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Infra-red stability of Yukawa and soft-breaking fixed points

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We investigate the infra-red stability of the fixed points in the evolution of the Yukawa couplings, A-parameters and soft scalar masses in a broad class of supersymmetric theories. We show that the issue of stability is essentially determined in all three cases by the eigenvalues of the same matrix. In a very wide range of physically interesting theories it follows that, in the asymptotically free case, the existence of stable infra-red fixed points for the Yukawa couplings implies stable infra-red fixed points for the A-parameters and soft scalar masses.

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The predictive power of the supersymmetric standard model and its extensions may be enhanced if the renormalisation group (RG) running of the parameters is dominated by infra-red (IR) stable fixed points. Typically these fixed points are for ratios; for instance, of the Yukawa coupling to the gauge coupling[1], or the A-parameter to the gaugino mass. Moreover, even if these ratios have not attained their fixed point values at the weak scale (which in fact is usually the case), the couplings may be determined by quasi-fixed point behaviour [2]. Here the value of the coupling at the weak scale is independent of its value at the unification scale. For this scenario, the Yukawa couplings at the unification scale need to be large. There is a considerable literature devoted to the consequences of fixed-point and quasi-fixed point behaviour in the standard model and the MSSM, and in extensions thereof [1]–[24]. A necessary condition for the success of both the fixed point and the quasi-fixed point scenarios is the existence of *stable* infra-red fixed points. Hence it is of interest to be able to determine for a given theory the nature of the infra-red fixed points, and this is the question we address in this paper. We consider $N = 1$ supersymmetric gauge theories, and investigate the IR stability of the fixed point structure of the Yukawa couplings, the soft-breaking A-parameters and the soft-breaking masses[2]. The stability of the fixed points in each case is determined by the positivity of the eigenvalues of a matrix $M - pQI$ where Q is the coefficient of the one-loop gauge β-function, and the matrix M and the parameter p depend on the type of coupling. Our main result is that for a theory for which the anomalous dimension matrix for the matter fields is diagonal, there are matrices R, S (which are $r \times s$ and $s \times r$ respectively) and D (which is $r \times r$ and diagonal) such that the matrix M is given by DRS for the Yukawa case, RSD for the A-parameter case and SDR for the soft mass case. It is easy to show that the non-zero eigenvalues of the three matrices DRS, RSD and SDR coincide, and hence the eigenvalues determining the stability of the fixed points for all the couplings in the theory are given in terms of one basic set. Moreover in a more restricted class of theories (but which still includes all cases commonly considered) we can show that these non-zero eigenvalues are all positive. This implies that in the case of negative Q, all the fixed points are infra-red stable.

For an $N = 1$ supersymmetric gauge theory with gauge group $\Pi_a G_a$, superpotential

$$W = \frac{1}{6} Y^{ijk} \Phi_i \Phi_j \Phi_k + \frac{1}{2} \mu^{ij} \Phi_i \Phi_j, \tag{1}$$

[1] often called Pendleton-Ross (PR) fixed points [1]

[2] Stability of the Yukawa couplings alone has been considered for a range of theories in Ref. [18]

and soft breaking terms given by

$$L_{\rm SB} = (m^2)^j_i \phi^i \phi_j + \left(\frac{1}{6} h^{ijk} \phi_i \phi_j \phi_k + \frac{1}{2} b^{ij} \phi_i \phi_j + \frac{1}{2} \sum_a M_a \lambda_a \lambda_a + \text{h.c.} \right) \tag{2}$$

(where ϕ is the scalar component of Φ, and λ is the gaugino), the β-functions for Y, h, b and m^2 are given by

$$16\pi^2 \beta_Y^{(1)ijk} = Y^{ijp} P^k{}_p + (k \leftrightarrow i) + (k \leftrightarrow j), \tag{3a}$$
$$16\pi^2 \beta_h^{(1)ijk} = h^{ijl} P^k{}_l + Y^{ijl} X^k{}_l + (k \leftrightarrow i) + (k \leftrightarrow j), \tag{3b}$$
$$16\pi^2 \beta_b^{(1)(ij)} = b^{il} P^k{}_l + \frac{1}{2} Y^{ijl} Y_{lmn} b^{mn} + \mu^{il} X^j{}_l, \tag{3c}$$
$$\begin{aligned} 16\pi^2 [\beta_{m^2}^{(1)}]^j{}_i = {} & \frac{1}{2} Y_{ipq} Y^{pqn} (m^2)^j{}_n + \frac{1}{2} Y^{jpq} Y_{pqn} (m^2)^n{}_i + 2 Y_{ipq} Y^{jpr} (m^2)^q{}_r \\ & + h_{ipq} h^{jpq} - 8 M_a M_a^* g_a^2 C_i^a \delta^i{}_j + 2 g_a^2 (R_A^a)^i{}_j \text{Tr}[R_A^a m^2], \end{aligned} \tag{3d}$$

where

$$\begin{aligned} P^i{}_j &= \tfrac{1}{2} Y^{ikl} Y_{jkl} - 2 g_a^2 C_i^a \delta^i{}_j, \\ X^i{}_j &= h^{ikl} Y_{jkl} + 4 M g_a^2 C_i^a \delta^i{}_j. \end{aligned} \tag{4}$$

Here g_a is the gauge coupling for the gauge group G_a and C_i^a is the quadratic Casimir for the representation R_A^a of Φ_i. The matrix P is related to the one-loop anomalous dimension by $\gamma^{(1)} = \frac{1}{16\pi^2} P$. Now suppose that the distinct, independent couplings in the superpotential are denoted Y_α, $\alpha = 1, \ldots r$, and $\mu_{\hat{\alpha}}$, $\hat{\alpha} = 1, \ldots q$, and that the distinct group multiplets are denoted Φ_I, $I = 1, \ldots s$. Let us assume that $P^i{}_j$ is diagonal, with $P^i{}_j = P_I \delta^i{}_j$ for all Φ_i in Φ_I. Further, let us assume that one of the couplings g_a (where $G_a \neq U(1)$) and its corresponding gaugino mass M_a are dominant in the RG evolution. Henceforth we denote these parameters by g, M and suppress the remaining gauge couplings and gaugino masses. We may write

$$P_I = \sum_\alpha S_{I\alpha} y_\alpha - 2 g^2 C_I \tag{5}$$

where $y_\alpha = Y_\alpha^2$. Here $S_{I\alpha}$ is related to the dimensionality of the Φ_I; clearly it is zero unless the superpotential term involving y_α contains Φ_I. The one-loop RG equation for Y_α is

$$\frac{d}{dt} Y_\alpha = Y_\alpha \sum_I R_{\alpha I} P_I \tag{6}$$

where $16\pi^2 t = \ln \mu$ and $R_{\alpha I}$ takes the value 0, 1, 2 or 3 according as the superpotential term involving y_α contains 0, 1, 2 or 3 Φ_Is. Note that the zero entries of $R_{\alpha I}$ coincide with those of $S_{I\alpha}$, and we have $\sum_I R_{\alpha I} = 3$ for all α. As an example, for the usual MSSM superpotential, retaining only 3rd generation Yukawas and g_3,

$$W = Y_t Q t H_2 + Y_b Q b H_1 + Y_\tau L \tau H_1, \tag{7}$$

we have

$$\begin{aligned} P_Q = Y_t^2 + Y_b^2 - \tfrac{8}{3}g_3^2, \quad P_L = Y_\tau^2, \quad P_t = 2Y_t^2 - \tfrac{8}{3}g_3^2, \\ P_b = 2Y_b^2 - \tfrac{8}{3}g_3^2, \quad P_\tau = 2Y_\tau^2, \quad P_{H_2} = 3Y_t^2, \quad P_{H_1} = 3Y_b^2 + Y_\tau^2, \end{aligned} \tag{8}$$

so that $R_{Y_tQ} = R_{Y_bQ} = 1, R_{Y_tt} = R_{Y_bb} = 1$, and $S_{QY_t} = S_{QY_b} = 1, S_{tY_t} = S_{bY_b} = 2$ etc.

For the moment we shall focus on the parameters Y^{ijk}, h^{ijk} and m^2; we shall discuss the μ and b parameters later. We assume that the distinct independent trilinear soft-breaking couplings are in one-to-one correspondence with the Y_α, and we define $A_\alpha = \frac{h_\alpha}{Y_\alpha}$. We then find

$$16\pi^2 \beta^{(1)}_{A_\alpha} = 2\sum_{I,\beta} R_{\alpha I} S_{I\beta} y_\beta A_\beta + 4\sum_I g^2 M R_{\alpha I} C_I. \tag{9}$$

Similarly, we find, assuming a common soft mass m_I for each multiplet Φ_I,

$$16\pi^2 \beta^{(1)}_{m_I^2} = 2\sum_{\alpha,J} S_{I\alpha} y_\alpha R_{\alpha J} m_J^2 + 2\sum_\alpha S_{I\alpha} y_\alpha A_\alpha A_\alpha^* - 8g^2 M M^* C_I. \tag{10}$$

The one-loop RG equations for g and M are

$$\frac{d}{dt} g = Q g^3, \qquad \frac{d}{dt} M = 2Qg^2 M, \tag{11}$$

and it follows from Eqs. (6), (11) that the RG evolution of $\tilde{y}_\alpha = \frac{y_\alpha}{g^2}$, $\tilde{A}_\alpha = \frac{A_\alpha}{M}$ and $\tilde{m}_I^2 = \frac{m_I^2}{MM^*}$ is given by

$$\begin{aligned} \frac{d}{dt}\tilde{y}_\alpha &= 2\tilde{y}_\alpha \left(\sum_I R_{\alpha I} P_I - g^2 Q \right) \\ \frac{d}{dt}\tilde{A}_\alpha &= 2g^2 \sum_{I,\beta} R_{\alpha I} S_{I\beta} \tilde{y}_\beta \tilde{A}_\beta - 2Qg^2 \tilde{A}_\alpha + 4\sum_I g^2 R_{\alpha I} C_I \\ \frac{d}{dt}\tilde{m}_I^2 &= 2g^2 \sum_{\alpha,J} S_{I\alpha} \tilde{y}_\alpha R_{\alpha J} \tilde{m}_J^2 - 4Qg^2 \tilde{m}_I^2 + 2g^2 \sum_\alpha S_{I\alpha} \tilde{y}_\alpha \tilde{A}_\alpha \tilde{A}_\alpha^* - 8g^2 C_I. \end{aligned} \tag{12}$$

These equations may be rewritten in the more compact form

$$\begin{aligned} g^{-2}\frac{d}{dt}\tilde{y} &= 2D(RS\tilde{y} - U) \\ g^{-2}\frac{d}{dt}\tilde{A} &= 2(RSD - QI)\tilde{A} + 4RC \\ g^{-2}\frac{d}{dt}\tilde{m}^2 &= 2(SDR - 2QI)\tilde{m}^2 + 2SD\tilde{C} - 8C, \end{aligned} \tag{13}$$

where R, S and D are $r \times s$, $s \times r$ and $r \times r$ matrices with $D_{\alpha\beta} = \tilde{y}_\alpha \delta_{\alpha\beta}$, and the column vectors U and $\tilde{C}$ are defined by

$$U_\alpha = 2(RC)_\alpha + Q, \qquad \tilde{C}_\alpha = \tilde{A}_\alpha \tilde{A}^*_\alpha. \tag{14}$$

We now see the fixed point structure quite clearly. Turning firstly to the Yukawa couplings, $\tilde{y}_\alpha = 0$ is always a possible fixed point for each α. The most general fixed point satisfies

$$\begin{aligned} \tilde{y}_\alpha = 0 \quad (\alpha \in J) \\ (RS\tilde{y} - U)_\alpha = 0, \quad (\alpha \in \{1, \ldots r\} \backslash J) \end{aligned} \tag{15}$$

where J is an arbitrary subset of $\{1, \ldots r\}$. If this condition is satisfied then we also see that

$$\begin{aligned} \tilde{A}_\alpha &= -1 \quad (\alpha \in \{1, \ldots r\} \backslash J) \\ \tilde{A}_\alpha &= \frac{1}{Q}\left(2(RC)_\alpha - \sum_\beta (RSD)_{\alpha\beta}\right) \quad (\alpha \in J) \end{aligned} \tag{16}$$

represents a fixed point. (If $Q = 0$ and $J \neq \emptyset$, then there are in general no fixed points for $\tilde{A}_\alpha$ with $\alpha \in J$.) The existence of this fixed point, corresponding to a universal A-parameter, $A_\alpha = -M$ (in the case $J = \emptyset$), was first remarked in Ref. [12] (and the importance of this value for A in the special case of a finite theory was realised earlier, in Ref. [25]). If $(SDR - 2QI)$ is invertible then the fixed point for $\tilde{m}^2$ is given by

$$\tilde{m}^2 = (SDR - 2QI)^{-1}(4C - SD\tilde{C}), \tag{17}$$

while otherwise there is no obvious closed form expression. It is easy to show from Eq. (13) that in the case $J = \emptyset$, the quantity $(R\tilde{m}^2)_\alpha$ has the fixed point $(R\tilde{m}^2)_\alpha = 1$ for all α. This "sum rule" has been investigated in Refs. [26]. We should mention also that in the special case where the Yukawa fixed points satisfy $(S\tilde{y} - 2C)_I = \frac{1}{3}Q$ for all I (corresponding to the so-called $P = \frac{1}{3}Q$ case[27]) then it is straightforward to show that $\tilde{m}^2_I = \frac{1}{3}$ is a fixed point for all I.

In any event, our main concern here is with the IR stability of the fixed points rather than their actual values. The stability of the fixed points of the evolution equations for a generic set of parameters λ_i is determined by linearising the evolution equations about the fixed points λ^*_i so that they adopt the form $\frac{d}{dt}\delta\lambda_i = \frac{\partial\beta_i}{\partial\lambda_j}\big|_{\lambda^*}\delta\lambda_j$, where $\delta\lambda = \lambda - \lambda^*$. The system is IR stable if all the eigenvalues of $\frac{\partial\beta_i}{\partial\lambda_j}\big|_{\lambda^*}$ have positive real parts. In the case at hand the task is fairly simple, the soft-breaking equations being linear already, and the only subtlety arises for the Yukawa coupling case. We find

$$\frac{\partial\beta_{\tilde{y}_\alpha}}{\partial\tilde{y}_\beta} = 2\delta_{\alpha\beta}(RS\tilde{y}-U)_\beta + (DRS)_{\alpha\beta}. \tag{18}$$

Let us first of all consider the simplest case, of a fixed point with all of the $\tilde{y}_\alpha$ non-zero, in other words $J = \emptyset$. The Yukawa fixed points then simply satisfy $RS\tilde{y}-U=0$ and it follows from Eq. (18) that the Yukawa stability matrix is just DRS. We then see immediately from Eq. (13) that the stability matrices for the $\tilde{A}$ and $\tilde{m}^2$ are $RSD-QI$ and $SDR-2QI$. Hence for IR stability we require DRS, $RSD-QI$ and $SDR-2QI$ to have positive eigenvalues. The appearance of combinations of R, S and D in each of these stability matrices seems quite remarkable. Even more interestingly, the three matrices DRS, RSD and SDR have the same set of non-zero eigenvalues. This follows from the easily-proven result that MN and NM have the same non-zero eigenvalues for any $m\times n$ and $n\times m$ matrices M, N.

We have some immediate and surprisingly powerful results. In an asymptotically free theory ($Q<0$) such that the Yukawa couplings flow to non-zero PR fixed points, the A-parameters and soft masses also flow to fixed points. A non-asymptotically-free theory ($Q>0$) cannot have completely stable fixed points for both A-parameters and soft masses unless the number of Yukawa couplings and the number of fields is the same. This is because unless the matrices R and S are square, then either RSD or SDR will have some zero eigenvalues (except possibly in pathological cases where, RSD (say) has fewer than r independent eigenvectors).

Now let us turn to the case of a fixed point where some of the $\tilde{y}_\alpha$ (p, say) are chosen to be zero. Let us re-order the $\tilde{y}_\alpha$ so that $J=\{r-p+1,\ldots r\}$, and let us partition the $\tilde{y}$ into $\tilde{y}_\alpha$, $\alpha=1,\ldots r-p$ and $\tilde{y}_{\bar{\alpha}}$, $\bar{\alpha}=r-p+1,\ldots r$. It follows from Eq. (15) that the stability matrix for the Yukawa couplings now takes the block form

$$\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}, \tag{19}$$

where

$$T_{1\alpha\beta} = (DRS)_{\alpha\beta}, \quad T_{2\alpha\bar{\beta}} = (DRS)_{\alpha\bar{\beta}}, \quad T_{3\bar{\alpha}\bar{\beta}} = \delta_{\bar{\alpha}\bar{\beta}}(RS\tilde{y}-U)_{\bar{\beta}}, \tag{20}$$

and (with some abuse of notation) R, S and D represent the relevant matrices after re-ordering the couplings. Hence for stability of the Yukawa fixed points we require $(RS\tilde{y}-$

$U)_{\bar{\beta}} > 0$ for $\bar{\beta} = r-p+1, \ldots r$ together with the positivity of the eigenvalues of the matrix T_1. The stability matrices for the $\tilde{A}$ and $\tilde{m}^2$ are still given by $RSD - QI$ and $SDR - 2QI$, and it still follows that the non-zero eigenvalues of DRS, RSD and SDR coincide–though there are now additional zero eigenvalues, since D contains p zeroes on the diagonal. The eigenvalues of $RSD - QI$ consist of $-Q$ (p times) together with the eigenvalues of $T_1 - QI$. Hence this fixed point can only be stable in the asymptotically free case.

Finally we should say a word about the μ and b parameters. The evolution of the μ parameter is governed by

$$g^{-2}\frac{d}{dt}\mu_{\hat{\alpha}} = \mu_{\hat{\alpha}}(\hat{R}S\tilde{y} - 2\hat{R}C)_{\hat{\alpha}}, \tag{21}$$

where $\hat{R}_{\hat{\alpha}I}$ takes the value 0, 1 or 2 according as the μ term contains 0, 1 or 2 Φs. The only fixed point for μ (or any combination such as $\frac{\mu}{g}$, $\frac{\mu}{M}$) is $\mu_{\hat{\alpha}} = 0$ for each $\hat{\alpha}$ (except in the special case of a one-loop finite theory with $P = 0$, for which any value of μ is a fixed point). The stability of $\mu_{\hat{\alpha}} = 0$ requires $(\hat{R}S\tilde{y} - 2\hat{R}C)_{\hat{\alpha}} > 0$ for each $\hat{\alpha}$. In the case of a theory with no gauge singlets, we have $Y_{lmn}b^{mn} = 0$ in Eq. (3c). In that case we have

$$g^{-2}\frac{d}{dt}\tilde{B} = -2Q\tilde{B} + 2\hat{R}SD\tilde{A} + 4\hat{R}C, \tag{22}$$

where $\tilde{B}_{\hat{\alpha}} = \frac{b_{\hat{\alpha}}}{\mu_{\hat{\alpha}}M}$. Eq. (22) clearly determines the fixed point for $\tilde{B}$ in terms of that for $\tilde{A}$. Moreover, the stability of the fixed point for $\tilde{B}_{\hat{\alpha}}$ simply requires that Q be negative. However, in a theory with gauge singlets, the stability of $\tilde{B}_{\hat{\alpha}}$ is determined by a new matrix which has no obvious connection with those which have arisen hitherto.

Let us illustrate our results with some simple examples. We begin with a simplified version of the NMSSM, with the superpotential[3]

$$W = Y_t Q t H_2 + \lambda N H_1 H_2 - \tfrac{k}{3}N^3. \tag{23}$$

The one-loop anomalous dimensions are given by

$$\begin{aligned} P_Q = Y_t^2 - \tfrac{8}{3}g_3^2, \quad P_t = 2Y_t^2 - \tfrac{8}{3}g_3^2, \quad P_{H_2} = 3Y_t^2 + \lambda^2, \\ P_{H_1} = \lambda^2, \quad P_N = 2(\lambda^2 + k^2). \end{aligned} \tag{24}$$

[3] Here (and also in the next example of the MSSM) we ignore all except the third generation Y_t coupling. It is straightforward to generalise to the case of diagonal couplings for the other generations.

Writing $\tilde{Y}_t = \frac{Y_t}{g}$, etc, we find that our matrices R, S, D are given in the NMSSM case by

$$R = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix} \qquad D = \begin{pmatrix} \tilde{Y}_t^2 & 0 & 0 \\ 0 & \tilde{\lambda}^2 & 0 \\ 0 & 0 & \tilde{k}^2 \end{pmatrix}. \tag{25}$$

The Yukawa coupling evolution in this model was analysed in Ref. [18]. The only non-trivial fixed point with $\tilde{Y}_t^2, \tilde{\lambda}^2, \tilde{k}^2 \geq 0$ is

$$\tilde{\lambda} = \tilde{k} = 0, \quad \tilde{Y}_t^2 = \frac{7}{18}$$

and is clearly stable, since the stability matrix of Eq. (19) is just $\begin{pmatrix} \frac{7}{3} & \frac{7}{18} & 0 \\ 0 & \frac{25}{6} & 0 \\ 0 & 0 & 3 \end{pmatrix}$. From Eq. (16) we find the fixed point for the $\tilde{A}$ to be

$$\tilde{A}_{Y_t} = -1, \quad \tilde{A}_\lambda = \frac{7}{18}, \quad \tilde{A}_k = 0, \tag{26}$$

and from Eq. (17) we find the fixed points for the $\tilde{m}^2$ to be

$$\tilde{m}_Q^2 = \frac{41}{54}, \quad \tilde{m}_t^2 = \frac{17}{27}, \quad \tilde{m}_{H_2}^2 = -\frac{7}{18}, \tag{27}$$

in agreement with Ref. [19]. Again these fixed points for the $\tilde{A}$ and $\tilde{m}^2$ are clearly stable, since Q is negative and $T_1 = \frac{7}{3}$. Our formalism excludes the possibility of determining fixed point values for A-parameters corresponding to mixing between the generations.

For our second example we take the model with the MSSM superpotential of Eq. (7). The matrices R and S are given (ordering the fields as Q, L, t, b, τ, H_2, H_1) by

$$R = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ 3 & 0 & 0 \\ 0 & 3 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} \tilde{Y}_t^2 & 0 & 0 \\ 0 & \tilde{Y}_b^2 & 0 \\ 0 & 0 & \tilde{Y}_\tau^2 \end{pmatrix}. \tag{28}$$

There is no fixed point with $\tilde{Y}_\tau^2 > 0$. However, in the case where we take $\tilde{Y}_\tau = 0$ and $\tilde{Y}_t$, $\tilde{Y}_b$ non-zero at the fixed point, we find a fixed point with

$$\begin{gathered} \tilde{Y}_t^2 = \tilde{Y}_b^2 = \frac{1}{3}, \quad \tilde{A}_t = \tilde{A}_b = -1, \quad \tilde{A}_\tau = \frac{1}{3}, \\ \tilde{m}_Q^2 = \tilde{m}_t^2 = \tilde{m}_b^2 = \frac{2}{3}, \quad \tilde{m}_{H_1}^2 = \tilde{m}_{H_2}^2 = -\frac{1}{3}, \quad \tilde{m}_L^2 = \tilde{m}_\tau^2 = 0. \end{gathered} \tag{29}$$

We find in Eq. (19) that at the fixed point $T_1 = \begin{pmatrix} 2 & \frac{1}{3} \\ \frac{1}{3} & 2 \end{pmatrix}$ and $T_3 = 1$. The non-vanishing eigenvalues of T_1 are $\frac{5}{3}$ and $\frac{7}{3}$. Since moreover $Q < 0$ for the MSSM, we deduce that the fixed points are stable. Evidently the above fixed point values for Y_t, Y_b correspond to large $\tan\beta$ (specifically $\tan\beta = m_t/m_b$); the RG evolution of the Yukawa couplings in this region was studied in Ref. [10]. It would be interesting to extend this analysis to include the soft–breaking terms.

For our third example we take a model with superpotential

$$W = \sum_{i,I}^{n_g} Y_{iI} Q_i t_I H_{iI}. \tag{30}$$

We have

$$\begin{gathered} R_{iI,i} = 1, \quad R_{iI,I} = 1, \quad R_{iI,iI} = 1, \\ S_{i,iI} = 1, \quad S_{I,iI} = 2, \quad S_{iI,iI} = 3, \\ D_{iI,iI} = Y_{iI}^2/g^2 = \tilde{y}_{iI}, \end{gathered} \tag{31}$$

with all other entries in R, S and D being zero. The Yukawa fixed point is $\tilde{y}_{iI} = \tilde{y}$, where $\tilde{y} = \frac{3Q+16}{9(n_g+1)}$, for all i, I (assuming we take all the $\tilde{y}$ non-zero). In this case the fixed point for every $\tilde{A}$ is $\tilde{A}_{iI} = -1$, and the fixed points for $\tilde{m}^2$ are

$$\tilde{m}_i^2 = \frac{1}{3Q}(3n_g\tilde{y} - 8), \quad \tilde{m}_I^2 = \frac{1}{3Q}(6n_g\tilde{y} - 8), \quad \tilde{m}_{iI}^2 = \frac{3\tilde{y}}{Q}. \tag{32}$$

For stability it is sufficient to consider RS. It is straightforward to compute the eigenvalues explicitly in this case. We find that $RS - \lambda I$ is the $n_g^{n_g} \times n_g^{n_g}$ matrix given by

$$RS - \lambda I = \begin{pmatrix} [5-\lambda]I + K & 2I & 2I & \dots & \dots & 2I & 2I \\ 2I & [5-\lambda]I + K & 2I & \dots & \dots & 2I & 2I \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 2I & 2I & 2I & \dots & \dots & 2I & [5-\lambda]I + K \end{pmatrix} \tag{33}$$

where K is an $n_g \times n_g$ matrix with every entry 1, and I is the $n_g \times n_g$ unit matrix. By subtracting the last column from every other column, and then adding every row to the last row, we conclude that

$$\det(RS - \lambda I) = \det[(3-\lambda)I + K]^{n_g - 1} \det[(3 + 2n_g - \lambda)I + K]. \tag{34}$$

By similar operations we can show that

$$\det[xI + K] = x^{n_g - 1}(x + n_g), \tag{35}$$

and it follows that the eigenvalues of RS are 3 ($[n_g-1]^2$ times), n_g+3 ($[n_g-1]$ times), $2n_g+3$ ($[n_g-1]$ times) and $3n_g+3$ (once). These are all positive. Hence again we find stability in the asymptotically free case. It is easy to combine our second and third examples to obtain the n_g generation theory which was considered as a model for strong unification in Ref. [28].

As our final example we will take a semi-realistic GUT with gauge group $SU_3 \otimes SU_3 \otimes SU_3$, and a matter content consisting of n sets each of the representations $X \equiv (3,3,1)$, $Y \equiv (1,\overline{3},3)$ and $Z \equiv (\overline{3},1,\overline{3})$. The superpotential for the theory is :

$$W = \frac{1}{3!}(\lambda_1 X^3 + \lambda_2 Y^3 + \lambda_3 Z^3) + \rho XYZ. \tag{36}$$

Here $\lambda_1 X^3 \equiv (\lambda_1)^{\alpha\beta\gamma} X_\alpha X_\beta X_\gamma$, where $\alpha, \beta \cdots = 1 \cdots n$. If we set the three gauge couplings all equal to g then it is easy to see that they remain equal under renormalisation. In what follows we will suppose that we have $(\lambda_i^2)^\alpha_\beta = (\lambda_i^2)\delta^\alpha_\beta$, and $(\rho^2)^\alpha_\beta = \rho^2\delta^\alpha_\beta$. In this case we have (note that this is a case with $s > r$)

$$R = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 2 & 0 & 0 & 3 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 2 & 3 \end{pmatrix} \qquad D = \begin{pmatrix} \tilde{\lambda}_1^2 & 0 & 0 & 0 \\ 0 & \tilde{\lambda}_2^2 & 0 & 0 \\ 0 & 0 & \tilde{\lambda}_3^2 & 0 \\ 0 & 0 & 0 & \tilde{\rho}^2 \end{pmatrix}. \tag{37}$$

At the fixed point we have $6\tilde{\lambda}_i^2 + 9\tilde{\rho}^2 = Q + 16$, and hence we must take $Q > -16$ to ensure physical fixed point values. We find the non-zero eigenvalues of DRS etc to be $Q + 16 - 9\tilde{\rho}^2$ (twice) and $Q + 16$, which are positive provided the $\tilde{\lambda}_i^2$ and $\tilde{\rho}^2$ are positive. The implications for the stability of the infra-red fixed points were discussed in Ref. [12]. Again stability is assured in the case $Q < 0$.

Each example we investigated had the property that the non-zero eigenvalues of DRS were positive (for physical fixed-point values for the $\tilde{y}$, in other words each diagonal entry of D non-negative at the fixed point) leading to IR stability of the fixed points for $\tilde{A}$, $\tilde{m}^2$ for an asymptotically free theory. The obvious question is whether this is true in general. We have so far been unable to answer this question completely; however, as a first step we can show that if S may be written $S = D_1 R^T D_2$, where D_1 and D_2 are diagonal matrices with positive entries along the diagonal, and if D has non-negative entries, then DRS has non-negative eigenvalues. All the examples we have considered have this property. For instance, in our first example, $D_1 = \mathrm{diag}(1,2,3,3,6)$, $D_2 = \mathrm{diag}(1,\frac{1}{3},\frac{1}{9})$; in our second example, $D_1 = \mathrm{diag}(1,3,2,2,6,3,3)$, $D_2 = \mathrm{diag}(1,1,\frac{1}{3})$; in our third example, $D_{1i,i} = 1$,

$D_{1I,I} = 2$, $D_{1iI,iI} = 3$, $D_2 = I$; and in our fourth example, $D_1 = \frac{2}{3}I$, $D_2 = \mathrm{diag}(1, 1, \frac{9}{2})$. It is possible to write down matrices R and S, subject to the constraints that $\sum_I R_{\alpha I} = 3$ and that R corresponds to a theory with P diagonal, which do not have the property above; but they do not correspond to any physically interesting model, and in any case we have not been able to construct any examples which lead to non-zero eigenvalues of DRS with negative real parts. It is tempting to speculate that this is a general result.

We thus conclude that the focusing of Yukawa couplings and soft breakings observed for example, in Ref. [19] for the MSSM is not in fact specific to that theory but a general phenomenon; the existence of stable infra-red fixed points for the Yukawa couplings implies (given asymptotic freedom) stable infra-red fixed points for the A-parameters and soft scalar masses. In Ref. [12] we proposed that universality of soft masses and couplings at the unification scale might be associated with IR fixed points. Our demonstration here that there exists a strong relationship between the IR stability of the fixed points for these masses and couplings with the stability of the fixed points in the Yukawa sector makes this scenario even more plausible.

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