On the $n \to 0$ limit of $\gamma_{gg}(a)$ in QCD

J.F. Bennett & J.A. Gracey,
Theoretical Physics Division,
Department of Mathematical Sciences,
University of Liverpool,
Peach Street,
Liverpool,
L69 7ZF,
United Kingdom.

Abstract. We consider the $n \to 0$ limit of the DGLAP splitting function $\gamma_{gg}(a)$ at all orders in the strong coupling constant, a, by analysing the leading order large N_f form of the associated d-dimensional critical exponent. We show that for unpolarized scattering the pole at $n = 0$ which appears in successive orders in perturbation theory is absent in the resummed expression.

The DGLAP equation, [1], is widely used to evolve the parton structure functions of the nucleon constituents over a large range of energy scales. Central to the equation are the splitting functions which depend on the variable x which represents the fraction of the momentum carried by that parton in the nucleon. Alternatively one can work with the anomalous dimension of the underlying twist-2 operators built out of the quark and gluon fields of QCD which depend on the variable n. They are related to the splitting functions via a Mellin transform restricted to the unit interval in x. It is generally accepted that the solution of the DGLAP evolution which effectively represents perturbative QCD, fits the data extremely well. See, for example, [2]. Moreover, it appears to transcend the region where the perturbative approximation ought not to be valid. Therefore, one would hope that either by resumming perturbation theory in some fashion or developing non-perturbative methods, such as the BFKL formalism [3], it might be possible to begin to explore phenomena in the more extreme non-perturbative regions. One recent issue, [4], has been the problem of understanding the $n \rightarrow 0$ behaviour of the DGLAP splitting functions in QCD. It is known that in perturbation theory each term of the expansion of the operator anomalous dimensions in the strong coupling constant has successively higher order poles at $n = 0$, [5, 6, 7]. However, other considerations suggest that the function is finite at $n = 0$, [4, 8], though there is some disagreement with this point of view, [9]. In this letter we provide some more insight into the $n \rightarrow 0$ limit of the gluon-gluon splitting function, $\gamma_{gg}(a)$, where $a = \alpha_s/(4\pi)$ is the strong coupling constant. This is achieved by examining the large N_f result for $\gamma_{gg}(a)$ which has been computed in [10] where N_f is the number of quark flavours. Essentially the $1/N_f$ expansion sums a different set of Feynman diagrams from those which are ordinarily computed in the loop or coupling constant expansion of perturbation theory. Moreover, the resummation is to all orders in a. This technique has been used in [11, 12, 10, 13] to determine the anomalous dimensions of all the twist-2 operators used in deep inelastic scattering for both unpolarized and polarized processes at $O(1/N_f)$. The all orders results are expressed as a function of d, where d is the dimension of spacetime, known as critical exponents. The coefficients of the corresponding renormalization group function are deduced from knowledge of the d-dimensional fixed point of the QCD β-function and properties of the critical renormalization group equation. The results have been shown to be in agreement with all known explicit two and three loop perturbative results, [5, 6, 7, 14, 15], which puts the validity of the large N_f results in relation to deep inelastic scattering on a firm footing. Hence, in this letter we will examine the $n \rightarrow 0$ limit of the singlet gluon splitting function at $O(1/N_f)$ and all orders in a.

First, we recall the basic formalism of the problem. As the flavour singlet gluonic twist-2 operator mixes with the fermionic operator under renormalization in perturbation theory, one has to deal with a matrix of anomalous dimensions, $\gamma_{ij}(a)$. Since we will be considering it in the large N_f expansion we define the coefficients of its perturbative expansion formally by

$$\gamma_{ij}(a) ~=~ \left(\begin{array}{cc} \gamma_{qq}(a) & \gamma_{gq}(a) \\ \gamma_{qg}(a) & \gamma_{gg}(a) \end{array} \right) \tag{1}$$

where

$$\begin{aligned} \gamma_{qq}(a) &= a_1 a + (a_{21} N_f + a_{22}) a^2 + (a_{31} N_f^2 + a_{32} N_f + a_{33}) a^3 + O(a^4) \\ \gamma_{gq}(a) &= b_1 a + (b_{21} N_f + b_{22}) a^2 + (b_{31} N_f^2 + b_{32} N_f + b_{33}) a^3 + O(a^4) \\ \gamma_{qg}(a) &= c_1 N_f a + c_2 N_f a^2 + (c_{31} N_f^2 + c_{32} N_f + c_{33}) a^3 + O(a^4) \\ \gamma_{gg}(a) &= (d_{11} N_f + d_{12}) a + (d_{21} N_f + d_{22}) a^2 + (d_{31} N_f^2 + d_{32} N_f + d_{33}) a^3 + O(a^4) ~. \end{aligned} \tag{2}$$

The explicit values of the coefficients to two loops as a function of n are given in [5, 6, 7] and exact values for the low operator moments at three loops are found in [14, 15]. In particular we note

$$d_{11} ~=~ \frac{4}{3} T(R) \tag{3}$$

where $T(R)$ is given by $\mathrm{tr}(T^aT^b) = T(R)\delta^{ab}$ and T^a are the generators of the colour group whose structure constants are f^{abc}. In [10] it was argued that the set of coefficients of the $C_2(G)$ sector of $\gamma_{gg}(a)$ at leading order in $1/N_f$, corresponding to $d_{l1}^{C_2(G)}$ at the lth loop, could be written compactly in the critical exponent form as

$$
\begin{aligned}
\lambda_{+,1}^{C_2(G)}(a_c) \;=\; & \Big[[32\mu^5n^2 + 32\mu^5n + 32\mu^5 + 8\mu^4n^4 + 16\mu^4n^3 - 120\mu^4n^2 - 128\mu^4n \\
& - 160\mu^4 - 32\mu^3n^4 - 64\mu^3n^3 + 160\mu^3n^2 + 192\mu^3n + 316\mu^3 \\
& + 48\mu^2n^4 + 96\mu^2n^3 - 78\mu^2n^2 - 126\mu^2n - 306\mu^2 - 31\mu n^4 - 62\mu n^3 \\
& + 31\mu n + 146\mu + 7n^4 + 14n^3 + 7n^2 - 28]\Gamma(n+2-\mu)\Gamma(\mu) \\
& /[8n(\mu-1)^3(n+2)(n^2-1)\Gamma(2-\mu)\Gamma(\mu+n)] \\
& - \; [32\mu^5n^2 + 32\mu^5n + 32\mu^5 - 144\mu^4n^2 - 144\mu^4n - 160\mu^4 - 4\mu^3n^4 \\
& \quad - 8\mu^3n^3 + 240\mu^3n^2 + 244\mu^3n + 316\mu^3 + 16\mu^2n^4 + 32\mu^2n^3 \\
& \quad - 180\mu^2n^2 - 196\mu^2n - 306\mu^2 - 20\mu n^4 - 40\mu n^3 + 59\mu n^2 \\
& \quad + 79\mu n + 146\mu + 8n^4 + 16n^3 - 6n^2 - 14n - 28] \\
& \quad /[8n(\mu-1)^3(n+2)(n^2-1)] \\
& + \; 2(\mu-1)S_1(n) \Big] \frac{\mu C_2(G)\eta^{\mathrm{O}}_1}{(2\mu-1)(\mu-2)T(R)}
\end{aligned}
\tag{4}
$$

where $S_l(n) = \sum_{r=1}^{\infty} 1/r^l$, $d = 2\mu$, $f^{acd}f^{bcd} = C_2(G)\delta^{ab}$ and

$$
\eta^{\mathrm{O}}_1 \;=\; \frac{(2\mu-1)(\mu-2)\Gamma(2\mu)}{4\Gamma^2(\mu)\Gamma(\mu+1)\Gamma(2-\mu)} \; . \tag{5}
$$

We recall that a feature of the large N_f approach to computing information on the perturbative coefficients of (2) was that the anomalous dimensions of the *eigen*-operators of (1) at criticality were determined and denoted by $\lambda_\pm(a_c) = \sum_{i=1}^{\infty} \lambda_{\pm,i}(a_c)/N_f^i$, [10]. That eigen-operator which was predominantly gluonic in content corresponds to the eigen-critical exponent $\lambda_+(a_c)$ whilst $\lambda_-(a_c)$ corresponds to the dimension of the mainly fermionic eigen-operator. The location of the fixed point, a_c, is given by the non-trivial zero of the d-dimensional QCD β-function and in the present notation, [16, 17, 18, 19, 20],

$$
\begin{aligned}
a_c \;=\; & \frac{3\epsilon}{4T(R)N_f} + \frac{1}{T^2(R)N_f^2} \left[\frac{33}{16}C_A\epsilon - \left(\frac{27}{16}C_F + \frac{45}{16}C_A \right)\epsilon^2 + \left(\frac{99}{64}C_F + \frac{237}{128}C_A \right)\epsilon^3 \right. \\
& \left. + \left(\frac{77}{64}C_F + \frac{53}{128}C_A \right)\epsilon^4 - \frac{3}{1024}[(288\zeta(3) + 214)C_F + (480\zeta(3) - 229)C_A]\,\epsilon^5 + O(\epsilon^6) \right]
\end{aligned}
\tag{6}
$$

to $O(1/N_f^3)$ in the large N_f expansion where $d = 4 - 2\epsilon$ and $\zeta(r)$ is the Riemann zeta function. From (4) and (6), we find, for instance,

$$
\begin{aligned}
d_{51}^{C_2(G)} \;=\; & \frac{128[9n^4 + 18n^3 + 79n^2 + 70n + 32]S_3(n)}{243(n+2)(n+1)^2(n-1)n^2} \\
& + \frac{256[9n^4 + 18n^3 + 79n^2 + 70n + 32]S_1^3(n)}{243(n+2)(n+1)^2(n-1)n^2} \\
& - \frac{128[63n^6 + 189n^5 + 821n^4 + 1327n^3 + 1176n^2 + 544n + 96]S_1^2(n)}{243(n+2)(n+1)^3(n-1)n^3}
\end{aligned}
$$

$$
\begin{aligned}
&- 256[4n^{10} + 20n^9 + 17n^8 - 52n^7 - 414n^6 - 976n^5 - 1521n^4 - 1516n^3 \\
&\quad - 930n^2 - 320n - 48]S_1(n)/[243(n+2)(n+1)^4(n-1)n^4] \\
&- \frac{2048[n^2+n+1]\zeta(4)}{27(n+2)(n^2-1)n} + \frac{1024S_1(n)\zeta(4)}{27} - \frac{10240S_1(n)\zeta(3)}{243} \\
&+ \frac{128[3n^6 + 9n^5 + 307n^4 + 599n^3 + 746n^2 + 448n + 96]\zeta(3)}{243(n+2)(n+1)^2(n-1)n^2} \\
&+ 4[155n^{12} + 930n^{11} + 1455n^{10} - 1250n^9 - 9879n^8 - 21786n^7 \\
&\quad - 47107n^6 - 80550n^5 - 97392n^4 - 78336n^3 - 40000n^2 \\
&\quad - 11776n - 1536]/[243(n+2)(n+1)^5(n-1)n^5] \ . \qquad (7)
\end{aligned}
$$

This illustrates earlier remarks in that this coefficient clearly has a fifth order pole at $n = 0$ which is a degree larger than the previous loop order, [10]. Clearly to all orders this degree of divergence will increase. However, since (4) encodes the structure of this particular coefficient of $\gamma_{gg}(a)$ to all orders in the coupling constant in leading order in $1/N_f$ as a function of n, we can examine the form of the exponent for $n = 0$. In other words prior to making the connection with explicit perturbation theory. Therefore setting $n = 0$ in (4) we can determine if the expression is divergent which would indicate that such a pole persists in $\gamma_{gg}(a)$ or if it is finite. The latter case would indicate consistency with the *supposition* from more general principles, [4, 8]. First, by inspection of (4) the $S_1(n)$ independent terms each have simple poles in n. However, by careful examination of the numerators of each term and in particular their n-independent terms, it is evident that the residue at $n = 0$ is the same for both whilst the relative sign between the terms means that overall the residue is zero for the simple pole at $n = 0$. Also for the term involving $S_1(n)$ one writes

$$
S_{l+1}(n) \ = \ \frac{(-1)^l}{l!} \left[\psi^{(l)}(n+1) \ - \ \psi^{(l)}(1) \right] \qquad (8)
$$

for general l, which vanishes in the limit $n \to 0$ where $\psi(x)$ is the derivative of the logarithm of the Euler Γ-function. Hence the exponent (4) is in fact finite at $n = 0$. This is consistent with the resummed splitting function being non-singular at this point. However, it is important to recall that we have only demonstrated this property for that part of the series which is *leading* order in the $1/N_f$ expansion. Moreover that this part is not inconsistent with general considerations is, we believe, an important observation and the central result of this article. Furthermore, we record that when $n = 0$ the exponent (4) is

$$
\begin{aligned}
\lambda^{C_2(G)}_{+,1}(a_c)\Big|_{n=0} \ = \ & \Big[[16\mu^4 - 64\mu^3 + 94\mu^2 - 59\mu + 14](\mu-1)(\mu-2)[\psi(\mu-1) - \psi(3-\mu)] \\
& + \ \mu(8\mu^4 - 26\mu^3 + 23\mu^2 - 4) \Big] \frac{\mu C_2(G)\eta^{\mathrm{O}}_1}{8(2\mu-1)(\mu-1)^3(\mu-2)^2} \ . \qquad (9)
\end{aligned}
$$

For completeness, we also comment on the $n \to 0$ limit of the critical exponents of the other twist-2 operators at $O(1/N_f)$. We recall that, [12],

$$
\begin{aligned}
\lambda_{-,1}(a_c) \ = \ & \frac{4\mu(\mu-1)C_2(R)\eta^{\mathrm{O}}_1}{(2\mu-1)(\mu-2)T(R)N_f} \left[\frac{(\mu-1)(n-1)(2\mu+n-2)}{2\mu(\mu+n-1)(\mu+n-2)} \ + \ [\psi(\mu-1+n) - \psi(\mu)] \right. \\
& - \ \frac{\Gamma(n-1)\Gamma(2\mu)}{4(\mu+n-1)(\mu+n-2)\Gamma(2\mu-1+n)} \\
& \left. \times \ \left[(n^2 + n + 2\mu - 2)^2 + 2(\mu-2)(n(n-1)(2\mu-3+2n) + 2(\mu-1+n)) \right] \right] \\
& \qquad (10)
\end{aligned}
$$

for the flavour singlet fermionic operator where $T^a T^a = C_2(R)$. The critical exponent for the non-singlet operator dimension corresponds to the first two terms of (10) and it is clear that in the limit we are interested in that that exponent is finite. In particular

$$\lambda^{\mbox{non-singlet}}_{-,1}(a_c)\Big|_{n=0} ~=~ - ~ \frac{4(2\mu^2 - 4\mu + 1)C_2(R)\eta^{\mbox{o}}_1}{(2\mu-1)(\mu-2)^2T(R)N_{\!f}} ~. \qquad (11)$$

By contrast, the singlet fermionic eigen-operator, which in perturbation theory corresponds to the combination of coefficients $(a_{l1} - b_{l1}c_1/d_{11})$ at lth loop, diverges at $n = 0$ having the behaviour

$$\lambda_{-,1}(a_c) ~\sim~ \frac{4\mu(\mu-1)(2\mu-3)C_2(R)\eta^{\mbox{o}}_1}{n(\mu-2)^2T(R)N_{\!f}} \qquad (12)$$

as $n \rightarrow 0$ with the singularity arising from the final term of (10) which corresponds to Feynman diagrams with the operator inserted in a closed quark loop.

To conclude with we have demonstrated that for unpolarized scattering the behaviour of $\gamma_{gg}(a)$ at $n = 0$ at leading order in the large $N_{\!f}$ expansion is consistent with the function being finite at this point. However, it is worth putting this result in context with other observations from the BFKL formalism which has also been used to study the anomalous dimensions discussed here. In [21] it was shown that the $C_2(G)$ sector of $\gamma_{gg}(a)$ in the BFKL approach not only does not contain any poles in n but has in fact three branch points. Indeed this does not appear to be just a feature of QCD. If one examines the toy model of scalar ϕ^3 theory in six dimensions the *complete* analogous anomalous dimension was resummed at leading order without reference to the small x limit and again only branch points and no poles were observed, [22]. For other anomalous dimensions in QCD, the non-singlet and polarized dimensions also do not have poles at $n = 0$ but branch points, [23]. (For a recent review of these issues see, for example, [24].) Therefore it is reassuring that our large $N_{\!f}$ observation is also not inconsistent with the BFKL formalism. In light of this it would be interesting to go beyond our leading order $1/N_{\!f}$ analysis to confirm the absence of the $n = 0$ pole at next order as well as being able to understand the branch point structure in the large $N_{\!f}$ point of view. To compute the $O(1/N_{\!f}^2)$ correction to $\lambda_+(a_c)$, though, would involve the evaluation of a large set of $1/N_{\!f}$ graphs which contain, for example, several six loop diagrams and is therefore, we believe, not attainable in the near future.

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References.

[1] V.N. Gribov & L.N. Lipatov, Sov. J. Nucl. Phys. **15** (1972), 438; D. Gross & F. Wilczek, Phys. Rev. **D8** (1973), 3633; **D9** (1974), 980; H. Georgi & H.D. Politzer, Phys. Rev. **D9** (1974), 416; L.N. Lipatov, Sov. J. Nucl. Phys. **20** (1975), 95; G. Altarelli & G. Parisi, Nucl. Phys. **B126** (1977), 298; Yu.L. Dokshitzer, Sov. Phys. JETP **46** (1977), 641.

[2] J. Breitweg et al, ZEUS Collaboration, Phys. Lett. **B407** (1997), 432; C. Adloff et al, H1 Collaboration, Nucl. Phys. **B497** (1997), 3.

[3] V.S. Fadin, E.A. Kuraev & L.N. Lipatov, Phys. Lett. **B60** (1975), 50; I.I. Balitsky & L.N. Lipatov, Sov. J. Nucl. Phys. **28** (1978), 822.

[4] J.R. Cudell, A. Donnachie & P.V. Landshoff, Phys. Lett. **B448** (1999, 281; P.V. Landshoff, Nucl. Phys. Proc. Suppl. **79** (1999), 204.

[5] E.G. Floratos, D.A. Ross & C.T. Sachrajda, Nucl. Phys. **B129** (1977), 66; **B139** (1978), 545(E); A. González-Arroyo, C. López & F.J. Ynduráin, Nucl. Phys. **B153** (1979), 161; G. Curci, W. Furmanski & R. Petronzio, Nucl. Phys. **B175** (1980), 27.

[6] E.G. Floratos, D.A. Ross & C.T. Sachrajda, Nucl. Phys. **B152** (1979), 493.

[7] A. González-Arroyo & C. López, Nucl. Phys. **B166** (1980), 429; C. López & F.J. Ynduráin, Nucl. Phys. **B183** (1981), 157; E.G. Floratos, C. Kounnas & R. Lacaze, Phys. Lett. **B98** (1981), 89, 285; Nucl. Phys. **B192** (1981), 417.

[8] T. Jaroszewicz, Phys. Lett. **B116** (1982), 291; S. Catani & F. Hautmann, Nucl. Phys. **B427** (1994), 475.

[9] W.L. van Neerven, Nucl. Phys. Proc. Suppl. **79** (1999), 36.

[10] J.F. Bennett & J.A. Gracey, Nucl. Phys. **B517** (1998), 241.

[11] J.A. Gracey, Phys. Lett. **B322** (1994), 141.

[12] J.A. Gracey, Nucl. Phys. **B480** (1996), 73.

[13] J.F. Bennett & J.A. Gracey, Phys. Lett. **B432** (1998), 209.

[14] S.A. Larin, T. van Ritbergen & J.A.M. Vermaseren, Nucl. Phys. **B427** (1994), 41.

[15] S.A. Larin, P. Nogueira, T. van Ritbergen & J.A.M. Vermaseren, Nucl. Phys. **B492** (1997), 338.

[16] D.J. Gross & F.J. Wilczek, Phys. Rev. Lett. **30** (1973), 1343; H.D. Politzer, Phys. Rev. Lett. **30** (1973), 1346.

[17] W.E. Caswell, Phys. Rev. Lett. **33** (1974), 244; D.R.T. Jones, Nucl. Phys. **B75** (1974), 531; E.S. Egorian & O.V. Tarasov, Teor. Mat. Fiz. **41** (1979), 26.

[18] O.V. Tarasov, A.A. Vladimirov & A.Yu. Zharkov, Phys. Lett. **93B** (1980), 429; S.A. Larin & J.A.M. Vermaseren, Phys. Lett. **B303** (1993), 334.

[19] T. van Ritbergen, J.A.M. Vermaseren & S.A. Larin, Phys. Lett. **B400** (1997), 379.

[20] J.A. Gracey, Phys. Lett. **B373** (1996), 178.

[21] J. Blümlein, hep-ph/9506446; R.K. Ellis, F. Hautmann & B.R. Webber, Phys. Lett. **B348** (1995), 582.

[22] J. Blümlein & W.L. van Neerven, Phys. Lett. **B450** (1999), 412.

[23] J. Blümlein & A. Vogt. Phys. Lett. **B386** (1996), 350; **B370** (1996), 149.

[24] J. Blümlein, hep-ph/9909449.