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Two loop effective potential for $\left\langle A_\mu^2 \right\rangle$ in the Landau gauge in quantum chromodynamics

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Abstract. We construct the effective potential for the dimension two composite operator $\frac{1}{2} {A_\mu^a}^2$ in QCD with massless quarks in the Landau gauge for an arbitrary colour group at two loops. For $SU(3)$ we show that an estimate for the effective gluon mass decreases as N_f increases.

1 Introduction.

Quantum chromodynamics, (QCD), is widely accepted as the quantum field theory of the nucleon partons known as quarks and gluons. At high energy, perturbation theory provides an excellent description of the phenomenology of hadron physics from, say, the point of view of deep inelastic scattering. Although the physics of QCD at low energies is not as well understood, it is generally accepted that quarks are confined and the perturbative vacuum, which is used for higher energy QCD calculations, is unstable, [1, 2, 3, 4]. To probe the infrared régime from the field theoretic point of view several formalisms have been developed. One in particular makes use of the operator product expansion and sum rules where the effect of dimension four operators such as $(G^a_{\mu\nu})^2$ are incorporated, [5]. Such operators have non-zero vacuum expectation values and can be regarded as probing the structure of the true vacuum. Such methods have been successful in revealing insights into the low energy structure of Yang-Mills theory and QCD. However, in this general context there has been a large amount of activity recently into the effects that dimension two operators have on this scenario, [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. For instance, the operator $\frac{1}{2}{A^a_\mu}^2$ and the related BRST invariant operator $\frac{1}{2}{A^a_\mu}^2 + \alpha\bar{c}^a c^a$ have received particular attention being motivated by early work on the Curci-Ferrari model, [20], where A^a_μ is the gluon field and c^a and $\bar{c}^a$ are the ghost and anti-ghost fields of the covariant gauge fixing which is parametrized by α. The BRST algebraic properties of these operators has been extensively studied, [21, 22, 23, 24, 25, 26, 27, 28], particularly in relation to unitarity. Unlike $(G^a_{\mu\nu})^2$ such operators are clearly not gauge invariant and therefore they can only arise in gauge variant quantities such as the strong coupling constant. (See, for example, [29, 30].) However, one can construct a dimension two gauge invariant operator which is non-local, [11], but is believed to reduce to $\frac{1}{2}{A^a_\mu}^2$ in the Landau gauge where it is clearly local. Therefore, various groups have examined the effect such an operator has in Yang-Mills theories both in the case of ordinary linear covariant gauge fixing and in a related non-linear covariant gauge fixing. The latter is known as the Curci-Ferrari gauge having its origin in the Curci-Ferrari model which was an early attempt to construct a field theory where the gluon and ghosts were classically massive, [20]. This gauge differs from the usual covariant gauge fixing due to the presence of a renormalizable four-ghost interaction and a different ghost-gluon interaction. Its ultra-violet behaviour is equivalent to that of the usual linear Landau gauge when the covariant gauge parameter is nullified.

One important study of the effects of the $\frac{1}{2}{A^a_\mu}^2$ operator has been provided in [11, 31]. There the effective potential of this operator has been constructed at two loops in Yang-Mills theory. This is not a straightforward exercise as one has to apply the local composite operator, (LCO), formalism which correctly accounts for a composite operator being included in the path integral formalism, [11, 31, 32, 33]. Early work with this technique centred on studying mass generation in the two dimensional Gross-Neveu model where estimates for the mass gap were obtained which were in good agreement with the known exact mass gap of the model, [32, 33]. Consequently the LCO technique provides an effective action which corresponds to the Yang-Mills action as well as a new contribution which involves an extra scalar field whose elimination by its equation of motion is related to the dimension two composite operator. In this respect it is similar to the Gross-Neveu model though in that case the analogous scalar field arises *naturally* in the Lagrangian. With the new action derived with the LCO technique the effective potential of the dimension two operator has been constructed at two loops in Yang-Mills theory, [11]. However, as the nucleon world possesses quarks there is clearly a need to repeat the analysis for full QCD. This is the main purpose of this article where we will extend the effective potential of [11] to include massless quarks. Whilst it may appear that our calculation follows [11], it should be noted that [11] considered only the $SU(N_c)$ colour groups and, moreover, required the evaluation of three loop massive vacuum bubble graphs using the tensor correction method developed in [31]. By contrast, we will consider arbitrary colour groups which allows one to study, in principle, the effect in grand unified theories. More importantly, though, we will bypass the need to compute three loop massive vacuum bubble diagrams. These were required to determine the explicit expression for a particular parameter, $\zeta(g)$, as a function of the coupling constant, g. This quantity, which plays a role similar to the coupling constant for the new scalar field of the new action, obeys a particular renormalization group equation and its explicit value, as we will demonstrate, only requires the evaluation of three loop *massless* Feynman diagrams. The importance of this observation is not to be underestimated since from a calculational point of view it reduces the work significantly and will be important in other similar applications of the LCO formalism for which the symbolic manipulation programmes developed here can be applied. One final motivation for this article is to ascertain the effect the quarks have on an effective gluon mass. In the construction of the two loop effective potential it transpires that for both Yang-Mills and QCD the naive perturbative vacuum is not stable, [11], but there is a stable vacuum where the gluon field develops a mass through the non-zero vacuum expectation value of the new scalar field arising from the LCO formalism. Clearly an estimate of such a mass is important when quarks are present. While our calculations are still effectively perturbation theory probing towards the true non-perturbative vacuum, the full vacuum expectation value of the scalar field will be comprised of two parts. One is the perturbative piece estimated from the effective potential but the other component will arise from purely non-perturbative phenomena such as instantons, [11]. In this context there has been various studies of dimension two operators and condensates on the lattice giving estimates of similar quantities, [29, 30]. Hence, it will be useful to provide information with quarks present ahead of lattice simulations of full QCD where it would be hoped the dimension two condensates could be measured at an energy scale compatible with the energy scale we will determine our effective gluon mass estimates.

The paper is organised as follows. In section two we will review the construction of the LCO formalism for full QCD and determine several quantities which are central to the construction of the two loop effective potential in the QCD case. The full two loop potential is discussed in section three whilst we provide a detailed analysis of the colour groups $SU(2)$ and $SU(3)$ in section four. Concluding remarks are given in section five.

2 Formalism.

In this section we review the basic formalism for constructing the effective potential of a composite operator using the LCO formalism, [11]. Our notation parallels that of [11] and we will focus on the main ingredients as well as where new features emerge. The starting point is the construction of an effective action which includes the composite operator we are interested in, $\frac{1}{2}A^{a\,2}_\mu$, in the Landau gauge, and is based on the usual QCD action. In particular the QCD Lagrangian is

$$L ~=~ -~ \frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} ~-~ \frac{1}{2\alpha} (\partial^\mu A^a_\mu)^2 ~-~ \bar{c}^a \partial^\mu D_\mu c^a ~+~ i \bar{\psi}^{iI} \not{D} \psi^{iI} \tag{2.1}$$

where $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$, f^{abc} are the colour group structure constants, ψ^{iI} is the quark field and $1 \leq a \leq N_A$, $1 \leq I \leq N_F$ and $1 \leq i \leq N_f$ with N_F and N_A the dimensions of the fundamental and adjoint representations respectively and N_f is the number of quark flavours. The covariant derivatives are defined by

$$D_\mu c^a ~=~ \partial_\mu c^a ~-~ g f^{abc} A^b_\mu c^c ~~,~~ D_\mu \psi^{iI} ~=~ \partial_\mu \psi^{iI} ~+~ i g T^a A^a_\mu \psi^{iI} ~. \tag{2.2}$$

Although most of the recent activity in this area has dwelt on Yang-Mills in the Curci-Ferrari gauge since it has properties in common with the maximal abelian gauge, we have chosen to fix the gauge with the usual linear covariant gauge fixing. This is because the potential in either this gauge or the Curci-Ferrari gauge is the same when $\alpha = 0$. For reasons which will become apparent later the quarks in (2.1) are massless. Moreover, we can regard (2.1) as the bare Lagrangian and introduce renormalized quantities via the usual definitions such as $A_{\mathrm{o}}^{a\,\mu} = \sqrt{Z_A}\, A^{a\,\mu}$ and $g_{\mathrm{o}} = Z_g g$ where the subscript, $_{\mathrm{o}}$, denotes a bare quantity.

To include the composite operator, $\frac{1}{2} A^{a\,2}_\mu$, in the path integral it would initially seem that one should include the term $\frac{1}{2} J A^{a\,2}_\mu$ in the Lagrangian where J is a source term and derive the usual path integral generating functional, $W[J]$. Ordinarily in such an approach one couples the fields themselves to each source to obtain a generating functional which is used to determine the Green's functions. However, in coupling to an object quadratic in the fields an immediate problem arises in that the new action ceases to be multiplicatively renormalizable, [11]. This is due to the generation of divergences proportional to J^2 which are related to vacuum energy divergences, [11]. Divergences involving one power of J are absorbed into the renormalization constant for the renormalization of the composite operator, $\frac{1}{2} A^{a\,2}_\mu$, itself, [31, 34]. Similar features are also present in the Gross-Neveu model, [32, 35]. To circumvent this lack of multiplicative renormalizability one must add a term quadratic in J with an appropriate counterterm in order to have a sensible generating functional, $W[J]$. Therefore, the generating functional which is the starting point of the LCO formalism is, [11],

$$e^{-W[J]} ~=~ \int \mathcal{D} A_{\mathrm{o}}^\mu \mathcal{D} \psi_{\mathrm{o}} \mathcal{D} \bar{\psi}_{\mathrm{o}} \mathcal{D} c_{\mathrm{o}} \mathcal{D} \bar{c}_{\mathrm{o}} \, \exp \left[\int d^d x \left(L_{\mathrm{o}} ~-~ \frac{1}{2} J_{\mathrm{o}} A^{a\,2}_{\mathrm{o}\,\mu} ~+~ \frac{1}{2} \zeta_{\mathrm{o}} J_{\mathrm{o}}^2 \right) \right] \qquad (2.3)$$

where all quantities are bare and the new quantity ζ_{o} has been introduced to ensure one will have a homogeneous renormalization group equation for $W[J]$, [11], whose conventions we follow throughout.

With the extra terms in the action additional renormalization constants are required over and above those needed to render the normal QCD Lagrangian, L, finite, [11]. However, it is important to realise that the additional terms do not affect the explicit values of the ordinary QCD renormalization group functions. This can be understood from the fact that the new cubic interaction can be regarded as a mass term for the gluon which, from the point of view of renormalization, will regularize infrared infinities and leave the ultraviolet structure unchanged. Therefore, we need to introduce two new renormalization constants which to be consistent with [11] we denote by Z_m and $\delta\zeta$ where the latter is strictly a counterterm. Therefore, using

$$J_{\mathrm{o}} ~=~ \frac{Z_m}{Z_A} J ~~~,~~~ \zeta_{\mathrm{o}} J_{\mathrm{o}}^2 ~=~ (\zeta ~+~ \delta\zeta)\, J^2 \qquad (2.4)$$

(2.3) becomes

$$e^{-W[J]} ~=~ \int \mathcal{D} A_\mu \mathcal{D} \psi \mathcal{D} \bar{\psi} \mathcal{D} c \mathcal{D} \bar{c} \, \exp \left[\int d^d x \left(L ~-~ \frac{1}{2} Z_m J A^{a\,2}_\mu ~+~ \frac{1}{2} (\zeta + \delta\zeta) J^2 \right) \right] ~. \qquad (2.5)$$

For the construction of the effective potential the explicit values of each renormalization constant will be required. It turns out that these are straightforward to compute. For instance, Z_m has already been determined in [26, 36, 37, 38, 39] at one and two loops and to three loops in [31, 34]. The method to achieve this is twofold. One can either use a Lagrangian with a gluon of squared mass J and renormalize with a massive gluon propagator in the Landau gauge or one can regard the operator $\frac{1}{2} J A^{a\,2}_\mu$ as a composite operator and renormalize it as an operator insertion in a gluon two-point function using *massless* propagators. Indeed it was the latter approach which was used to deduce the three loop operator anomalous dimension [34]. However, a key property that emerged from that work, which has subsequently been proved to all orders in [17], is that the renormalization of $\frac{1}{2} A^{a\,2}_\mu$ is not independent. In the Landau gauge the anomalous dimension is given by the sum of the gluon and ghost wave function anomalous dimensions. As this will play an important role in the construction of the effective potential we note that in the Landau gauge the explicit value in the $\overline{\mbox{MS}}$ scheme is

$$\begin{aligned} \gamma_m(g)|_{\alpha=0} &= \left[\frac{35}{6} C_A - \frac{8}{3} T_F N_f \right] \frac{g^2}{16\pi^2} ~+~ \left[\frac{449}{24} C_A^2 - 8 C_F T_F N_f - \frac{35}{3} C_A T_F N_f \right] \frac{g^4}{(16\pi^2)^2} \\ &+ \left[\left(\frac{75607}{864} - \frac{9}{16} \zeta(3) \right) C_A^3 + \frac{88}{9} T_F^2 N_f^2 C_F + \frac{386}{27} T_F^2 N_f^2 C_A + 4 T_F N_f C_F^2 \right. \\ &\quad \left. + ~ \left(18 \zeta(3) - \frac{5563}{54} \right) T_F N_f C_A^2 - \left(24 \zeta(3) + \frac{415}{18} \right) T_F N_f C_F C_A \right] \frac{g^6}{(16\pi^2)^3} \\ &+ O(g^8) \end{aligned} \qquad (2.6)$$

where the Casimirs are defined by $T^a T^a = C_F I$, $f^{acd} f^{bcd} = C_A \delta^{ab}$ and $\mbox{Tr} \left(T^a T^b \right) = T_F \delta^{ab}$.

Therefore all that remains is to evaluate the counterterm $\delta \zeta$. In [11] the source, J, was considered to be constant giving the gluon a mass. The counterterm was then deduced by renormalizing three loop massive vacuum bubble diagrams generated from the interactions of L. This required the development of the tensor correction method, [31], to efficiently handle the tensor reduction and algebra associated with massive Feynman diagrams. Then the divergences of the vacuum diagrams, which were proportional to J^2, were absorbed into $\delta \zeta$. In our approach we have considered the source, J, to be a field which interacts with the gluon leaving no candidate for a mass term for the gluon. It therefore remains massless. By viewing the source like an interaction rather than a mass, we have avoided the need to evaluate massive vacuum bubbles. Instead we proceed by considering the vacuum Green's function

$$\langle 0 | \mathcal{O}(x) \, \mathcal{O}(y) | 0 \rangle \qquad (2.7)$$

using (2.1) where $\mathcal{O}(x) = \frac{1}{2} J A^{a\,2}_\mu$. It is the divergences of (2.7) which must be cancelled by $\delta \zeta$. We carried out the three loop renormalization by using the Mincer algorithm, [40], as implemented in Form, [41, 42], with the relevant Feynman diagrams generated by Qgraf, [43]. The converter files mapping the Qgraf output into input Form notation have already been used in [34]. However, the Mincer algorithm evaluates the divergences of two-point functions whereas our vacuum Green's function has no external legs. It was therefore necessary to cut each diagram through the one internal J propagator and consider the resulting J two-point function. This is valid because the propagator of this source field is a constant and is thus common to each graph. It is important to note that this is not equivalent to considering the two-point source Green's function for the complete Lagrangian, (2.5), as this would contain extra diagrams with internal J propagators which do not contribute to (2.7). From a calculational perspective it would seem that renormalizing (2.7) in this way is more efficient since one needs only to use *massless* propagators. Indeed this alternative could probably be applied to similar computations of the effective potentials of other dimension two composite operators. One benefit, for instance, of using the Mincer algorithm, [40, 41], is that the tedious tensor reductions are automatically implemented. To check that this interpretation is consistent with the three loop massive vacuum bubble calculation of [11, 31] we have renormalized (2.7) for arbitrary N_f and found in dimensional regularization with $d = 4 - 2\epsilon$ using $\overline{\mbox{MS}}$, that

$$\begin{aligned} \delta \zeta &= N_A \left[~-~ \frac{3}{2\epsilon} ~+~ \left(\left(\frac{35}{8} C_A - 2 T_F N_f \right) \frac{1}{\epsilon^2} + \left(\frac{8}{3} T_F N_f - \frac{139}{12} C_A \right) \frac{1}{\epsilon} \right) \frac{g^2}{16\pi^2} \right. \\ &\quad + ~ \left(\left(\frac{73}{6} T_F N_f C_A - \frac{8}{3} T_F^2 N_f^2 - \frac{665}{48} C_A^2 \right) \frac{1}{\epsilon^3} \right. \end{aligned}$$

$$
\begin{aligned}
&+ \left(\frac{32}{9}T_F^2N_f^2 - 4T_FN_fC_F - \frac{535}{18}T_FN_fC_A + \frac{6629}{144}C_A^2\right)\frac{1}{\epsilon^2} \\
&+ \left(\frac{40}{27}T_F^2N_f^2 + \left(\frac{115}{3} - 32\zeta(3)\right)T_FN_fC_F + \left(\frac{4381}{216} + 32\zeta(3)\right)T_FN_fC_A \right. \\
&- \left.\left(\frac{71551}{864} + \frac{231}{32}\zeta(3)\right)C_A^2\right)\frac{1}{\epsilon}\Bigg)\frac{g^4}{(16\pi^2)^2}\Bigg] ~+~ O(g^6) ~. \qquad (2.8)
\end{aligned}
$$

When $N_f = 0$ (2.8) corresponds *exactly* with the result of [11]*.

With (2.8) we need to construct the associated renormalization group function denoted by $\delta(g)$ and defined by

$$\mu\frac{\partial\zeta}{\partial\mu} ~=~ 2\gamma_m(g)\zeta ~+~ \delta(g) ~. \qquad (2.9)$$

The renormalization group equation which $W[J]$ satisfies is, [11],

$$\left[\mu\frac{\partial}{\partial\mu} ~+~ \beta(g)\frac{\partial}{\partial g^2} ~-~ \gamma_m(g)\int_x J\frac{\delta}{\delta J} ~+~ [\delta(g) + 2\zeta\gamma_m(g)]\frac{\partial}{\partial\zeta}\right]W[J] ~=~ 0 ~. \qquad (2.10)$$

Thus from

$$\delta(g) ~=~ \left[2\epsilon ~+~ 2\gamma_m(g) ~-~ \beta(g)\frac{\partial}{\partial g^2}\right]\delta\zeta \qquad (2.11)$$

we find

$$
\begin{aligned}
\delta(g) ~=~ \frac{N_A}{16\pi^2}\Bigg[&- 3 ~+~ \left(\frac{32}{3}T_FN_f - \frac{139}{3}C_A\right)\frac{g^2}{16\pi^2} \\
&+ \left(\left(\frac{4381}{36} + 192\zeta(3)\right)T_FN_fC_A + (230 - 192\zeta(3))\,T_FN_fC_F + \frac{80}{9}T_F^2N_f^2\right. \\
&- \left.\left(\frac{71551}{144} + \frac{693}{16}\zeta(3)\right)C_A^2\right)\frac{g^4}{(16\pi^2)^2}\Bigg] ~+~ O(g^6) ~. \qquad (2.12)
\end{aligned}
$$

The next stage, [11], is to choose $\zeta(g)$ to be the solution of the differential equation

$$\beta(g)\frac{d}{dg}\zeta(g) ~=~ 2\gamma_m(g)\zeta(g) ~+~ \delta(g) \qquad (2.13)$$

so that the running of the coupling coupling constant means that $\zeta(g)$ will also run according to its renormalization group equation, (2.13), and which ensures that one can construct a consistent effective action which includes $\frac{1}{2}JA^{a\,2}_\mu$ as a composite operator. Given the nature of this differential equation its solution is singular at the origin and has the expansion, in powers of the coupling constant,

$$\zeta(g) ~=~ \sum_{n\,=\,-\,1}^\infty c_n g^{2n} ~. \qquad (2.14)$$

With the explicit expressions we have quoted we can deduce that

$$
\begin{aligned}
\frac{1}{g^2\zeta(g)} ~=~ &\Bigg[\frac{(13C_A - 8T_FN_f)}{9N_A} \\
&+ \Big(2685464C_A^3T_FN_f - 1391845C_A^4 - 213408C_A^2C_FT_FN_f - 1901760C_A^2T_F^2N_f^2 \\
&\quad +~ 221184C_AC_FT_F^2N_f^2 + 584192C_AN_f^3T_F^3 - 55296C_FT_F^3N_f^3 \\
&\quad -~ 65536T_F^4N_f^4\Big)\frac{g^2}{5184\pi^2N_A(35C_A - 16T_FN_f)(19C_A - 8T_FN_f)}
\end{aligned}
$$

*In [11] the convention $d = 4 - \epsilon$ was used.

$$
\begin{aligned}
&+ \Big(\Big(62228252520 C_A^6 N_f T_F - 8324745975 C_A^7 - 42525100800 C_A^5 C_F N_f T_F \\
&\quad - 123805256256 C_A^5 N_f^2 T_F^2 + 105262940160 C_A^4 C_F N_f^2 T_F^2 \\
&\quad + 112398515712 C_A^4 N_f^3 T_F^3 - 103719518208 C_A^3 C_F N_f^3 T_F^3 \\
&\quad - 52888043520 C_A^3 N_f^4 T_F^4 + 50866421760 C_A^2 C_F N_f^4 T_F^4 \\
&\quad + 12606898176 C_A^2 N_f^5 T_F^5 - 12419334144 C_A C_F N_f^5 T_F^5 \\
&\quad - 1207959552 C_A N_f^6 T_F^6 + 1207959552 C_F N_f^6 T_F^6\Big)\zeta(3) - 13223737800 C_A^7 \\
&\quad + 5886241060 C_A^6 N_f T_F + 52585806000 C_A^5 C_F N_f T_F + 41351916768 C_A^5 N_f^2 T_F^2 \\
&\quad + 522849600 C_A^4 C_F^2 N_f T_F - 130596636288 C_A^4 C_F N_f^2 T_F^2 \\
&\quad - 67857620736 C_A^4 N_f^3 T_F^3 - 1286267904 C_A^3 C_F^2 N_f^2 T_F^2 \\
&\quad + 128750638080 C_A^3 C_F N_f^3 T_F^3 + 46700324864 C_A^3 N_f^4 T_F^4 \\
&\quad + 1180127232 C_A^2 C_F^2 N_f^3 T_F^3 - 63001780224 C_A^2 C_F N_f^4 T_F^4 \\
&\quad - 16782753792 C_A^2 N_f^5 T_F^5 - 475987968 C_A C_F^2 N_f^4 T_F^4 \\
&\quad + 15308685312 C_A C_F N_f^5 T_F^5 + 3106406400 C_A N_f^6 T_F^6 \\
&\quad + 70778880 C_F^2 N_f^5 T_F^5 - 1478492160 C_F N_f^6 T_F^6 \\
&\quad - 234881024 N_f^7 T_F^7\Big) \frac{g^4}{995328\pi^4 N_A (35C_A - 16T_F N_f)^2 (19C_A - 8T_F N_f)^2}\Bigg] \\
&+ O(g^6)
\end{aligned}
\qquad (2.15)
$$

where the $N_f = 0$ case agrees *exactly* with the corrected result of the erratum of [11].

Finally, for practical purposes it is more appropriate to transform the action, $W[J]$, to a situation where it depends linearly on the source, J, which will allow one to apply the usual path integral machinery to construct the effective potential itself. In [11] this was achieved by the Hubbard-Stratonovich transformation where one includes the redundant term

$$
1 ~=~ \int \mathcal{D}\sigma \, \exp\left(- \, \left[a_1 \sigma ~+~ a_2 A^{a\,2}_\mu ~+~ a_3 J \right]^2 \right) \qquad (2.16)
$$

in the path integral where σ is a scalar auxiliary field. The arbitrary coefficients, $\{a_i\}$, are chosen appropriately to cancel off the $\frac{1}{2}J^2$ and $\frac{1}{2}JA^{a\,2}_\mu$ terms to leave a term proportional to σJ as the only source dependence. These coefficients will depend on g, $\zeta(g)$, Z_m and Z_ζ. This results in a new Lagrangian, involving the field σ, which includes the original QCD Lagrangian with its gauge fixing and a new set of interactions

$$
\begin{aligned}
L^\sigma ~&=~ -\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} ~-~ \frac{1}{2\alpha} (\partial^\mu A^a_\mu)^2 ~-~ \bar{c}^a \partial^\mu D_\mu c^a ~+~ i \bar{\psi}^{iI} \not{D} \psi^{iI} \\
&\quad -~ \frac{\sigma^2}{2g^2 \zeta(g) Z_\zeta} ~+~ \frac{Z_m}{2g\zeta(g) Z_\zeta} \sigma A^a_\mu A^{a\,\mu} ~-~ \frac{Z_m^2}{8\zeta(g) Z_\zeta} \left(A^a_\mu A^{a\,\mu} \right)^2
\end{aligned}
\qquad (2.17)
$$

where

$$
e^{-W[J]} ~=~ \int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}c \mathcal{D}\bar{c} \mathcal{D}\sigma \, \exp\left[\int d^d x \left(L^\sigma ~-~ \frac{\sigma J}{g} \right) \right] \qquad (2.18)
$$

which is now linear in the source term. Here the quantity $\zeta(g)$ denotes the explicit expression (2.15) and the extra terms again do not affect the renormalization of (2.1) which retains the usual renormalization constants. The sigma field corresponds to the composite operator from its equation of motion and is similar to the σ field of the Gross-Neveu model where it is introduced to remove the four fermion interaction. In (2.17) a new renormalization constant, Z_ζ, appears which is defined to be

$$Z_\zeta ~=~ 1 ~+~ \frac{\delta\zeta}{\zeta(g)} \tag{2.19}$$

with $\zeta(g)$ also given by (2.15). To the order we will require we note the value of the inverse, since that is what appears in (2.17), is

$$\begin{aligned} Z_\zeta^{-1} ~&=~ 1 ~+~ \left(\frac{13}{6} C_A - \frac{4}{3} T_F N_f \right) \frac{g^2}{16\pi^2\epsilon} \\ &+~ \left[\left(1464 C_A^2 T_F N_f - 1365 C_A^3 - 384 C_A T_F^2 N_f^2 \right) \frac{1}{\epsilon^2} \right. \\ &\quad +~ \left(5915 C_A^3 - 6032 C_A^2 T_F N_f - 1248 C_A C_F T_F N_f + 1472 C_A T_F^2 N_f^2 \right. \\ &\quad\quad \left. \left. +~ 768 C_F T_F^2 N_f^2 \right) \frac{1}{\epsilon} \right] \frac{g^4}{6144\pi^4 (35 C_A - 16 T_F N_f)} ~+~ O(g^6) ~. \end{aligned} \tag{2.20}$$

We have now extended the LCO formalism of [11] in the context of QCD and L^σ, (2.17), has emerged as the effective action to compute the effective potential as a function of σ. If σ has a zero vacuum expectation value then one is in the usual perturbative QCD situation where the vacuum is known to be unstable, [1, 2, 3, 4]. However, if σ has a non-zero vacuum expectation value it generates a gluon mass in a non-perturbative fashion in a vacuum whose stability properties need to be checked. This requires the explicit form of the effective potential.

3 Effective potential.

With the previous formalism we are now in a position to construct the effective potential of the σ field. It is derived by using standard techniques since the source for the operator is now implemented linearly in the generating functional. Beyond the tree approximation the calculation breaks into two pieces. Briefly the one loop correction is determined by summing a set of n-point diagrams with constant σ external field. Hence, to one loop the potential is

$$V(\sigma) ~=~ \frac{\sigma^2}{2g^2\zeta(g)Z_\zeta} ~+~ \frac{(d-1)N_A}{2} \int \frac{d^dk}{(2\pi)^d} \ln \left(k^2 ~+~ \frac{\sigma}{g\zeta(g)} \right) \tag{3.1}$$

where the logarithm emerges from summing the appropriate diagrams based on (2.17). However, the momentum integral is divergent and the infinity which cancels it arises from the counterterm available from Z_ζ in the tree term as indicated in our notation. Hence, expanding to the finite part one is left with the one loop effective potential

$$\begin{aligned} V(\sigma) ~&=~ \frac{9N_A}{2} \lambda_1 \sigma'^2 \\ &+~ \left[\frac{3}{64} \ln \left(\frac{g\sigma'}{\bar{\mu}^2} \right) + C_A \left(- \frac{351}{8} C_F \lambda_1 \lambda_2 + \frac{351}{16} C_F \lambda_1 \lambda_3 - \frac{249}{128} \lambda_2 + \frac{27}{64} \lambda_3 \right) \right. \\ &\quad +~ C_A^2 \left(- \frac{81}{16} \lambda_1 \lambda_2 + \frac{81}{32} \lambda_1 \lambda_3 \right) \\ &\quad \left. + \left(- \frac{13}{128} - \frac{207}{32} C_F \lambda_2 + \frac{117}{32} C_F \lambda_3 \right) \right] \frac{g^2 N_A \sigma'^2}{\pi^2} ~+~ O(g^4) \end{aligned} \tag{3.2}$$

where we have set

$$\lambda_1 ~=~ [13C_A - 8T_F N_f]^{-1} ~~,~~ \lambda_2 ~=~ [35C_A - 16T_F N_f]^{-1} ~~,~~ \lambda_3 ~=~ [19C_A - 8T_F N_f]^{-1} \tag{3.3}$$

and

$$\sigma = \frac{9N_A}{(13C_A - 8T_F N_f)} \sigma' . \tag{3.4}$$

As a check on the result we note that (3.2) agrees with [11] in the $N_f \rightarrow 0$ limit. To go to the next order we must piece in not only the higher loop Feynman diagrams but also the counterterms arising from the one loop integral of (3.1). The latter is achieved by replacing $\sigma/(g\zeta(g))$ in the logarithm in (3.1) by $\sigma Z_m/(g\zeta(g)Z_\zeta)$. The resulting double and simple poles in ϵ will cancel the same poles in the two loop corrections. These are determined by repeating the resummation of the one loop calculation with n-point σ Green's function but now with one radiative correction included on each topology. However, it is well known that this procedure is equivalently reproduced by computing two loop vacuum bubble graphs constructed from a new effective Lagrangian founded on (2.17). This is given by shifting the σ field to a new field, $\tilde{\sigma}$, which has a zero vacuum expectation value, and is defined by

$$\sigma = \langle \sigma \rangle + \tilde{\sigma} \tag{3.5}$$

and then dropping terms linear in $\tilde{\sigma}$ and the overall additive constant [44]. One, in principle does this for all fields in (2.17). However, the gluon, ghost and quark fields all carry either a Lorentz or colour index which means that their vacuum expectation values are each zero and thus the shifted fields are equivalent to the original ones. The net result for (2.17) is that one has a massive gluon with a mass proportional to $\langle \sigma \rangle$ and the interactions of this new action are effectively the same as before. Therefore, the two loop effective potential, in the Landau gauge, is deduced from (2.17) with a gluon whose propagator is

$$- \frac{1}{(k^2 - m^2)} \left[\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right] \tag{3.6}$$

where

$$m^2 = \frac{\sigma}{g\zeta(g)} . \tag{3.7}$$

In our case there are five graphs to compute. Two purely gluonic and three which involve either ghost, quark or $\tilde{\sigma}$ propagators. From (2.17) the propagator of the latter is momentum independent.

The explicit calculation of each graph involves the evaluation of two loop vacuum Feynman diagrams with both massive and massless propagators. However, the exact expressions for such two loop topologies with three different masses is already known and given in, for example, [45]. We have written a FORM programme to handle the substitution and determination to the finite part for each of the five diagrams. The result for the sum, after substituting for $\zeta(g)$ is

$$\begin{aligned} \frac{N_A \sigma'^2 g^4}{\pi^4} \Bigg[& - \frac{9C_A}{4096\epsilon^2} + \left(\frac{N_f T_F}{192} - \frac{37C_A}{1536} + \frac{9C_A}{2048} \ln \left(\frac{g\sigma'}{\bar{\mu}^2} \right) \right) \frac{1}{\epsilon} + \left(\frac{29}{2304} + \frac{\zeta(2)}{128} \right) N_f T_F \\ & + \left(\frac{891}{8192} s_2 - \frac{2023}{36864} - \frac{5\zeta(2)}{2048} \right) C_A - \frac{N_f T_F}{96} \ln \left(\frac{g\sigma'}{\bar{\mu}^2} \right) \\ & + \frac{37C_A}{768} \ln \left(\frac{g\sigma'}{\bar{\mu}^2} \right) - \frac{9C_A}{2048} \left(\ln \left(\frac{g\sigma'}{\bar{\mu}^2} \right) \right)^2 \Bigg] \end{aligned} \tag{3.8}$$

where $s_2 = (4\sqrt{3}/3) Cl_2(\pi/3)$ and $Cl_2(x)$ is the Clausen function, $\zeta(n)$ is the Riemann zeta function and μ is the renormalization scale introduced to ensure the coupling constant remains dimensionless in d-dimensions. It is related to the usual $\overline{\text{MS}}$ scale $\bar{\mu}$ by $\bar{\mu}^2 = 4\pi e^{-\gamma} \mu^2$ where γ is the Euler-Mascheroni constant. Extending the one loop calculation to the finite part of the next order in the coupling constant and adding to (3.8) we find the finite two loop effective potential for non-zero N_f is

$$
\begin{aligned}
V(\sigma) \; = \; & \frac{9N_A}{2}\lambda_1\sigma'^2 \\
& + \left[\frac{3}{64}\ln\left(\frac{g\sigma'}{\bar{\mu}^2}\right) + C_A\left(- \frac{351}{8}C_F\lambda_1\lambda_2 + \frac{351}{16}C_F\lambda_1\lambda_3 - \frac{249}{128}\lambda_2 + \frac{27}{64}\lambda_3 \right) \right. \\
& \quad \left. + \; C_A^2\left(- \frac{81}{16}\lambda_1\lambda_2 + \frac{81}{32}\lambda_1\lambda_3 \right) + \left(- \frac{13}{128} - \frac{207}{32}C_F\lambda_2 + \frac{117}{32}C_F\lambda_3 \right) \right] \frac{g^2N_A\sigma'^2}{\pi^2} \\
& + \left[C_A\left(- \frac{593}{16384} - \frac{255}{16}C_F\lambda_2 + \frac{36649}{4096}C_F\lambda_3 - \frac{1053}{64}C_F^2\lambda_1\lambda_2 + \frac{1053}{128}C_F^2\lambda_1\lambda_3 \right. \right. \\
& \qquad - \; \frac{5409}{1024}C_F^2\lambda_2^2 + \frac{1053}{1024}C_F^2\lambda_3^2 + \frac{891}{8192}s_2 - \frac{1}{4096}\zeta(2) - \frac{3}{64}\zeta(3) \\
& \qquad \left. + \; \frac{585}{16}\zeta(3)C_F\lambda_2 - \frac{4881}{256}\zeta(3)C_F\lambda_3 \right) \\
& \quad + \; C_A^2\left(- \frac{11583}{128}C_F\lambda_1\lambda_2 + \frac{11583}{256}C_F\lambda_1\lambda_3 + \frac{72801}{2048}C_F\lambda_2^2 + \frac{11583}{2048}C_F\lambda_3^2 \right. \\
& \qquad + \; \frac{3159}{128}C_F^2\lambda_1\lambda_2^2 + \frac{3159}{512}C_F^2\lambda_1\lambda_3^2 + \frac{372015}{16384}\lambda_2 - \frac{189295}{16384}\lambda_3 \\
& \qquad + \; \frac{3159}{16}\zeta(3)C_F\lambda_1\lambda_2 - \frac{3159}{32}\zeta(3)C_F\lambda_1\lambda_3 - \frac{1053}{16}\zeta(3)C_F\lambda_2^2 \\
& \qquad \left. - \; \frac{3159}{256}\zeta(3)C_F\lambda_3^2 - \frac{6885}{256}\zeta(3)\lambda_2 + \frac{116115}{8192}\zeta(3)\lambda_3 \right) \\
& \quad + \; C_A^3\left(\frac{34749}{256}C_F\lambda_1\lambda_2^2 + \frac{34749}{1024}C_F\lambda_1\lambda_3^2 + \frac{64071}{512}\lambda_1\lambda_2 - \frac{64071}{1024}\lambda_1\lambda_3 \right. \\
& \qquad - \; \frac{694449}{16384}\lambda_2^2 - \frac{64071}{8192}\lambda_3^2 - \frac{9477}{32}\zeta(3)C_F\lambda_1\lambda_2^2 - \frac{9477}{128}\zeta(3)C_F\lambda_1\lambda_3^2 \\
& \qquad \left. - \; \frac{37179}{256}\zeta(3)\lambda_1\lambda_2 + \frac{37179}{512}\zeta(3)\lambda_1\lambda_3 + \frac{12393}{256}\zeta(3)\lambda_2^2 + \frac{37179}{4096}\zeta(3)\lambda_3^2 \right) \\
& \quad + \; C_A^4\left(- \frac{192213}{1024}\lambda_1\lambda_2^2 - \frac{192213}{4096}\lambda_1\lambda_3^2 + \frac{111537}{512}\zeta(3)\lambda_1\lambda_2^2 + \frac{111537}{2048}\zeta(3)\lambda_1\lambda_3^2 \right) \\
& \quad + \left(- \frac{247}{4096}C_F + \frac{1185}{1024}C_F^2\lambda_2 - \frac{615}{1024}C_F^2\lambda_3 + \frac{1}{128}\zeta(2)N_fT_F + \frac{3}{64}\zeta(3)C_F \right) \\
& \quad + \; \left[C_A\left(+ \frac{75}{4096} - \frac{315}{1024}C_F\lambda_2 \right) + C_A^2\left(+ \frac{315}{4096}\lambda_2 \right) + \frac{9}{1024}C_F \right] \ln\left(\frac{g\sigma'}{\bar{\mu}^2}\right) \\
& \quad \left. - \; \frac{9}{4096}C_A\left(\ln\left(\frac{g\sigma'}{\bar{\mu}^2}\right)\right)^2 \right] \frac{g^4N_A\sigma'^2}{\pi^4} \; + \; O(g^6)
\end{aligned}
\tag{3.9}
$$

which is the main result of this article. Again it agrees with the corrected potential of [11] when $N_f = 0$ which is a strong check on our computations and renormalization though it is more complicated.

With the explicit form we can make some general observations. One issue which always arises with potentials is that of boundedness. For a well defined physically interesting potential it must be bounded from below. However, the tree term is clearly dependent on the parameter λ_1 and therefore before any analysis can proceed we require

$$\lambda_1 \; = \; [13C_A - 8T_FN_f]^{-1} \; > \; 0 \tag{3.10}$$

as $N_A > 0$. Interestingly this combination of group parameters is proportional to the one loop coefficient of the gluon wave function anomalous dimension in the Landau gauge. This would suggest that it has to be positive for a bounded potential and clearly this *cannot* occur in the case of quantum electrodynamics, (QED). Moreover, there has been renormalization group studies of the relation of this particular coefficient of the gluon anomalous dimension in connection with confinement, [46, 47]. Thus, it is curious that a positivity condition emerges on the one loop Landau gauge anomalous dimension in the effective potential of a composite operator which has the dimensions of mass since the presence of a mass gap in QCD is indicative of confinement. The other main point about (3.9) is that we have computed the two loop corrections with *massless* quarks in the one two loop diagram where quarks can appear. As quarks are massive in the real world including a mass in the quark propagator in fact leads to divergent terms proportional to the quark mass which cannot be cancelled off by terms in the current effective potential. One might expect that a mechanism similar to that of the LCO technique could be used to introduce mass terms for the quarks. However, that approach runs into immediate difficulties since a four quark interaction, analogous to the four gluon term of (2.17), would emerge which would clearly lead to a non-renormalizable effective action.

4 Analysis.

With the explicit two loop effective potential for arbitrary colour group we can now analyse various theories of interest. In particular we will focus on the $SU(2)$ and $SU(3)$ colour groups. As the latter is strictly the theory of quarks in the nucleons we will provide this analysis in detail. First, to appreciate how our analysis is carried out we consider the one loop potential for an arbitrary gauge group. From (3.2) the first task is to determine the stationary points defined by the solution of

$$\frac{dV}{d\sigma} ~=~ 0 \tag{4.1}$$

which implies

$$\begin{aligned} 0 ~=~ N_A \sigma' \Bigg[\frac{9}{2}\lambda_1 ~+~ & \left(\frac{3}{64} \ln \left(\frac{g\sigma'}{\bar{\mu}^2} \right) + \frac{3}{128} + \left(- \frac{13}{128} - \frac{207}{32} C_F \lambda_2 + \frac{117}{32} C_F \lambda_3 \right) \right. \\ & + C_A \left(- \frac{351}{8} C_F \lambda_1 \lambda_2 + \frac{351}{16} C_F \lambda_1 \lambda_3 - \frac{249}{128} \lambda_2 + \frac{27}{64} \lambda_3 \right) \\ & \left. + ~ C_A^2 \left(- \frac{81}{16} \lambda_1 \lambda_2 + \frac{81}{32} \lambda_1 \lambda_3 \right) \right) \frac{g^2}{\pi^2} \Bigg] ~+~ O(g^4) ~. \end{aligned} \tag{4.2}$$

Setting the scale to be that given where the logarithm vanishes, $g\sigma' = \bar{\mu}^2$, then there are stationary points at

$$\sigma' ~=~ 0 \tag{4.3}$$

and

$$\sigma' ~\equiv~ \sigma'_0 ~=~ \frac{\bar{\mu}^2}{g_0} \tag{4.4}$$

where

$$y_0 ~\equiv~ \frac{C_A g_0^2}{16\pi^2} ~=~ \frac{36 C_A (35 C_A - 16 T_F N_f)}{(6545 C_A^2 - 6008 C_A T_F N_f + 864 C_F T_F N_f + 1280 T_F^2 N_f^2)} ~. \tag{4.5}$$

This particular combination of the coupling constant on the left hand side has been chosen in order to compare with [11]. Clearly the trivial solution is the usual perturbative vacuum which has $V(0) = 0$ whereas the latter induces a mass term for the gluon and is a global minimum provided (3.10) is satisfied. With (4.5) we can obtain an N_f dependent estimate for the *effective* gluon mass, m_{eff}, defined in [11] as

$$m_{\mathrm{eff}}^2 ~=~ g\sigma' ~. \tag{4.6}$$

First, we need to convert our parameters to a reference mass scale, which is taken to be $\Lambda_{\overline{\rm MS}}$ and is introduced from the solution of the one loop QCD β-function which determines the μ dependence of the running coupling constant, $g(\mu)$,

$$\frac{g^2(\mu)}{16\pi^2} ~=~ \left[2\beta_0 \ln \left[\frac{\mu^2}{\Lambda^2_{\overline{\rm MS}}} \right] \right]^{-1} \tag{4.7}$$

where

$$\beta_0 ~=~ \frac{11}{3} C_A ~-~ \frac{4}{3} T_F N_f ~. \tag{4.8}$$

Hence, at one loop we obtain

$$m_{\rm eff} ~=~ \Lambda^{(N_f)}_{\overline{\rm MS}} \exp \left[\frac{\left(6545 C_A^2 - 6008 C_A T_F N_f + 864 C_F T_F N_f + 1280 T_F^2 N_f^2 \right)}{24(11C_A - 4T_F N_f)(35C_A - 16 T_F N_f)} \right] \tag{4.9}$$

which agrees with the one loop expression of [11] for the unitary colour groups when $N_f = 0$. It is worth noting that this formula is *not* valid in the QED case where $C_A = 0$, $C_F = 1$ and $T_F = 1$ because the original one loop potential is then unbounded from below.

Repeating this analysis for the potential at two loops for an arbitrary colour group is extremely cumbersome and it is more appropriate to concentrate on specific interesting cases. For instance, for $SU(3)$ the two loop potential becomes

$$\begin{aligned} V(\sigma)|_{SU(3)} ~=~& \frac{36}{(39-4N_f)} \sigma^{\prime\,2} \\ &+ \left[\frac{3}{8} \ln \left(\frac{g\sigma^{\prime\,2}}{\bar{\mu}^2} \right) - \frac{13}{16} + \frac{3537}{4(39-4N_f)(57-4N_f)} - \frac{3537}{2(39-4N_f)(105-8N_f)} \right. \\ &\left. \quad + \frac{393}{8(57-4N_f)} - \frac{1851}{16(105-8N_f)} \right] \frac{g^2\sigma^{\prime\,2}}{\pi^2} \\ &+ \left[\frac{2673}{1024} s_2 - \frac{9289}{6144} - \frac{3}{512}\zeta(2) + \frac{1}{32}\zeta(2) N_f - \frac{5}{8}\zeta(3) \right. \\ &\quad - \frac{1129005}{128(39-4N_f)(57-4N_f)} + \frac{397305}{64(39-4N_f)(57-4N_f)}\zeta(3) \\ &\quad - \frac{10161045}{512(39-4N_f)(57-4N_f)^2} + \frac{3575745}{256(39-4N_f)(57-4N_f)^2}\zeta(3) \\ &\quad + \frac{1129005}{64(39-4N_f)(105-8N_f)} - \frac{397305}{32(39-4N_f)(105-8N_f)}\zeta(3) \\ &\quad - \frac{10161045}{128(39-4N_f)(105-8N_f)^2} + \frac{3575745}{64(39-4N_f)(105-8N_f)^2}\zeta(3) \\ &\quad - \frac{3404293}{6144(57-4N_f)} + \frac{420267}{1024(57-4N_f)}\zeta(3) - \frac{1129005}{1024(57-4N_f)^2} \\ &\quad + \frac{397305}{512(57-4N_f)^2}\zeta(3) + \frac{7012085}{6144(105-8N_f)} - \frac{24525}{32(105-8N_f)}\zeta(3) \\ &\quad - \frac{12222795}{2048(105-8N_f)^2} + \frac{132435}{32(105-8N_f)^2}\zeta(3) \\ &\quad + \left(\frac{273}{512} - \frac{2205}{512(105-8N_f)} \right) \ln \left(\frac{g\sigma^{\prime\,2}}{\bar{\mu}^2} \right) \\ &\quad \left. - \frac{27}{512} \left(\ln \left(\frac{g\sigma^{\prime\,2}}{\bar{\mu}^2} \right) \right)^2 \right] \frac{g^4\sigma^{\prime\,2}}{\pi^4} ~+~ O(g^6) ~. \end{aligned} \tag{4.10}$$

With (4.10) we can determine the value of the coupling constant where the minimum occurs.

N_f	$y_0^{(1)}$	$y_0^{(2)}$
0	0.1925134	0.1394790
2	0.2219195	0.1832721
3	0.2398224	0.2225696

Table 1. One and two loop estimates of the coupling constant at the minimum of $V(\sigma)$ for $SU(3)$.

These are recorded in Table 1 where we concentrate on the values $N_f = 2$ and 3 and the superscript here and elsewhere denotes the loop order. To consider more flavours one would be in the situation where the charm quark is present but for the case where it is treated as massless. As its mass is more comparable with the one loop effective gluon mass than the three light quarks and since we have neglected quark mass effects it is not clear how it would influence the approximations we have made.

We can now determine the effect the two loop corrections have on the effective gluon mass, m_{eff}. This requires the inclusion of the two loop correction to the solution of the β-function for the coupling constant as a function of μ which gives

$$\frac{g^2(\mu)}{16\pi^2} ~=~ \left[2\beta_0 \ln \left[\frac{\mu^2}{\Lambda^2_{\overline{\mathrm{MS}}}} \right] \right]^{-1} \left[1 ~-~ \beta_1 \left[2\beta_0^2 \ln \left[\frac{\mu^2}{\Lambda^2_{\overline{\mathrm{MS}}}} \right] \right]^{-1} \ln \left[2 \ln \left[\frac{\mu^2}{\Lambda^2_{\overline{\mathrm{MS}}}} \right] \right] \right] \tag{4.11}$$

where

$$\beta_1 ~=~ \frac{34}{3} C_A^2 ~-~ 4 C_F T_F N_f ~-~ \frac{20}{3} C_A T_F N_f ~. \tag{4.12}$$

Converting the coupling constant at the minimum of the potential, g_0, when $g\sigma' = \bar{\mu}^2$ and

N_f	$m^{(1)}_{\mathrm{eff}}/\Lambda^{(N_f)}_{\overline{\mathrm{MS}}}$	$m^{(2)}_{\mathrm{eff}}/\Lambda^{(N_f)}_{\overline{\mathrm{MS}}}$
0	2.03	2.12
2	2.01	1.99
3	2.00	1.89

Table 2. One and two loop estimates of the gluon effective mass for $SU(3)$.

substituting into (4.6) we obtain the one and two loop estimates given in Table 2. We have expressed our estimates in terms of the N_f-dependent $\Lambda^{(N_f)}_{\overline{\mathrm{MS}}}$ which for low N_f can be taken to be 237 MeV giving a two loop estimate of around 450 MeV for $N_f = 3$. From the table, aside from the Yang-Mills result of [11] at two loops it is clear this effective gluon mass decreases slowly with the increase in the number of quark flavours. However, it is not clear whether this a real feature since quark mass effects, which should not be significant for light flavours, have been neglected. Moreover, it is worth stressing that m_{eff} is *not* the physical gluon mass in the new vacuum. The physical mass can only be deduced by computing the pole of the gluon propagator at *two* loops using the effective action (2.17). This is clearly beyond the scope of the present article.

We have repeated the above analysis for the case of $SU(2)$ which follows the same path and the results are summarized in Tables 3 and 4. For $N_f = 3$ a second positive root emerged for the value for which the potential was a minimum at two loops. This was $y_0^{(2)} = 0.7559862$

N_f	$y_0^{(1)}$	$y_0^{(2)}$
0	0.1925134	0.1394790
2	0.2416107	0.2239704
3	0.2758161	0.4342486

Table 3. One and two loop estimates of the coupling constant at the minimum of $V(\sigma)$ for $SU(2)$.

which we have neglected in the table as its value is significantly different from the one loop case. Though since the value given in Table 3 seems to be out of step with the pattern of Table 1 it is not clear how reliable that coupling is. The effective gluon mass values given in Table 4 are derived using the couplings of Table 3. Overall for most of the $SU(2)$ and $SU(3)$ cases we have analysed the two loop corrections to the effective gluon mass are no more than about 5% of the one loop result which is encouraging since it indicates a degree of stability of the approximation. However, the exception is the $N_f = 3$ $SU(2)$ case which gives a gluon mass estimate which is 20% different from the one loop estimate. As in this case a second positive root emerged for the coupling constant at the minimum, it may require a three loop effective potential to establish which if either is spurious.

N_f	$m^{(1)}_{\mathrm{eff}}/\Lambda^{(N_f)}_{\overline{\mathrm{MS}}}$	$m^{(2)}_{\mathrm{eff}}/\Lambda^{(N_f)}_{\overline{\mathrm{MS}}}$
0	2.03	2.12
2	1.99	1.88
3	1.97	1.58

Table 4. One and two loop estimates of the gluon effective mass for $SU(2)$.

Finally, following the method of [31], we have computed the value of the one and two loop effective potentials at the non-trivial minimum and presented the numerical values in Table 5 for $SU(3)$ where they are all clearly negative. As the potentials are zero at the trivial minimum this justifies our remark that the non-trivial minimum corresponds to the stable vacuum. Clearly the inclusion of quarks increases the value of the minimum but the two loop correction is significantly larger.

N_f	$V^{(1)}_{\mathrm{min}}/\left(\Lambda^{(N_f)}_{\overline{\mathrm{MS}}}\right)^4$	$V^{(2)}_{\mathrm{min}}/\left(\Lambda^{(N_f)}_{\overline{\mathrm{MS}}}\right)^4$
0	$-$ 0.32300	$-$ 0.76451
2	$-$ 0.31145	$-$ 0.66966
3	$-$ 0.30617	$-$ 0.61079

Table 5. One and two loop estimates of the minimum of the effective potential for $SU(3)$.

5 Discussion.

We conclude with various remarks. First, we have included quarks in the effective potential of the dimension two operator, $\frac{1}{2}A^{a\,2}_\mu$ in the Landau gauge and shown that the potential is bounded from below for QCD but not for QED. Also, we have demonstrated that an effective gluon mass decreases in value slowly as N_f increases. However, this is for the unrealistic case of massless quarks, though for the values of N_f we have studied the quarks are light and their mass effects ought not to be significant. Clearly, for a more pragmatic study it would be better if quark masses could be included but this would require the removal of extra infinities which are quark mass dependent from the *two* loop effective potential. At present this will require an extension of the LCO formalism or another mechanism. Moreover, whilst our gluon effective mass estimates are essentially for a mass which from L^σ corresponds to a classical mass, a much better quantity to study would be the physical gluon mass itself which is defined as the pole of the gluon propagator derived from the effective action by computing the radiative corrections to the gluon two-point function. In the application of the LCO technique to the Gross-Neveu model and after improving convergence of the series, [32, 33], it was shown that such a physical mass was accurate to a few percent with the known exact mass gap. It would be hoped that a similar calculation for QCD would give a reliable estimate for the gluon mass. Finally, given that we have now derived a two loop effective potential for a particular dimension two composite operator there is scope for examining the potentials of other related dimension two operators. For instance, ghost number breaking operators have been considered by various authors either in the maximal abelian gauge or in the Curci-Ferrari gauge which acts as a testbed for understanding gluon mass generating problems in the former gauge. Since we have simplified the way of computing the quantity $\zeta(g)$ without having to resort to the computation of three loop massive vacuum bubble graphs then the construction of such potentials for other operators or combination of operators can proceed in an efficient manner.

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