$$

which can be explicitly verified by means of the three loop computations available in [6]. The operator  $A_\mu^a A_\mu^a$  turns out to be multiplicatively renormalizable also in the linear covariant gauges [7]. Its condensation and the ensuing dynamical gluon mass generation in this gauge have been discussed in [8].

Moreover, the operator  $A_\mu^a A_\mu^a$  in the Landau gauge can be generalized to other gauges such as the Curci-Ferrari and maximal Abelian gauges. Indeed, as was shown in [9, 10], the mixed gluon-ghost operator\*  $(\frac{1}{2}A_\mu^a A_\mu^a + \alpha \bar{c}^a c^a)$  turns out to be BRST invariant on-shell, where  $\alpha$  is the gauge parameter. In both gauges, the operator  $(\frac{1}{2}A_\mu^a A_\mu^a + \alpha \bar{c}^a c^a)$  turns out to be multiplicatively renormalizable to all orders of perturbation theory and, as in the case of the Landau gauge, its anomalous dimension is not an independent parameter of the theory [11]. A detailed study of the analytic evaluation of the effective potential for the condensate  $\langle \frac{1}{2}A_\mu^a A_\mu^a + \alpha \bar{c}^a c^a \rangle$  in these gauges can be found in [12, 13]. In particular, it is worth emphasizing that in the case of the maximal Abelian gauge, the off-diagonal gluons become massive due to the gauge condensate  $\langle \frac{1}{2}A_\mu^a A_\mu^a + \alpha \bar{c}^a c^a \rangle$ , a fact that can be interpreted as evidence for the Abelian dominance hypothesis underlying the dual superconductivity mechanism for color confinement.

The aim of this work is to analyse the renormalization properties of the operator  $A_\mu^a A_\mu^a$  in three dimensional Yang-Mills theories in the Landau gauge. This investigation might be useful in order to study by analytical methods the formation of the condensate  $\langle A_\mu^a A_\mu^a \rangle$  in three dimensions, whose relevance for the Yang-Mills theories at high temperatures has been pointed out long ago [14]. Furthermore, the possibility of a dynamical gluon mass generation related to the operator  $A_\mu^a A_\mu^a$  could provide a suitable infrared cutoff which would prevent three dimensional Yang-Mills theory from the well known infrared instabilities [15], due to its superrenormalizability.

The organization of the paper is as follows. In Sect.2 we discuss the renormalizability of the three dimensional Yang-Mills theory in the Landau gauge, when the operator  $A_\mu^a A_\mu^a$  is added to the starting action in the form of a mass term,  $m^2 \int d^3x A_\mu^a A_\mu^a$ . We shall be able to prove that the renormalization factor  $Z_{m^2}$  of the mass parameter  $m^2$  can be expressed in terms of the renormalization factors  $Z_A$  and  $Z_g$  of the gluon field and of the gauge coupling constant, according to

$$Z_{m^2} = Z_g Z_A^{-1/2}. \quad (2)$$

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\*In the case of the maximal Abelian gauge, the color index  $a$  runs only over the  $N(N-1)$  off-diagonal components.

This relation represents the analogue in three dimensions of the eq.(1). In Sect.3 we give an explicit verification of the relation (2) by using the large  $N_f$  expansion method. In Sect.4 we present the generalization to the nonlinear Curci-Ferrari gauge.

## 2 Renormalizability of massive three dimensional Yang-Mills theory in the Landau gauge.

### 2.1 Ward identities.

In order to analyze the renormalizability of three dimensional Yang-Mills theory, in the presence of the mass term  $\frac{1}{2}m^2 \int d^3x A_\mu^a A_\mu^a$ , we start from the following gauge fixed action

$$S = \int d^3x \left( -\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{1}{2} m^2 A_\mu^a A_\mu^a + b^a \partial_\mu A_\mu^a + \bar{c}^a \partial_\mu (D_\mu c)^a \right), \quad (3)$$

with

$$(D_\mu c)^a = \partial_\mu c^a + g f^{abc} A_\mu^b c^c, \quad (4)$$

where  $b^a$  is the Lagrange multiplier enforcing the Landau gauge condition,  $\partial_\mu A_\mu^a = 0$ , and  $\bar{c}^a$ ,  $c^a$  are the Faddeev-Popov ghosts. Concerning the mass term in expression (3), two remarks are in order. The first one is that, although in three dimensions the gauge field might become massive due to the introduction of the Chern-Simons topological action [16], one should note that the mass term considered here is of a different nature. In fact, unlike the Chern-Simons term, the mass term  $m^2 A_\mu^a A_\mu^a$  does not break parity. As a consequence, the starting action (3) is parity preserving. Therefore, the parity breaking Chern-Simons term cannot show up due to radiative corrections. The second remark is related to the superrenormalizability of three dimensional Yang-Mills theories, as expressed by the dimensionality of the gauge coupling  $g$ . As shown in [15], a standard perturbation theory would be affected by infrared singularities in the massless case. However, the presence of the mass term prevents the theory from this infrared instability, allowing one to define an infrared safe perturbative expansion.

Following [17], the action (3) is left invariant by a set of modified BRST transformations, given by

$$\begin{aligned} s A_\mu^a &= - (D_\mu c)^a, & s c^a &= \frac{g}{2} f^{abc} c^b c^c, \\ s \bar{c}^a &= b^a, & s b^a &= -m^2 c^a, \end{aligned} \quad (5)$$

and

$$s S = 0. \quad (6)$$

Notice that, due to the introduction of the mass term  $m$ , the operator  $s$  is not strictly nilpotent, *i.e.*

$$\begin{aligned} s^2 \Phi &= 0, & (\Phi &= A^{a\mu}, c^a), \\ s^2 \bar{c}^a &= -m^2 c^a, \\ s^2 b^a &= -m^2 \frac{g}{2} f^{abc} c^b c^c. \end{aligned} \quad (7)$$

Therefore, setting

$$s^2 \equiv -m^2 \delta, \quad (8)$$

we have

$$\delta S = 0. \quad (9)$$

The operator  $\delta$  is related to a global  $SL(2, \mathbb{R})$  symmetry [17], which is known to be present in the Landau, Curci-Ferrari and maximal Abelian gauges [18]. Finally, in order to express the BRST and  $\delta$  invariances in a functional way, we introduce the external action [19]

$$\begin{aligned} S_{\text{ext}} &= \int d^3x \left( \Omega_\mu^a sA_\mu^a + L^a sc^a \right) \\ &= \int d^3x \left( -\Omega_\mu^a (D_\mu c)^a + L^a \frac{g}{2} f^{abc} c^b c^c \right), \end{aligned} \quad (10)$$

where  $\Omega_\mu^a$  and  $L^a$  are external sources invariant under both BRST and  $\delta$  transformations, coupled to the nonlinear variations of the fields  $A_\mu^a$  and  $c^a$ . It is easy to check that the complete classical action,

$$\Sigma = S + S_{\text{ext}}, \quad (11)$$

is invariant under BRST and  $\delta$  transformations

$$s\Sigma = 0, \quad \delta\Sigma = 0. \quad (12)$$

When translated into functional form, the BRST and the  $\delta$  invariances give rise to the following Ward identities for the complete action  $\Sigma$ , namely

- the Slavnov-Taylor identity

$$\mathcal{S}(\Sigma) = 0, \quad (13)$$

- with

$$\mathcal{S}(\Sigma) = \int d^3x \left( \frac{\delta\Sigma}{\delta\Omega_\mu^a} \frac{\delta\Sigma}{\delta A_\mu^a} + \frac{\delta\Sigma}{\delta L^a} \frac{\delta\Sigma}{\delta c^a} + b^a \frac{\delta\Sigma}{\delta \bar{c}^a} - m^2 c^a \frac{\delta\Sigma}{\delta b^a} \right), \quad (14)$$

- the  $\delta$  Ward identity

$$\mathcal{W}(\Sigma) = 0, \quad (15)$$

with

$$\mathcal{W}(\Sigma) = \int d^3x \left( c^a \frac{\delta\Sigma}{\delta \bar{c}^a} + \frac{\delta\Sigma}{\delta L^a} \frac{\delta\Sigma}{\delta b^a} \right). \quad (16)$$

In addition, the following Ward identities holds in the Landau gauge [19], *i.e.*

- the gauge fixing condition and the antighost equation

$$\frac{\delta\Sigma}{\delta b^a} = \partial_\mu A_\mu^a, \quad \frac{\delta\Sigma}{\delta \bar{c}^a} + \partial_\mu \frac{\delta\Sigma}{\delta \Omega_\mu^a} = 0, \quad (17)$$

- the integrated ghost equation [20, 19]

$$\mathcal{G}^a \Sigma = \Delta_{\text{cl}}^a, \quad (18)$$

with

$$\mathcal{G}^a = \int d^3x \left( \frac{\delta}{\delta c^a} + g f^{abc} \bar{c}^b \frac{\delta}{\delta b^c} \right), \quad (19)$$

and

$$\Delta_{\text{cl}}^a = g \int d^3x f^{abc} \left( A_\mu^b \Omega_\mu^c - L^b c^c \right). \quad (20)$$

Notice that the breaking term  $\Delta_{\text{cl}}^a$  in the right-hand side of eq.(18), being linear in the quantum fields, is a classical breaking, not affected by quantum corrections [20, 19].

## 2.2 Algebraic characterization of the invariant counterterm.

Having established all Ward identities obeyed by the classical action  $\Sigma$ , we can now proceed with the characterization of the most general local counterterm compatible with the identities (13), (15), (17) and (18). Let us begin by displaying the quantum numbers of all fields, sources and parameters

	$A_\mu^a$	$c^a$	$\bar{c}^a$	$b^a$	$L^a$	$\Omega_\mu^a$	$g$	$s$	$m$
Gh. number	0	1	-1	0	-2	-1	0	1	0
Dimension	1/2	0	1	3/2	5/2	2	1/2	1/2	1

(21)

In order to characterize the most general invariant counterterm which can be freely added to all orders of perturbation theory, we perturb the classical action  $\Sigma$  by adding an arbitrary integrated, parity preserving, local polynomial  $\Sigma^{\text{count}}$  in the fields and external sources of dimension bounded by three and with zero ghost number, and we require that the perturbed action  $(\Sigma + \eta\Sigma^{\text{count}})$  satisfies the same Ward identities and constraints as  $\Sigma$  to first order in the perturbation parameter  $\eta$ , which are

$$\begin{aligned}
\mathcal{S}(\Sigma + \eta\Sigma^{\text{count}}) &= 0 + O(\eta^2), \\
\mathcal{W}(\Sigma + \eta\Sigma^{\text{count}}) &= 0 + O(\eta^2), \\
\frac{\delta(\Sigma + \eta\Sigma^{\text{count}})}{\delta b^a} &= \partial_\mu A_\mu^a + O(\eta^2), \\
\left(\frac{\delta}{\delta \bar{c}^a} + \partial_\mu \frac{\delta}{\delta \Omega_\mu^a}\right)(\Sigma + \eta\Sigma^{\text{count}}) &= 0 + O(\eta^2), \\
\mathcal{G}^a(\Sigma + \eta\Sigma^{\text{count}}) &= \Delta_{\text{cl}}^a + O(\eta^2).
\end{aligned}
\tag{22}$$

This amounts to imposing the following conditions on  $\Sigma^{\text{count}}$

$$\mathcal{B}_\Sigma \Sigma^{\text{count}} = 0, \tag{23}$$

with

$$\begin{aligned}
\mathcal{B}_\Sigma &= \int d^3x \left( \frac{\delta \Sigma}{\delta A_\mu^a} \frac{\delta}{\delta \Omega^{a\mu}} + \frac{\delta \Sigma}{\delta \Omega^{a\mu}} \frac{\delta}{\delta A_\mu^a} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta}{\delta c^a} + \frac{\delta \Sigma}{\delta c^a} \frac{\delta}{\delta L^a} \right. \\
&\quad \left. + b^a \frac{\delta}{\delta \bar{c}^a} - m^2 c^a \frac{\delta}{\delta b^a} \right),
\end{aligned}
\tag{24}$$

$$\mathcal{W}_\Sigma \Sigma^{\text{count}} = \int d^3x \left( c^a \frac{\delta \Sigma^{\text{count}}}{\delta \bar{c}^a} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta \Sigma^{\text{count}}}{\delta b^a} + \frac{\delta \Sigma}{\delta b^a} \frac{\delta \Sigma^{\text{count}}}{\delta L^a} \right) = 0, \tag{25}$$

$$\frac{\delta \Sigma^{\text{count}}}{\delta b^a} = 0, \quad \frac{\delta \Sigma^{\text{count}}}{\delta \bar{c}^a} + \partial_\mu \frac{\delta \Sigma^{\text{count}}}{\delta \Omega_\mu^a} = 0, \tag{26}$$

and

$$\mathcal{G}^a \Sigma^{\text{count}} = 0. \tag{27}$$

Following the algebraic renormalization procedure [19], it turns out that the most general local, parity preserving, invariant counterterm  $\Sigma^{\text{count}}$  compatible with all constraints (23), (25), (26) and (27), contains only two independent free parameters  $\sigma$  and  $a_1$ , and is given by

$$\begin{aligned}
\Sigma^{\text{count}} &= \int d^3x \left( -\frac{(\sigma + 4a_1)}{4} F_{\mu\nu}^a F_{\mu\nu}^a + a_1 F_{\mu\nu}^a \partial_\mu A_\nu^a + \frac{a_1}{2} m^2 A_\mu^a A_\mu^a \right. \\
&\quad \left. + a_1 (\Omega_\mu^a + \partial_\mu \bar{c}^a) \partial_\mu c^a \right).
\end{aligned}
\tag{28}$$

### 2.3 Stability and renormalization of the mass parameter.

It remains now to discuss the stability of the classical action [19], *i.e.* to check that  $\Sigma^{\text{count}}$  can be reabsorbed in the classical action  $\Sigma$  by means of a multiplicative renormalization of the coupling constant  $g$ , the mass parameter  $m^2$ , the fields  $\{\phi = A, c, \bar{c}, b\}$  and the sources  $L, \Omega$ , namely

$$\Sigma(g, m^2, \phi, L, \Omega) + \eta \Sigma^{\text{count}} = \Sigma(g_0, m_0^2, \phi_0, L_0, \Omega_0) + O(\eta^2), \quad (29)$$

with the bare fields and parameters defined as

$$\begin{aligned} A_{0\mu}^a &= Z_A^{1/2} A_\mu^a, & \Omega_{0\mu}^a &= Z_\Omega \Omega_\mu^a, \\ c_0^a &= Z_c^{1/2} c^a, & L_0^a &= Z_L L^a, \\ g_0 &= Z_g g, & m_0^2 &= Z_{m^2} m^2, \\ \bar{c}_0^a &= Z_{\bar{c}}^{1/2} \bar{c}^a, & b_0^a &= Z_b^{1/2} b^a. \end{aligned} \quad (30)$$

The parameters  $\sigma$  and  $a_1$ , are easily seen to be related to the renormalization of the gauge coupling constant  $g$  and of the gauge field  $A_\mu^a$ , according to

$$\begin{aligned} Z_g &= 1 - \eta \frac{\sigma}{2}, \\ Z_A^{1/2} &= 1 + \eta \left( \frac{\sigma}{2} + a_1 \right). \end{aligned} \quad (31)$$

Concerning the other fields and the sources  $\Omega_\mu^a, L^a$ , it can be verified that they are renormalized as

$$Z_{\bar{c}} = Z_c = Z_g^{-1} Z_A^{-1/2}, \quad (32)$$

$$Z_b = Z_A^{-1}, \quad Z_\Omega = Z_c^{1/2}, \quad Z_L = Z_A^{1/2}. \quad (33)$$

Finally, for the mass parameter  $m^2$ ,

$$Z_{m^2} = Z_g Z_A^{-1/2}, \quad (34)$$

which, due to eq.(32), can be rewritten as

$$Z_{m^2} = Z_c^{-1} Z_A^{-1}. \quad (35)$$

Equation (32) expresses the well known nonrenormalization property of the ghost-antighost-gluon vertex in the Landau gauge. As shown in [20], this is a direct consequence of the ghost Ward identity (18). Also, as anticipated, equation (34) shows that the renormalization of the mass parameter  $m^2$  can be expressed in terms of the gauge field and coupling constant renormalization factors. It is worth mentioning here that eqs.(32), (34) are in complete agreement with the results obtained in the case of the four dimensional Yang-Mills theory in the Landau gauge [5].

Although we did not consider matter fields in the previous analysis, it can be easily checked that the renormalizability of the mass operator  $A_\mu^a A_\mu^a$  and the relations (34), (35) remain unchanged if massless spinor fields are included, namely

$$S_{\text{matter}} = \int d^3x \left( i \bar{\psi}^i \not{\partial} \psi^i + g A_\mu^a \bar{\psi}^i \gamma^\mu T^a \psi^i \right), \quad (36)$$

with  $i = 1, \dots, N_f$ . In fact, as was pointed out in [15], the addition of massless fermions does not break the parity invariance of the starting action (3). Of course, the inclusion of the matter action (36) requires the introduction of a suitable renormalization factor  $Z_\psi$  for the spinor fields.

## 2.4 Absence of one loop ultraviolet divergences.

In the previous section we have proven that the massive three dimensional Yang-Mills action (3) is multiplicatively renormalizable to all orders of perturbation theory, displaying interesting renormalization features, as expressed by equations (32) and (34). Only two renormalization constants,  $Z_g$  and  $Z_A$ , are needed at the quantum level. These factors should be computed order by order by means of a suitable regularization, which in the present case could be provided by dimensional regularization. Due to the absence of parity breaking terms, this would give an invariant regularization scheme. Furthermore, we recall that Yang-Mills theory in three dimensions is a superrenormalizable theory, a property which reduces the number of divergent integrals. It is thus worth looking at the Feynman diagrams of the theory. Let us begin with the one loop ghost-antighost self-energy. It is almost trivial to check that, due to the transversality of the gluon propagator in the Landau gauge, the Feynman integral for the ghost self-energy

$$g^2 \int \frac{d^3k}{(2\pi)^3} \frac{p_\mu(p-k)_\nu}{(p-k)^2} \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \frac{1}{k^2 + m^2}, \quad (37)$$

where  $p_\mu$  stands for the external momentum, is free from ultraviolet divergences. As a consequence we have that, at one loop order in  $\overline{\text{MS}}$ ,

$$Z_c = Z_{\bar{c}} = 1, \quad \text{at one loop order.} \quad (38)$$

Analogously, by simple inspection, it turns out that the one loop correction to the ghost-antighost-gluon vertex is also finite. The same feature holds for the one loop Feynman diagrams contributing to the four gluon vertex, from which it follows that in  $\overline{\text{MS}}$

$$Z_g^2 Z_A^2 = 1, \quad \text{at one loop order.} \quad (39)$$

Moreover, from equation (32), we have

$$Z_A = 1, \quad \text{at one loop order,} \quad (40)$$

so that

$$Z_g = 1, \quad \text{at one loop order} \quad (41)$$

in  $\overline{\text{MS}}$ . We see therefore that, at one loop order, the theory is completely free from ultraviolet divergences, a feature which also holds in the presence of massless fermions. At higher orders, ultraviolet divergences could show up.

To provide a non-trivial check of the validity of the relation (1) from another point of view, we shall make use of the large  $N_f$  expansion, given the existence of a fixed point in the  $\beta$ -function. Within this large  $N_f$  expansion technique, it is commonly known that this fixed point can be obtained by analytic continuation of the one existing in  $d = 4 - 2\epsilon$  dimensions. This will be considered in the following section.

## 3 Large $N_f$ verification.

Having established the renormalizability of the mass operator in the Landau gauge, we verify the result in QCD using the large  $N_f$  critical point method developed in [22, 23] for the non-linear  $\sigma$  model and extended to QED and QCD in [24, 25, 26, 27]. Briefly, this method allows one to compute the critical exponents associated with the renormalization of the fields, coupling constants or composite operators at the  $d$ -dimensional fixed point of the QCD  $\beta$ -function. The

critical exponents encode all orders information on the respective anomalous dimensions,  $\beta$ -function and operator anomalous dimensions and are more fundamental than their associated renormalization group functions in that they are renormalization group invariant. Knowing the explicit location of the  $d$ -dimensional fixed point allows one to convert the information encoded in the exponents to the explicit coefficients in the four dimensional perturbative expansion of the renormalization group functions. Since we are interested in the renormalization of  $\frac{1}{2}A_\mu^a A_\mu^a$  in the Landau gauge and its connection with the gluon and ghost wave function renormalization we will show that, in agreement with eqs.(34), (35), the critical exponent associated with the Landau gauge renormalization of  $A_\mu^a A_\mu^a$  at leading order in large  $N_f$  is simply the sum of the gluon and ghost wave function critical exponents. The latter have already been determined in [26]. Moreover, since the computation is in  $d$ -dimensions,  $2 < d < 4$ , the three dimensional result of the previous sections will emerge *naturally*.

To fix notation for this section, we recall that the  $d$ -dimensional  $\overline{\text{MS}}$  QCD  $\beta$ -function, [28], is

$$\begin{aligned} \beta(a) = & (d-4)a + \left[ \frac{2}{3}T_F N_f - \frac{11}{6}C_A \right] a^2 + \left[ \frac{1}{2}C_F T_F N_f + \frac{5}{6}C_A T_F N_f - \frac{17}{12}C_A^2 \right] a^3 \\ & - \left[ \frac{11}{72}C_F T_F^2 N_f^2 + \frac{79}{432}C_A T_F^2 N_f^2 + \frac{1}{16}C_F^2 T_F N_f + \frac{2857}{1728}C_A^3 \right. \\ & \left. - \frac{205}{288}C_F C_A T_F N_f - \frac{1415}{864}C_A^2 T_F N_f \right] a^4 + O(a^5), \end{aligned} \quad (42)$$

where the group Casimirs are defined by  $T^a T^a = C_F I$ ,  $f^{acd} f^{bcd} = C_A \delta^{ab}$  and  $\text{Tr}(T^a T^b) = T_F \delta^{ab}$ . The leading  $O(a)$  term corresponds to the dimension of the coupling in  $d$ -dimensions and is necessary to deduce the location of the non-trivial  $d$ -dimensional fixed point  $a_c$ . Expanding in powers of  $1/N_f$  it is given by

$$\begin{aligned} a_c = & \frac{3\epsilon}{T_F N_f} + \frac{1}{4T_F^2 N_f^2} \left[ 33C_A \epsilon - (27C_F + 45C_A) \epsilon^2 \right. \\ & \left. + \left( \frac{99}{4}C_F + \frac{237}{8}C_A \right) \epsilon^3 + O(\epsilon^4) \right] + O\left( \frac{1}{N_f^3} \right), \end{aligned} \quad (43)$$

where  $d = 4 - 2\epsilon$ . QCD is in the same universality class as the non-Abelian Thirring model (NATM), [29], which has the Lagrangian

$$\mathcal{L}^{\text{NATM}} = i\bar{\psi}^i \not{\partial} \psi^i + \frac{\lambda^2}{2} (\bar{\psi}^i \gamma^\mu T^a \psi^i)^2, \quad (44)$$

or rewriting it in terms of an auxiliary vector field,  $\tilde{A}_\mu^a$ ,

$$\mathcal{L}^{\text{NATM}} = i\bar{\psi}^i \not{\partial} \psi^i + \tilde{A}_\mu^a \bar{\psi}^i \gamma^\mu T^a \psi^i - \frac{(\tilde{A}_\mu^a)^2}{2\lambda^2}, \quad (45)$$

where the coupling constant  $\lambda$  is dimensionless in two dimensions. By analogy the NATM plays the same role as the  $O(N)$  nonlinear  $\sigma$  model in the  $d$ -dimensional critical point equivalence with the 4-dimensional  $O(N)$   $\phi^4$  theory at the  $d$ -dimensional Wilson-Fisher fixed point. One feature of the universality criterion at criticality is that the interactions of the fields play the major role. Hence, comparing the QCD and NATM Lagrangians where for this section we take

$$\mathcal{L}^{\text{QCD}} = i\bar{\psi}^i \not{\partial} \psi^i + \tilde{A}_\mu^a \bar{\psi}^i \gamma^\mu T^a \psi^i - \frac{(F_{\mu\nu}^a)^2}{4g^2}, \quad (46)$$



the quark-gluon 3-point interaction of both models is dominant in the large  $N_f$  critical point method. In QCD the field strength of the Lagrangian is infrared irrelevant and drops out of the large  $N_f$  analysis. However, in practice the triple and quartic gluon interactions emerge in diagrams with closed quark loops with respectively three and four external  $\tilde{A}_\mu^a$  fields, [29, 27]. It is worth noting that in this section alone we have redefined the gluon field and incorporated a power of the QCD coupling constant into its definition,  $\tilde{A}_\mu^a = gA_\mu^a$  which is the origin of the power of  $g^2$  factor with the field strength term. This rescaling is necessary for the application of the critical point large  $N_f$  programme which requires a unit coupling constant for the quark gluon interaction and therefore defines the canonical scaling dimensions in such a way as to make the calculational tool of uniqueness applicable which was used extensively in the original large  $N_f$  critical point method of [22, 23]. As we are interested in the critical exponents and

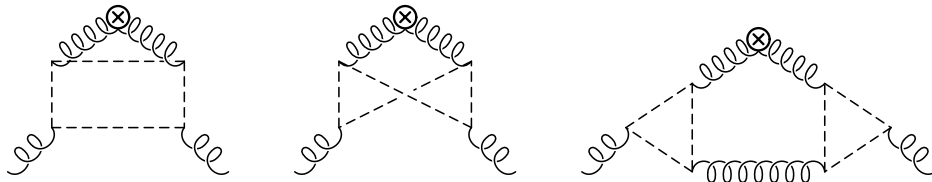


Figure 1:  $O(1/N_f)$  diagrams contributing to  $\eta_{A^2}$ .

therefore the anomalous dimensions of the composite operator  $\frac{1}{2}A_\mu^a A_\mu^a$  in the large  $N_f$  expansion, we follow the method of [27]. There the critical exponent  $\omega$  associated with the QCD  $\beta$ -function was computed at  $O(1/N_f)$  in  $d$ -dimensions by inserting the composite operator  $(F_{\mu\nu}^a)^2$  into a gluon 2-point function and applying the method of [30] to determine the critical dimension of its associated coupling. For the anomalous dimension of  $\frac{1}{2}A_\mu^a A_\mu^a$  we follow the same approach and note that the appropriate  $O(1/N_f)$  large  $N_f$  diagrams are given in figure 1 where a gluon line counts one power of  $1/N_f$  which is why there are two *and* three loop Feynman diagrams at this order. The latter three loop graph in fact contains the relevant contribution from the triple gluon vertex which is absent in the NATM Lagrangian. Unlike perturbation theory the propagators of figure 1 are not the usual ones. Their asymptotic scaling forms are deduced from dimensional analysis and consistency with Lorentz symmetry. In the Landau gauge we have, [25, 26],

$$\psi(k) \sim \frac{Ak}{(k^2)^{\mu-\alpha}} \quad , \quad A_{\mu\nu}(k) \sim \frac{B}{(k^2)^{\mu-\beta}} \left[ \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right] \quad , \quad c(k) \sim \frac{C}{(k^2)^{\mu-\gamma}} \quad , \quad (47)$$

in momentum space at leading order as  $k^2 \rightarrow \infty$  as one approaches the  $d$ -dimensional fixed point. We have given the ghost propagator asymptotic scaling form for completeness and to define its scaling dimension even though it is not needed at  $O(1/N_f)$  for the explicit computation of the critical exponent of  $\frac{1}{2}A_\mu^a A_\mu^a$ . The powers of the propagators are defined as

$$\alpha = \mu + \frac{1}{2}\eta \quad , \quad \beta = 1 - \eta - \chi \quad , \quad \gamma = \mu - 1 + \frac{1}{2}\eta_c \quad , \quad (48)$$

where  $A$ ,  $B$  and  $C$  are the momentum independent amplitudes though only the combinations  $z = A^2 B$  and  $y = C^2 B$  appear in calculations, [26]. We use  $\mu = d/2$  for shorthand,  $\eta$  is the critical exponent of the quark field,  $\chi$  is the critical exponent of the quark-gluon vertex anomalous dimension and  $\eta_c$  is the ghost critical exponent. We note that the explicit  $O(1/N_f)$  values of the critical exponents in  $d$ -dimensions in the Landau gauge are, [26],

$$\begin{aligned} \eta_1 &= \frac{(2\mu - 1)(\mu - 2)\Gamma(2\mu)C_F}{4\Gamma^2(\mu)\Gamma(\mu + 1)\Gamma(2 - \mu)T_F} \equiv \eta_1^O \frac{C_F}{T_F} \quad , \\ \chi_1 &= - \left[ C_F + \frac{C_A}{2(\mu - 2)} \right] \frac{\eta_1^O}{T_F} \quad , \quad \eta_{c1} = - \frac{C_A \eta_1^O}{2(\mu - 2)T_F} \quad , \end{aligned} \quad (49)$$

where we will use the notation  $\eta = \sum_{i=1}^{\infty} \eta_i/N_f^i$ . The expression for the ghost anomalous dimension follows from the usual Slavnov-Taylor identity as expressed in exponent language,

$$\eta_c = \eta + \chi - \chi_c , \quad (50)$$

where  $\chi_c$  is the anomalous dimension of the ghost-gluon vertex and was shown in [26] to vanish in the Landau gauge at  $O(1/N_f)$ .

The explicit computation of the exponent associated with the renormalization of  $\frac{1}{2}A_\mu^a A_\mu^a$ , which we will call  $\eta_{A^2}$ , is deduced by inserting (47) into the diagrams of figure 1 and applying the procedure of [30] to determine the scaling dimension of the operator insertion,  $\eta_{\mathcal{O}}$ . The value of  $\eta_{A^2}$  is deduced from the relation

$$\eta_{A^2} = \eta + \chi + \eta_{\mathcal{O}} , \quad (51)$$

where the first two terms correspond to the anomalous part of the gluon critical dimension or wave function renormalization. For completeness we note that the corresponding critical exponent in the Thirring model,  $\omega^{\text{NATM}}$ , is deduced by dimensionally analysing the final term of (45) giving

$$\omega^{\text{NATM}} = \mu - 1 + \eta + \chi + \eta_{\mathcal{O}} . \quad (52)$$

In practice a regularization has to be introduced for the Feynman integrals which is obtained by shifting the exponent of the vertex renormalization,  $\chi$ , to the new value of  $\chi + \Delta$ . Here  $\Delta$  plays a role akin to  $\epsilon$  in dimensional regularization. Though it should be stressed that we are working in fixed dimensions,  $d$ , and not dimensionally regularizing here. The actual contribution to  $\eta_{\mathcal{O}}$  is determined from the residue of the simple pole in  $\Delta$  from the sum of all the diagrams of figure 1. In [27] the two and three loop diagrams were computed using various techniques such as integration by parts and uniqueness, [23], after the regularized Feynman integrals were broken up into a set of basic integrals which were straightforward to determine and a set which required a substantial amount of effort particularly in the case of the three loop diagram. We have used the same integrals here but supplemented with an extra set since the operator insertion of  $\frac{1}{2}A_\mu^a A_\mu^a$  alters the power of the internal gluon line containing the operator insertion. An example of one

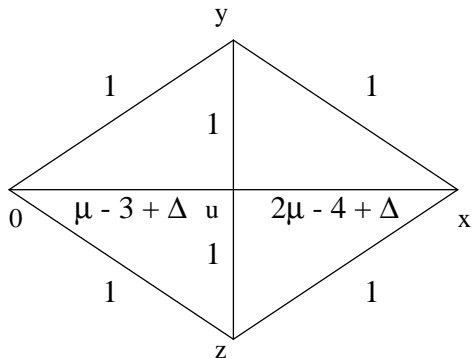


Figure 2: Basic three loop Feynman diagram.

of the tedious graphs in this respect is that illustrated in figure 2 where we have indicated the power of the propagator beside the line. We have used coordinate space representation where one integrates over the location of the internal vertices,  $u$ ,  $y$  and  $z$ , but with  $x$  corresponding to the external coordinate or momentum of the diagram. To determine the residue with respect to the  $\Delta$ -pole we convert the integral to momentum representation, [23], which produces the first

diagram of figure 3. There we have nullified the regularization since the associated factor from the transformation is

$$\frac{a^6(1)a(\mu-3)a(2\mu-4)}{a(-1+2\Delta)}, \quad (53)$$

which, due to the denominator factor, is clearly divergent as  $\Delta \rightarrow 0$  since  $a(\alpha) = \Gamma(\mu-\alpha)/\Gamma(\alpha)$ . To proceed we use the language of [23] and apply a conformal transformation to the first diagram of figure 3 based on the left external point. Then integrating the unique triangle and subsequent unique vertex before undoing the original conformal transformation finally produces the second diagram of figure 3. The factor associated with these manipulations from the first diagram of figure 3 is  $a^4(\mu-1)/a(2\mu-4)$ . To deduce the value of the final diagram which is  $\Delta$ -finite we integrate by parts on the top right internal vertex based on the line with exponent 1. This produces four two loop diagrams. However, these intermediate diagrams are in fact divergent though their sum is finite. To ensure the correct finite part emerges, one introduces a temporary intermediate regularization prior to integrating by parts by shifting the exponent of the line labeled  $3-\mu$  to an exponent of  $3-\mu+\delta$ . In fact two of the resulting diagrams then cancel exactly, leaving two integrals which are related to the function  $\text{ChT}(\alpha, \beta)$  defined in [23] and evaluated exactly in [31, 23]. Explicitly one has the difference of  $\text{ChT}(-1-\delta, 3-\mu)$  and  $\text{ChT}(\mu-3-\delta, 3-\mu)$  and expanding in powers of  $\delta$  a finite expression emerges. Accumulating all the contributions the final contribution of the integral of figure 2 to the critical exponent computation is

$$\frac{(2\mu-3)(\mu-1)^2(2\mu^2-7\mu+4)}{4\Gamma(\mu+1)\Delta}. \quad (54)$$

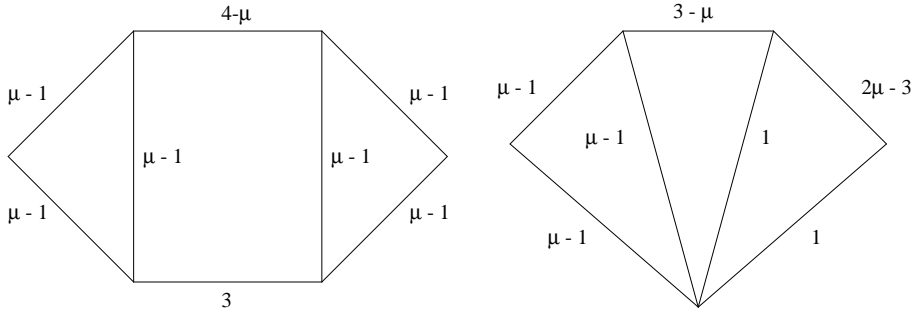


Figure 3: Intermediate three loop Feynman diagrams.

Having completed the computation of all the intermediate basic integrals we note that the transverse contribution of each of the four diagrams of figure 1 to  $\eta_{\mathcal{O}}$  are respectively,

$$-\frac{(2\mu-1)(2\mu-3)C_F\eta_1^{\mathcal{O}}}{2(\mu-2)T_F}, \quad \frac{(2\mu-1)(2\mu-3)[C_F-\frac{1}{2}C_A]\eta_1^{\mathcal{O}}}{2(\mu-2)T_F},$$

$$\frac{(\mu-1)^2C_A\eta_1^{\mathcal{O}}}{(\mu-2)T_F}. \quad (55)$$

Hence,

$$\eta_{A^2} = -\frac{C_A\eta_1^{\mathcal{O}}}{4(\mu-2)T_F N_f} + O\left(\frac{1}{N_f^2}\right), \quad (56)$$

in  $d$ -dimensions. Clearly this is equivalent to the sum of anomalous dimension parts of the Landau gauge gluon and ghost critical exponents at  $O(1/N_f)$ . More explicitly, from (48) and (49),

$$\eta_{A^2 1} = \eta_1 + \chi_1 - \frac{1}{2}\eta_{c1}, \quad (57)$$

which due to our choice of conventions and notation was the way this identity was originally uncovered in [6] prior to the all orders proof of [5] and its subsequent expression in the form of (1). Therefore, (57) is an explicit  $d$ -dimensional verification of the all orders result of the previous section. Moreover, it nicely recovers the  $d$ -dimensional case of [6, 5, 32].

As three dimensional QCD is of interest in other problems, we note that the explicit three dimensional value of  $\omega^{\text{NATM}}$  is

$$\omega^{\text{NATM}}\Big|_{d=3} = \frac{1}{2} - \frac{4C_A}{3\pi^2 T_F N_f} + O\left(\frac{1}{N_f^2}\right). \quad (58)$$

In two dimensions, interestingly the critical exponent does not run to its mean field value and one has

$$\omega^{\text{NATM}}\Big|_{d=2} = -\frac{C_A}{16T_F N_f} + O\left(\frac{1}{N_f^2}\right). \quad (59)$$

## 4 Generalization to other gauges: the example of the Curci-Ferrari gauge.

The mass operator  $A_\mu^a A_\mu^a$  in the Landau gauge can be generalized to other gauges, such as the Curci-Ferrari and the maximal Abelian gauge. In this case the mixed gluon-ghost mass operator ( $\frac{1}{2}A_\mu^a A_\mu^a + \alpha \bar{c}^a c^a$ ) has to be considered, where  $\alpha$  stands for the gauge parameter. Let us consider here the case of the Curci-Ferrari nonlinear gauge. For the gauge fixed action we have

$$\begin{aligned} S_{\text{CF}} = & \int d^3x \left( -\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + b^a \partial_\mu A_\mu^a + \frac{\alpha}{2} b^a b^a + \bar{c}^a \partial_\mu (D_\mu c)^a - \frac{\alpha}{2} g f^{abc} b^a \bar{c}^b c^c \right. \\ & \left. - \frac{\alpha}{8} g^2 f^{abc} f^{cde} \bar{c}^a \bar{c}^b c^d c^e + m^2 \left( \frac{1}{2} A_\mu^a A_\mu^a + \alpha \bar{c}^a c^a \right) \right). \end{aligned} \quad (60)$$

Notice that in this case also the Faddeev-Popov ghosts  $\bar{c}^a, c^a$  are massive. Moreover, the Curci-Ferrari gauge reduces to the Landau gauge in the limit  $\alpha \rightarrow 0$ . The action (60) is invariant under the BRST and  $\delta$  transformations of eqs.(5), (8). Introducing the external action

$$S_{\text{ext}} = \int d^3x \left( -\Omega_\mu^a (D_\mu c)^a + L^a \frac{g}{2} f^{abc} c^b c^c \right),$$

it follows that the complete classical action

$$\Sigma_{\text{CF}} = S_{\text{CF}} + S_{\text{ext}}, \quad (61)$$

turns out to be constrained by the Slavnov-Taylor identity

$$\mathcal{S}(\Sigma) = \int d^3x \left( \frac{\delta \Sigma}{\delta \Omega_\mu^a} \frac{\delta \Sigma}{\delta A_\mu^a} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta \Sigma}{\delta c^a} + b^a \frac{\delta \Sigma}{\delta \bar{c}^a} - m^2 c^a \frac{\delta \Sigma}{\delta b^a} \right) = 0, \quad (62)$$

and by the  $\delta$  Ward identity

$$\mathcal{W}(\Sigma) = \int d^3x \left( c^a \frac{\delta \Sigma}{\delta \bar{c}^a} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta \Sigma}{\delta b^a} \right) = 0. \quad (63)$$

Due to the presence of the quartic ghost-antighost term  $g^2 f^{abc} f^{cde} \bar{c}^a \bar{c}^b c^d c^e$  and of  $g f^{abc} b^a \bar{c}^b c^c$  the additional Ward identities (17) and (18) of the Landau gauge do not hold in the present case. Nevertheless, identities (62) and (63) ensure the multiplicative renormalizability of the model.

Proceeding as in the previous section, it turns out that the most general invariant counterterm contains five free independent parameters,  $\sigma$ ,  $a_1$ ,  $a_2$ ,  $a_3$ ,  $a_5$  and is given by

$$\begin{aligned}
\Sigma_{\text{CF}}^{\text{count}} &= \int d^3x \left( -\frac{(\sigma + 4a_1)}{4} F_{\mu\nu}^a F_{\mu\nu}^a + a_1 F_{\mu\nu}^a \partial_\mu A_\nu^a + (a_1 - a_2) b^a \partial_\mu A_\mu^a \right. \\
&+ a_1 (\partial_\mu \bar{c}^a + \Omega_\mu^a) \partial_\mu c^a + (a_1 - a_2) \bar{c}^a \partial_\mu (D_\mu c)^a + a_5 (\partial_\mu \bar{c}^a + \Omega_\mu^a) (D_\mu c)^a \\
&- a_3 \frac{\alpha}{2} b^a b^a + \frac{(a_3 + a_5)}{2} \alpha g f^{abc} b^a \bar{c}^b c^c + \frac{(a_3 + 2a_5)}{8} \alpha g^2 f^{abc} f^{cde} \bar{c}^a \bar{c}^b c^d c^e \\
&\left. - \frac{a_5}{2} g f^{abc} L^a c^b c^c + m^2 \left( (a_1 - \frac{a_2}{2} + \frac{a_5}{2}) A_\mu^a A_\mu^a - \alpha a_3 \bar{c}^a c^a \right) \right). \tag{64}
\end{aligned}$$

The parameters  $\sigma$ ,  $a_1$ ,  $a_2$ ,  $a_3$ ,  $a_5$  are easily seen to correspond to a multiplicative renormalization of the fields, sources and parameters, according to

$$\begin{aligned}
Z_g &= 1 - \eta \frac{\sigma}{2}, \\
Z_A^{1/2} &= 1 + \eta \left( \frac{\sigma}{2} + a_1 \right), \\
Z_c^{1/2} &= Z_{\bar{c}}^{1/2} = 1 - \eta \left( \frac{(a_2 + a_5)}{2} \right), \\
Z_L &= 1 + \eta \left( \frac{\sigma}{2} + a_2 \right), \\
Z_\alpha &= 1 + \eta (-a_3 + 2a_2 + \sigma), \tag{65}
\end{aligned}$$

and

$$\begin{aligned}
Z_\Omega &= Z_A^{-1/2} Z_c^{1/2} Z_L, \\
Z_b^{1/2} &= Z_L^{-1}, \\
Z_{m^2} &= Z_L^{-2} Z_c^{-1}. \tag{66}
\end{aligned}$$

In particular, from eqs.(66) it follows that the renormalization factor  $Z_{m^2}$  is not independent, being expressed in terms of the ghost renormalization factor  $Z_c$  and of the renormalization factor  $Z_L$  of the source  $L^a$  coupled to the composite ghost operator  $\frac{1}{2} g f^{abc} c^b c^c$ . Again, these results are in complete agreement with those obtained in the four dimensional case [11].

## 5 Conclusion.

In this paper we have analysed the renormalization properties of the mass operator  $A_\mu^a A_\mu^a$  in three dimensional Yang-Mills theories in the Landau gauge. In analogy with the four dimensional case, the renormalization factor  $Z_{m^2}$  is not an independent parameter of the theory, as expressed by the relations (34) and (35), which have been explicitly verified in the large  $N_f$  expansion method. These results will be used in order to investigate by analytical methods the possible formation of the gauge condensate  $\langle A_\mu^a A_\mu^a \rangle$ . This would provide a dynamical generation of a parity preserving mass for the gluons in three dimensions, a topic which has been extensively investigated in recent years. For instance, see [33, 34, 35, 36].

Finally, we underline that the Curci-Ferrari gauge allows one to study the generalized mixed gluon-ghost condensate  $\langle \frac{1}{2} A_\mu^a A_\mu^a + \alpha \bar{c}^a c^a \rangle$ . In particular, as discussed in the four dimensional case, the presence of the gauge parameter  $\alpha$  could be useful to investigate the gauge independence of the vacuum energy, due to the formation of the aforementioned condensates.

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