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Three loop $\overline{\text{MS}}$ renormalization of the Curci-Ferrari model and the dimension two BRST invariant composite operator in QCD

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Abstract. The massless Curci-Ferrari model with N_f flavours of quarks is renormalized to three loops in the $\overline{\text{MS}}$ scheme in an arbitrary covariant gauge with parameter α. The renormalization of the BRST invariant dimension two composite operator, $\frac{1}{2}{A^a_\mu}^2 - \alpha \bar{c}^a c^a$, which corresponds to the mass operator in the massive Curci-Ferrari model, is determined by renormalizing the Green's function where the operator is inserted in a ghost two-point function. Consequently the anomalous dimension of the QCD Landau gauge operator, $\frac{1}{2}{A^a_\mu}^2$, and the (gauge independent) photon mass anomalous dimension in QED are both deduced at three loops.

There has been a renewed interest recently in trying to understand the origin of confinement in Yang-Mills theories and QCD from the point of view of the existence of a non-zero vacuum expectation value of the dimension two composite operator $\frac{1}{2}A^{a\,2}_\mu$ where A^a_μ is the gluon field. For instance, lattice gauge theory studies by Boucaud et al, for example, [1] and references therein, have shown evidence that this operator has a non-zero vacuum expectation value. Whilst it might appear that the operator itself can have no physical relevance due to its lack of gauge invariance, it has been argued for instance in, [2], that in the Landau gauge it is gauge invariant since it is one term of a more general dimension two operator which is non-local in non-Landau gauges. With the assumption that the non-zero vacuum expectation value $\langle \frac{1}{2}A^{a\,2}_\mu \rangle$ is present in Yang-Mills theories it has been the subject of various analytic and field theoretic investigations either using the operator product expansion, [3, 4, 5], or other methods, [2, 6, 7, 8, 9, 10, 11, 12, 13]. One interesting approach has been that of [2]. There the effective potential of the composite operator itself is computed at two loops in $SU(N)$ Yang-Mills theory. It is demonstrated that the associated auxiliary field develops a non-zero vacuum expectation value in the true vacuum. The classical vacuum where the vacuum expectation value remains zero is energetically unstable. This calculation developed the early work of [14, 15] to compute the effective potential of a composite operator in a field theory to two loops which is a non-trivial exercise. One of the reasons for this is that non-perturbative terms arise at leading order due to the presence of a $1/g^2$ term where g is the basic coupling constant. The upshot of this is that one has to compute several quantities to *three* loops including the anomalous dimension of the gluonic dimension two local composite operator, [2]. This was achieved by the tensor correction technique developed in [16] to handle massive and massless Feynman integrals to three loop order in automatic Feynman diagram computer programmes. For example, the method of [2] produced a numerical estimate for the vacuum expectation value of $\frac{1}{2}A^{a\,2}_\mu$ in Yang-Mills theory in the context of this essentially perturbative probing of a non-perturbative phenomenon. Clearly those non-perturbative contributions from instantons are not included in this approach but this does not detract from its success and potential application to other situations. Indeed in this context there has been recent studies of similar vacuum expectation value generation problems in Yang-Mills theories in different gauges. For instance, in [6, 7, 8, 9, 10, 11, 12, 13] the condensation of ghost number breaking vacuum expectation values has been investigated at one loop in the Landau gauge as well as the more interesting maximal abelian gauge. In the latter case the off-diagonal ghosts gain a mass in contrast to the diagonal ghosts remaining massless. One hope is that the same feature occurs in the gluon sector, indicating that the centre of the group is special for confinement since abelian monopoles are believed to drive this mechanism.

Given this current interest in this area it is worth noting that it can be pursued in several directions. Clearly a two loop extension of [12, 13], for instance, would be interesting. However, all the current investigations have been for Yang-Mills theories. For more realistic studies of the effective potential approach one needs to include N_f flavours of quarks. Therefore, the aim of this article is to provide the first stage in this problem which is the determination of the three loop anomalous dimension of the $\frac{1}{2}A^{a\,2}_\mu$ composite operator including quarks, thereby extending the result given in [2]. To achieve this we will in fact deduce it as a consequence of the renormalization of a model similar to QCD in its ultraviolet properties but differing from it in the infrared. Known as the Curci-Ferrari model, [17], it was believed it could be central to understanding massive vector bosons as an alternative to the Higgs mechanism. However, as it is also not a unitary model, [18, 19, 20, 21, 22, 23] it has only received renewed interest due to its relation to the ghost condensation problem since the Curci-Ferrari model has a feature similar to the maximal abelian gauge and QCD in a class of nonlinear gauges, [24, 25], which is the presence of a four ghost interaction which is a crucial ingredient for the phenomenon. As the Curci-Ferrari model has a natural dimension two BRST invariant mass term it can be used to determine the ultraviolet structure of the renormalization group functions of QCD itself since the mass acts as an infrared regulator, [17, 20, 21, 22, 23, 26, 27]. The model is renormalizable and the renormalization group functions are in agreement with those of QCD in the Landau gauge. As this BRST dimension two operator is the one required for the earlier discussion we will determine its anomalous dimension in the Curci-Ferrari model in an arbitrary covariant gauge and then specialize to the Landau gauge. Whilst this may appear a roundabout method we can exploit several technical points to ease our computation. Moreover, we will additionally renormalize the Curci-Ferrari model itself at three loops and extend the lower order calculations of [17, 21, 28, 29] thereby addressing several problems simultaneously.

We begin by recalling the Curci-Ferrari model which includes a BRST invariant mass term for the gluon. The Lagrangian is, [17],

$$\begin{aligned} L^{\mathrm{CF}} &= -\frac{1}{4}G^a_{\mu\nu}G^{a\,\mu\nu} - \frac{1}{2\alpha}(\partial^\mu A^a_\mu)^2 + \frac{1}{2}m^2 A^a_\mu A^{a\,\mu} - \bar{c}^a\partial^\mu D_\mu c^a - \alpha m^2 \bar{c}^a c^a \\ &\quad + \frac{g}{2}f^{abc}\partial^\mu A^a_\mu\,\bar{c}^b c^c + \frac{\alpha g^2}{8}f^{eab}f^{ecd}\bar{c}^a c^b \bar{c}^c c^d + i\bar{\psi}^{iI}\not{D}\psi^{iI} - \sqrt{\beta}m\bar{\psi}^{iI}\psi^{iI} \end{aligned} \quad (1)$$

where A^a_μ is the gluon field, c^a and $\bar{c}^a$ are the ghost and antighost fields, ψ^{iI} is the quark field and α is the covariant gauge fixing parameter. The covariant derivatives are given by $D_\mu\psi = \partial_\mu\psi + igA^a_\mu T^a\psi$ and $D_\mu c^a = \partial_\mu c^a - gf^{abc}A^b_\mu c^c$ implying $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu$ where f^{abc} are the structure constants of the colour group whose generators are T^a, $1 \leq a \leq N_A$, $1 \leq I \leq N_F$ and $1 \leq i \leq N_f$ with N_A and N_F are the dimensions of the respective adjoint and fundamental representations. The gluon mass is m and the quark mass is expressed in terms of this basic scale with the parameter β introduced to indicate that the masses are different. The model is renormalizable and the renormalization group functions are known at two loops, [28, 29]. In the case when $m^2 = 0$ one has QCD fixed in a nonlinear gauge which unlike the Curci-Ferrari model is a unitary theory. The presence of the non-zero mass in (1) breaks both unitarity and the nilpotency of the BRST transformation, [18, 19, 20, 21, 22, 23, 24, 25]. To compute the renormalization constants of (1) we follow a two stage approach. First, we determine the basic renormalization constants of the fields and parameters and then deduce the renormalization of the mass operator treated as a composite operator inserted in a Green's function based on the *massless* Lagrangian. This method has been used, for example, to determine the renormalization of the quark mass in QCD at three loops in [30, 31]. The advantage of considering the massless Lagrangian is that one can apply the Mincer algorithm [32], as written in version 2.0 of the symbolic manipulation language Form, [33, 34]. This is an efficient automatic Feynman diagram package which determines the ultraviolet structure of massless three loop two point functions with respect to dimensional regularization in $d = 4 - 2\epsilon$ dimensions. More specifically to achieve this for each of the Green's functions we need to examine, we generate the appropriate set of Feynman diagrams using the Qgraf package, [35], in a format which is readable in Form, [34]. These are then integrated using the appropriate Mincer routine after the basic topology of the diagrams has been identified. As the first stage in the computation concerns the wave function renormalization which derive from the two point functions we note that the renormalization constants are defined by

$$\begin{aligned} A^{a\,\mu}_{\mathrm{o}} &= \sqrt{Z_A}\,A^{a\,\mu} ~,~~ c^a_{\mathrm{o}} = \sqrt{Z_c}\,c^a ~,~~ \bar{c}^a_{\mathrm{o}} = \sqrt{Z_c}\,\bar{c}^a ~,~~ \psi_{\mathrm{o}} = \sqrt{Z_\psi}\psi \\ g_{\mathrm{o}} &= Z_g\,g ~,~~ m_{\mathrm{o}} = Z_m\,m ~,~~ \alpha_{\mathrm{o}} = Z^{-1}_\alpha Z_A\,\alpha ~,~~ \beta_{\mathrm{o}} = Z_\beta\beta ~. \end{aligned} \quad (2)$$

where the subscript $_{\mathrm{o}}$ denotes the bare quantity. We have, using the modified minimal subtraction scheme,

$$Z_A = 1 + \left[\left(\frac{13}{6} - \frac{\alpha}{2}\right)C_A - \frac{4}{3}T_F N_f\right]\frac{a}{\epsilon}$$

$$
\begin{aligned}
&+ \left[\left(\left(\frac{3\alpha^2}{16} - \frac{17\alpha}{24} - \frac{13}{8} \right) C_A^2 + C_A T_F N_f \left(\frac{2}{3}\alpha + 1 \right) \right) \frac{1}{\epsilon^2} \right. \\
&\quad - \left. \left(\left(\frac{\alpha^2}{16} + \frac{11\alpha}{16} - \frac{59}{16} \right) C_A^2 + 2 C_F T_F N_f + \frac{5}{2} C_A T_F N_f \right) \frac{1}{\epsilon} \right] a^2 \\
&+ \left[\left(\left(\frac{403}{144} + \frac{47\alpha}{48} + \frac{\alpha^2}{8} - \frac{\alpha^3}{16} \right) C_A^3 - C_A^2 T_F N_f \left(\frac{22}{9} + \frac{5\alpha}{6} + \frac{\alpha^2}{4} \right) + \frac{4}{9} C_A T_F^2 N_f^2 \right) \frac{1}{\epsilon^3} \right. \\
&\quad + \left(\left(\frac{5\alpha^3}{96} + \frac{97\alpha^2}{192} - \frac{143\alpha}{96} - \frac{7957}{864} \right) C_A^3 + C_A^2 T_F N_f \left(\frac{481}{54} + \frac{19\alpha}{12} + \frac{\alpha^2}{12} \right) \right. \\
&\qquad - \left. \frac{50}{27} C_A T_F^2 N_f^2 - \frac{8}{9} C_F T_F^2 N_f^2 + C_A C_F T_F N_f \left(\frac{31}{9} + \alpha \right) \right) \frac{1}{\epsilon^2} \\
&\quad + \left(\left(\frac{9965}{864} - \frac{167\alpha}{96} - \frac{101\alpha^2}{384} - \frac{\alpha^3}{64} - \left(\frac{\alpha}{4} + \frac{3}{16} \right) \zeta(3) \right) C_A^3 \right. \\
&\quad - C_A^2 T_F N_f \left(\frac{911}{54} - 6\zeta(3) - \frac{2\alpha}{3} \right) - C_A C_F T_F N_f \left(\frac{5}{54} + 8\zeta(3) \right) + \frac{76}{27} C_A T_F^2 N_f^2 \\
&\quad + \left. \left. \frac{44}{27} C_F T_F^2 N_f^2 + \frac{2}{3} C_F^2 T_F N_f \right) \frac{1}{\epsilon} \right] a^3 \; + \; O(a^4) \qquad (3)
\end{aligned}
$$

$$
\begin{aligned}
Z_\alpha \;=\; & 1 \; - \; \left(\frac{\alpha}{4} \right) C_A \frac{a}{\epsilon} \; + \; C_A^2 \left[\left(\frac{\alpha^2}{16} + \frac{3\alpha}{16} \right) \frac{1}{\epsilon^2} \; - \; \left(\frac{\alpha^2}{32} + \frac{5\alpha}{32} \right) \frac{1}{\epsilon} \right] a^2 \\
& - \left[\left(\left(\frac{31\alpha}{96} + \frac{3\alpha^2}{32} + \frac{\alpha^3}{64} \right) C_A^3 - \frac{\alpha}{12} C_A^2 T_F N_f \right) \frac{1}{\epsilon^3} \right. \\
& \quad - \left(\left(\frac{7\alpha^3}{384} + \frac{11\alpha^2}{64} + \frac{115\alpha}{192} \right) C_A^3 - \frac{5\alpha}{24} C_A^2 T_F N_f \right) \frac{1}{\epsilon^2} \\
& \quad + \left. \left(\left(\frac{67\alpha}{128} + \frac{13\alpha^2}{128} + \frac{\alpha^3}{128} \right) C_A^3 - \frac{5\alpha}{16} C_A^2 T_F N_f \right) \frac{1}{\epsilon} \right] a^3 \; + \; O(a^4) \qquad (4)
\end{aligned}
$$

$$
\begin{aligned}
Z_c \;=\; & 1 \; + \; \left(\frac{3}{4} - \frac{\alpha}{4} \right) C_A \frac{a}{\epsilon} \; + \; \left[\left(\left(\frac{\alpha^2}{16} - \frac{35}{32} \right) C_A^2 + \frac{1}{2} C_A T_F N_f \right) \frac{1}{\epsilon^2} \right. \\
& \qquad - \left. \left(\left(\frac{\alpha^2}{32} - \frac{\alpha}{32} - \frac{95}{96} \right) C_A^2 + \frac{5}{12} C_A T_F N_f \right) \frac{1}{\epsilon} \right] a^2 \\
& + \left[\left(\left(\frac{2765}{1152} + \frac{35\alpha}{384} - \frac{3\alpha^2}{64} - \frac{\alpha^3}{64} \right) C_A^3 - C_A^2 T_F N_f \left(\frac{149}{72} + \frac{\alpha}{24} \right) + \frac{4}{9} C_A T_F^2 N_f^2 \right) \frac{1}{\epsilon^3} \right. \\
& \quad + \left(\left(\frac{7\alpha^3}{384} + \frac{11\alpha^2}{128} + \frac{5\alpha}{96} - \frac{15587}{3456} \right) C_A^3 + C_A^2 T_F N_f \left(\frac{1405}{432} - \frac{\alpha}{48} \right) \right. \\
& \qquad + \left. C_A C_F T_F N_f - \frac{10}{27} C_A T_F^2 N_f^2 \right) \frac{1}{\epsilon^2} \\
& \quad + \left(\left(\frac{15817}{5184} - \frac{17\alpha}{96} - \frac{55\alpha^2}{768} - \frac{\alpha^3}{128} + \left(\frac{\alpha}{8} + \frac{3}{32} \right) \zeta(3) \right) C_A^3 \right. \\
& \qquad - C_A^2 T_F N_f \left(\frac{97}{324} + 3\zeta(3) - \frac{7\alpha}{24} \right) - C_A C_F T_F N_f \left(\frac{15}{4} - 4\zeta(3) \right) \\
& \qquad - \left. \left. \frac{35}{81} C_A T_F^2 N_f^2 \right) \frac{1}{\epsilon} \right] a^3 \; + \; O(a^4) \qquad (5)
\end{aligned}
$$

$$
\begin{aligned}
Z_\psi &= 1 ~-~ \alpha C_F \frac{a}{\epsilon} ~+~ \left[\left(C_F C_A \left(\frac{\alpha^2}{8} + \frac{3\alpha}{4} \right) + \frac{\alpha^2}{2} C_F^2 \right) \frac{1}{\epsilon^2} \right. \\
& \qquad \left. -~ \left(C_F C_A \left(\alpha + \frac{25}{8} \right) ~-~ C_F T_F N_f ~-~ \frac{3}{4} C_F^2 \right) \frac{1}{\epsilon} \right] a^2 \\
& ~+~ \left[\left(\frac{\alpha}{3} C_A C_F T_F N_f - \left(\frac{3\alpha^2}{4} + \frac{\alpha^3}{8} \right) C_A C_F^2 - C_A^2 C_F \left(\frac{31\alpha}{24} + \frac{3\alpha^2}{16} + \frac{\alpha^3}{48} \right) - \frac{\alpha}{6} C_F^3 \right) \frac{1}{\epsilon^3} \right. \\
& \quad ~+~ \left(\frac{8}{9} C_F T_F^2 N_f^2 - \frac{3\alpha}{4} C_F^3 + \left(\frac{2}{3} - \alpha \right) C_F^2 T_F N_f - \left(\frac{47}{9} + \alpha \right) C_A C_F T_F N_f \right. \\
& \qquad ~+~ \left. \left(\alpha^2 + \frac{25\alpha}{8} - \frac{11}{6} \right) C_A C_F^2 + \left(\frac{275}{36} + \frac{73\alpha}{24} + \frac{7\alpha^2}{16} + \frac{\alpha^3}{48} \right) C_A^2 C_F \right) \frac{1}{\epsilon^2} \\
& \quad ~+~ \left(- \frac{20}{27} C_F T_F^2 N_f^2 - \frac{1}{2} C_F^3 - C_F^2 T_F N_f + \left(\frac{287}{27} + \frac{17\alpha}{12} \right) C_A C_F T_F N_f \right. \\
& \qquad ~-~ \left(\frac{\alpha^3}{32} + \frac{5\alpha^2}{16} + \frac{263\alpha}{96} + \frac{9155}{432} - \left(\frac{23}{8} - \frac{\alpha}{4} \right) \zeta(3) \right) C_A^2 C_F \\
& \qquad ~+~ \left. \left. \left(\frac{143}{12} - 4\zeta(3) \right) C_A C_F^2 \right) \frac{1}{\epsilon} \right] a^3 ~+~ O(a^4)
\end{aligned} \tag{6}
$$

where $T^a T^a = C_F I$, $\mathrm{Tr} \left(T^a T^b \right) = T_F \delta^{ab}$, $f^{acd} f^{bcd} = C_A \delta^{ab}$, $\zeta(n)$ is the Riemann zeta function and $a = g^2/(16\pi^2)$. We have checked our routines and programmes which determine these expressions by first running the files for the usual QCD Lagrangian as input before replacing the file defining the Feynman rules with those for the Curci-Ferrari model. We found exact agreement with all known QCD expressions for arbitrary α, including the more recently determined ghost anomalous dimension, [36, 37, 38, 39, 40]. In all our calculations of the renormalization constants we follow the technique of [39] by computing the Green's function in terms of *bare* parameters and then rescaling them at the end via (2) which is equivalent to the subtraction procedure. The remaining infinities are removed by fixing the associated renormalization constant for that particular Green's function.

To proceed further we need to confirm that with these wave function anomalous dimensions, the correct coupling constant renormalization emerges in the Curci-Ferrari model. As this ought to be gauge independent for all α this will provide a stringent check on (3) to (6). However, to achieve this one must consider several of the three point vertices of (1) which, to apply the Mincer algorithm, requires an external leg momentum to be nullified. In this case there is the potential difficulty that a spurious infrared infinity can emerge in the answer due to infrared infinities at the nullified vertex. Therefore, it is appropriate to either consider those Green's functions where this problem never arises in the first place or introduce the infrared rearrangement procedure, [41, 42], which is difficult to automate. We have chosen the former which can be made automatic for the coupling constant renormalization but note that the method will also be central in deducing the gluon mass renormalization. We have computed the quark-gluon and ghost-gluon vertex renormalization at three loops and found that for *both* QCD and the Curci-Ferrari model the same gauge invariant coupling constant renormalization constant emerged which agrees with the original result of [43]. We found

$$
\begin{aligned}
Z_g &= 1 ~+~ \left(\frac{2}{3} T_F N_f - \frac{11}{6} C_A \right) \frac{a}{\epsilon} ~+~ \left[\left(\frac{121}{24} C_A^2 + \frac{2}{3} T_F^2 N_f^2 - \frac{11}{3} C_A T_F N_f \right) \frac{1}{\epsilon^2} \right. \\
& ~+~ \left. \left(C_F T_F N_f + \frac{5}{3} C_A T_F N_f - \frac{17}{6} C_A^2 \right) \frac{1}{\epsilon} \right] a^2 \\
& ~+~ \left[\left(\frac{605}{36} C_A^2 T_F N_f - \frac{55}{9} C_A T_F^2 N_f^2 + \frac{20}{27} T_F^3 N_f^3 - \frac{6655}{432} C_A^3 \right) \frac{1}{\epsilon^3} \right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(\frac{22}{9} C_F T_F^2 N_f^2 - \frac{121}{18} C_A C_F T_F N_f - \frac{979}{54} C_A^2 T_F N_f + \frac{110}{27} C_A T_F^2 N_f^2 + \frac{2057}{108} C_A^3 \right) \frac{1}{\epsilon^2} \\
&+ \left(\frac{205}{54} C_A C_F T_F N_f - \frac{22}{27} C_F T_F^2 N_f^2 + \frac{1415}{162} C_A^2 T_F N_f \right. \\
&\quad \left. - \frac{79}{81} C_A T_F^2 N_f^2 - \frac{1}{3} C_F^2 T_F N_f - \frac{2857}{324} C_A^3 \right) \frac{1}{\epsilon} \Big] a^3 + O(a^4) . \qquad (7)
\end{aligned}
$$

This together with (6) means the basic renormalization group functions for the *massless* Curci-Ferrari model at three loops are

$$
\begin{aligned}
\gamma_A(a) &= [(3\alpha - 13) C_A + 8 T_F N_f] \frac{a}{6} \\
&+ \left[\left(\alpha^2 + 11\alpha - 59 \right) C_A^2 + 40 C_A T_F N_f + 32 C_F T_F N_f \right] \frac{a^2}{8} \\
&+ \left[\left(54\alpha^3 + 909\alpha^2 + (6012 + 864\zeta(3))\alpha + 648\zeta(3) - 39860 \right) C_A^3 \right. \\
&\quad - (2304\alpha + 20736\zeta(3) - 58304) C_A^2 T_F N_f + (27648\zeta(3) + 320) C_A C_F T_F N_f \\
&\quad \left. - 9728 C_A T_F^2 N_f^2 - 2304 C_F^2 T_F N_f - 5632 C_F T_F^2 N_f^2 \right] \frac{a^3}{1152} + O(a^4) \\
\gamma_\alpha(a) &= - [(3\alpha - 26) C_A + 16 T_F N_f] \frac{a}{12} \\
&- \left[\left(\alpha^2 + 17\alpha - 118 \right) C_A^2 + 80 C_A T_F N_f + 64 C_F T_F N_f \right] \frac{a^2}{16} \\
&- \left[\left(27\alpha^3 + 558\alpha^2 + (4203 + 864\zeta(3))\alpha + (648\zeta(3) - 39860) \right) C_A^3 \right. \\
&\quad - (1224\alpha + 20736\zeta(3) - 58304) C_A^2 T_F N_f + (27648\zeta(3) + 320) C_A C_F T_F N_f \\
&\quad \left. - 9728 C_A T_F^2 N_f^2 - 2304 C_F^2 T_F N_f - 5632 C_F T_F^2 N_f^2 \right] \frac{a^3}{1152} + O(a^4) \\
\gamma_c(a) &= (\alpha - 3) C_A \frac{a}{4} + \left[\left(3\alpha^2 - 3\alpha - 95 \right) C_A^2 + 40 C_A T_F N_f \right] \frac{a^2}{48} \\
&+ \left[\left(162\alpha^3 + 1485\alpha^2 + (3672 - 2592\zeta(3))\alpha - (1944\zeta(3) + 63268) \right) C_A^3 \right. \\
&\quad - (6048\alpha - 62208\zeta(3) - 6208) C_A^2 T_F N_f - (82944\zeta(3) - 77760) C_A C_F T_F N_f \\
&\quad \left. + 9216 C_A T_F^2 N_f^2 \right] \frac{a^3}{6912} + O(a^4) \\
\beta(a) &= - \left[\frac{11}{3} C_A - \frac{4}{3} T_F N_f \right] a^2 - \left[\frac{34}{3} C_A^2 - 4 C_F T_F N_f - \frac{20}{3} C_A T_F N_f \right] a^3 \\
&+ \left[2830 C_A^2 T_F N_f - 2857 C_A^3 + 1230 C_A C_F T_F N_f - 316 C_A T_F^2 N_f^2 \right. \\
&\quad \left. - 108 C_F^2 T_F N_f - 264 C_F T_F^2 N_f^2 \right] \frac{a^4}{54} + O(a^5) . \qquad (8)
\end{aligned}
$$

It is worth noting that as a consequence we have verified the three loop gauge independence of the β-function by computing in this particular nonlinear covariant gauge. We have chosen not to re-compute the quark mass anomalous dimension since from the two loop calculation, [29], it is clear that the same three loop gauge independent expression as [30, 31] will emerge. For completeness, we note the anomalous dimension of the gauge parameter, α, in our conventions, (2), is

$$
\begin{aligned}
\gamma_A(a) + \gamma_\alpha(a) &= \alpha \left[\frac{a}{4} C_A + (\alpha + 5) C_A^2 \frac{a^2}{16} \right. \\
&\quad \left. + 3 C_A^2 \left[\left(\alpha^2 + 13\alpha + 67\alpha \right) C_A - 40 T_F N_f \right] \frac{a^3}{128} \right] + O(a^4) \qquad (9)
\end{aligned}
$$

which implies that the Landau gauge remains as a fixed point of this renormalization group function at three loops.

Armed with these basic renormalization constants we have deduced the mass renormalization by considering the composite operator

$$\mathcal{O} ~=~ \frac{1}{2} A^a_\mu A^{a\,\mu} ~-~ \alpha \bar{c}^a c^a \tag{10}$$

in the massless Curci-Ferrari model and renormalizing it by inserting it in an appropriate two point function. In [44] it was verified that this operator is multiplicatively renormalizable at two loops where the one loop check was established in [5]. We expect this is an all orders property intimately related to the fact that it is BRST invariant. Indeed the interplay of renormalizability and BRST invariance of this operator has been explored at two loops in [44]. Clearly, we need to be careful which two point function $\mathcal{O}$ is inserted into, due to the problems noted earlier. There is an additional potential difficulty in this case in that the operator must not be inserted at zero momentum. In other words a momentum must flow through the operator, otherwise an incorrect result could be obtained for the operator anomalous dimension. An excellent example of such pitfalls has been elegantly expounded in the context of the *one* loop renormalization of $(G^a_{\mu\nu})^2$ in QCD in [45]. The upshot is that with $\mathcal{O}$ inserted in a two point function with a non-zero momentum that Green's function is in fact a three point function. To apply the MINCER algorithm an external momentum must be nullified which clearly cannot be that passing through the operator. Instead it must be the external momentum associated with the field in which the operator is inserted. To determine the three loop result the only possibilities are the gluon and ghost fields. Inserting in a quark two point function would require a four loop calculation due to the absence of a tree term. Gluon external legs would inevitably lead to an infrared problem even at one loop which we are trying to a avoid so we are forced us to consider a ghost two point function. It transpires, by considering the way the momentum is nullified in this case, that spurious infrared infinities cannot arise, in much the same way that they do not in the three point function evaluations for the earlier coupling constant renormalizations. Hence, we have renormalized $\langle c^a(p_1) \mathcal{O}(p_3) \bar{c}^b(p_2) \rangle$ with MINCER using this procedure where $p_1 + p_2 + p_3 = 0$ and $p_1 = 0$. Allowing for the ghost wave function renormalization of the external fields we find

$$\begin{aligned}
Z_{\mathcal{O}} ~=~ & 1 ~+~ \left[\left(\frac{\alpha}{4} - \frac{35}{12} \right) C_A + \frac{4}{3} T_F N_f \right] \frac{a}{\epsilon} \\
& +~ \left[\left(\left(\frac{2765}{288} - \frac{11\alpha}{12} \right) C_A^2 ~+~ \frac{16}{9} T_F^2 N_f^2 ~+~ \left(\frac{\alpha}{3} - \frac{149}{18} \right) C_A T_F N_f \right) \frac{1}{\epsilon^2} \right. \\
& \left. \quad +~ \left(\left(\frac{\alpha^2}{32} + \frac{11\alpha}{32} - \frac{449}{96} \right) C_A^2 ~+~ 2 C_F T_F N_f ~+~ \frac{35}{12} C_A T_F N_f \right) \frac{1}{\epsilon} \right] a^2 \\
& +~ \left[\left(\frac{64}{27} T_F^3 N_f^3 + \left(\frac{493}{12} - \frac{173\alpha}{72} \right) C_A^2 T_F N_f \right. \right. \\
& \quad \left. -~ \left(\frac{154}{9} - \frac{4\alpha}{9} \right) C_A T_F^2 N_f^2 + \left(\frac{3767\alpha}{1152} - \frac{113365}{3456} \right) C_A^3 \right) \frac{1}{\epsilon^3} \\
& \quad +~ \left(\frac{56}{9} C_F T_F^2 N_f^2 + \frac{85}{9} C_A T_F^2 N_f^2 + \left(\frac{\alpha}{2} - \frac{263}{18} \right) C_A C_F T_f N_f \right. \\
& \qquad +~ \left(\frac{\alpha^2}{24} + \frac{71\alpha}{48} - \frac{5407}{144} \right) C_A^2 T_F N_f \\
& \qquad \left. +~ \left(\frac{41579}{1152} - \frac{99\alpha}{32} - \frac{59\alpha^2}{384} - \frac{\alpha^3}{384} \right) C_A^3 \right) \frac{1}{\epsilon^2} \\
& \quad +~ \left(- \frac{44}{27} C_F T_F^2 N_f^2 - \frac{193}{81} C_A T_F^2 N_f^2 + \left(\frac{415}{108} + 4\zeta(3) \right) C_A C_F T_f N_f \right.
\end{aligned}$$

$$
\begin{aligned}
&+ \left(\frac{5563}{324} - \frac{\alpha}{3} - 3\zeta(3)\right) C_A^2 T_F N_f - \frac{2}{3} C_F^2 T_F N_f \\
&- \left(\frac{75607}{5184} - \frac{167\alpha}{192} - \frac{101\alpha^2}{768} - \frac{\alpha^3}{128} - \left(\frac{3}{32} + \frac{\alpha}{8}\right)\zeta(3)\right) C_A^3\right) \frac{1}{\epsilon}\Bigg] a^3 + O(a^4)
\end{aligned}
\tag{11}
$$

where

$$
\mathcal{O}_{\mathrm{o}} = Z_{\mathcal{O}} \mathcal{O} \, . \tag{12}
$$

Clearly the one and two loop terms agree with the known results for the mass renormalization in the Curci-Ferrari model itself, [2, 5, 21, 28, 29]. Moreover, the three loop answer is derived using the *same* converter routines used for the coupling constant renormalization. Also it is straightforward to check that like (3) to (6) the triple and double pole terms with respect to ϵ at three loops can be predicted from the one and two loop terms and these values in (11) are in *exact* agreement for all α. Hence we find

$$
\begin{aligned}
\gamma_{\mathcal{O}}(a) &= \left[16 T_F N_f + (3\alpha - 35) C_A\right] \frac{a}{24} \\
&+ \left[280 C_A T_F N_f + (3\alpha^2 + 33\alpha - 449) C_A^2 + 192 C_F T_F N_f\right] \frac{a^2}{96} \\
&+ \Big[\left((2592\alpha + 1944)\zeta(3) + 162\alpha^3 + 2727\alpha^2 + 18036\alpha - 302428\right) C_A^3 \\
&\quad - (62208\zeta(3) + 6912\alpha - 356032) C_A^2 T_F N_f + (82944\zeta(3) + 79680) C_A C_F T_F N_f \\
&\quad - 49408 C_A T_F^2 N_f^2 - 13824 C_F^2 T_F N_f - 33792 C_F T_F^2 N_f^2\Big] \frac{a^3}{13824} + O(a^4) \, .
\end{aligned}
\tag{13}
$$

The only other check on this result is the Yang-Mills expression computed in [2] for $SU(N_c)$ and $\alpha = 0$ with the tensor correction method. Unfortunately we do not find agreement with [2] for two reasons. First, the term of (13) involving 302428 is 377452 from equation (24) of [2] when compared over the same denominator. (The term $1944\zeta(3)$ is in agreement with that given in [2].) However, it is implicit in [2] and explicit in [16] that to deduce the corresponding $Z_{\mathcal{O}}$ an incorrect value of the three loop Landau gauge gluon anomalous dimension has been used. Allowing for the correct value, (3), the expressions still differ by the quantity 19/24. We are confident, however, that our result (13) is in fact correct for the following simple reason, which to our knowledge has not been noted before. If one considers for the moment the one and two loop Yang-Mills results for $\alpha = 0$ it transpires that the coefficients are given by the sum of the gluon and ghost anomalous dimensions in the Landau gauge. Indeed it is also apparent that this holds for $N_f \neq 0$. Turning to the three loop expression, (13), it is straightforward to observe that the same feature emerges there too. Of course such a property could be accidental but it justifies our confidence in the veracity of (13). Moreover, it would also suggest the existence of some underlying Slavnov-Taylor identity. If so then $Z_{\mathcal{O}}$ is not an independent renormalization constant and this would reduce the number of such constants required to render (1) finite. The search for and construction of such a Slavnov-Taylor identity is beyond the scope of this article but clearly its prior existence would have rendered a certain amount of our computation unnecessary. However, if constructed it might play a crucial role in simplifying the calculation of similar anomalous dimensions in other gauges such as the maximal abelian gauge.

Having deduced (13) in the Curci-Ferrari model we can determine the Landau gauge value which will correspond to the QCD value in the same gauge. We found

$$
\begin{aligned}
\gamma_{\mathcal{O}}(a)\big|_{\alpha=0} &= \left[16 T_F N_f - 35 C_A\right] \frac{a}{24} \\
&+ \left[280 C_A T_F N_f - 449 C_A^2 + 192 C_F T_F N_f\right] \frac{a^2}{96}
\end{aligned}
$$

$$
\begin{aligned}
&+ \Big[(486\zeta(3) - 75607) C_A^3 - (15552\zeta(3) - 89008)\, C_A^2 T_F N_f \\
&\quad + (20736\zeta(3) + 19920)\, C_A C_F T_F N_f - 12352 C_A T_F^2 N_f^2 \\
&\quad - \; 3456 C_F^2 T_F N_f - 8448 C_F T_F^2 N_f^2 \Big] \frac{a^3}{3456} \; + \; O(a^4)
\end{aligned} \tag{14}
$$

which also corresponds to the mass renormalization in the Curci-Ferrari model, (1), at three loops and together with (8) completes the full three loop renormalization. One consequence of our calculation is that we can quote the value for the operator $\mathcal{O}$ in QED. Setting $T_F = C_F = 1$ and $C_A = 0$ we find, from (13),

$$
\gamma_{\mathcal{O}}(a)|_{\mathrm{QED}} \;=\; \frac{2}{3} N_f a \;+\; 2 N_f a^2 \;-\; N_f \left[22 N_f \;+\; 9 \right] \frac{a^3}{9} \;+\; O(a^4) \tag{15}
$$

for all α.

To conclude with we note that the *new* expression for (13) will alter the two loop predictions made in [2] for the numerical estimate of the operator vacuum expectation value generated by the effective potential method. Whilst not detracting from the achievement of that tour-de-force it would be interesting to repeat those calculations to explore the effect the inclusion of quarks has on the vacuum expectation value estimates for the more realistic case of QCD. This will require (13) but also needs the full two loop effective potential of the composite operator with quarks computed in the Landau gauge.

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References.

[1] Ph. Boucaud, A. Le Yaounac, J.P. Leroy, J. Micheli, O. Pene & J. Rodriguez-Quintero, Phys. Rev. **D63** (2001), 114003.

[2] H. Verschelde, K. Knecht, K. van Acoleyen & M. Vanderkelen, Phys. Lett. **B516** (2001), 307.

[3] K.-I. Kondo, Phys. Lett. **B514** (2001), 335.

[4] K.-I. Kondo, hep-ph/0110013.

[5] K.-I. Kondo, T. Murakami, T. Shinohara & T. Imai, Phys. Rev. **D65** (2002), 085034.

[6] K.-I. Kondo, Phys. Lett. **B491** (2000), 263.

[7] M. Schaden, hep-th/9909011.

[8] M. Schaden, hep-th/0108034.

[9] U. Ellwanger & N. Wschebor, hep-th/0205057.

[10] D. Dudal & H. Verschelde, hep-th/0209025.

[11] V.E.R. Lemes, M.S. Sarandy & S.P. Sorella, hep-th/0206251.

[12] V.E.R. Lemes, M.S. Sarandy & S.P. Sorella, hep-th/0210036.

[13] V.E.R. Lemes, M.S. Sarandy & S.P. Sorella, hep-th/0210077.

[14] H. Verschelde, Phys. Lett. **B351** (1995), 242.

[15] H. Verschelde, S. Schelstraete & M. Vanderkelen, Z. Phys. **C76** (1997), 161.

[16] K. Knecht, 'Algoritmische multiloop berekeningen in massieve kwantumveldentheorie' University of Gent PhD thesis.

[17] G. Curci & R. Ferrari, Nuovo Cim. **A32** (1976), 151.

[18] G. Curci & R. Ferrari, Nuovo Cim. **A35** (1976), 1; G. Curci & R. Ferrari, Nuovo Cim. **A35** (1976), 273; Nuovo Cim. **A47** (1978), 555.

[19] I. Ojima, Z. Phys. **C13** (1982), 173.

[20] L. Baulieu, Phys. Rep. **13** (1985), 1; R. Delbourgo, S. Twisk & G. Thompson, Int. J. Mod. Phys. **A3** (1988), 435; T. Hurth, Helv. Phys. Acta **70** (1997), 406.

[21] J. de Boer, K. Skenderis, P. van Nieuwenhuizen & A. Waldron, Phys. Lett. **B367** (1996), 175.

[22] F. Delduc & S.P. Sorella, Phys. Lett. **B231** (1989), 408.

[23] A. Blasi & N. Maggiore, Mod. Phys. Lett. **A11** (1996), 1665.

[24] L. Baulieu & J. Thierry-Mieg, Nucl. Phys. **B197** (1982), 477.

[25] R. Delbourgo & P.D. Jarvis, J. Phys. **A15** (1982), 611.

[26] V. Periwal, hep-th/9509084.

[27] E. Andriyash & K. Stepanyantz, hep-th/0204211.

[28] J.A. Gracey, Phys. Lett. **B525** (2002), 89.

[29] R.E. Browne & J.A. Gracey, Phys. Lett. **B540** (2002), 68.

[30] O.V. Tarasov, JINR preprint P2-82-900.

[31] S.A. Larin, Phys. Lett. **B303** (1993), 113.

[32] S.G. Gorishny, S.A. Larin, L.R. Surguladze & F.K. Tkachov, Comput. Phys. Commun. **55** (1989), 381.

[33] S.A. Larin, F.V. Tkachov & J.A.M. Vermaseren, "The Form version of Mincer", NIKHEF-H-91-18.

[34] J.A.M. Vermaseren, math-ph/0010025.

[35] P. Nogueira, J. Comput. Phys. **105** (1993), 279.

[36] D.J. Gross & F.J. Wilczek, Phys. Rev. Lett. **30** (1973), 1343; H.D. Politzer, Phys. Rev. Lett. **30** (1973), 1346.

[37] D.R.T. Jones, Nucl. Phys. **B75** (1974), 531; W.E. Caswell, Phys. Rev. Lett. **33** (1974), 244.

[38] O.V. Tarasov & A.A. Vladimirov, Sov. J. Nucl. Phys. **25** (1977), 585; E. Egorian & O.V. Tarasov, Theor. Math. Phys. **41** (1979), 863.

[39] S.A. Larin & J.A.M. Vermaseren, Phys. Lett. **B303** (1993), 334.

[40] K.G. Chetyrkin & A. Rétey, hep-ph/0007088.

[41] A.A. Vladimirov, Theor. Math. Phys. **43** (1980), 417.

[42] K.G. Chetyrkin, A.L. Kataev & F.V. Tkachov, Nucl. Phys. **B174** (1980), 345.

[43] O.V. Tarasov, A.A. Vladimirov & A.Yu. Zharkov, Phys. Lett. **B93** (1980) 429.

[44] R.E. Browne & J.A. Gracey, paper in preparation.

[45] J.C. Collins, *Renormalization* (Cambridge University Press, 1984).