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Dynamical gluon mass generation from $\left\langle A_\mu^2 \right\rangle$ in linear covariant gauges

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Abstract

We construct the multiplicatively renormalizable effective potential for the mass dimension two local composite operator $A_\mu^a A^{\mu a}$ in linear covariant gauges. We show that the formation of $\left\langle A_\mu^a A^{\mu a} \right\rangle$ is energetically favoured and that the gluons acquire a dynamical mass due to this gluon condensate. We also discuss the gauge parameter independence of the resultant vacuum energy.

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1 Introduction.

In a previous paper [1], we took the first step towards constructing a renormalizable effective potential for the local composite operator (LCO) $A_\mu^2 \equiv A_\mu^a A^{\mu a}$ in linear covariant gauges. It was shown within the algebraic renormalization formalism [2, 3] that A_μ^2 is multiplicatively renormalizable to all orders in perturbation theory. At the same time, the anomalous dimension of A_μ^2 was explicitly computed to 2-loops in the $\overline{\text{MS}}$ scheme as a function of the gauge fixing parameter, α, where $\alpha = 0$ corresponds to the Landau gauge. The computation exploited the fact that in linear covariant gauges the operator does not mix, for example, with ghost operators of dimension two with the same quantum numbers.

The operator A_μ^2 has recently received widespread interest in Yang-Mills theory in the Landau gauge. Its relevance has been advocated both from a theoretical point of view as well as from lattice simulations [4, 5, 6, 7, 8]. Analytic results in favour of a non-zero value for the condensate $\left\langle A_\mu^2 \right\rangle$ in the Landau gauge have been obtained recently, [9, 10]. Further, the inclusion of quarks has been considered in [11]. Motivated by the result of [12] it has been shown in [13] that A_μ^2 is multiplicatively renormalizable to all orders in the Landau gauge, but its anomalous dimension is given by a combination of the gauge beta function, $\beta(a)$, and the anomalous dimension, $\gamma_A(a)$, of the gluon field, according to the relation [12, 13]

$$\gamma_{A^2}(a) \;=\; -\;\left(\frac{\beta(a)}{a} \;+\; \gamma_A(a)\right), \qquad\qquad a \;=\; \frac{g^2}{16\pi^2}\,. \tag{1}$$

An important consequence of the formation of the $\left\langle A_\mu^2 \right\rangle$ condensate in the Landau gauge is the dynamical generation of a gluon mass $m_{\text{gluon}} \approx 500\text{MeV}$ [9]. Lattice simulations of the $SU(2)$ gluon propagator in the Landau gauge report $m_{\text{gluon}} \approx 600\text{MeV}$, [14]. Gluon masses have also been extracted from lattice methods in the Laplacian [15, 16] and Maximal Abelian gauges [17, 18]. Earlier the pairing of gluons was discussed in connection with mass generation as a result of the instability of the perturbative Yang-Mills vacuum [19, 20, 21, 22, 23]. A dynamical gluon mass might also be important, for example, in connection with the glueball spectra, [24, 25]. A dimension two gluon condensate $\langle A_i^2 \rangle$ was already introduced in [25], where the Coulomb gauge was considered. Furthermore, a dynamical gluon mass is part of a certain criterion for confinement introduced by Kugo and Ojima, [26, 27]. For a recent review see [28].

It is no coincidence that the Landau gauge is employed in the search for a gluon condensate of mass dimension two. As is well known, there does not exist a local, gauge-invariant operator of mass dimension two in Yang-Mills theories. However, a *non-local* gauge invariant dimension two operator can be constructed by minimizing A_μ^2 along each gauge orbit [29, 30, 31], $A_{\text{min}}^2 \equiv (VT)^{-1} \min_U \int d^4x \left(A_\mu^U\right)^2$ where VT is the space time volume and U is a generic $SU(N)$ transformation. This operator A_{min}^2 is related to the so-called fundamental modular region (FMR), which is the set of absolute minima of $(VT)^{-1} \int d^4x \left(A_\mu^U\right)^2$. In the Landau gauge, $\partial_\mu A^{a\mu} = 0$, so that A_{min}^2 and A_μ^2 coincide within the FMR. As such, a gauge invariant meaning can indeed be attached to $\left\langle A_\mu^2 \right\rangle$ in the Landau gauge, as it was also expressed in [9].

Another interesting property of the Landau gauge is that the operator A_μ^2 is BRST invariant on-shell. If one considered alternative gauges to the Landau gauge, one could search for a class of gauges in which the operator A_μ^2 can be generalized to a mass dimension two operator while maintaining the on-shell BRST invariance. Doing so, one should consider a class of non-linear covariant gauges, which are the so-called Curci-Ferrari gauges [32, 33], where A_μ^2 is generalized to the mixed gluon-ghost operator $\left(\frac{1}{2}A_\mu^a A^{\mu a} + \alpha \overline{c}^a c^a\right)$ [34, 35]. The latter operator is indeed

BRST invariant on-shell [34, 35], and has been proven to be multiplicatively renormalizable to all orders [36], and to give rise to a dynamical gluon mass in the Curci-Ferrari gauge [37]. Moreover, in [38], the physical meaning of $\left(\frac{1}{2}A^a_\mu A^{\mu a} + \alpha\overline{c}^a c^a\right)$ was discussed, based on on-shell BRST invariance.

In the Maximal Abelian gauge, which is a renormalizable gauge in the continuum [39, 40], one should consider the operator $\left(\frac{1}{2}A^\beta_\mu A^{\mu\beta} + \xi\overline{c}^\beta c^\beta\right)$ where the group index β labels the off-diagonal generators of $SU(N)$ with $\beta = 1, \ldots, N(N-1)$ and ξ is the gauge parameter of the Maximal Abelian gauge. This operator also enjoys the property of being both BRST invariant on-shell [34, 35] and multiplicatively renormalizable to all orders [36, 41]. Although the effective potential for the condensate $\left\langle\frac{1}{2}A^\beta_\mu A^{\mu\beta} + \xi\overline{c}^\beta c^\beta\right\rangle$ has not yet been obtained, we expect it to have a non-vanishing vacuum expectation value, which would result in a dynamical mass for the off-diagonal gluons. This is based on the close similarity between the Maximal Abelian gauge and the Curci-Ferrari gauge and hence the results of [37].

More commonly, the Landau gauge is a special case of the well known linear covariant gauges. Although the operator A^2_μ is not even BRST invariant on-shell in these gauges, it is still renormalizable to any order in perturbation theory [1]. This is due to the fact that, thanks to the additional shift symmetry, $\overline{c} \to \overline{c} + \text{const}$, of the antighost in the linear covariant gauges, the composite operator A^2_μ does not mix into the dimension two ghost operator $\overline{c}c$. In this article, we will construct the effective potential for the dimension two condensate $\left\langle A^2_\mu\right\rangle$ in linear gauges and show that a non-vanishing value of $\left\langle A^2_\mu\right\rangle$ is energetically favourable, resulting in dynamical gluon mass generation.

The paper is organized as follows. In section 2, we briefly review the local composite operators formalism and explicitly calculate the 1-loop effective potential. In section 3, we discuss the gauge parameter independence of the vacuum energy which requires an extension of the LCO formalism. The behaviour of the gluon propagator is discussed briefly in section 4 whilst we provide concluding comments in section 5.

2 LCO formalism and effective potential for A^2_μ.

2.1 Construction of a renormalizable effective action for A^2_μ.

We begin with the Yang-Mills action in linear covariant gauges

$$\begin{aligned} S &= S_{YM} + S_{GF+FP} \\ &= -\frac{1}{4}\int d^4x F^a_{\mu\nu}F^{a\mu\nu} + \int d^4x \left(b^a\partial_\mu A^{a\mu} + \frac{\alpha}{2}b^a b^a + \overline{c}^a\partial^\mu D^{ab}_\mu c^b\right) , \end{aligned} \tag{2}$$

where

$$D^{ab}_\mu \equiv \partial_\mu\delta^{ab} - gf^{abc}A^c_\mu , \tag{3}$$

is the covariant derivate in the adjoint representation. In order to study the local composite operator A^2_μ, we introduce it into the action by means of a BRST doublet [2] of external sources (J, λ), namely

$$S_J = s\int d^4x \left(\frac{1}{2}\lambda A^2_\mu + \frac{\zeta}{2}\lambda J\right) = \int d^4x \left(\frac{1}{2}JA^2_\mu + \lambda A^a_\mu\partial^\mu c^a + \frac{\zeta}{2}J^2\right) , \tag{4}$$

where s denotes the BRST nilpotent operator acting as

$$sA^a_\mu = -D^{ab}_\mu c^b ,$$

$$
\begin{aligned}
sc^a &= \frac{1}{2} g f^{abc} c^b c^c \ , \\
s\overline{c}^a &= b^a \ , \\
sb^a &= 0 \ , \\
s\lambda &= J \ , \\
sJ &= 0 \ .
\end{aligned} \tag{5}
$$

According to the local composite operator technique [9, 42, 43, 44], the dimensionless parameter ζ is needed to account for the divergences present in the vacuum Green function $\left\langle A^2_\mu(x) A^2_\nu(y) \right\rangle$, which turn out to be proportional to J^2. As is apparent from the expressions (2) and (4), the action $(S_{YM} + S_{GF+FP} + S_J)$ is BRST invariant

$$
s\left(S_{YM} + S_{GF+FP} + S_J\right) = 0 \ . \tag{6}
$$

As was shown in [1], the action $(S_{YM} + S_{GF+FP} + S_J)$ enjoys the property of being multiplicatively renormalizable to all orders of perturbation theory.

To obtain the effective potential, we set the source λ to zero and consider the renormalized generating functional

$$
\exp -i\mathcal{W}(J) = \int [D\varphi] \exp iS(J) \ , \tag{7}
$$

with

$$
S(J) = S_{YM} + S_{GF+FP} + S_{CT} + \int d^4x \left(Z_2 J \frac{A^2_\mu}{2} + (\zeta + \delta\zeta) \frac{J^2}{2} \right) \ , \tag{8}
$$

where φ denotes the relevant fields and S_{CT} is the usual counterterm contribution. Also, $\delta\zeta$ is the counterterm accounting for the divergences proportional to J^2. The bare quantities are given by [1]

$$
\begin{aligned}
A^{\mu a}_o &= Z_A^{1/2} A^{\mu a} \ , \qquad c^a_o = Z_c^{1/2} c^a \ , \qquad \overline{c}^a_o = Z_c^{1/2} \overline{c}^a \ , \qquad b^a_o = Z_A^{-1/2} b^a \ , \\
g_o &= Z_g g \ , \qquad \alpha_o = Z_A \alpha \ , \qquad \zeta_o = Z_\zeta \zeta \ , \qquad J_o = Z_J J \ ,
\end{aligned} \tag{9}
$$

where $Z_\zeta \zeta = \zeta + \delta\zeta$ and $Z_J = \frac{Z_2}{Z_A}$. The functional $\mathcal{W}(J)$ obeys the renormalization group equation (RGE)

$$
\left(\mu \frac{\partial}{\partial\mu} + \beta(g^2) \frac{\partial}{\partial g^2} + \alpha\gamma_\alpha(g^2) \frac{\partial}{\partial\alpha} - \gamma_{A^2}(g^2) \int d^4x J \frac{\delta}{\delta J} + \eta(g^2, \zeta) \frac{\partial}{\partial\zeta} \right) \mathcal{W}(J) = 0 \ , \tag{10}
$$

where

$$
\begin{aligned}
\beta(g^2) &= \mu \frac{\partial}{\partial\mu} g^2 \ , \\
\gamma_\alpha(g^2) &= \mu \frac{\partial}{\partial\mu} \ln\alpha = \mu \frac{\partial}{\partial\mu} \ln Z_A^{-1} = -2\gamma_A(g^2) \ , \\
\gamma_{A^2}(g^2) &= \mu \frac{\partial}{\partial\mu} \ln Z_J \ , \\
\eta(g^2, \zeta) &= \mu \frac{\partial}{\partial\mu} \zeta \ .
\end{aligned} \tag{11}
$$

From the bare Lagrangian, we infer that

$$
\zeta_o J^2_o = \mu^{-\varepsilon} (\zeta + \delta\zeta) J^2 \ , \tag{12}
$$

where we will use dimensional regularization throughout with the convention that $d = 4 - \varepsilon$. Hence

$$\mu\frac{\partial}{\partial\mu}\zeta = \eta(g^2,\zeta) = 2\gamma_{A^2}(g^2)\zeta + \delta(g^2,\alpha) \; , \tag{13}$$

with

$$\delta(g^2,\alpha) = \left(\varepsilon + 2\gamma_{A^2}(g^2,\alpha) - \beta(g^2)\frac{\partial}{\partial g^2} - \alpha\gamma_\alpha(g^2,\alpha)\frac{\partial}{\partial\alpha}\right)\delta\zeta \; . \tag{14}$$

Now, we are faced with the problem of the hitherto arbitrary parameter ζ. As explained in [9, 42, 43, 44], setting $\zeta = 0$ would give rise to an inhomogeneous RGE for $\mathcal{W}(J)$

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g^2)\frac{\partial}{\partial g^2} + \alpha\gamma_\alpha(g^2)\frac{\partial}{\partial\alpha} - \gamma_{A^2}(g^2)\int d^4x J\frac{\delta}{\delta J}\right)\mathcal{W}(J) = \delta(g^2,\alpha)\int d^4x\frac{J^2}{2} \; , \tag{15}$$

and a non-linear RGE for the effective action Γ for the composite operator A_μ^2. This problem can be overcome by making ζ a function of g^2 and α so that, if g^2 runs according to $\beta(g^2)$ and α according to $\gamma_\alpha(g^2)$, $\zeta(g^2,\alpha)$ will run according to (13). This is accomplished by setting ζ equal to the solution of the differential equation

$$\left(\beta(g^2)\frac{\partial}{\partial g^2} + \alpha\gamma_\alpha(g^2,\alpha)\frac{\partial}{\partial\alpha}\right)\zeta(g^2,\alpha) = 2\gamma_{A^2}(g^2)\zeta(g^2,\alpha) + \delta(g^2,\alpha) \; . \tag{16}$$

Since $\zeta(g^2,\alpha)$ now automatically runs according to its RGE, $\mathcal{W}(J)$ obeys the homogeneous renormalization group equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g^2)\frac{\partial}{\partial g^2} + \alpha\gamma_\alpha(g^2)\frac{\partial}{\partial\alpha} - \gamma_{A^2}(g^2)\int d^4x J\frac{\delta}{\delta J}\right)\mathcal{W}(J) = 0 \; . \tag{17}$$

The final step in the formal construction of the effective potential for $\left\langle A_\mu^2\right\rangle$ is the removal of the J^2 terms from the Lagrangian by means of a renormalized Hubbard-Stratonovich transformation. By this procedure, the energy interpretation of the effective action is made explicit again and the conventional 1PI machinery applies. We insert unity written as

$$1 = \frac{1}{N}\int[D\sigma]\exp\left[i\int d^4x\left(-\frac{1}{2Z_\zeta\zeta}\left(\frac{\sigma}{g} - Z_2\frac{A_\mu^2}{2} - Z_\zeta\zeta J\right)^2\right)\right] \; , \tag{18}$$

with N the appropriate normalization factor, in (7) to arrive at the Lagrangian

$$\mathcal{L}(A_\mu,\sigma) = -\frac{1}{4}F_{\mu\nu}^2 + \mathcal{L}_{GF+FP} + \mathcal{L}_{CT} - \frac{\sigma^2}{2g^2Z_\zeta\zeta} + \frac{1}{2}\frac{Z_2}{g^2Z_\zeta\zeta}g\sigma A_\mu^2 - \frac{1}{8}\frac{Z_2^2}{Z_\zeta\zeta}\left(A_\mu^2\right)^2 \; , \tag{19}$$

while

$$\begin{aligned} \exp -i\mathcal{W}(J) &= \int[D\varphi]\exp iS_\sigma(J) \; , & (20)\\ S_\sigma(J) &= \int d^4x\left(\mathcal{L}(A_\mu,\sigma) + J\frac{\sigma}{g}\right) \; . & (21)\end{aligned}$$

Now, the source J appears as a linear source term for $\frac{\sigma}{g}$. From (7) and (20), one has the following identification

$$\left.\frac{\delta\mathcal{W}(J)}{\delta J}\right|_{J=0} = -\left\langle\frac{A_\mu^2}{2}\right\rangle = -\left\langle\frac{\sigma}{g}\right\rangle \; , \tag{22}$$

where we will not write the renormalization factors from now on. This equation states that the gauge condensate $\left\langle A_\mu^2\right\rangle$ is related to the expectation value of the field σ, evaluated with the new action, $\int d^4x\mathcal{L}(A_\mu,\sigma)$, of (19).

Although we have not considered the contribution from (massless) quark fields in the previous analysis, it can be checked that the results remain unchanged if matter fields are included.

2.2 Explicit calculation of the 1-loop effective potential.

Firstly, we will determine the renormalization group function $\delta(g^2,\alpha)$ as defined in (14). All the following results will be within the $\overline{\text{MS}}$ scheme. The value for $\beta(g^2)$ can be found in the literature. In d dimensions, one has

$$\begin{aligned}
\beta(g^2) &= -\varepsilon g^2 - 2\left(\beta_0 g^4 + \beta_1 g^6 + O(g^8)\right) , \\
\beta_0 &= \frac{1}{16\pi^2}\left(\frac{11}{3}C_A - \frac{4}{3}T_F N_f\right) , \\
\beta_1 &= \frac{1}{(16\pi^2)^2}\left(\frac{34}{3}C_A^2 - 4C_F T_F N_f - \frac{20}{3}C_A T_F N_f\right) .
\end{aligned} \tag{23}$$

where the Casimirs of the colour group are defined by $\mathrm{Tr}(T^aT^b) = T_F\delta^{ab}$, $T^aT^a = C_F I$, $f^{acd}f^{bcd} = C_A\delta^{ab}$ and N_f is the number of quark flavours. For $\gamma_\alpha(g^2)$, we use the relation $\gamma_\alpha(g^2) = -2\gamma_A(g^2)$. The anomalous dimension $\gamma_A(g^2)$ of the gluon field in linear covariant gauges was calculated at three loops in $\overline{\text{MS}}$ in [45]. Adapting that result to our convention, the anomalous dimension of the gauge parameter is

$$\begin{aligned}
\gamma_\alpha(g^2) &= a_0 g^2 + a_1 g^4 + O(g^6) , \\
a_0 &= \frac{1}{16\pi^2}\left(C_A\left(\frac{13}{3} - \alpha\right) - \frac{8}{3}T_F N_f\right) , \\
a_1 &= \frac{1}{(16\pi^2)^2}\left(C_A^2\left(\frac{59}{4} - \frac{11}{4}\alpha - \frac{1}{2}\alpha^2\right) - 10C_A N_f T_F - 8C_F N_f T_F\right) .
\end{aligned} \tag{24}$$

The anomalous dimension, $\gamma_{A^2}(g^2)$, of the composite operator A_μ^2 was calculated in [1] and reads

$$\begin{aligned}
\gamma_{A^2}(g^2) &= \gamma_0 g^2 + \gamma_1 g^4 + O(g^6) , \\
\gamma_0 &= \frac{1}{6}\frac{1}{16\pi^2}\left[(35+3\alpha)\,C_A - 16T_F N_f\right] , \\
\gamma_1 &= \frac{1}{24}\frac{1}{(16\pi^2)^2}\left[\left(449 + 33\alpha + 18\alpha^2\right)C_A^2 - 280C_A T_F N_f - 192C_F T_F N_f\right] .
\end{aligned} \tag{25}$$

In order to determine $\delta(g^2,\alpha)$, we still require the counterterm $\delta\zeta$. In principle, this can be directly calculated from the divergences in $\mathcal{W}(J)$ when the propagator for a gluon with mass J is used. However, a less cumbersome way to compute $\delta\zeta$ was described in [11]. It is based on the fact that the divergences arise in the $O(J^2)$ term and therefore that part of the Green's function which contains these divergences is equivalent to the Green's function with a double insertion of the JA_μ^2 operator. More specifically, one has two external J insertions with a non-zero momentum flowing into one insertion where the only internal couplings are those of the usual QCD action. Moreover, one does not require massive propagators but instead can use massless fields which simplifies the calculation. Therefore one is reduced to computing a massless two-point function for which the MINCER algorithm, [46], was designed. We used the version written in FORM, [47, 48], where the Feynman diagrams are generated by QGRAF, [49], to determine the divergence structure to three loops. Although we only require the result to two loops the extra loop evaluation in fact acts as a non-trivial check on the two loop result. This is because the emergence of the correct double and triple poles in ε at three loops, in a way which is consistent with the renormalization group, verifies that the single and double poles of the two loop expression for $\delta\zeta$ are correct. We found

$$\begin{aligned}
\delta\zeta &= \frac{2}{\varepsilon}\frac{N_A}{16\pi^2}\left(-\frac{3}{2} - \frac{\alpha^2}{2}\right) + \frac{N_A g^2}{(16\pi^2)^2}\left[\frac{4}{\varepsilon^2}\left(C_A\left(\frac{35}{8} + \frac{3}{8}\alpha + \frac{3}{8}\alpha^2 + \frac{3}{8}\alpha^3\right) - 2T_F N_f\right)\right. \\
&+ \left.\frac{2}{\varepsilon}\left(C_A\left(-\frac{139}{12} - \frac{5}{8}\alpha - \frac{1}{2}\alpha^2 - \frac{1}{8}\alpha^3\right) + \frac{8}{3}T_F N_f\right)\right] + O(g^4) ,
\end{aligned} \tag{26}$$

where N_A is the dimension of the adjoint representation of the colour group. Assemblying our results leads to

$$\begin{aligned}
\delta(g^2,\alpha) &= \delta_0 + \delta_1 g^2 + O(g^4) \; , \\
\delta_0 &= \frac{N_A}{16\pi^2}\left(-3-\alpha^2\right) \; , \\
\delta_1 &= \frac{1}{6}\frac{N_A}{(16\pi^2)^2}\left(C_A\left(-278-15\alpha-12\alpha^2-3\alpha^3\right)+64T_F N_f\right) \; .
\end{aligned} \tag{27}$$

As a check we see that $\delta(g^2,\alpha)$ contains no poles for $\varepsilon \to 0$. Further, the expressions (27) lead to the same results which were obtained earlier in the case of the Landau gauge ($\alpha = 0$), as can be inferred from [9] without quarks and [11] with quarks.

From the renormalization group functions (23), (24), (25) and (27), it is easy to see that the equation (16) can be solved for by expanding $\zeta(g^2,\alpha)$ in a Laurent series as

$$\zeta(g^2,\alpha) = \frac{\zeta_0(\alpha)}{g^2} + \zeta_1(\alpha) + O(g^2) \; . \tag{28}$$

Substituting this expression in equation (16), we obtain

$$2\beta_0\zeta_0 + \alpha a_0 \frac{\partial\zeta_0}{\partial\alpha} = 2\gamma_0\zeta_0 + \delta_0 \; , \tag{29}$$

$$2\beta_1\zeta_0 + \alpha a_0 \frac{\partial\zeta_1}{\partial\alpha} + \alpha a_1 \frac{\partial\zeta_0}{\partial\alpha} = 2\gamma_0\zeta_1 + 2\gamma_1\zeta_0 + \delta_1 \; . \tag{30}$$

Thus (29) gives

$$\zeta_0(\alpha) = \frac{2\alpha C_0 + 3\left(78-26\alpha^2+3\alpha^3+18\alpha\ln|\alpha|\right)C_A N_A + 48\left(\alpha^2-3\right)N_A N_f T_F}{2\left((3\alpha-13)C_A+8N_f T_F\right)^2} \; , \tag{31}$$

with C_0 a constant of integration. As a consequence of the already rather complicated structure of ζ_0, we will determine ζ_1 without quarks present corresponding to $N_f = 0$ since the expression for ζ_1 with $N_f \neq 0$ is several pages long. Using `Mathematica`, we find

$$\begin{aligned}
\zeta_1(\alpha) = & -\frac{1}{1220736\pi^2(13-3\alpha)^4}\Bigg(-1220736\pi^2\alpha^{35/13}\left|-13+3\alpha\right|^{4/13}C_1 \\
& + 12716 C_0\alpha^2\left(-442-132\alpha+54\alpha^2-1287\left(1-\frac{3\alpha}{13}\right)^{4/13}\alpha\,{}_2F_1\left[\frac{4}{13},\frac{4}{13};\frac{17}{13};\frac{3\alpha}{13}\right]\right)C_A \\
& + \Bigg(1697175909\left(1-\frac{3\alpha}{13}\right)^{4/13}\alpha^3\,{}_3F_2\left[\frac{4}{13},\frac{4}{13},\frac{4}{13};\frac{17}{13},\frac{17}{13};\frac{3\alpha}{13}\right] \\
& + 3335904\left(1-\frac{3\alpha}{13}\right)^{4/13}\alpha^4\,{}_3F_2\left[\frac{17}{13},\frac{17}{13},\frac{17}{13};\frac{30}{13},\frac{30}{13};\frac{3\alpha}{13}\right] \\
& + 17\Big(-396870474+368850105\alpha-48761440\alpha^2+2066214\alpha^3+1928718\alpha^4 \\
& - 1004751\alpha^5+60588\alpha^6-12894024\left(1-\frac{3\alpha}{13}\right)^{4/13}\alpha^2\,{}_2F_1\left[-\frac{9}{13},-\frac{9}{13};\frac{4}{13};\frac{3\alpha}{13}\right] \\
& - 833976\left(1-\frac{3\alpha}{13}\right)^{4/13}\alpha^4\,{}_2F_1\left[\frac{4}{13},\frac{17}{13};\frac{30}{13};\frac{3\alpha}{13}\right]-8926632\alpha^2\ln|\alpha| \\
& + 2059992\alpha^3\ln|\alpha|+833976\left(1-\frac{3\alpha}{13}\right)^{4/13}\alpha^4\,{}_2F_1\left[\frac{4}{13},\frac{17}{13};\frac{30}{13};\frac{3\alpha}{13}\right]\ln|\alpha| \\
& - 43758\alpha^3\left(-1961+702\ln|\alpha|\right)\Big)\Bigg)N_A\Bigg) \; ,
\end{aligned} \tag{32}$$

where C_1 is a constant of integration and the (generalized) hypergeometric function is

$$ {}_pF_q\left[a_1,\cdots,a_p;b_1,\cdots,b_q;z\right] = \sum_{k=0}^{+\infty} \frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!} \;, \tag{33} $$

where

$$ (a)_k = a(a+1)\cdots(a+k-1) \;, \tag{34} $$

is the Pochhammer symbol. We note that $\zeta_0(\alpha=0) = \frac{9N_A}{13C_A}$ and $\zeta_1(\alpha=0) = \frac{161}{832}\frac{N_A}{\pi^2}$, which recovers the Landau gauge results of [9, 11]. Further, the constants of integration C_0 and C_1 do no enter the Landau gauge results.

From expression (19), we deduce that the tree level gluon mass is provided by

$$ m^2 = \frac{g\sigma}{\zeta_0} \;, \tag{35} $$

while the 1-loop effective potential becomes

$$ \begin{aligned} V_1(\sigma) &= \frac{\sigma^2}{2\zeta_0}\left(1-\frac{\zeta_1}{\zeta_0}g^2\right) + \frac{1}{2}\ln\det\left[\delta^{ab}\left(\delta^{\mu\nu}(\partial^2+m^2) - \left(1-\frac{1}{\alpha}\right)\partial^\mu\partial^\nu\right)\right] \\ &= \frac{\sigma^2}{2\zeta_0}\left(1-\frac{\zeta_1}{\zeta_0}g^2\right) + \frac{N_A}{2}\left[(d-1)\mathrm{tr}\ln\left(\partial^2+m^2\right) + \mathrm{tr}\ln\left(\partial^2+\alpha m^2\right)\right] \;. \end{aligned} \tag{36} $$

In dimensional regularization and using the $\overline{\mathrm{MS}}$ scheme, one finds

$$ \begin{aligned} V_1(\sigma) &= \frac{\sigma^2}{2\zeta_0}\left(1-\frac{\zeta_1}{\zeta_0}g^2\right) + \frac{3N_A}{64\pi^2}\frac{g^2\sigma^2}{\zeta_0^2}\left(-\frac{5}{6}+\ln\frac{g\sigma}{\zeta_0\overline{\mu}^2}\right) \\ &+ \frac{N_A}{64\pi^2}\frac{\alpha^2g^2\sigma^2}{\zeta_0^2}\left(-\frac{3}{2}+\ln\frac{\alpha g\sigma}{\zeta_0\overline{\mu}^2}\right) \;, \end{aligned} \tag{37} $$

where $\overline{\mu}$ is the renormalization scale. It can be easily checked that the infinities in the effective potential cancel when the counterterms are included.

Next, we look for a non-trivial minimum of the effective potential, which amounts to solving the gap equation $\frac{dV}{d\sigma}=0$. To avoid possibly large logarithms, we will set $\overline{\mu}^2 = m^2 = \frac{g\sigma}{\zeta_0}$ in the gap equation,

$$ \begin{aligned} \left.\frac{dV}{d\sigma}\right|_{\overline{\mu}^2=\frac{g\sigma}{\zeta_0}} &= \frac{\sigma}{\zeta_0}\left(1-\frac{\zeta_1}{\zeta_0}g^2\right) + \frac{3N_A}{32\pi^2}\frac{g^2\sigma}{\zeta_0^2}\left(-\frac{5}{6}\right) + \frac{3N_A}{64\pi^2}\frac{g^2\sigma}{\zeta_0^2} \\ &+ \frac{N_A}{32\pi^2}\frac{\alpha^2g^2\sigma}{\zeta_0^2}\left(-\frac{3}{2}+\ln\alpha\right) + \frac{N_A}{64\pi^2}\frac{\alpha^2g^2\sigma}{\zeta_0^2} = 0, \end{aligned} \tag{38} $$

and use the RGE to sum leading logarithms. Defining $y \equiv \frac{g^2N}{16\pi^2}$, we find as a solution of (38)

$$ \sigma = 0 \text{ or } y = \frac{C_A\zeta_0}{16\pi^2\zeta_1 + \frac{N_A}{2}\left(1+\alpha^2-\alpha^2\ln|\alpha|\right)} \;. \tag{39} $$

The first solution corresponds to the trivial vacuum, while the second one leads to

$$ m = \Lambda_{\overline{\mathrm{MS}}}e^{\frac{3}{22y}} \;, \tag{40} $$

where the 1-loop formula for the coupling constant

$$ g^2(\overline{\mu}) = \frac{1}{\beta_0\ln\frac{\overline{\mu}^2}{\Lambda^2_{\overline{\mathrm{MS}}}}} \;, \tag{41} $$

was used. The vacuum energy is given by

$$E_{\text{vac}} = -\frac{1}{2}\frac{N_A}{64\pi^2}\left(3+\alpha^2\right)m^4\ . \tag{42}$$

We now consider the numerical evaluation of our results and restrict ourselves to the colour group $SU(3)$. For $SU(N)$ one has $T_F = \frac{1}{2}$, $C_F = \frac{N^2-1}{2N}$, $C_A = N$ and $N_A = N^2-1$. For completeness, we quote the results for the Landau gauge $\alpha = 0$.

$$y^{\text{Landau}} = \frac{36}{187} \approx 0.193\ , \tag{43}$$

$$m^{\text{Landau}} = e^{\frac{17}{24}}\Lambda_{\overline{\text{MS}}} \approx 2.031\Lambda_{\overline{\text{MS}}}\ , \tag{44}$$

$$E_{\text{vac}}^{\text{Landau}} = -\frac{3}{16\pi^2}e^{\frac{17}{6}}\Lambda_{\overline{\text{MS}}}^4 \approx -0.323\Lambda_{\overline{\text{MS}}}^4\ . \tag{45}$$

The results for general α are displayed in the Figures 1-3. For the moment, we have set

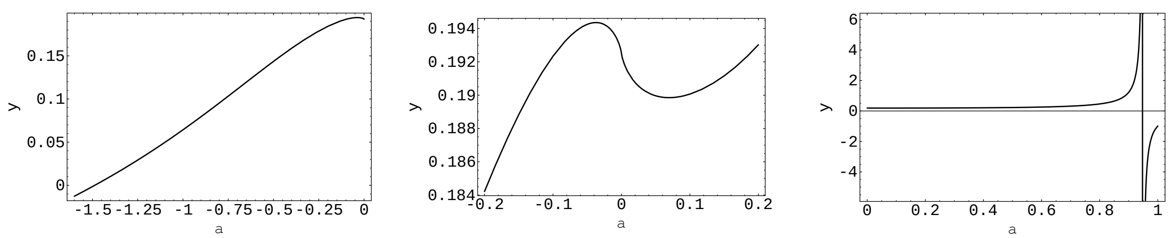

Figure 1: y as a function of α.

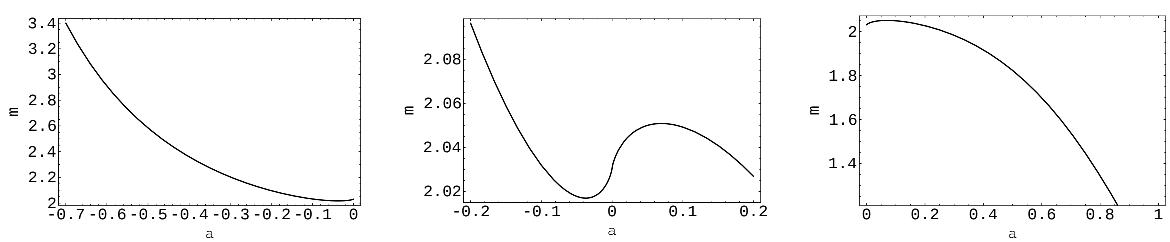

Figure 2: m as a function of α.

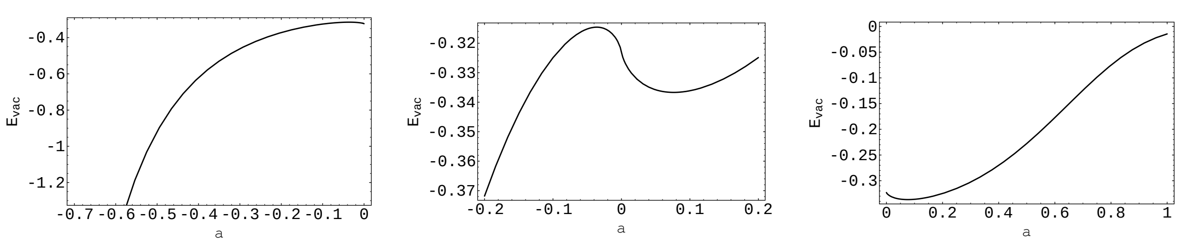

Figure 3: E_{vac} as a function of α.

$C_0 = C_1 = 0$. Evidently, y should certainly be positive and also relatively small to have a sensible expansion. Hence, we conclude from Figure 1 that we should restrict the range of values for α further. We also see that m becomes rapidly larger and E_{vac} becomes rapidly more and more negative as α gets more negative. A more urgent problem is the fact that the vacuum energy E_{vac} depends on the gauge parameter α. Since E_{vac} is a physical quantity, it should be independent on the gauge parameter α. In the next section, we shall give a detailed account of this gauge parameter dependence. We shall see that it is related to the impossibility of evaluating the effective potential to arbitrary high loop orders. Further, we shall provide a simple way to circumvent this problem and obtain a vacuum energy which is independent of α.

3 Investigation of the gauge parameter dependence.

One possible explanation as to why $E_{\rm vac}$ depends on α could reside in the values of the constants of integration C_0 and C_1 we have chosen. With another choice for these constants, it could be that $E_{\rm vac}$ does not depend in α, or equivalently $E_{\rm vac} = E_{\rm vac}^{\rm Landau}$. This can be investigated by considering the expression (42) for $E_{\rm vac}(\alpha, C_0, C_1)$. In order to have the same $E_{\rm vac}$ for each value of α, we should solve the following equation

$$\begin{aligned} \frac{dE_{\rm vac}}{d\alpha} = 0 \quad &\Leftrightarrow \quad 2\alpha m^4 + 4(\alpha^2+3)m^3\frac{dm}{d\alpha} = 0 \\ &\Leftrightarrow \quad \alpha - \frac{3}{11y^2}(3+\alpha^2)\left(\frac{\partial y}{\partial \alpha} + \frac{\partial y}{\partial \zeta_0}\frac{\partial \zeta_0}{\partial \alpha} + \frac{\partial y}{\partial \zeta_1}\frac{\partial \zeta_1}{\partial \alpha}\right) = 0 \;, \end{aligned} \tag{46}$$

in terms of C_0 and C_1. However, the solutions of this equation depend on α, and this is not allowed since C_0 and C_1 should be α independent constants. This means that the α-dependence of $E_{\rm vac}$ cannot be eliminated by a suitable choice of C_0 and C_1.

3.1 BRST symmetry and gauge parameter independence.

Let us now turn to a more general analysis. Consider again the generating functional (20). We have the following identification, ignoring the overall normalization factors

$$\begin{aligned} \exp -i\mathcal{W}(J) &= \int [D\varphi]\exp iS_\sigma(J) \\ &= \frac{1}{N}\int [D\varphi D\sigma]\exp i\left[S(J) + \int d^4x\left(-\frac{1}{2\zeta}\left(\frac{\sigma}{g} - \frac{A_\mu^2}{2} - \zeta J\right)^2\right)\right] \;, \end{aligned} \tag{47}$$

where $S(J)$ and $S_\sigma(J)$ are given respectively by (8), and (21). Since

$$\frac{d}{d\alpha}\frac{1}{N}\int [D\sigma]\exp\left[i\int d^4x\left(-\frac{1}{2\zeta}\left(\frac{\sigma}{g} - \frac{A_\mu^2}{2} - \zeta J\right)^2\right)\right] = \frac{d}{d\alpha}1 = 0 \;, \tag{48}$$

we find

$$-\frac{d\mathcal{W}(J)}{d\alpha} = \left\langle s\int d^4x\left(\frac{\overline{c}b}{2}\right)\right\rangle_{J=0} + \text{terms proportional to } J \;, \tag{49}$$

which follows by noticing that

$$\begin{aligned} \frac{dS(J)}{d\alpha} &= \int d^4x\left(\frac{b^a b^a}{2} + \frac{\partial\zeta}{\partial\alpha}\frac{J^2}{2}\right) \\ &= s\int d^4x\left(\frac{\overline{c}b}{2}\right) + \text{terms proportional to } J \;. \end{aligned} \tag{50}$$

We see that the first term in the right hand side of (50) is an exact BRST variation. As such, its vacuum expectation value vanishes. This is the usual argument to prove the gauge parameter independence in the BRST framework [2]. Of course, this is based on the assumption that the BRST symmetry is not broken. Notice therefore that there does not exist an operator $\mathcal{G}$ with

$A_\mu^2 = s\mathcal{G}$, so that a non-vanishing vacuum expectation value for the condensate $\left\langle A_\mu^2 \right\rangle$ does not break the BRST invariance. Indeed, from

$$s\sigma = \frac{g}{2} s A_\mu^2 = -g A_\mu^a \partial^\mu c^a \ , \tag{51}$$

one can easily check that

$$s \int d^4x \mathcal{L}(A_\mu, \sigma) = 0 \ , \tag{52}$$

so that we have a BRST invariant σ-action.

The rest of the argument is based on the fact that $J = 0$ when the vacuum is considered, so that we are left with only the BRST exact term in (49). More formally, the effective action $\Gamma(\sigma) \equiv \Gamma\left(\frac{\sigma}{g}\right)$ is related to $\mathcal{W}(J)$ through a Legendre transformation

$$\Gamma\left(\frac{\sigma}{g}\right) = -\mathcal{W}(J) - \int d^4y J(y) \frac{\sigma(y)}{g} \ . \tag{53}$$

The effective potential $V(\sigma)$ is then defined as

$$-V(\sigma) \int d^4x = \Gamma\left(\frac{\sigma}{g}\right) \ . \tag{54}$$

Let σ_{min} be the solution of

$$\frac{dV(\sigma)}{d\sigma} = 0 \ . \tag{55}$$

Since

$$\frac{\delta}{\delta\left(\frac{\sigma}{g}\right)} \Gamma = -J \ , \tag{56}$$

one finds

$$\sigma = \sigma_{min} \Rightarrow J = 0 \ , \tag{57}$$

and invoking (57), from (53) and (54) we derive

$$\left.\frac{d}{d\alpha} V(\sigma)\right|_{\sigma=\sigma_{min}} \int d^4x = \left.\frac{d}{d\alpha} \mathcal{W}(J)\right|_{J=0} \ . \tag{58}$$

Finally, combining (49) and (58)

$$\left.\frac{d}{d\alpha} V(\sigma)\right|_{\sigma=\sigma_{min}} = 0 \ . \tag{59}$$

From this, we conclude that the vacuum energy $E_{\rm vac}$ should be independent of the gauge parameter α.

Apparently, our explicit result (42) for $E_{\rm vac}$ is not in agreement with the above proof that $E_{\rm vac}$ is the same for each α. If we examine the proof in more detail we notice that a key argument is that J becomes zero at the end of the calculation. In practice, this is achieved by solving the gap equation. Now, in a power series expansion in the coupling constant, the derivative of the effective potential with respect to σ is something of the form

$$\left(v_0 + v_1 g^2 + 0(g^4)\right) \sigma \ , \tag{60}$$

where we assume that we work up to order g^2 and that we have chosen $\overline{\mu}$ so that the logarithms vanish. Then, the gap equation corresponding to (60) reads

$$v_0 + v_1 g^2 + O(g^4) = 0 \ . \tag{61}$$

Due to (54) and (56), one also has

$$J = g\left(v_0 + v_1 g^2 + O(g^4)\right)\sigma \ . \tag{62}$$

This means that, if we solve the gap equation (61) up to certain order, we have

$$J = g\left(0 + O(g^4)\right)\sigma \ . \tag{63}$$

We also have

$$\frac{\partial\zeta}{\partial\alpha} = \frac{\partial\zeta_0}{\partial\alpha}\frac{1}{g^2} + \frac{\partial\zeta_1}{\partial\alpha} + O(g^4) \ . \tag{64}$$

So, working to the order we are considering

$$\frac{\partial\zeta}{\partial\alpha}J^2 = \left(\frac{\partial\zeta_0}{\partial\alpha}v_0^2 + \left(\frac{\partial\zeta_0}{\partial\alpha}2v_0v_1 + \frac{\partial\zeta_1}{\partial\alpha}v_0^2\right)g^2 + O(g^4)\right)\sigma^2 \ . \tag{65}$$

From the square of the gap equation (61),

$$v_0^2 + 2v_1v_0g^2 + O(g^4) = 0 \ , \tag{66}$$

it follows that

$$\frac{\partial\zeta}{\partial\alpha}J^2 = \left(\frac{\partial\zeta_1}{\partial\alpha}v_0^2g^2 + O(g^4)\right)\sigma^2 \ . \tag{67}$$

We see that, if one consistently works to the order we are considering, terms such as $\frac{\partial\zeta}{\partial\alpha}J^2$ do not equal zero although $J = 0$ to that order. Terms like those on the right hand side of (67) are cancelled by terms which are formally of higher order. This has its consequences for the terms proportional to J in (49). If one were able to work to infinite order, the problem would not arise. However, we do not have this ability, and we are faced with a gauge parameter dependence slipping into E_{vac}.

3.2 Circumventing the gauge parameter dependence.

We could resolve this issue by saying that the gauge parameter dependence of the vacuum energy should become less and less severe as we go to higher orders, and that eventually it will drop out if we go to infinite order. However, this is not very satisfactory, especially since we can surely never calculate the potential up to infinite order. Also as is clear from the quite complicated expression for $\zeta_1(\alpha)$, which will enter the differential equation for $\zeta_2(\alpha)$, a 2-loop evaluation of the effective potential is already out of the question.

Therefore, we could try to modify the LCO formalism in order to circumvent the gauge parameter dependence of E_{vac}. Therefore, we consider the following action

$$\widetilde{S}(\widetilde{J}) \;=\; S_{YM} + S_{GF+FP} + \int d^4x\left[\widetilde{J}\mathcal{F}(\alpha)\frac{A_\mu^2}{2} + \frac{\zeta}{2}\mathcal{F}^2(\alpha)\widetilde{J}^2\right] \ , \tag{68}$$

instead of (8) where, for the moment, $\mathcal{F}(\alpha)$ is an arbitrary function of α of the form

$$\mathcal{F}(\alpha) = 1 + f_0(\alpha)g^2 + f_1(\alpha)g^4 + O(g^6) \ , \tag{69}$$

and $\widetilde{J}$ is now the source. The generating functional becomes

$$\exp -i\widetilde{\mathcal{W}}(\widetilde{J}) = \int [D\phi]\exp i\widetilde{S}(\widetilde{J}) \ . \tag{70}$$

Taking the functional derivative of $\widetilde{\mathcal{W}}(\widetilde{J})$ with respect to $\widetilde{J}$, we obtain

$$\left.\frac{\delta\widetilde{\mathcal{W}}(\widetilde{J})}{\delta\widetilde{J}}\right|_{\widetilde{J}=0} = -\mathcal{F}(\alpha)\left\langle\frac{A_\mu^2}{2}\right\rangle . \tag{71}$$

Again, we insert unity via

$$1 = \frac{1}{N}\int[D\widetilde{\sigma}]\exp\left[i\int d^4x\left(-\frac{1}{2\zeta}\left(\frac{\widetilde{\sigma}}{g\mathcal{F}(\alpha)} - \frac{A_\mu^2}{2} - \zeta\widetilde{J}\mathcal{F}(\alpha)\right)^2\right)\right] , \tag{72}$$

to arrive at the following renormalized Lagrangian

$$\widetilde{\mathcal{L}}(A_\mu,\widetilde{\sigma}) = -\frac{1}{4}F_{\mu\nu}^2 + \mathcal{L}_{GF+FP} - \frac{\widetilde{\sigma}^2}{2g^2\mathcal{F}^2(\alpha)Z_\zeta\zeta} + \frac{1}{2}\frac{Z_2}{g^2\mathcal{F}(\alpha)Z_\zeta\zeta}g\widetilde{\sigma}A_\mu^2 - \frac{1}{8}\frac{Z_2^2}{Z_\zeta\zeta}\left(A_\mu^2\right)^2 + \widetilde{J}\frac{\widetilde{\sigma}}{g} . \tag{73}$$

From the generating functional

$$\exp -i\widetilde{\mathcal{W}}(\widetilde{J}) = \int[D\phi]\exp i\int d^4x\widetilde{\mathcal{L}}(A_\mu,\widetilde{\sigma}) , \tag{74}$$

it follows that

$$\left.\frac{\delta\widetilde{\mathcal{W}}(\widetilde{J})}{\delta\widetilde{J}}\right|_{\widetilde{J}=0} = -\left\langle\frac{\widetilde{\sigma}}{g}\right\rangle \Rightarrow \langle\widetilde{\sigma}\rangle = g\mathcal{F}(\alpha)\left\langle\frac{A_\mu^2}{2}\right\rangle , \tag{75}$$

where the anomalous dimension of $\widetilde{\sigma}$ equals

$$\gamma_{\widetilde{\sigma}}(g^2) = \frac{\mu}{\widetilde{\sigma}}\frac{\partial\widetilde{\sigma}}{\partial\mu} = \frac{\beta(g^2)}{2g^2} + \gamma_{A^2}(g^2) + \mu\frac{\partial\ln\mathcal{F}(\alpha)}{\partial\mu} . \tag{76}$$

The lowest order gluon mass is now provided by

$$m^2 = \frac{g\widetilde{\sigma}}{\zeta_0} , \tag{77}$$

and the vacuum configurations are now determined by solving

$$\frac{d\widetilde{V}(\widetilde{\sigma})}{d\widetilde{\sigma}} = 0 . \tag{78}$$

with $\widetilde{V}(\widetilde{\sigma})$ the effective potential. In the $\overline{\mathrm{MS}}$ scheme, the 1-loop effective potential reads

$$\begin{aligned}
\widetilde{V}_1(\widetilde{\sigma}) &= \frac{\widetilde{\sigma}^2}{2\zeta_0}\left(1 - \left(2f_0 + \frac{\zeta_1}{\zeta_0}\right)g^2 + \frac{2}{\varepsilon}\frac{N_A}{16\pi^2}\frac{g^2}{\zeta_0}\left(\frac{3}{2} + \frac{\alpha^2}{2}\right)\right) \\
&+ \frac{3\,(N_A)}{64\pi^2}\frac{g^2\widetilde{\sigma}^2}{\zeta_0^2}\left(-\frac{2}{\varepsilon} - \frac{5}{6} + \ln\frac{g\widetilde{\sigma}}{\zeta_0\overline{\mu}^2}\right) + \frac{N_A}{64\pi^2}\frac{\alpha^2 g^2\widetilde{\sigma}^2}{\zeta_0^2}\left(-\frac{2}{\varepsilon} - \frac{3}{2} + \ln\frac{\alpha g\widetilde{\sigma}}{\zeta_0\overline{\mu}^2}\right) \\
&= \frac{\widetilde{\sigma}^2}{2\zeta_0}\left(1 - \left(2f_0 + \frac{\zeta_1}{\zeta_0}\right)g^2\right) + \frac{3N_A}{64\pi^2}\frac{g^2\widetilde{\sigma}^2}{\zeta_0^2}\left(-\frac{5}{6} + \ln\frac{g\widetilde{\sigma}}{\zeta_0\overline{\mu}^2}\right) \\
&+ \frac{N_A}{64\pi^2}\frac{\alpha^2 g^2\widetilde{\sigma}^2}{\zeta_0^2}\left(-\frac{3}{2} + \ln\frac{\alpha g\widetilde{\sigma}}{\zeta_0\overline{\mu}^2}\right) .
\end{aligned} \tag{79}$$

We included the counterterm contribution here to illustrate explicitly that $\widetilde{V}_1(\widetilde{\sigma})$ is finite. With (76), it can also be checked that

$$\mu\frac{d}{d\mu}\widetilde{V}_1(\widetilde{\sigma}) = 0 + O(g^4) . \tag{80}$$

Now, we can continue with the determination of the 1-loop vacuum energy, which will not only depend on α, C_0 and C_1, but also on $f_0(\alpha)$. We will determine an expression for $f_0(\alpha)$ so that $E_{\text{vac}}(\alpha, C_0, C_1, f_0(\alpha))$ does not depend on α. In the meantime, we could also absorb the constants of integration C_0 and C_1 in $f_0(\alpha)$ so that E_{vac} does not depend on them either. Based on this, we will immediately set $C_0 = C_1 = 0$. As usual, we put $\overline{\mu}^2 = \frac{g\widetilde{\sigma}}{\zeta_0}$ in the gap equation, which now reads

$$
\begin{aligned}
\left.\frac{d\widetilde{V}}{d\widetilde{\sigma}}\right|_{\overline{\mu}^2=\frac{g\widetilde{\sigma}}{\zeta_0}} &= \frac{\widetilde{\sigma}}{\zeta_0}\left(1-\left(2f_0+\frac{\zeta_1}{\zeta_0}\right)g^2\right)+\frac{3N_A}{32\pi^2}\frac{g^2\widetilde{\sigma}}{\zeta_0^2}\left(-\frac{5}{6}\right)+\frac{3N_A}{64\pi^2}\frac{g^2\widetilde{\sigma}}{\zeta_0^2} \\
&+ \frac{N_A}{32\pi^2}\frac{\alpha^2 g^2\widetilde{\sigma}}{\zeta_0^2}\left(-\frac{3}{2}+\ln\alpha\right)+\frac{N_A}{64\pi^2}\frac{\alpha^2 g^2\widetilde{\sigma}}{\zeta_0^2}=0, \qquad (81)
\end{aligned}
$$

and use the RGE to sum the leading logarithms. One finds, in addition to the trivial solution $\widetilde{\sigma} = 0$,

$$
\begin{aligned}
y &= \frac{C_A\zeta_0}{16\pi^2\left(2f_0\zeta_0+\zeta_1\right)+\frac{N_A}{2}\left(1+\alpha^2-\alpha^2\ln|\alpha|\right)}\ , && (82)\\
m &= \Lambda_{\overline{\rm MS}}e^{\frac{3}{22y}}\ , && (83)\\
E_{\text{vac}} &= -\frac{1}{2}\frac{N_A}{64\pi^2}\left(3+\alpha^2\right)m^4\ . && (84)
\end{aligned}
$$

In principle, the analytic solution for $f_0(\alpha)$ can be obtained by solving the following differential equation

$$
\begin{aligned}
\frac{dE_{\text{vac}}}{d\alpha}=0 \quad &\Leftrightarrow \quad 2\alpha m^4+4(\alpha^2+3)m^3\frac{dm}{d\alpha}=0 \\
&\Leftrightarrow \quad \alpha-\frac{3}{11y^2}(3+\alpha^2)\left(\frac{\partial y}{\partial\alpha}+\frac{\partial y}{\partial\zeta_0}\frac{\partial\zeta_0}{\partial\alpha}+\frac{\partial y}{\partial\zeta_1}\frac{\partial\zeta_1}{\partial\alpha}+\frac{\partial y}{\partial f_0}\frac{\partial f_0}{\partial\alpha}\right)=0\ . \qquad (85)
\end{aligned}
$$

The quantity $f_0(\alpha)$ constructed in this fashion will ensure $E_{\text{vac}}(\alpha)$ is independent of the gauge parameter α. However, we still have the freedom of choosing an initial condition. We will determine $f_0(\alpha)$ so that $E_{\text{vac}}(\alpha) = E_{\text{vac}}(0) \equiv E_{\text{vac}}^{\text{Landau}}$. This amounts to choosing $f_0(\alpha = 0) = 0$. We can justify this choice based on our remark in the introduction, which is that A_μ^2 coincides with the gauge invariant quantity A^2_{min} in the Landau gauge in the FMR. Since our calculation is based on a perturbative expansion around $A_\mu^a = 0$, which lies within the FMR, we stay within the FMR [29, 30, 31].

Unfortunately, the differential equation (85) is very hard to solve analytically. We could solve (85) and consequently y, m and E_{vac} numerically. However, there is a more elegant way to obtain the analytical solution for $f_0(\alpha)$. Considering the colour group $SU(3)$ for simplicity, then since we know that by construction that $E_{\text{vac}} = E_{\text{vac}}^{\text{Landau}}$, we are able to write down the analytical solution for m as

$$
m=\left(\frac{3e^{17/6}}{3+\alpha^2}\right)^{1/4}\Lambda_{\overline{\rm MS}}\ , \qquad (86)
$$

where use was made of (45) and (84). Putting (86) in (83), we deduce that

$$
y=\frac{36}{66\ln\frac{3}{3+\alpha^2}+187}\ . \qquad (87)
$$

Combining (82) and (87) finally gives the analytic expression for $f_0(\alpha)$

$$
f_0(\alpha)=\frac{\frac{\zeta_0}{12}\left(66\ln\frac{3}{3+\alpha^2}+187\right)-4\left(1+\alpha^2-\alpha^2\ln|\alpha|\right)-16\pi^2\zeta_1}{32\pi^2\zeta_0}\ . \qquad (88)
$$

We have displayed $f_0(\alpha)$, $y(\alpha)$ and $m(\alpha)$ for the range of values $-\frac{13}{3} < \alpha < \frac{13}{3}$ in Figures 4-6. As a check, we have also plotted, in Figure 7, $E_{\text{vac}}(\alpha, f_0(\alpha))$ as given in (84) to verify that $E_{\text{vac}}(\alpha, f_0(\alpha)) = E_{\text{vac}}^{\text{Landau}}$. We observe several features. Firstly, although $f_0(\alpha)$ has some singularities in $\left]-\frac{13}{3}, \frac{13}{3}\right[$, the quantities y, m and E_{vac} are completely regular functions of α. Secondly, the expansion parameter y remains relatively small, which makes our numerical predictions at least qualitatively trustworthy. Thirdly, we also see that the value for the tree level mass does not change spectacularly in the considered region. In the Feynman gauge $\alpha = 1$, we have $m^{\text{Feynman}} = 1.89\Lambda_{\overline{\text{MS}}}$.

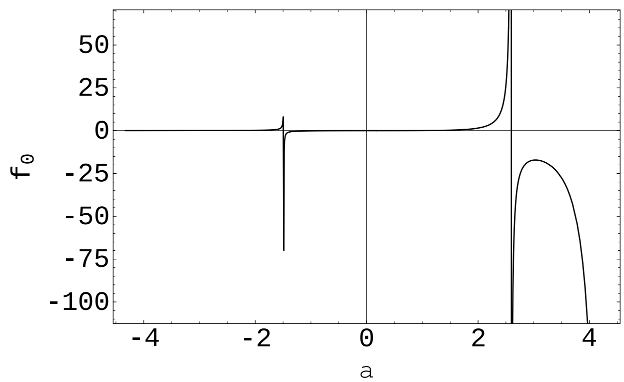

Figure 4: f_0 as a function of α with $-\frac{13}{3} < \alpha < \frac{13}{3}$.

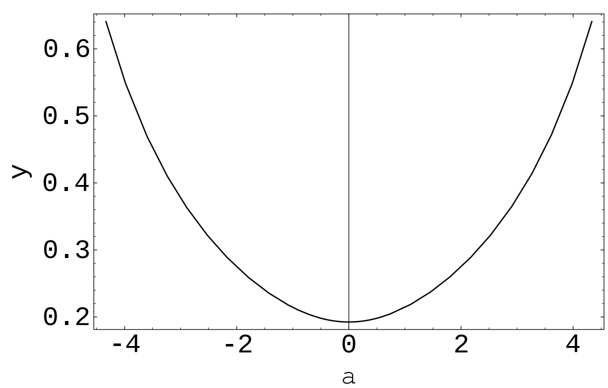

Figure 5: y as a function of α with $-\frac{13}{3} < \alpha < \frac{13}{3}$.

Before ending this section, there are several other points. We have determined $\mathcal{F}(\alpha)$ with the renormalization scale $\overline{\mu}$ chosen in such a way that the logarithms vanish. Other choices of $\overline{\mu}$ are of course also valid. We did not explicitly write this $\overline{\mu}$ dependence of $\mathcal{F}(\alpha)$ in (69).

Also, the procedure we have described here applies of course at higher order. For example, at 2-loops, $f_1(\alpha)$ will be required to remove the α dependence. If we were to work to infinite order in g^2, we could transform the action $\widetilde{S}(\widetilde{J})$ (68) *exactly* into the action $S(J)$ (8) by means of the transformation

$$\widetilde{J} = \frac{J}{\mathcal{F}(\alpha)} \ . \tag{89}$$

The corresponding transformation for the σ and $\widetilde{\sigma}$ fields reads

$$\widetilde{\sigma} = \mathcal{F}(\alpha)\sigma \ , \tag{90}$$

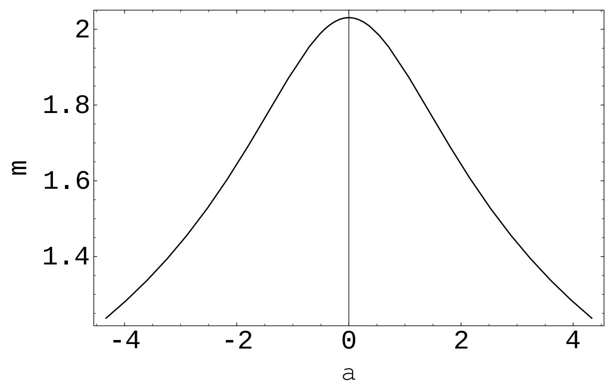

Figure 6: m as a function of α with $-\frac{13}{3} < \alpha < \frac{13}{3}$.

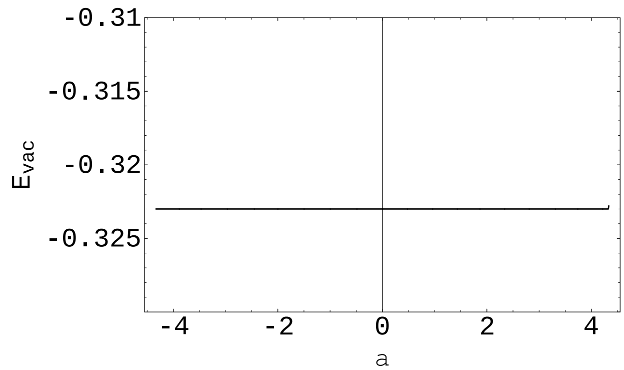

Figure 7: E_{vac} as a function of α with $-\frac{13}{3} < \alpha < \frac{13}{3}$.

which will transform the effective potential $\widetilde{V}_\infty(\widetilde{\sigma})$ *exactly* into $V_\infty(\sigma)$. As such, the constructed vacuum energy will be the same in both cases and independent of the choice of α.

4 Gluon propagator in linear covariant gauges.

In [14], the gluon propagator in the Landau was investigated, and a fit of the lattice results gave evidence for a gluon mass. In the Landau gauge, the lattice also gives evidence for the existence of a non-zero $\left\langle A_\mu^2 \right\rangle$ condensate, based on the discrepancy in the 10 GeV region, between the behaviour of the observed lattice gluon propagator and strong coupling constant and the expected perturbative behaviour. The results could be matched together using an operator product expansion analysis with a non-zero $\left\langle A_\mu^2 \right\rangle$ condensate [6, 7, 8]. A combined lattice fit resulted in $\left\langle A_\mu^2 \right\rangle_{\text{OPE}} \approx (1.64\text{GeV})^2$. This quantity was obtained at a scale of 10 GeV in the MOM renormalization scheme. Later, it was argued that this $\left\langle A_\mu^2 \right\rangle_{\text{OPE}}$ condensate could be explained with instantons [8].

One will notice that we did not give the estimate for $\langle A^2 \rangle$ itself. From the identification (22) and using the relation (35) and the explicit result (44), one finds

$$\left\langle A_\mu^2 \right\rangle = -\frac{187}{52\pi^2} e^{\frac{17}{12}} \Lambda_{\overline{\text{MS}}}^2 \approx -\left(0.29\text{GeV}\right)^2 \tag{91}$$

The extra minus sign arises because we have rotated from Minkowskian to Euclidean space time to make possible a comparison with the lattice. We used $\Lambda_{\overline{\mathrm{MS}}} = 0.233\mathrm{GeV}$, which was the value obtained in [7]. We should be careful not to misinterpret the relatively big difference between $\left\langle A_\mu^2 \right\rangle_{\mathrm{OPE}}$ and (91). Although our result is non-perturbative in nature, it is still obtained in perturbation theory and as such it only gives information from the high energy region (or short range), while the OPE approach of Boucaud *et al* can only describe the low energy (or long range) content of $\left\langle A_\mu^2 \right\rangle$. It was already argued in [5] that $\left\langle A_\mu^2 \right\rangle$ can receive long and short range contributions. The minus sign in front of our result has to do with the regularization and renormalization of the quantity $\left\langle A_\mu^2 \right\rangle$. We refer to [9] for more details.

To our knowledge, there has been little attention on the lattice to the gluon propagator in a general linear covariant gauge. Giusti *et al* managed to put the linear covariant gauge on the lattice [50, 51, 52, 53]. The tree level gluon propagator of Euclidean Yang-Mills theory with a linear covariant gauge fixing is given by

$$D_{\mu\nu}(q) = \frac{1}{q^2}\left(\delta_{\mu\nu} - (1-\alpha)\frac{q_\mu q_\nu}{q^2}\right) . \tag{92}$$

This can be decomposed into the transverse and longitudinal parts as

$$D_{\mu\nu}(q) = \frac{1}{q^2}\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) D^T(q) + \frac{q_\mu q_\nu}{q^2}\frac{D^L(q)}{q^2} , \tag{93}$$

where $D^T(q^2)$ is q^2 times the one used in [50, 51, 52, 53]. In general, one determines $D^L(q)$ via the projector

$$P^L_{\mu\nu}(q) = q_\mu q_\nu . \tag{94}$$

If there is a tree level gluon mass m present, as in (73), the Euclidean gluon propagator in linear covariant gauges reads

$$D_{\mu\nu}(q) = \frac{1}{q^2+m^2}\left(\delta_{\mu\nu} - (1-\alpha)\frac{q_\mu q_\nu}{q^2+\alpha m^2}\right) , \tag{95}$$

with the value of m given in (86). The longitudinal part of this propagator is

$$D^L(q) = P^L_{\mu\nu}(q) D_{\mu\nu}(q) = \frac{1}{q^2+m^2}\left(q^2 - (1-\alpha)\frac{q^4}{q^2+\alpha m^2}\right) . \tag{96}$$

$D^L(q)$ is plotted in Figure 8, again using $\Lambda_{\overline{\mathrm{MS}}} = 0.233\mathrm{GeV}$. Of course, we should not attach any firm meaning to this plot, since we are only considering the tree level propagator and do not include any renormalization effects. If we could calculate the form factors, we would also inevitably encounter the problem of a diverging perturbation theory in the infrared region. We cannot make any conclusion about the behaviour of the propagator in the IR from the above. Many other (non-perturbative) effects can influence the propagators form in the IR. Nevertheless, it might be worth noticing that the longitudinal part $D^L(q)$ is not proportional to the gauge parameter. A similar behaviour was found by Giusti *et al*, see e.g. Figure 4 of [52]. This is already different from the perturbative prediction of massless Yang-Mills theory with a linear covariant gauge fixing [28].

5 Conclusion.

We have considered Yang-Mills theories in linear covariant gauges and constructed a renormalizable effective potential by means of the local composite operator formalism for A_μ^2. The

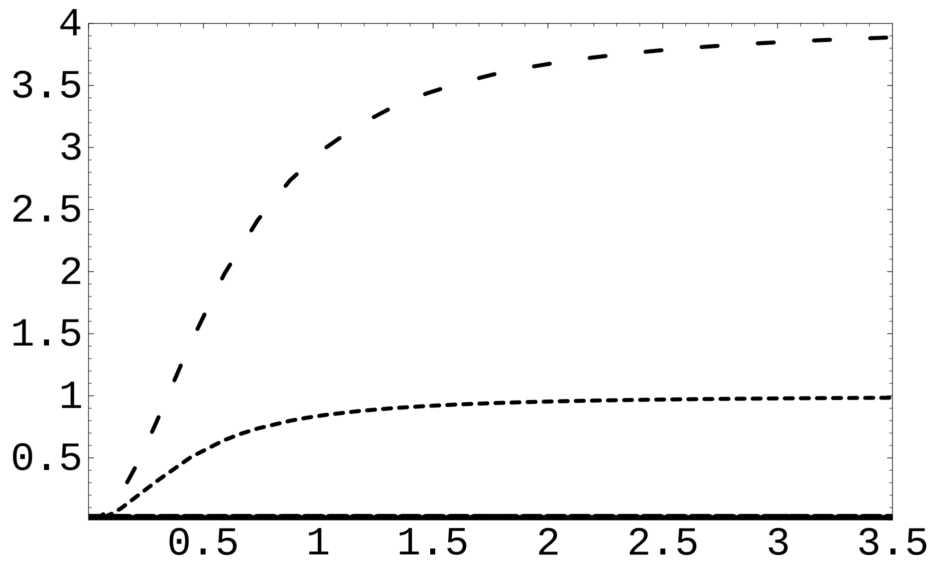

Figure 8: $D_L(q)$ as a function of q with $0 < q < 3.5\text{GeV}$ for $\alpha = 0$ (fat), $\alpha = 1$ (dashed) and $\alpha = 4$ (wide dashed).

formation of the gluon condensate of mass dimension two is favoured since it lowers the vacuum energy. As a result, the gluons acquire a dynamical mass m. We discussed the gauge parameter dependence of the resultant vacuum energy and observed that this is due to the fact that we do not work up to infinite order precision, but have to truncate the perturbative expansion at a finite order. We explained how this gauge parameter dependence can be avoided by a modification of our method.

Although there is limited lattice data available for the general linear covariant gauges compared with the Landau gauge, it would be interesting to calculate the form factor of the longitudinal and transverse part of the gluon propagator to make a more detailed comparison possible with the lattice results of [50, 51, 52, 53]. It would also be useful to have direct evidence from the lattice community that the $\left\langle A_\mu^2 \right\rangle$ condensate exists and that the gluons become massive, in analogy with the Landau gauge. A further point worth investigating is the possible existence of ghost condensates in the linear covariant gauges, as is the case in the Landau gauge [54, 55]. These condensates can modify the gluon propagator further.

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