Transversal Homotopy Theory of Whitney Stratified Manifolds

Thesis submitted in accordance with the requirements of
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For Mam.

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Abstract

We modify the theory of homotopy groups to obtain invariants of Whitney stratified spaces by considering smooth maps which are transversal to all strata, and smooth homotopies through such maps. Using this idea we obtain transversal homotopy monoids with duals for any Whitney stratified space. Just as in ordinary homotopy theory we may also define higher categorical invariants of spaces. Here instead of groupoids we obtain categories with duals. We concentrate on examples involving the sphere, stratified by a point and its complement, and complex projective space stratified in a natural way. We also suggest a definition for n-category with dual, which we call a Whitney category. This is defined as a presheaf on a certain category of Whitney stratified spaces, that resticts to a sheaf on a certain subcategory. We show in detail that this definition matches the accepted notion of n-category with duals, at least for small n. It also allows us to prove a version of the Tangle Hypothesis, due to Baez and Dolan, which states that "The n-category of framed codimension k-tangles is equivalent to the free k-tuply monoidal n-category with duals on one object."

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Chapter 1

Introduction

In his article *Analysis Situs* [19], Henri Poincaré associated to a based topological space (X, x), its fundamental group $\pi_1(X, x)$. The elements of $\pi_1(X, x)$ are homotopy classes of continuous maps $S^1 \to X$, based at x. This was later generalised by Čech and then Hurewicz to higher homotopy groups, whose elements were homotopy classes of based maps $S^n \to X$. In contrast with homology groups, which are difficult to define but easy to calculate, homotopy groups are "terribly easy to define but essentially impossible to calculate" – Peter Hilton [11]. That said, some progress has been made calculating the homotopy groups of spheres, for instance, although this still remains an important open problem in general. An important theorem by Hurewicz tells us that for a simply-connected space X, the first non-zero homotopy group $\pi_k(X)$, with $k > 0$, is isomorphic to the first non-zero homology group $H_k(X)$. For the n-sphere, this immediately implies that for $n > 0$, $\pi_n(S^n) \cong H_n(S^n) \cong \mathbb{Z}$. But, whereas in homology $H_i(S^n) = 0$ for $i > n$, this is certainly not true for $\pi_i(S^n)$ for $n \geqslant 2$. A prime example of this comes from the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ which is the generator of $\pi_3(S^2) \cong \mathbb{Z}$.

We use homotopy groups to detect topological invariants of a space, i.e. features that are invariant under a continuous deformation of the spaces, such as 'holes of different dimensions'. While homotopy may perhaps also be used to detect singularities in a space, we encounter difficulty when we try to use homotopies of continuous maps to detect invariants of topological spaces with extra data specified. An example of this is a topological space equipped with a stratification, i.e. certain submanifolds which are singled out as being 'special'. In this thesis we study a class of such spaces known as Whitney stratified spaces. These are collections of submanifolds, known as strata, of an ambient manifold, which 'fit together' smoothly. We will make this notion precise in Section 2. Whitney stratified spaces arise naturally in a number of areas of mathematics and physics. A manifold with boundary is naturally stratified by its boundary and complement. The configuration space of some physical system may be stratified, where crossing a stratum may correspond to a significant event such as the annihilation of two particles or, crossing in the opposite direction, their creation (see discussion in [3]). More interestingly perhaps, any complex analytic variety admits a Whitney stratification [25].

Consider a loop in a based Whitney stratified space W, i.e. a continuous map $f : ([0,1], \partial[0,1]) \to (W, x)$, mapping the boundary to x. If the loop crosses a codimension 1 stratum transversally and then turns and crosses back over the same stratum, it is homotopic through continuous maps to a loop that does not cross the stratum, i.e. ordinary homotopy, using continuous maps, does not distinguish between these two loops. At some time though, a 'slice' of such a homotopy will not be transversal to the stratum; the time at which we pull the loop 'off' the stratum. If we insist that maps must also be smooth and transversal to all strata, and further that homotopy must be through transversal maps, we develop a new tool that allows us to detect strata. Observe that we can not expect to have an inverse for each element, but by reversing the parametrisation of a path we obtain a dual path. This suggests that we obtain transversal homotopy monoids with duals, as opposed to groups.

The idea to use transversal paths is due to John Baez and James Dolan. The discussion [3] where John Baez introduced the idea of a 'fundamental n-category with duals' associated to a smooth stratified space was the original inspiration for this thesis. In this discussion, not only were the basic ideas laid down but also the key examples. These invariants mimic the fundamental groupoid of a topological space X, i.e. a category whose objects are the points of X and whose morphisms are homotopy classes of paths between such points. The fundamental groupoid is a categorical generalisation of the fundamental group of a space. We take the same approach by first defining transversal homotopy monoids associated to a smooth stratified space, and then generalising to transversal homotopy categories. We give details of the interesting example, original to this thesis, of the transversal homotopy monoids of $\mathbb{CP}^k$.

We also examine 'The Tangle Hypothesis' [1], again due to Baez and Dolan which states "The n-category of framed codimension k-tangles is equivalent to the free k-tuply monoidal n-category with duals on one object." This is a vast generalisation of Shum's Theorem [22] which states, in our language, that the category of framed tangles in 3 dimensions is the "free braided monoidal category with duals on one object". An obstruction to proving the Tangle Hypothesis has been the difficulty in defining the notion of n-category with duals. This we would argue is a reason it is stated as hypothesis i.e. something to be tested, as opposed to a conjecture. To see the difficulty one only has to consider the explicit, but already very complicated definition of a braided monoidal 2-categories with duals in [2] where Laurel Langford and John Baez also show that the 2-category of 2-tangles in 4 dimensions is the free braided monoidal 2-category with duals on an unframed self-dual object. Further progress has been made in the work of Jacob Lurie. His definition of n-category is more fundamental than ours, in the sense that it does not need stratified spaces to define it, but it is quite a lot more complicated. For details we direct the reader to Lurie's 700 page book, [15].

To overcome this, we use a new notion of n-category with duals called a Whitney n-category, introduced in [23]. These are defined to be presheaves on a certain category of Whitney stratified spaces that restrict to sheaves on a certain subcategory. Using this notion the Tangle Hypothesis becomes almost tautological. Whitney n-categories also allow us to define the 'fundamental n-category with duals' of a smooth stratified space. We give constructions that show these Whitney categories are closely related to notions of n-category for $n = 1, 2$. These constructions in Chapters 8 and 9 are also original and indeed central to this thesis.

We conclude this section with a brief summary of the remaining chapters.

In *Chapter 2* we recall background material on Whitney stratified manifolds, i.e. decompositions of a manifold into submanifolds which fit together smoothly. A Whitney stratified space is a closed union of these submanifolds. A *stratified space* is then the germ of a Whitney stratified manifold at such a closed union. In Section 2.2 we recall what it means for a map to be transversal and give details of some important results about maps of this kind. Section 2.3 consists of some key definitions of maps between stratified spaces. For instance, since a Whitney stratified space is not in general a manifold, we must define what it means for maps between such spaces to be smooth and likewise, transversal. Here we also define stratified maps and stratified normal submersions. These are maps that preserve transversality under post- and pre-composition respectively. We will deal with a number of categories that have stratified spaces as objects. Key to the thesis will be the categories $Strat_n$ and $Prestrat_n$. The objects in both these categories are stratified spaces of dimension $\leqslant n$, and morphisms are stratified maps and prestratified maps respectively. A map is prestratified if it becomes stratified after a refinement of the stratification of the source.

In *Chapter 3* we describe some of the algebraic tools we will be using later in the thesis. As already mentioned, transversal homotopy produces monoids with duals, also known as dagger monoids. We describe some properties of this type of monoid and relate the category of dagger monoids to the category of groups. The results proven about dagger monoids are original to this thesis as the author was unable to find reference to such results anywhere else. We also give the definition of dagger pivotal categories. These categories will be integral to describing the categorical extension to transversal homotopy theory. We conclude this chapter with a description of what a graphical language for a category should be. Here we borrow heavily from Peter Selinger's Survey of Graphical Languages for Monoidal Categories [20]. These graphical languages will allow us to prove certain facts about dagger pivotal categories by manipulating diagrams representing equations between morphisms.

In *Chapter 4* we recall the details of the Pontryagin Thom construction. Using this construction, Lev Pontrjagin established an isomorphism between the homotopy group $\pi_n(S^p)$ and the group $\Omega^{fr}_{n-p}(S^n)$ of cobordism classes of differentiable $(n-p)$ dimensional submanifolds of S^n, which are 'framed', i.e. have a trivialized normal bundle. This gives a geometric interpretation of certain homotopy groups. We recall the details of this construction here and use a generalised version of it later to give a geometric interpretation of certain transversal homotopy monoids. It will also be of central importance in our proof of the Tangle Hypothesis. We follow Milnor's treatment in [17] closely.

Part II of the thesis concerns transversal homotopy theory.

We begin in *Chapter 5* by defining the nth transversal homotopy monoid $\psi_n(X)$, of a Whitney stratified manifold X, as equivalence classes of based transversal maps of the n-cube into X, where equivalence is defined up to homotopy through transversal maps. Here we also introduce the concept of the pullback stratification which, given a map into a Whitney stratified space, is a stratification of the domain by preimages of strata from the target space. If our maps are transversal to all strata, these preimages are manifolds and have the same codimension as their image strata. We use this idea of the pullback stratification to give a geometric interpretation of transversal homotopy monoids. For example, we find that the third transversal homotopy monoid of the 2-sphere stratified by a point and its complement, $\psi_3(\mathbb{S}^2)$, is equivalent to the collection of framed tangles in 3 dimensions up to ambient isotopy. Although transversal homotopy monoids do not, in general, easily admit either an algebraic or geometric description there are some interesting examples involving spheres. For example, again taking $\mathbb{S}^n$ to be the n-sphere stratified by a point and its complement, $\psi_n(\mathbb{S}^n)$ can be described algebraically as the free monoid with duals on one generator which is commutative for $n \geqslant 2$. It can also be viewed geometrically as isotopy classes of framed codimension n submanifolds of the n-sphere. Taking another example, $\psi_{n+1}(\mathbb{S}^n)$ can be described as framed codimension n submanifolds inside $n+1$ space, up to a suitable equivalence. For instance $\psi_2(\mathbb{S}^1)$, that is classes of based transversal maps of S^2 into $\mathbb{S}^1$ (up to homotopy through such maps), can be represented as nested circles in $\mathbb{R}^2$ up to ambient isotopy; but an algebraic description is not so apparent.

We then give a geometric description of $\psi_n(\mathbb{CP}^k)$, the nth transversal homotopy monoid of k-dimensional complex projective space, where $\mathbb{CP}^k$ has the stratification induced from the filtration $\mathbb{CP}^0 \subset \mathbb{CP}^1 \subset \ldots \subset \mathbb{CP}^k$. This is a way of generalising $\mathbb{S}^2$, i.e. $\mathbb{CP}^1 \cong \mathbb{S}^2$, and so we would like to find a nice geometric description of its transversal homotopy monoids $\psi_n(\mathbb{CP}^k)$ by relating transversal maps of the sphere into $\mathbb{CP}^k$, with the pullback stratifications of the sphere. Answering the question: "What are the necessary and sufficient conditions for a stratification of S^n to be the pullback stratification of a transversal map $f : S^n \to \mathbb{CP}^k$?" allows us to give just such a geometric description of $\psi_n(\mathbb{CP}^k)$.

In *Chapter 6* we describe an extension from transversal homtopy monoids to transversal homotopy categories. This is analogous to the previously mentioned, well-known categorical extension of the theory of homotopy groups to that of homotopy groupoids. The objects in these transversal homotopy categories are transversal maps of the sphere into a Whitney stratified space and the morphisms are homotopy classes of such maps. We then give details of some of the more computable transversal homotopy categories of the sphere, stratified in the usual way.

In *Part III*, which contains our main results, we introduce *Whitney n-categories*, a new notion of 'n-category with duals'. This notion and the other concepts in this thesis were jointly worked out from suggestions proposed by the first author in [23], inspired in turn by ideas of Baez and Dolan, and Morrison and Walker's definition of n-category with duals. There are two motivations for our definition. First, this definition of an n-category with duals makes the Baez-Dolan Tangle Hypothesis almost tautological. Second, Whitney n-categories allow us to construct 'fundamental n-categories with duals' for each smooth stratified space. This answers another question posed posed by John Baez in [3].

In *Chapter 7* we define the notion of a Whitney n-category. The idea is to define a Whitney n-category as a presheaf, i.e a contravariant functor

$$Prestrat_n \to Set$$

from the category of n-dimensional cellular stratified spaces and prestratified maps, i.e maps that become stratified after a refinement of the stratification of the source, to the category of sets. We further insist that when restricted to the category $Strat_n$ which has the same objects but whose morphisms are stratified maps the functor

$$Strat_n \to Set$$

is a sheaf for a certain Grothendieck topology on $Strat_n$. This ensures that the set it assigns to a space X is determined by the set it assigns to the cellular strata of that space. This may strike the reader as quite different from the usual notion of n-categories. The idea goes as follows: given a Whitney n-category $\mathscr{A}$ and a compact cellular stratified space X of dimension $\leqslant n$, the set $\mathscr{A}(X)$ consists of morphisms of shape X in $\mathscr{A}$. For example, $\mathscr{A}$ assigns to a point, the set of objects $\mathscr{A}(pt)$, and to the interval $[0,1]$ it assigns the set of 1-morphisms $\mathscr{A}([0,1])$. This description borrows from the ideas of Morrison and Walker in [18]. They promote the point of view that one should consider morphisms of general shape not simply globules, simplices, or cubes. We then give three motivating examples of such categories. These are, the representable Whitney n-category, the transversal homotopy Whitney n-categories, and the Whitney category of tangles. Finally we define what it means for two Whitney n-categories to be equivalent and define a k-tuply monoidal Whitney n-category $\mathscr{A}$ to be a Whitney $(n+k)$-category such that $\mathscr{A}(X)$ is a one element set for all X of dimension strictly less than k.

In *Chapter 8* we show in detail that Whitney 1-categories match the definition of a 1-category with duals from [21], i.e. a dagger category. In particular we give a construction that, from a Whitney 1-category, produces a dagger category and vice versa. These are functors between the category of Whitney 1-categories, $1Whit$, and the category of dagger categories, Dag. The subsequent sections show these constructions are inverse to each other, i.e. the functors

$$C : 1Whit \to Dag$$
$$W : Dag \to 1Whit$$

are inverse to each other in the sense that there is a natural isomorphism of dagger categories,

$$\mathscr{C} \to C(W(\mathscr{C}))$$

and a natural isomorphism of Whitney 1-categories,

$$\mathscr{A} \to W(C(\mathscr{A})).$$

In *Chapter 9* we take this idea further and show how one-object Whitney 2-categories relate to the notion of a one-object 2-category with duals from [20]. These are dagger pivotal category, i.e. dagger categories in which right and left duals exist and are in fact the same element. We do this by constructing an adjunction between the categories of one-object Whitney 2-categories, $2Whit_1$, and the category $DagPiv$, of dagger pivotal categories. We first define a construction that takes a one-object Whitney 2-category and produces a dagger pivotal category.

Constructing a functor in the opposite direction is more complicated. We detail three such constructions, each with their own virtues. In each case we construct a presheaf on $Prestrat_2$ which then restricts to a sheaf on $Strat_2$. Given a dagger pivotal category $\mathscr{D}$, we define three closely related one-object Whitney 2-categories: $W(\mathscr{D})$, $\widetilde{G}(\mathscr{D})$ and $\widetilde{R}(\mathscr{D})$ by the following constructions:

- The *labelling construction* W, where for a stricly cellular space $X \in Strat_2$ we define an element of the set $W(\mathscr{D})(X)$ to be an equivalence class of labellings of X by the objects and morphisms of a category $\mathscr{D}$. This generalises the Whitney 1-category construction and clearly shows how the dagger pivotal structure relates to the underlying Whitney 2-category. But, we find it impossible to show satisfactorily that we obtain a functor $W : DagPiv \to 2Whit_1$ and so we try a different approach. Nevertheless we include the construction as it clearly shows the algebraic structure appearing.

- The *graph construction* $\widetilde{G}$, where $\widetilde{G}(\mathscr{D})$ assigns to a space X, the set of $\mathscr{D}$-labelled transversal graphs in X, up to isotopy. A $\mathscr{D}$-labelled transversal graph is defined to be the intersection of a space $X \in Strat_2$ with a codimension 1 submanifold of the ambient manifold to X. We then label these intersections by the objects and morphisms of $\mathscr{D}$. These are more geometrically natural, are the simplest to define, and are the easiest to see as one-object Whitney 2-categories.

- The *ribbon construction* $\widetilde{R}$, where $\widetilde{R}(\mathscr{D})$ assigns to a space X, the set of $\mathscr{D}$-labelled ribbon graphs in X, up to isotopy. We obtain a $\mathscr{D}$-labelled ribbon graph by 'fattening' a transversal graph, replacing edges with ribbons and vertices with boxes. We then label with objects and morphisms of $\mathscr{D}$.

The motivation for the ribbon graph construction is that it allows us to construct the functor $\widetilde{R} : DagPiv \to 2Whit$ which is left adjoint to
$C : 2Whit \to DagPiv$.

We then show that the three constructions are equivalent as Whitney 2-categories. Here we are assuming that W can be well-defined but the equivalence between the other constructions does not depend on this.

Next, we use these constructions to recover our notion of transversal homotopy theory from Whitney n-categories as evidence that they are a viable candidate to answer the question, posed by Baez, "Given a smooth stratified space, can we assign to it an n-category with duals?".

In *Chapter 10* we gather together these concepts to give a proof of the Tangle Hypothesis which in our language can now be stated as follows "The Whitney category $nTang_k^{fr}$, of framed tangles is equivalent to the free k-tuply monoidal Whitney n-category on one $\mathbb{S}^k$-morphism."

We prove this by first showing, by the Yoneda Lemma, that $Rep(\mathbb{S}^k)$ is free on one $\mathbb{S}^k$ morphism, this being the identity map on the sphere. We then establish an equivalence between $Rep(\mathbb{S}^k)$, the Whitney $(n+k)$-category represented by the sphere, and the k-tuply monoidal Whitney n-category of framed tangles, $nTang_k^{fr}$. This equivalence arises from the Pontryagin-Thom collapse map construction which relates framed codimension k tangles in X to prestratified maps $X \to \mathbb{S}^k$.

Finally we establish an equivalence, again using Pontryagin-Thom, between $\Psi_{k,n+k}(\mathbb{S}^k)$ and $nTang_k^{fr}$. Thus we have equivalences of k-tuply monoidal Whitney n-categories

$$Rep(\mathbb{S}^k) \cong \Psi_{k,n+k}(\mathbb{S}^k) \cong nTang_k^{fr}$$

yielding three descriptions of the same object which we can think of respectively as algebraic (in the sense that the representable Whitney category is free on one $\mathbb{S}^k$ -morphism), homotopy-theoretic, and geometric.

Part I

Background

Chapter 2

Whitney Stratified Manifolds

2.1 Whitney Stratified Spaces

We begin with some motivating definitions and examples of the spaces that will concern us.

Definition 2.1. [25] A *Whitney stratification of a manifold* X is a disjoint decomposition $X = \bigcup_\alpha X_\alpha$ of X into submanifolds (which are not necessarily connected, but which must each have a fixed dimension for all of their connected components) that satisfies the following four axioms:

1. *Local finiteness.* The decomposition is locally finite, i.e. every point $x \in X$ has a neighbourhood U with the property that $U \cap X_\alpha$ is empty for all but a finite number of strata X_α.

2. *The axiom of the frontier.* If one stratum X_α has a non-empty intersection with the closure $\overline{X_\beta}$ of another stratum X_β, then X_α lies entirely within $\overline{X_\beta}$.

3. *Whitney's condition A.* Suppose X_α lies entirely in the closure of X_β. Suppose that $x_1, x_2, x_3, \dots$ is a sequence of points in X_β which converges to a point y in X_α. Suppose further that the sequence of tangent spaces $T_{x_1}X_\beta$, $T_{x_2}X_\beta$, $T_{x_3}X_\beta, \dots$ converges, as a sequence of subspaces of the tangent spaces TX to X, to a "limiting tangent space" $\tau \subseteq T_yX$. Then the tangent space T_yX_α is contained in the limiting space τ.

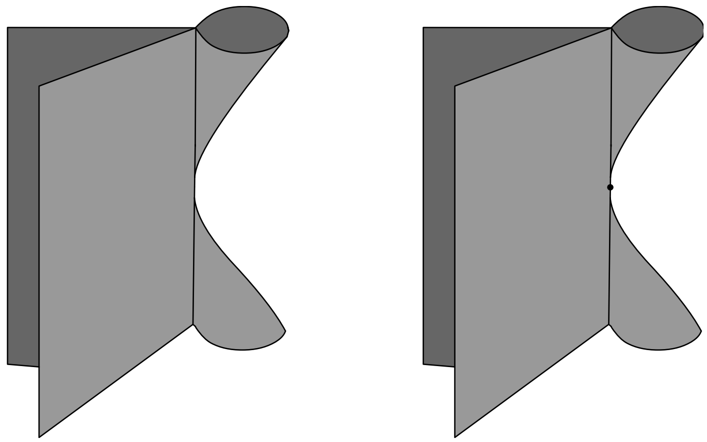


Figure 2.1: On the left is the Whitney cusp, an example of a non-Whitney stratified space. On the right is a refinement of the stratification of the Whitney cusp which makes it a Whitney stratified space.

4. *Whitney's condition B.* Suppose that X_α lies in the closure of X_β. Suppose $x_1, x_2, x_3 \ldots$ is a sequence of points in X_β which converge to a point y in X_α, and $y_1, y_2, y_3 \ldots$ is a sequence of points in X_β which also converge to y. Suppose as before the sequence of tangent spaces $T_{x_1}X_\beta$, $T_{x_2}X_\beta$, $T_{x_3}X_\beta, \ldots$ converges, as a sequence of subspaces of the tangent spaces TX to X, to a "limiting tangent space" $\tau \subseteq T_yX$. Suppose further that the sequence of secant lines $\overline{x_1y_1}, \overline{x_2y_2}, \overline{x_3y_3}, \ldots$ converges to a limiting line $l \subseteq T_yX$. Then the limiting line is contained in the limiting tangent space τ.

Remark 2.2. In fact, Whitney's condition B implies condition A but condition A is used more in applications. To make sense of the secant lines in B, choose a local coordinate system around y. The condition holds, independently of such a choice.

Remark 2.3. Whitney conditions A and B ensure that the geometry of X looks the same at every point of a given stratum. This allows us to avoid examples such as the Whitney cusp i.e. the variety defined by $x^3 + z^2x^2 - y2 = 0$ and stratified by $x = y = 0$ and its complement. The origin looks different from all other points in that stratum and so to avoid this we would specify the origin as a 0-dimensional stratum, see Figure 2.1.

Examples of Whitney stratified manifolds abound, for instance any manifold with a trivial stratification, i.e. a single stratum for each connected component, is a Whitney stratified manifold. An important example in what follows will be the sphere stratified by a point and its complement which we will denote $\mathbb{S}^k$. We will also repeatedly refer to $\mathbb{I}$, which is the interval $[0,1]$ stratified by its endpoints and their complement. This generalises to $\mathbb{I}_n$, for $n \in \mathbb{Z}$, which is the interval $[0,n]$ stratified by the integers and the connected components of their complement. We will also define a stratification on real projective space, $\mathbb{RP}^n = \mathbb{R}^{n-1} \cup \mathbb{R}^{n-2} \cup \cdots \cup \mathbb{R}^0$ and complex projective space $\mathbb{CP}^k = \mathbb{C}^{k-1} \cup \mathbb{C}^{k-2} \cup \cdots \cup \mathbb{C}^0$, the later being of central importance in Section 5.3. In fact, any real or complex analytic variety admits a Whitney stratification [25].

Definition 2.4. A *Whitney stratified space* is a closed union of strata X in a Whitney stratified manifold M.

Remark 2.5. Given two stratified spaces X and Y, we stratify the product $X \times Y$ by taking the products of strata in X with strata in Y. Take as an example the space $\mathbb{I}_2 \times \mathbb{I}$. The intervals are stratified as usual by the integer points and their complement. The resulting stratification is shown in Figure 2.2.

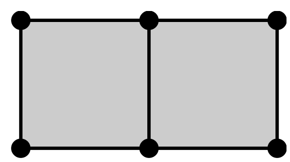


Figure 2.2: The product stratification on $\mathbb{I}_2 \times \mathbb{I}$.

Next we define an equivalence relation:

Two Whitney stratified spaces $X \subset M$ and $X' \subset M'$ are equivalent if and only if there are open neighbourhoods U of X in M and U' of X' in M' and a diffeomorphism $U \to U'$ which restricts to a homeomorphism $X \to X'$ which preserves strata. The resulting equivalence classes are *germs* of Whitney stratified spaces.

We say two such germs, $[X \subset M]$ and $[X' \subset M']$ are stably equivalent if and only if

$$[X \subset (M \times \mathbb{R}^m)] = [X' \subset (M' \times \mathbb{R}^{m'})]$$

where we embed X into $M \times \mathbb{R}^m$ via

$$\begin{aligned} X \hookrightarrow M \hookrightarrow M \times \mathbb{R}^m \\ p \mapsto (p, 0). \end{aligned}$$

Definition 2.6. For convenience we use the term *stratified space* to mean a stable germ, in the above sense, i.e. an equivalence class of Whitney stratified spaces under the relation generated by

$$(X \subset M) \sim (X' \subset M')$$

if and only if there are neighbourhoods V of X in $M \times \mathbb{R}^m$ and V' of X' in $M' \times \mathbb{R}^{m'}$ and a diffeomorphism $V \to V'$ which restricts to a stratum preserving homeomorphism $X \to X'$ where

$$X \hookrightarrow M \to M \times \mathbb{R}^m$$

$$x \mapsto x \mapsto (x, 0).$$

Remark 2.7. If we wish to refer to Whitney stratified spaces, rather than to stable germs thereof, we will we will say so explicitly.

2.2 Transversality

We recall some well-known definitions and properties of transversal maps, collected from [17], [10] and [26].

Two smooth submanifolds A and B of a smooth manifold M are said to *intersect transversally* if for any point $x \in A \cap B$ we have $T_xA + T_xB = T_xM$. The sum need not be direct and if the intersection is empty we say that A and B are vacuously transversal.

More generally, smooth maps $f : A \to M$ and $g : B \to M$ are transversal if for each $p \in A$ and $q \in B$ where $f(p) = g(q)$ we have

$$df(T_pA) + dg(T_qB) = T_{f(p)=g(q)}M.$$

This is equivalent to the composite

$$T_pA \xrightarrow{df} T_{f(p)}M \to T_{f(p)}M/dg(T_qB)$$

being surjective. Transversality of submanifolds corresponds to the case where f is the inclusion of A and g is the inclusion of B.

Example 2.8. In $\mathbb{R}^2$ take the x-axis and the graph of $y = x^2$. These submanifolds do not intersect transversally, whereas the x-axis does intersect $y = x^2 - 1$ transversally. See Figure 2.3.

Example 2.9. Consider arcs on a surface. Two arcs intersect transversally if and only if they are not tangent to each other. From this we get the idea that transversality is in some sense "opposite" to tangency, see Figure 2.5.

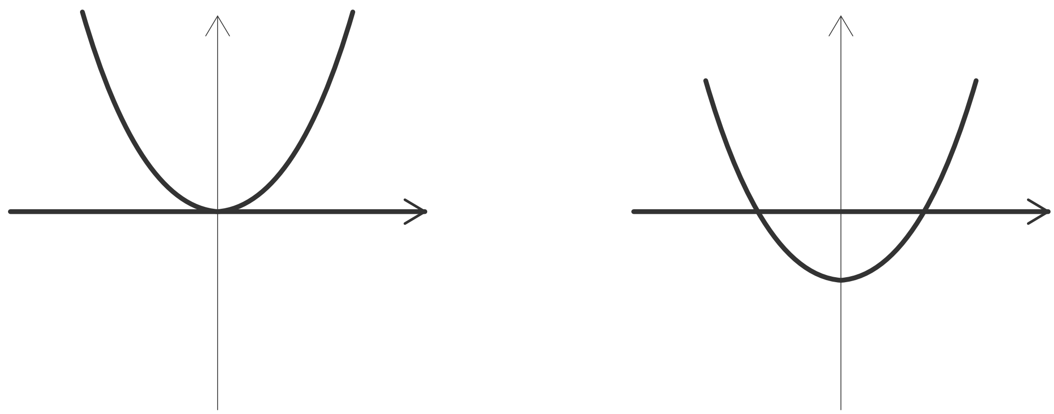


Figure 2.3: On the left, the non-transversal intersection of the $y = 0$ and $y = x^2$; on the right the transversal intersection of $y = 0$ and $y = x^2 - 1$.

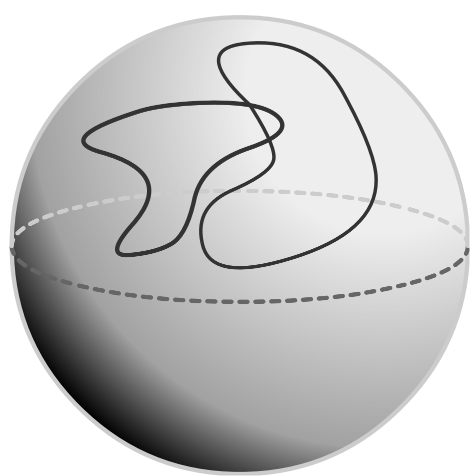


Figure 2.4: Transversal intersection of submanifolds of the 2-sphere

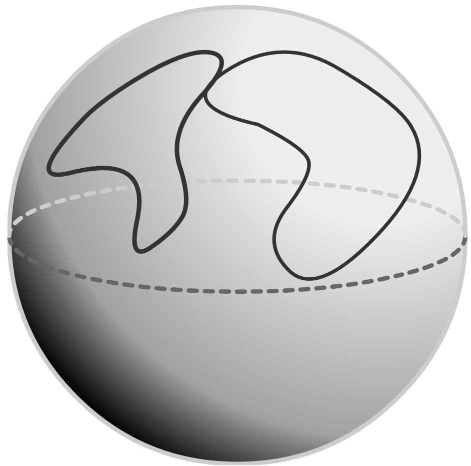


Figure 2.5: Non-transversal intersection of submanifolds on the 2-sphere

Theorem 2.10 (The Local Submersion Theorem, see for example [10, Chapter 1, §4])**.** If $f : X \to Y$ is a submersion at x and $y = f(x)$, then f is locally equivalent to a canonical submersion near x. That is there exist local coordinates around x and y such that $f(x_1, \dots, x_k) = (x_1, \dots, x_l)$ where $k \geqslant l$.

Proof. Choose local parameterisations ϕ and ψ around x and y such that $\phi(0) = x$ and $\psi(0) = y$.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \phi \uparrow & & \uparrow \psi \\ U & \xrightarrow{g} & V \end{array}$$

As $dg_0 : \mathbb{R}^k \to \mathbb{R}^l$ is surjective, by linear changes of coordinates we may assume it has $l \times k$ matrix $(I_l|0)$. Now define $G : U \to \mathbb{R}^k$ by

$$G(a) = (g(a), a_{l+1}, \dots, a_k)$$

where $a = (a_1, \dots, a_k)$. The matrix of dG_0 is then I_k which has non-zero determinant and is therefore an isomorphism. The Inverse Function Theorem, see for example [10, Chapter 1, §3], then tells us that G^{-1} exists as a local diffeomorphism of some open neighbourhood U' of 0 into U. By construction, $g =$ (canonical submersion) $\circ\, G$ and rearranging we see that $g \circ G^{-1}$ is the canonical submersion. Then the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \phi\circ G^{-1}}\uparrow & & \uparrow{\scriptstyle \psi} \\ U' & \xrightarrow[\text{Submersion}]{\text{Canonical}} & V \end{array}$$

commutes. □

Note that if f is a submersion at x, then it is a submersion in a whole neighbourhood of x.

Definition 2.11. For a smooth map of manifolds $f : X \to Y$ a point $y \in Y$ is called a regular value for f if $df_x : T_xX \to T_yY$ is surjective for every point x such that $f(x) = y$. In this case x is called a critical point. If df_x fails to be surjective then y is called a critical value and x a critical point.

Theorem 2.12 (Preimage Theorem, see for example [10, Chapter 1, §4])**.** If y is a regular value of a smooth map of manifolds $f : X \to Y$, then the preimage $f^{-1}(y)$ is a submanifold of X.

Proof. From the Local Submersion Theorem we know that since f is a submersion at the point $x \in f^{-1}(y)$ we can choose local coordinates around x and y so that

$$f(x_1, \ldots, x_k) = (x_1, \ldots, x_l),$$

and y corresponds to $(0, \ldots, 0)$. Thus near $x, f^{-1}(y)$ is just the set of points $(0, \ldots, 0, x_{l+1}, \ldots, x_k)$. More precisely, let V denote the neighbourhood of x on which the coordinate system $(x_1, \ldots, x_k)$ is defined. Then $f^{-1}(y) \cap V$ is the set of points where $x_1 = 0, \ldots, x_l = 0$. The functions $x_{l+1}, \ldots, x_k$ therefore form a coordinate system on the set $f^{-1}(y) \cap V$, which is an open subset of $f^{-1}(y)$. □

Proposition 2.13. (see for example [10, Chapter 2, §3]) If a smooth map of manifolds $f : X \to Y$ is transversal to a submanifold $Z \subset Y$, then the preimage $f^{-1}(Z)$ is a submanifold of X. Moreover the codimension of $f^{-1}(Z)$ in X equals the codimension of Z in Y.

Proof. Let $y = f(x)$, we may write Z in a neighbourhood of y as the zero set of a collection of independent functions $g_1, \ldots, g_l$ where l is the codimension of Z in Y. Then near x the preimage, $f^{-1}(Z)$ is the zero set of the functions $g_1 \circ f, \ldots, g_l \circ f$. Let g denote the submersion $(g_1, \ldots, g_l)$ defined around y. Now to the map $g \circ f : W \to \mathbb{R}^l$ we apply the preimage theorem. This shows $(g \circ f)^{-1}(0)$ is guaranteed to be a manifold when 0 is a regular value of $g \circ f$.

Since

$$d(g \circ f)_x = dg_y \circ df_x,$$

the linear map $d(g \circ f)_x : T_x(X) \to \mathbb{R}^l$ is surjective if and only if dg_y carries the image of df_x onto $\mathbb{R}^l$. But $dg_y : T_y(Y) \to \mathbb{R}^l$ is a surjective linear transformation whose kernel is $T_y(Z)$. Thus dg_y carries a subspace of $T_y(Y)$ onto $\mathbb{R}^l$ precisely if that subspace and $T_y(Z)$ together span all of $T_y(Y)$. In other words, $g \circ f$ is a submersion at the point $x \in f^{-1}(Z)$ if and only if

$$\text{Image}(df_x) + T_y(Z) = T_y(Y).$$

Also locally we have written $f^{-1}(Z)$ as the zero set of l independent functions $g_1\circ, f \ldots, g_l \circ f$. Therefore the codimension of $f^{-1}(Z)$ in X is l, which is the codimension of Z in Y. □

Proposition 2.14. If $f : A \to M$ and $g : B \to M$ are transversal maps then the fibre product $A \times_M B = \{(a, b) \in A \times B | f(a) = g(b)\}$

$$\begin{array}{ccc} A \times_M B & \dashrightarrow & B \\ \downarrow & & \downarrow g \\ A & \xrightarrow{f} & M \end{array}$$

is a smooth manifold and $\dim(A \times_M B) = \dim A + \dim B - \dim M$.

Proof. Both f and g are smooth maps so we have a smooth map $f \times g : A \times B \to M \times M$. Now consider the following diagram

$$\begin{array}{ccc} A \times B & \xrightarrow{f \times g} & M \times M \\ \uparrow & & \uparrow \\ A \times_M B & \dashrightarrow & \Delta \end{array}$$

where $\Delta = \{(m, m) \in M \times M\}$. The preimage $(f \times g)^{-1}(\Delta) = \{(f^{-1}(m), g^{-1}(m)) \in A \times B\} = \{(a, b) \in A \times B | f(a) = g(b)\} = A \times_M B$, so by Proposition 2.13, it suffices to show that $f \times g$ is transversal to the inclusion $i : \Delta \hookrightarrow M \times M$.

Since f and g are transversal, any tangent vector $w \in T_m M$ can be decomposed as $(df(u), df(v)) \in df(T_a A) + dg(T_b B)$. It then follows that $(w_1, w_2) \in T_{m,m}(M \times M)$ can be decomposed as

$$\begin{aligned} df(u_1, u_2) + dg(v_1, v_2) \quad &= \quad (df(u_2), df(u_2)) + (dg(v_1), dg(v_1)) \\ &\quad + (df(u_1 - u_2), dg(v_2 - v_1)) \end{aligned}$$

The first part is a vector in $T_{(m,m)}\Delta$ and the second is a vector in $d(f \times g)(T_{(x,y)}(A \times B))$ and so $f \times g$ is transversal $\Delta \subset M \times M$.

Finally, Proposition 2.13 also tells us that the codimension of $(f \times g)^{-1}(\Delta)$ is equal to the codimension of Δ in $M \times M$, i.e. $\dim(M)$. So we have,

$$\begin{aligned} \dim(A \times_M B) &= \dim(A \times B) - \operatorname{codim}(A \times_M B) \\ &= \dim A + \dim B - \dim M. \end{aligned}$$

□

Remark 2.15. If f and g are inclusions of submanifolds that intersect transversally then

- $A \times_M B \cong A \cap B$ is a smooth submanifold of M,
- $\operatorname{codim} A \cap B = \operatorname{codim} A + \operatorname{codim} B$.

Also $NA \cap B = NA \oplus NB$ where NA and NB are the normal bundles of A and B and $NA \cap B$ is the normal bundle of $A \cap B$. Also, note that if $A \xrightarrow{f} M$ is transversal to $B \subset M$ then

$$N_a(f^{-1}B) \cong f^* N_b B.$$

In Example 2.8, the x-axis intersects the graph of $y = x^2 - 1$ transversally at two points, i.e. a 0-dimensional manifold which has codimension 2 in $\mathbb{R}^2$. This equals the sum of the codimensions of the x-axis and the graph of $y = x^2$, both of which are 1. Likewise the transversal intersection of arcs on the sphere in Example 2.9, again gives a 0-dimensional manifold which has codimension 2 in S^2. The converse is not necessarily true, as in Example 2.8,

$$\operatorname{codim}(x\text{-axis} \cap \text{graph of } y = x^2) = \operatorname{codim}(x\text{-axis}) + \operatorname{codim}(\text{graph of } y = x^2)$$

but the curves do not intersect transversally.

Theorem 2.16 (Sard's Theorem, see for example [10, Chapter 1, §7])**.** If $f : X \to Y$ is any smooth, i.e. C^∞, map of manifolds, then almost every point in Y is a regular value of f. Here we follow Spivak in [24] and say a subset V of Y has measure zero if there is a sequence of coordinate charts (x_i, U_i) whose union contains V and such that $x_i(U_i \cap V)$ has measure 0 (in the usual sense) in $\mathbb{R}^m$ for all i.

The following theorem tells us that almost every continuous map is transversal, i.e. transversality is a generic condition. This allows us to take a non-transversal map and "tweak" it so it becomes transversal.

Theorem 2.17 (Transversality Theorem, see [10, Chapter 2, §3], for example)**.** Suppose that $F : X \times S \to Y$ is a smooth map of manifolds, and let Z be a submanifold of Y. If F is transversal to Z, then for almost every $s \in S$, f_s is transversal to Z, where $f_s = F(-, s)$.

Proof. By Proposition 2.13, the preimage $F^{-1}(Z)$ is a submanifold of $X \times S$. Let $\pi : X \times S \to S$ be the natural projection. First we shall show that whenever s is a regular value for the restriction of π to $F^{-1}(Z)$ then f_s is transversal to Z.

$$\begin{array}{ccc} F^{-1}(Z) = (X \times S) \times_Y Z & \longrightarrow & Z \\ \big\downarrow & & \big\downarrow \\ X \times S & \xrightarrow{\quad F \quad} & Y \\ \big\downarrow{\scriptstyle \pi} & & \\ S & & \end{array}$$

If (x, s) is a regular point of π then

$$T_{(x,s)}(X \times S) = T_x X + T_{(x,s)} F^{-1}(Z).$$

Since F is transversal to Z we have

$$dF(T_{(x,s)}(X \times S)) + T_{F(x,t)} Z = T_{F(x,s)} Y.$$

Combining the two we get

$$dF(T_x X + T_{(x,s)} F^{-1}(Z)) + T_{F(x,t)} Z = T_{F(x,s)} Y.$$

Now using the fact that dF is linear and that $T_{(x,s)} F^{-1}(Z) = dF^{-1}(T_{F(x,s)} Z)$ we have

$$df_s(T_x X) + T_{F(x,t)} Z = T_{F(x,s)} Y.$$

Hence, for a regular point f_s is transversal to Z. Finally, using Sard's theorem we see that since π is a smooth map, almost every point in $X \times S$ is a regular point, so the theorem follows. □

2.3 Maps Between Stratified Spaces

We remind the reader that by a Whitney stratified space we mean the stable germ of a closed union X, of strata of a Whitney stratified manifold M, see Definition 2.6.

Definition 2.18. A *smooth map* $f : X \to Y$ *of Whitney stratified spaces* is the germ at X of a smooth map of the ambient manifolds. The germ of f at X is an equivalence class generated by the equivalence, $f \sim g$ if $f = g$ on some open neighbourhood U of X.

Definition 2.19. A map $f : W \to X$ of Whitney stratified spaces is transversal if and only if for each stratum $S \subset W$ the restriction

$$f|_s : S \to X$$

is transversal to each stratum of X.

Definition 2.20. A smooth map $g : X \to Y$ of Whitney stratified spaces is a *stratified submersion* if

- for any stratum B of Y, the inverse image $g^{-1}B$ is a union of strata of X.
- for each stratum B of Y and stratum $A \subset g^{-1}B$ the restriction $g|_A : A \to B$ is a submersion.

Remark 2.21. For ease of reading we will generally refer to germs of stratified submersions as stratified maps. A map that satisfies only the first condition in Definition 2.20, is known as a weakly stratified map.

Definition 2.22. A smooth map $g : X \to Y$ of Whitney stratified manifolds is a *stratified normal submersion* if the induced map $\phi : N_xA \to N_{g(x)}B$ of normal spaces is surjective, for each $x \in X$ where A is the stratum in which x lies.

$$\begin{array}{ccccc}
T_xA & \hookrightarrow & T_xX & \longrightarrow & N_xA = T_xX/T_xA \\
\downarrow & & \downarrow{\scriptstyle df} & & \downarrow{\scriptstyle \phi} \\
T_{f(x)}B & \hookrightarrow & T_{f(x)}Y & \longrightarrow & N_{fx}B = T_{f(x)}Y/T_{f(x)}B
\end{array}$$

The induced map of normal spaces is defined as $\phi([v]) = [dg(v)]$. This requires that f be weakly stratified. We must check that this is independent of the representative v. If $[v] = [w]$ then $v - w \in T_xA$ and $dg(v-w) = dg(v) - dg(w) = 0$, thus proving the claim.

Our motivation in defining stratified normal submersions, is that transversality is preserved by post-composition with stratified normal submersions and by pre-composition with stratified submersions.

Lemma 2.23. Suppose we have smooth maps $f : X \to Y$ and $g : Y \to Z$. If $g : Y \to Z$ is a stratified normal submersion then the composition $(g \circ f) : X \to Z$ is transversal whenever $f : X \to Y$ is transversal. Also if f is a stratified submersion then the composite $f \circ g$ is transversal whenever g is transversal.

Proof. Take a point $x \in X$. Let A be the stratum of X containing x, B be the stratum of Y containing $f(x)$ and C be the stratum of Z containing $(g \circ f)(x)$. Consider the diagram

$$
\begin{array}{ccccc}
T_x A & \hookrightarrow & T_x X & & \\
 & & \downarrow{\scriptstyle df} & & \\
T_{f(x)} B & \hookrightarrow & T_{f(x)} Y & \longrightarrow & N_{f(x)} B = T_{f(x)}/T_{f(x)} B \\
\vdots\downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle dg} & & \vdots\downarrow{\scriptstyle \alpha} \\
T_{gf(x)} C & \hookrightarrow & T_{gf(x)} Z & \longrightarrow & N_{gf(x)} C = T_{gf(x)} Z / T_{gf(x)} C
\end{array}
$$

Since f is transversal the composition $T_x A \to N_{fx} B$ is surjective. Since g is stratified we know β exists and α is the map induced by dg. When g is a stratified normal submersion α is naturally a surjection, making the bottom right square commute. Composing $T_x A \to N_{fx} B$ with α then gives a surjection $T_x A \to N_{gfx} C$ which shows that $g \circ f$ is also transversal.

The proof of the second statement is similar. □

Example 2.24. Consider a map that includes the x-axis, stratified by the origin and its complement, into $\mathbb{R}^2$ stratified by the y-axis and its complement. This map is not a submersion but is a stratified normal submersion.

Example 2.25. The map sending $\mathbb{R}$ with trivial stratification to $\mathbb{S}^1$, i.e. the circle stratified by a point and its complement, via $s \mapsto (\cos 2\pi s, \sin 2\pi s)$, is a submersion but since the preimage of the zero stratum is a disjoint union of generic points which is not the union of strata of $\mathbb{R}$ it is not even a stratified map let alone a stratified normal submersion. However, if we stratify $\mathbb{R}$ by the preimages of the strata this map becomes a stratified normal submersion, and a stratified submersion.

Lemma 2.26. Whitney stratified manifolds and stratified normal submersions form a category which we call $WhitMan$.

Proof. The identity map on a Whitney stratified manifold is always a stratified normal submersion and, since the induced maps ϕ_f and ϕ_g are surjective, composing two stratified normal submersions gives a stratified normal submersion.

This is verified by the following commuting diagram.

$$
\begin{array}{ccccc}
T_xA & \hookrightarrow & T_xX & \longrightarrow & N_xA = T_x/T_xA \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle df} & & \downarrow{\scriptstyle \phi_f} \\
T_{fx}B & \hookrightarrow & T_{fx}Y & \longrightarrow & N_{fx}B = T_{fx}/T_{fx}B \\
\downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle dg} & & \downarrow{\scriptstyle \phi_g} \\
T_{gfx}C & \hookrightarrow & T_{gfx}Z & \longrightarrow & N_{gfx}C = T_{gfx}Z/T_{gfx}C
\end{array}
$$

□

A map from any X to Y is a stratified normal submersion if Y has the trivial stratification so $WhitMan$ contains the category of manifolds and smooth maps as a full subcategory.

Chapter 3

Algebraic Structure

In this chapter we outline the algebraic machinery that we will call upon later. Transversal homotopy theory does not produce groups, in general, unlike 'ordinary' homotopy theory. Instead we obtain dagger monoids, i.e. monoids that, for each element, there exists a dual element. We would like to relate dagger monoids to groups and so in the first section, along with the basic definitions and results, we outline a method of producing a group from a dagger monoid. This will allow us to compare transversal homotopy with ordinary homotopy in later chapters. In ordinary homotopy, one can define higher categorical analogues of homotopy groups, i.e. homotopy groupoids, see Section 6.1. We will find similar structures in transversal homotopy theory but just as in the previous example we cannot hope to produce groupoids in general. In the low dimensional cases that we examine in Chapters 8 and 9, we obtain dagger categories and dagger pivotal categories, both of which we define in Section 3.2.

3.1 Dagger Monoids

A *dagger monoid* is a monoid M with an involution $a \mapsto a^\dagger$ such that $1^\dagger = 1$ and $(ab)^\dagger = b^\dagger a^\dagger$. A *homomorphism of dagger monoids* is a map $\phi : M \to M'$ which preserves the unit, product and involution.

Definition 3.1. A dagger submonoid $N \leqslant M$ is *inner-normal* if for any $a, x, y \in M$,

$$xaa^\dagger y \in N \Rightarrow xy \in N.$$

It is *outer-normal* if for any $n \in N$ and any $a \in M$,

$$ana^\dagger \in N.$$

A dagger submonoid is said to be *well-balanced* if it is both inner-normal and outer-normal. In particular, any group is a dagger monoid where the dual of any element is its inverse.

Remark 3.2. If M is commutative then the notions of inner and outer-normality coincide.

Example 3.3. Let M be the free dagger monoid on one generator a. Consider the homomorphism of dagger monoids

$$f : M \to G$$

where G is a group and where $f(a^\dagger)$. Here, $f^{-1}(1)$ is the collection of words where the number of times a appears is the same as the number of times $a^\dagger$ appears. It is a well-balanced submonoid and

$$M/f^{-1}(1) \cong \mathrm{im}(f).$$

We will make this more precise later when we define quotients of dagger monoids by well-balanced submonoids.

Lemma 3.4. If N is a well-balanced dagger submonoid then

$$ab \in N \Rightarrow ba \in N.$$

Proof. Since N is outer-normal,

$$ab \in N \Rightarrow b(ab)b^\dagger \in N.$$

But N is also inner-normal so

$$ba(bb^\dagger) \in N \Rightarrow ba \in N.$$

□

Note if N is in fact a normal subgroup, it clearly follows

$$\begin{aligned} fg \in N &\Rightarrow g(fg)g^{-1} \in N \\ &\Rightarrow gf \in N. \end{aligned}$$

Definition 3.5. Let M be a dagger monoid and N be a dagger submonoid which is well-ballanced, then we define the relation $a \sim b$ if

$$ab^\dagger \in N.$$

Lemma 3.6. The relation defined in Definition 3.5 is an equivalence relation.

Proof. The relation is easily seen to be reflexive since

$$a \sim a \Leftrightarrow aa^\dagger \in N.$$

It is symmetric since

$$\begin{aligned} a \sim b &\Leftrightarrow ab^\dagger \in N \\ &\Leftrightarrow (ab^\dagger)^\dagger \in N \\ &\Leftrightarrow ba^\dagger \in N \\ &\Leftrightarrow b \sim a. \end{aligned}$$

Finally, it is transitive since

$$\begin{aligned} a \sim b, b \sim c &\Leftrightarrow ab^\dagger, bc^\dagger \in N \\ &\Rightarrow ab^\dagger bc^\dagger \in N \\ &\Rightarrow ac^\dagger \in N \end{aligned}$$

by inner normality. □

We denote the class of a by $[a]$ and the quotient by M/N. Note that we have only used the fact that N is an inner-normal dagger submonoid to show that N induces an equivalence relation. We will however need outer-normality to define the monoid structure on M/N.

Lemma 3.7. Given a dagger monoid M and a well-balanced submonoid N, the quotient M/N using the above equivalence relation is a monoid with unit $[1]$, i.e. the associative map

$$M/N \times M/N \to M/N : ([a], [b]) \to [ab]$$

is well-defined.

Proof. If $[a] = [a']$ and $[b] = [b']$ then $aa'^\dagger, bb'^\dagger \in N$. Since $bb'^\dagger \in N$, by outer-normality we have $abb'^\dagger a^\dagger \in N$ and since $aa'^\dagger \in N$ we have $(abb'^\dagger a^\dagger)(aa'^\dagger) \in N$. So by inner normality $abb'^\dagger a'^\dagger \in N$. This shows $[ab] = [a'b']$, i.e. the map is well defined. □

Lemma 3.8. If $N \leqslant M$ is a well balanced dagger submonoid then,

$$f : M \to M/N : a \mapsto [a]$$

is a homomorphism of dagger monoids with kernel $[1] = N$.

Proof. Define $[a]^\dagger = [a^\dagger]$ and take two representatives a and a' of the same equivalence class $[a]$. Then using Lemma 3.4,

$$\begin{aligned}[a] = [a'] &\Rightarrow aa'^\dagger \in N \\ &\Rightarrow a'a^\dagger \in N \\ &\Rightarrow a^\dagger a' \in N \\ &\Rightarrow [a^\dagger] = [a'^\dagger].\end{aligned}$$

Also

$$\begin{aligned}a \in [1] &\Leftrightarrow a1^\dagger \in N \\ &\Leftrightarrow a \in N.\end{aligned}$$

□

Remark 3.9. It is not true that $[a] = aN$ in general. If we again consider Example 3.3, where the kernel N of the map was the equivalence class of words with one more a than $a^\dagger$. This is not the same as the coset aN. It is clear that

$$aN \subset [a],$$

but the converse is not true. To see this consider the element $a^\dagger aa$ of $[a]$. This element cannot be realised as a member of N pre-multiplied by a, i.e. a member of aN.

Proposition 3.10. The quotient monoid M/N is a group if and only if N is a well-balanced submonoid of M.

Proof. Assume M/N is a group. We begin by showing that any surjective homomorphism $f : M \to G$ from a dagger monoid to a group has a well balanced kernel. We define the kernel as follows,

$$\ker(f) = \{m \in M | f(m) = 1_G\}.$$

Now if we take any $a \in M$,

$$\begin{aligned}f(a(\ker(f))a^\dagger) &= f(a)f(\ker(f))f(a^\dagger) \\ &= f(a)1_G f(a)^{-1} \\ &= f(a)f(a)^{-1} \\ &= 1_G\end{aligned}$$

This shows that $a(\ker(f))a^\dagger$ is in the kernel of f so $\ker(f)$ is outer-normal.

Now take any $a \in M$ and $xy \in \ker(f)$.

$$\begin{aligned} 1_G &= f(xy) \\ &= f(x)f(y) \\ &= f(x)f(a)f(a)^{-1}f(y) \\ &= f(xaa^\dagger y), \end{aligned}$$

i.e. $xaa^\dagger y$ is in the kernel also thus we have $\ker(f)$ is inner-normal and so $\ker(f)$ is well-balanced. Now since

$$\begin{aligned} [a] = [b] \in \ker(f) &\Leftrightarrow ab^\dagger \in \ker(f) \\ &\Leftrightarrow f(ab^\dagger) = 1 \\ &\Leftrightarrow f(a)f(b)^{-1} = 1 \\ &\Leftrightarrow f(a) = f(b) \end{aligned}$$

and taking f to be the quotient map in Lemma 3.8 which is surjective, then we have shown $ker(f) = N$ is a well-balanced dagger submonoid of M.

Next we will show that if N is a well-balanced dagger submonoid then M/N is a group. We have previously shown that M/N is a dagger monoid so all that is left is to show the existence of inverses. To see this, choose any class $[a] \in M/N$ and note that $[a][a]^\dagger = [a][a^\dagger] = [aa^\dagger] = [1]$. □

Proposition 3.11. Any arbitrary intersection of well-balanced submonoids, $\bigcap_i N_i$, of M is also well-balanced.

Proof. Choose any $a \in \bigcap_i N_i$ and any $m \in M$. Since each N_i is well balanced we know that $mam^\dagger$ is also contained in each N_i and so is contained in $\bigcap_i N_i$ which is therefore outer-normal. Similarly for any $a \in M$,

$$xaa^\dagger y \in N_i \Rightarrow xy \in N_i.$$

and so

$$xaa^\dagger y \in \bigcap_i N_i \Rightarrow xy \in \bigcap_i N_i$$

for all i. This shows $\bigcap_i N_i$ is inner-normal and thus well-balanced. □

Lemma 3.12. Any dagger monoid has a minimal well-balanced submonoid (i.e. a well-balanced submonoid which is contained in any well-balanced submonoid of M).

Proof. By Proposition 3.11, any intersection of well balanced dagger submonoids is itself a well balanced dagger submonoid. So the intersection of all such submonoids will be a minimal well balanced dagger submonoid. □

Remark 3.13. If $N \leqslant M$ is well-balanced then $aa^\dagger \in N$ for any $a \in M$ (by outer normality applied to $1 \in M$). Hence a minimal well-balanced submonoid contains $\langle aa^\dagger | \, a \in M \rangle$. This is an inner normal submonoid but is not necessarily outer normal, unless M is commutative. In the non-commutative case they can be more complicated. For example take

$$M = \langle x, y, z \rangle$$

to be the free dagger monoid on three generators x, y and z. Then

$$xx^\dagger yy^\dagger zz^\dagger \in \langle aa^\dagger | \, a \in M \rangle$$

but

$$zxx^\dagger yy^\dagger z^\dagger \notin \langle aa^\dagger | \, a \in M \rangle$$

and clearly must be in the kernel of any homomorphism to a group since daggers get sent to inverses.

Lemma 3.14. If $f : M \to M'$ is a homomorphism of dagger monoids with respective minimal well-balanced submonoids N and N' then $N \subset f^{-1}N'$ and f induces a group homomorphism

$$Qf : M/N \to M'/N'.$$

Proof. Suppose $xaa^\dagger y \in f^{-1}N'$ then

$$\begin{aligned} &\Rightarrow \quad f(x)f(a)f(a)^\dagger f(y) \in N' \\ &\Rightarrow \quad f(x)f(y)^\dagger \in N' \\ &\Rightarrow \quad xy \in f^{-1}N', \end{aligned}$$

i.e. $f^{-1}N'$ is inner normal.

Next suppose $n \in f^{-1}N'$, i.e. that $f(n) \in N'$ and consider $ana^\dagger$. We have

$$\begin{aligned} f(ana^\dagger) \quad &= \quad f(a)f(n)f(a)^\dagger \in N' \\ &\Rightarrow \quad ana^\dagger \in f^{-1}N'. \end{aligned}$$

Therefore $f^{-1}N'$ is outer normal and so is well-balanced. Since N is the minimal well-balanced submonoid of M, it must therefore be contained in $f^{-1}N'$.

We claim that there is a well-defined group homomorphism $f : M/N \to M'/N' : [a] \mapsto [f(a)]$. Suppose that $[a] = [b]$. Then $ab^\dagger \in N$. So $f(a)f(b)^\dagger \in f(N) \subset N'$. And so $[f(a)] = [f(b)]$. Therefore f is well-defined.

Finally since

$$\begin{aligned} f([a][b]) &= f([ab]) \\ &= [f(ab)] \\ &= [f(a)f(b)] \\ &= [f(a)][f(b)] \end{aligned}$$

group structure is preserved. □

Proposition 3.15. There is a functor Q from the category $DagMon$ of dagger monoids to Grp the category of groups, which is left adjoint to the inclusion functor $Grp \hookrightarrow DagMon$, i.e. there is a natural isomorphism

$$\mathrm{Hom}_{DagMon}(M, G) \cong \mathrm{Hom}_{Grp}(QM, G).$$

Proof. From Lemma 3.12 we can quotient any dagger monoid M by a minimal well-balanced submonoid N. This allows us to construct a functor

$$\begin{aligned} Q : DagMon &\rightarrow Grp \\ M &\mapsto M/N \\ f &\mapsto Qf \end{aligned}$$

where $Qf : M/N \rightarrow M'/N'$.

We need a bijection $\mathrm{Hom}(M, G) \cong \mathrm{Hom}(QM, G)$. Consider the following,

$$Q : \mathrm{Hom}(M, G) \rightarrow \mathrm{Hom}(QM, G)$$

$$(f : M \rightarrow G) \mapsto (Qf : QM \rightarrow QG = G),$$

$$(- \circ q) : \mathrm{Hom}(QM, G) \rightarrow \mathrm{Hom}(M, G)$$

$$(g : QM \rightarrow G) \mapsto (g \circ q : M \rightarrow G),$$

where $q : M \rightarrow QM$.

It remains to check that we have a bijection, i.e. $Qf \circ q = f$. This follows since $(Qf \circ q)(M) = Qf(QM) = QG = G$.

Next suppose we have a homomorphism $f : M \rightarrow M'$ of dagger monoids. We must verify that the following square

$$\begin{array}{ccc} \mathrm{Hom}(M', G) & \xrightarrow{\cong} & \mathrm{Hom}(QM', G) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(M, G) & \xrightarrow[\cong]{} & \mathrm{Hom}(QM, G). \end{array}$$

commutes, where the vertical maps are induced by composition respectively with f and Qf. Given a homomorphism of dagger monoids $g : M' \to G$ (i.e. an element in the top left corner of above square) we obtain, by going each way round the diagram, the maps $Q(f \circ g)$ and $Qf \circ Qg$. These are the same since for $a \in M'$,

$$Qf(Qg[a]) = Qf[g(a)] = [fg(a)] = Q(fg)[a].$$

There is another similar square

$$\begin{array}{ccc} \mathrm{Hom}(M, G) & \xrightarrow{\cong} & \mathrm{Hom}(QM, G) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(M, G') & \xrightarrow[\cong]{} & \mathrm{Hom}(QM, G'). \end{array}$$

arising from a group homomorphism $g : G \to G'$. Here given a homomorphism of dagger monoids $f : M \to G$ we obtain, by going each way round the diagram, the maps $Q(g \circ f)$ and $Qg \circ Qf$. These are the same since for $a \in M$,

$$Qg(Qf[a]) = Qg[f(a)] = [gf(a)] = Q(gf)[a]$$

and so the isomorphism is natural in both M and G. □

3.2 Dagger Pivotal Categories

Definition 3.16. [21] We define a (small) *dagger category* $\mathscr{C}$ as having two sets, Obj and Mor, and four maps:

i) $s : Mor \to Obj$, sending morphisms to their source,

ii) $t : Mor \to Obj$, sending morphisms to their target,

iii) $id : Obj \to Mor$, sending an object to the identity morphism on that object,

iv) $c : \{(f, g) \in Mor \times Mor : t(f) = s(g)\} \to Mor$, composing morphisms, i.e. $c(f, g) = g \circ f$.

These maps must satisfy the following three rules:

- Source and target of composites: $s(c(f, g)) = s(f)$ and $t(c(f, g)) = t(g)$.
- Associativity of composition: $c(c(f, g), h) = c(f, c(g, h))$.
- Identities act as units for composition: $c(id_x, f) = f = c(f, id_y)$, where $(f : x \to y) \in Mor$.

There is also an involutive, identity-on-object functor $\dagger : \mathscr{C}^{op} \to \mathscr{C}$. That is, one which associates to every morphism $f\colon x \to y$ in $\mathscr{C}$ its adjoint $f^\dagger\colon y \to x$ such that for all $f\colon x \to y$ and $g\colon y \to z$,

- $id_x = id_x^\dagger\colon x \to x$
- $(g \circ f)^\dagger = f^\dagger \circ g^\dagger\colon z \to x$
- $\mathrm{f}^{\dagger\dagger} = f\colon x \to y$

Example 3.17. The category $FdHilb$ of finite dimensional Hilbert spaces and linear maps is a dagger category. Given a linear map $f : H \to K$, the map $f^\dagger : K \to H$ is just its adjoint in the usual sense.

Definition 3.18. [7] An object a in a monoidal category $(\mathscr{C}, \otimes)$, is *left rigid* if it has a dual object a^*, with morphisms $\eta_a : a \otimes a^* \to 1$ and $\epsilon_a : 1 \to a^* \otimes a$ such that the composites,

$$a \xrightarrow{\eta_a \otimes id_a} (a \otimes a^*) \otimes a \to a \otimes (a^* \otimes a) \xrightarrow{id_a \otimes \epsilon_a} a$$

and $$a^* \xrightarrow{id_a * \otimes \eta_a} a^* \otimes (a \otimes a^*) \to (a^* \otimes a) \otimes a^* \xrightarrow{\epsilon_x \otimes id_x} a^*$$

are identities. These are known as the *triangle identities.* A right rigid object, ${}^*a$, is defined similarly. A monoidal category is said to be *rigid monoidal* if all objects have both left and right duals. We refer to the morphisms η_a and ϵ_a as the unit and counit of a, respectively.

Remark 3.19. Observe that $\epsilon_{a^*} = (\eta_a)^* : 1 \to a^* \otimes a$ and $\eta_{a^*} = (\epsilon_a)^*$.

Lemma 3.20. [4, Chapter 2, Lemma 2.1.6] Suppose that b has a dual b^*. Then there exists canonical isomorphisms

$$\mathrm{Hom}(a \otimes b, c) \cong \mathrm{Hom}(a, c \otimes b^*) \tag{3.1}$$

$$\mathrm{Hom}(a, b \otimes c) \cong \mathrm{Hom}(b^* \otimes a, c). \tag{3.2}$$

Proof. To $\psi \in \mathrm{Hom}(a \otimes b, c)$ we associate the composition

$$a \xrightarrow{a \otimes \eta_b} a \otimes b \otimes b^* \xrightarrow{\psi \otimes b^*} c \otimes b^*$$

which is an element of $\mathrm{Hom}(a, c \otimes b^*)$. Here we abuse notation by denoting by a the object and the identity morphism on that object. Similarly, to $\phi \in \mathrm{Hom}(a, c \otimes b^*)$ we assign

$$a \otimes b \xrightarrow{\phi \otimes b} c \otimes b^* \otimes b \xrightarrow{c \otimes \epsilon_b} c.$$

One can easily check that these two maps are inverse to each other, establishing (3.1). The proof of (3.2) is similar. □

Remark 3.21. In particular, if both a and b have duals, then by Lemma 3.20

$$\mathrm{Hom}(a,b) = \mathrm{Hom}(b^*,a^*) = \mathrm{Hom}(1, b \otimes a^*).$$

Definition 3.22. A *pivotal category* is a rigid category equipped with a monoidal natural isomorphism $a \to (a^*)^*$. Pivotal categories have also been called sovereign categories. See [9] for details.

Gathering together Definitions 3.16 and 3.22 we have the notion of a *dagger pivotal category.*

Example 3.23. Since we can take the dual of a finite dimensional Hilbert space and a linear map $f : A \to B$, has a dual map $f^* : B^* \to A^*$. So $FdHilb$ is in fact a dagger pivotal category.

3.3 Graphical Language for Categories

In [20], Peter Selinger describes what a graphical language for categories should consist of. We reproduce that description here as well as some of the results from that paper.

Definition 3.24. In a *graphical language of categories* objects are represented by edges (or wires) and morphisms by boxes (or vertices). An identity morphism is represented as a continuing edge and composition by connecting the outgoing edge of one diagram to the incoming edge of another, see Figure 3.1. We will give two examples of graphical languages in Chapter 9.

Definition 3.25 (Coherence)**.** The three defining axioms of categories, e.g. $id_B \circ f = f$, are automatically satisfied "up to isomorphism" in a graphical language. This property is known as *soundness.* Also, every equation that holds in the graphical language is a consequence of the axioms. This property is called *completeness.* We refer to a soundness and completeness theorem as a *coherence theorem* and the graphical language as coherent.

Lemma 3.26 (Coherence for categories)**.** A well-formed equation between two morphism terms in the language of categories follows from the axioms of categories if and only if it holds in the graphical language up to isomorphism of diagrams.

Proof. By associativity and unit axioms, each morphism term is uniquely equivalent to a term of the form $((f_n \circ \cdots) \circ f_2) \circ f_1$ for $n \geqslant 0$, with corresponding diagram in Figure 3.2. □

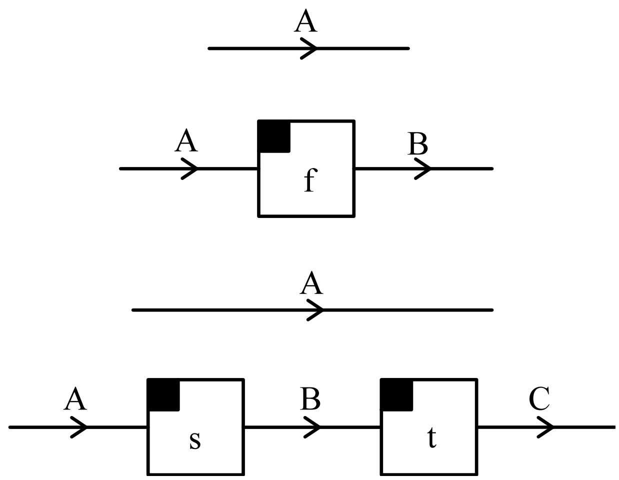


Figure 3.1: From top to bottom: An object A, a morphism $f : A \to B$, The identity morphism on A and the composition of morphisms $t \circ s$. Boxes are oriented so as to have a well-defined source and target.

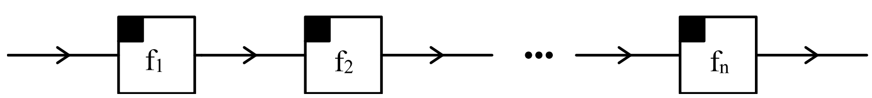


Figure 3.2:

Here two diagrams are isomorphic if the edges and boxes of the first are in bijective correspondence with the second, preserving the connection between boxes and edges.

We refer the reader to [20, §4 and §7] for similar coherence theorems for pivotal and dagger categories respectively.

Chapter 4

The Pontryagin-Thom Construction

The Pontryagin-Thom construction was originally conceived to provide a geometric interpretation of the homotopy groups $\pi_n(S^k)$ of spheres. We will use a variation of it later to give a geometric interpretation to the transversal homotopy monids of both spheres and complex projective space. In this chapter we we give details of the construction using theorems and results of Milnor's, from [17].

4.1 Reminder of Classical Homotopy Groups

Given a topological space X with basepoint x_0, we define a loop based at x_0 to be a map $f : (I, \partial I) \to (X, x_0)$. The fundamental group $\pi_1(X, x_0)$ is the set of homotopy classes of loops based at x_0, where $f_t(\partial I) = \{x_0\}$ for all t. We define composition $[f][g] = [f \cdot g]$ where

$$f \cdot g = \begin{cases} f(2t) & \text{if} \quad t \in [0, \frac{1}{2}] \\ g(2t-1) & \text{if} \quad t \in [\frac{1}{2}, 1] \end{cases}$$

is the concatenation of paths. The inverse is defined as $[f]^{-1} = [f^{-1}]$ where $f^{-1}(t) = f(1-t)$ is the same path traversed in the opposite direction.

The natural higher dimensional analogue of the fundamental group of a space X is the nth homotopy group, $\pi_n(X)$, which consists of maps $f : (I^n, \partial I^n) \to (X, x_0)$ up to homotopy relative to ∂I^n. Composition and inverses are defined in a similar way to the $n = 1$ case but for $n > 1$, $\pi_n(X)$ is abelian so we use additive notation. Define $[f] + [g] = [f + g]$ where,

$$(f+g)(t_1, t_2, \cdots, t_n) = \begin{cases} f(t_1, t_2, \cdots, 2t_n) & \text{if} \quad t_n \in [0, \frac{1}{2}] \\ g(t, t_2, \cdots, 2t_n - 1) & \text{if} \quad t_n \in [\frac{1}{2}, 1] \end{cases}$$

and $-f(t_1, t_2, \cdots, t_n) = f(t_1, t_2, \cdots, 1 - t_n)$.

4.2 Framed Cobordism

Definition 4.1. Two closed n-dimensional manifolds A and A' are said to be *cobordant* if their disjoint union $A \coprod A'$ is the boundary ∂W of a compact $n+1$ dimensional manifold W. A *cobordism* is a manifold W with boundary whose boundary is partitioned in two, $\partial W = A \coprod A'$.

The first non-trivial example is the pair of pants, see Figure 4.1. Any closed manifold is the boundary of the non-compact manifold $M \times [0, 1)$, so we only consider compact manifolds.

Definition 4.2. There is also a notion of *cobordism within an ambient manifold* M. Suppose A and A' are submanifolds of M. Then A is cobordant to A' within M if the subset

$$A \times [0, \epsilon) \cup A' \times (1 - \epsilon, 1]$$

of $M \times [0, 1]$ can be extended to a compact submanifold $X \subset M \times [0, 1]$ so that

$$\partial X = A \times 0 \cup A' \times 1$$

and also X does not intersect $M \times 0 \cup M \times 1$ except at the points of ∂X. From here on we will use the ambient notion of cobordism.

Definition 4.3. A *framing* of a submanifold A^n of M^m is a trivialisation of the normal bundle NA, i.e. a bundle isomorphism $NA \cong A \times \mathbb{R}^{m-n}$, where m and n are the dimensions of M and A respectively. We can explicitly describe a framing as follows: We choose a metric on M so that $NA \cong TA^{\perp}$. A *framing* of A is a section v which assigns to each $x \in A$ a basis

$$v(x) = (v^1(x), \ldots, v^{m-n}(x))$$

for the normal space $T_xA^{\perp} \subset T_xM$ of normal vectors to A in M at x and where $m - n$ is the codimension of the submanifold. The pair (A, v) is called a *framed submanifold* of M.

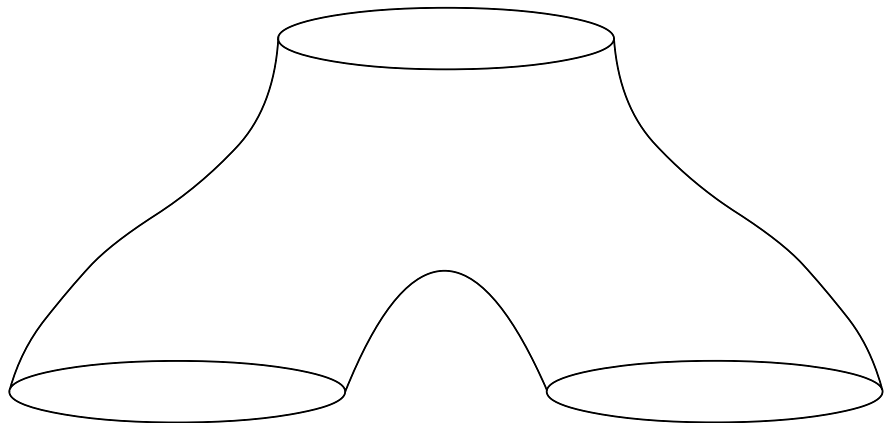


Figure 4.1: The cobordism between a single circle (at the top) and a pair of disjoint circles (at the bottom) known as "A Pair of Pants".

Definition 4.4. Two framed submanifolds (A, v) and (A', w) are *framed cobordant* if there exists a cobordism $X \subset M \times [0, 1]$ between A and $A'$and a framing u of X, so that

$$u^i(x,t) = (v^i(x), 0) \quad \text{for} \quad (x,t) \in A \times [0, \epsilon)$$
$$u^i(x,t) = (w^i(x), 0) \quad \text{for} \quad (x,t) \in A \times (1-\epsilon, 1]$$

where $(x, t) \in X \subset M \times [0, 1]$.

Framed cobordism generates an equivalence relation on framed submanifolds.

Definition 4.5. The set of equivalence classes of framed codimension p submanifolds of an n-dimensional manifold M is denoted $\Omega^{fr}_{n-p}(M)$.

Definition 4.6. A *tubular neighbourhood* of a submanifold A in M is an embedding of the normal bundle NA into M, i.e. $f : NA \to M$, where the zero section of the normal bundle is mapped identically onto A. The Tubular Neighbourhood Theorem (see for example[17, page 46]) says that a closed submanifold always has an open neighbourhood diffeomorphic to NA.

4.3 Pontryagin Manifolds

Definition 4.7. Consider a smooth map $f : M \to S^p$ from an arbitrary compact, boundaryless manifold to the p-dimensional sphere, and a regular value $y \in S^p$. The map f induces a framing of the manifold $f^{-1}(y)$ as follows:

Choose a positively oriented basis $v = (v^1, \ldots, v^p)$ for the tangent space $T_y S^p$. For each $x \in f^{-1}(y)$,

$$df_x : T_x M \to T_y S^p$$

maps the subspace $T_x f^{-1}(y)$ to zero and maps its orthogonal complement $N_x f^{-1}(y)$ isomorphically onto $T_y S^p$. Hence there is a unique vector

$$w^i(x) \in N_x f^{-1}(y) \subset T_x M$$

that maps to v^i under df_x. The map $x \mapsto (w^1(x), \cdots, w^p(x))$ gives a framing w of $f^{-1}(y)$. We will refer to $(w^1(x), \ldots, w^p(x))$ as f^*v. The framed manifold $(f^{-1}(y), f^*v)$ is known as the *Pontryagin manifold* associated with f.

Lemma 4.8. If y is a regular value of f and z is sufficiently close to y, then $f^{-1}(z)$ is framed cobordant to $f^{-1}(y)$.

Proof. Theorem 2.16 (Sard's Theorem) tells us that the set $f(C)$ of critical values is compact (as the image of a closed set in a compact space) so we can choose $\epsilon > 0$ so that the ϵ neighbourhood of y contains only regular values. Given z with $||z - y|| < \epsilon$, choose a smooth one-parameter family of rotations $r_t : S^p \to S^p$ so that $r_1(y) = z$, and so that

1. r_t is the identity for $0 \leqslant t < \epsilon'$,
2. r_t equals r_1 for $1 - \epsilon' < t \leqslant 1$, and
3. each $r_t^{-1}(z)$ lies on the great circle from y to z, and hence is a regular value of f.

Define the homotopy

$$F : M \times [0, 1] \to S^p$$

by $F(x, t) = r_t f(x)$. For each t note that z is a regular value of the composition

$$r_t \circ f : M \to S^p.$$

It follows *a fortiori* that z is a regular value for the mapping F. Hence

$$F^{-1}(z) \subset M \times [0, 1]$$

is a framed manifold and provides a framed cobordism between the framed manifolds $f^{-1}(z)$ and$(r_1 \circ F)^{-1}(z) = f^{-1} r_1^{-1} = f^{-1}(y)$. □

Lemma 4.9. If f and g are smoothly homotopic maps from $M \to S^p$ and y is a regular value for both, then $f^{-1}(y)$ is framed cobordant to $g^{-1}(y)$.

Proof. Choose a homotopy F with

$$F(x,t) = f(x) \qquad 0 \leqslant t \leqslant \epsilon$$
$$F(x,t) = g(x) \qquad 1-\epsilon \leqslant t \leqslant 1$$

Choose a regular value z for F which is close enough to y so that $f^{-1}(z)$ is framed cobordant with $f^{-1}(y)$ and so that $g^{-1}(z)$ is framed cobordant to $g^{-1}(y)$. Then $F^{-1}(z)$ is a framed manifold and provides a framed cobordism between $f^{-1}(z)$ and $g^{-1}(z)$. □

Lemma 4.10. If y and z are regular values of f and v and w are positively oriented bases for T_yS^p and T_zS^p, then the framed manifold $(f^{-1}(z), f^*w)$ is cobordant to $(f^{-1}(y), f^*v)$.

Proof. Given any two regular values y and z for f, we can choose rotations

$$r_t : S^p \to S^p$$

so that r_0 is the identitiy and $r_1(y) = z$. Thus f is homotopic to $r_1 \circ f$; hence $f^{-1}(z)$ is framed cobordant to

$$(r_1 \circ f)^{-1}(z) = f^{-1}r_1^{-1}(z) = f^{-1}(y).$$

□

Lemma 4.11. Any compact framed submanifold (A, w) of codimension p in M, occurs as a Pontryagin manifold for some smooth mapping $f : M \to S^p$.

Proof. Because we have a framing, i.e. a trivialisation of the normal bundle, the tubular neighbourhood theorem tells us there exists a diffeomorphism $g : V \to A \times \mathbb{R}^p$ from some neighbourhood $V \subset M$ of A to

$$A \times \mathbb{R}^p.$$

Furthermore, g can be choosen so that each $x \in A$ corresponds to $(x, 0) \in A \times \mathbb{R}^p$ and so that each normal frame corresponds to the standard basis of $\mathbb{R}^p$. If we now define a projection

$$\pi : V \to \mathbb{R}^p$$

by $\pi(g(x,y)) = y$, then 0 is a regular value for this map and $\pi^{-1}(0)$ is precisely A with its given framing.

Next we choose a smooth map $\phi : \mathbb{R}^p \to S^p$ which maps every x with $||x|| \geqslant 1$ into a base point s_0, and maps the open unit ball in $\mathbb{R}^p$ diffeomorphically onto

$S^p - s_0$. For example, $\phi(x) = h^{-1}(x/\lambda(||x||^2))$, where h is the stereographic projection from s_0 and where λ is a smooth monotone decreasing function with $\lambda(t) > 0$ for $t < 1$ and $\lambda(t) = 0$ for $t \geqslant 1$.

Define

$$f : M \to S^p$$

by $f(x) = \phi(\pi(x))$ for $x \in V$ and $f(x) = s_0$ for $x \notin V$. Clearly f is smooth, and the point $\phi(0)$ is a regular value of f. So we see that the corresponding Pontryagin manifold

$$f^{-1}(\phi(0)) = \pi^{-1}(0)$$

is precisely equal to the framed manifold A. □

Now we show that the Pontryagin manifold of a map determines its homotopy class. Let $f, g : M \to S^p$ be smooth maps with a common regular value y.

Lemma 4.12. If the framed manifold $(f^{-1}(y), f^*v)$ is equal to $(g^{-1}(y), g^*v)$ then f is smoothly homotopic to g.

Proof. It will be convenient to set $A = f^{-1}(y)$. Since $f^*v = g^*v$ we have $df_x = dg_x$ for all $x \in A$. First suppose that f actually coincides with g in a entire neighbourhood V of A. Let $h : (S^p - y) \to \mathbb{R}^p$ be stereographic projection. Then the homotopy

$$H(x,t) = \begin{cases} f(x) & \text{for} \quad x \in V \\ h^{-1}[t \cdot h(f(x)) + (1-t) \cdot h(g(x))] & \text{for} \quad x \in M - A \end{cases}$$

proves that f is smoothly homotopic to g.

Thus it suffices to deform f so that it coincides with g in some small neighbourhood of A, being careful not to map any new points into y during the deformation. Choose a product representation

$$A \times \mathbb{R}^p \to V \subset M$$

for a neighbourhood V of A, where V is small enough so that $f(V)$ and $g(V)$ do not contain the antipode y' of y. Identifying V with $A \times \mathbb{R}^p$ and identifying $S^p - y'$ with $\mathbb{R}^p$, we obtain corresponding mappings

$$F, G : A \times \mathbb{R}^p \to \mathbb{R}^p,$$

with

$$F^{-1}(0) = G^{-1}(0) = A \times 0,$$

and with

$$dF_{(x,0)} = dG_{(x,0)} = (\text{projection to } \mathbb{R}^p)$$

for all $x \in A$. We will first find a constant c so that

$$F(x, u) \cdot u > 0 \quad \text{and} \quad G(x, u) \cdot u > 0$$

for $x \in A$ and $0 < ||u|| < c$. That is, the points $F(x, u)$ and $G(x, u)$ belong to the same open half-space in $\mathbb{R}^p$. So the homotopy

$$(1 - t)F(x, u) + tG(x, u)$$

between F and G will not map any new points into 0, at least for $||u|| < c$.

By Taylor's theorem, for $||u|| \leqslant 1$,

$$||F(x, u) - u|| \leqslant c_1 ||u||^2.$$

Hence

$$|(F(x, u) - u) \cdot u| \leqslant c_1 ||u||^3$$

and

$$F(x, u) \cdot u \geqslant ||u||^2 - c_1 ||u||^3 > 0$$

for $0 < ||u|| < c = \mathrm{Min}(\mathrm{c}_1^{-1}, 1)$, with a similar inequality for G.

To avoid moving distant points we select a smooth map $\lambda : \mathbb{R}^p \to \mathbb{R}$ with

$$\begin{aligned} \lambda(u) &= 1 \quad \text{for} \quad ||u|| \leqslant c/2 \\ \lambda(u) &= 0 \quad \text{for} \quad ||u|| \geqslant c. \end{aligned}$$

Now the homotopy

$$F_t(x, u) = [1 - \lambda(u)t]F(x, u) + \lambda(u)tG(x, u)$$

deforms $F = F_0$ into a mapping F_1 that

1. coincides with G in the region $||u|| < c/2$,
2. coincides with F for $||u|| \geqslant c$, and
3. has no new zeros.

Finally, we make a corresponding deformation of the original mapping f. □

Lemma 4.13. Two mappings from M to S^p are smoothly homotopic if and only if the associated Pontryagin manifolds are framed cobordant.

Proof. If f and g are smoothly homotopic, then from Lemma 4.9 we see that the Pontryagin manifolds $f^{-1}(y)$ and $g^{-1}(y)$ are framed cobordant. Conversely, given a framed cobordism (X, w) between $f^{-1}(y)$ and $g^{-1}(y)$ we can construct a homotopy

$$F : M \times [0,1] \to S^p$$

whose Pontryagin manifold $(F^{-1}(y), F^*v)$ is precisely equal to (X, w). Setting $F_t(x) = F(x,t)$, note that the maps F_0 and f have exactly the same Pontryagin manifold. Hence, by Lemma 4.12, $F_0 \sim f$ and similarly $F_1 \sim g$. Therefore $f \sim g$. □

Remark 4.14. If in the above theorems we choose our manifold M to be an n-dimensional sphere, then we have constructed a one to one correspondence between the n^{th} homotopy group of a p dimensional sphere and cobordism classes of framed codimension p submanifolds of the n dimensional sphere. Lemma 4.11 shows a smooth map $f : S^n \to S^p$ gives a well defined element

$$[f] \in \Omega^{fr}_{n-p}(S^n).$$

Lemma 4.13 tells us that $f \simeq g$ if and only if $[f] = [g]$. In particular there is a map

$$\pi_n(S^p) \to \Omega^{fr}_{n-p}(S^n)$$

which is injective. Theorem 4.11 showed the surjectivity of such a map. Thus we have

$$\pi_n(S^p) \cong \Omega^{fr}_{n-p}(S^n).$$

The sum of elements in $\pi_n(S^p)$ corresponds to the disjoint union of suitable representatives of corresponding classes in $\Omega^{fr}_{n-p}(S^n)$, and $0 \in \pi_n(S^p)$ corresponds to $[\varnothing] \in \Omega^{fr}_{n-p}(S^n)$.

Part II

Transversal Homotopy Theory

Chapter 5

Transversal Homotopy Monoids

Up until now we have seen how continuous maps and homotopies of such, can be used to detect invariants of topological spaces. We would like to use a similar tool to probe Whitney stratified spaces. Unfortunately, continuous maps do not interact with strata in an interesting way; they do not record crossing strata as a significant event. The idea is to instead use transversal maps. As an example, recall the space $\mathbb{S}^1$ and choose an arbitrary basepoint p in the open stratum. A transversal path $[0,1] \to \mathbb{S}^1$ based at p which crosses the point stratum is not homotopic through transversal maps to a map that does not cross it. At some time, a 'slice' of this homotopy will not be transversal to the point stratum.

5.1 Transversal Homotopy Monoids

Definition 5.1. [26] Suppose W is a compact Whitney stratified manifold. Stratify $W \times [0,1]$ as a product. A *homotopy through transversal maps* from W to a Whitney stratified manifold X, is a smooth map $h : W \times [0,1] \to X$ such that

$$h_t : W \times \{t\} \to X$$

is transversal for all t. Homotopy through transversal maps is an equivalence relation on transversal maps from W to X.

Remark 5.2. The Eckmann-Hilton Argument, [8], tells us the following. Given a set X equipped with two binary operations, which we will write . and $*$, and suppose:

1. $*$ and $.$ are both unital, with the same unit e, and,
2. for all $a, b, c, d \in X$, $(a * b).(c * d) = (a.c) * (b.d)$,

then $*$ and $.$ are the same and in fact commutative and associative. The proof of the Eckmann-Hilton argument is essentially formal manipulation. We write

$$x * y = (x.e) * (e.y) = (x * e).(e * y) = x.y$$

Hence, from that formal manipulation, $*$ and $.$ are equal. Now let us prove commutativity:

$$x * y = (e.x) * (y.e) = (e * y).(x * e) = y.x = y * x$$

from what has already been proved.

Definition 5.3. [26] Let X be a Whitney stratified manifold and fix a generic basepoint $x_0 \in X$, i.e. a base point in an open stratum. For $n \in \mathbb{N}$ we define the n^{th} *transversal homotopy monoid,* $\psi_n(X)$, to be the set of equivalence classes of transversal maps $[0,1]^n \to X$ which map some neighbourhood of the boundary to the basepoint x_0, up to the equivalence relation generated by homotopy through transversal maps. We denote the class of a transversal map f by $[f]$.

For $n = 0$ the boundary condition is vacuous, we do not obtain a monoid but the set $\psi_0(X)$ which is in bijection with the set of open strata of X. For $n \geqslant 1$, juxtaposition gives $\psi_n(X)$ the structure of a monoid: $[f]\cdot[g] = [f\cdot g]$ where

$$f \cdot g : [0,1]^n \to X = \begin{cases} f(t_1, \ldots, 2t_n) & t_n \in [0, \frac{1}{2}] \\ g(t_1, \ldots, 2t_n - 1) & t_n \in [\frac{1}{2}, 1]. \end{cases}$$

Insisting that a neighbourhood of the boundary maps to x_0 ensures that the juxtaposition is smooth; it is clearly transversal. The class of the constant map to x_0 is the unit for juxtaposition. For $n \geqslant 2$ we could choose to juxtapose in any of the coordinate directions. Here we employ the Eckmann-Hilton argument described in Remark 5.2, where juxtaposing in each direction corresponds to a binary product where the common unit is the empty square. This tells us that these monoidal structures are the same and in fact commutative commutative and associative.

Remark 5.4. By the Whitney Approximation Theorem, any continuous map of manifolds $f : X \to Y$ is homotopic to a smooth one, see for example [14, Page 138, Theorem 6.15]. Therefore, if X is trivially stratified then

$$\psi_n(X) \cong \pi_n(X).$$

Definition 5.5. [26] A transversal map $f : [0,1]^n \to X$ determines an induced stratication of $[0,1]^n$, which we will call *the pullback stratification* or the *induced stratification*. The strata are of the form $f^{-1}(S)$ for a stratum $S \subset X$, (or are the boundary strata of $[0,1]^n$). Each stratum A, of this induced stratification maps to a stratum in X, which we denote by S_A, so that f is weakly stratified with respect to this stratification. The restriction of f to A determines a homotopy class $[f|_A] \in [A, S_A]$. It follows from Proposition 5.7 that this class only depends on the class of f in ψ_n. Furthermore, using the fact that f is transversal to S, the derivative of f induces an isomorphism of bundles

$$(f|_A)^* NS_A \cong NA$$

where NS_A is the normal bundle of the stratum S_A in X. Up to isomorphism the pullback bundle on the left depends only on $[f|_A] \in [A, S_A]$.

Lemma 5.6 (Thom's First Isotopy Lemma)**.** Suppose X is a Whitney stratified subset of a manifold M and $f : M \to \mathbb{R}^n$ a smooth map whose restriction to X is proper and a stratified submersion. Then there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & (f^{-1}(0) \cap X) \times \mathbb{R}^n \\ & \searrow^{f} & \downarrow \pi \\ & & \mathbb{R}^n \end{array}$$

in which h is a stratified homeomorphism (i.e. a continuous stratified map with continuous stratified inverse) whose restriction to each stratum is smooth.

Proposition 5.7. [26] Suppose $h : W \times [0,1] \to X$ is a homotopy through transversal maps. Then there is a (continuous) ambient isotopy of $W \times [0,1]$ starting at the identity and ending at a stratum-preserving homeomorphism from the stratification induced by h to the product of the stratification on W induced from $h_0 : W \to X$ and the trivial stratification on $[0,1]$.

This is a restatement of Thom's First Isotopy Lemma. For details see, [26, Proposition 2.13, Remark 2.14, Proposition 2.15].

Remark 5.8. In special cases we obtain a smooth isotopy. For instance this is so when the stratication of X has strata $X_i - X_{i-1}$ for $i = 0, \cdots, k$ where $\varnothing = X_{-1} \subset X_1 \subset \cdots \subset X_k = X$ is a filtration by closed submanifolds. For details we again point the reader to [26, Proposition 2.13, Remark 2.14, Proposition 2.15].

A consequence of Proposition 5.7 is that, for any stratum A of W and B of X, the class of the map h_t in the set of homotopy classes $[A \cap f^{-1}B, B]$ is independent of t.

Lemma 5.9. [26] An element $[f] \in \psi_n(X)$ where $n \geqslant 1$ is invertible if and only if the stratification induced by f is trivial.

Proof. It follows from Proposition 5.7 that if the induced stratification of a transversal map $f : [0,1]^n \to X$ is not trivial then the induced stratification of any other representative is non-trivial too. Hence the condition is invariant under homotopies through transversal maps. Furthermore if the stratification induced by f is non-trivial then so is that induced by any composite $f \cdot g$ and so $[f]$ cannot be invertible. Conversely if the stratification induced by f is trivial then f maps $[0,1]^n$ into the open stratum containing the basepoint and the usual inverse of homotopy theory provides an inverse in $\psi_n(X)$. □

So in general $\psi_n(X)$ is not a group but a dagger monoid. The involution on $\psi_n(X)$ for $n > 0$ is given by $[f]^\dagger = [f^\dagger]$ where

$$f^\dagger : [0,1]^n \to X : (t_1, \cdots, t_n) \mapsto f(t_1, \cdots, t_{n-1}, 1 - t_n).$$

A stratified normal submersion $g : X \to Y$ induces a well-defined map

$$g^* : \psi_n(X) \to \psi_n(Y) : [f] \mapsto [g \circ f]$$

For $n > 0$ it is a map of dagger monoids. Recall, that since g is a stratified normal submersion, $g \circ f$ is transversal.

Lemma 5.10. [26] The constructions ψ_n give rise to functors from the category of Whitney Stratified manifolds and stratified normal submersions to the category of dagger monoids. We will abuse notation by calling the functors ψ_n also.

Proof. From what has already been shown, it is clear that our functor ψ_n takes the empty space to the trivial monoid, it is also clear that a composite of stratified normal submersions gets sent to a composite of corresponding maps of monoids. That is given Whitney stratified manifolds X, Y and Z and stratified normal submersions f and g such that

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

then $\psi_n g \circ \psi_n f = \psi_n(g \circ f)$. □

There is a natural map $\psi_n(X) \to \pi_n(X)$ obtained by simply forgetting the stratification of X. There is also the map $\psi_n(X) \to Q\psi_n(X)$, where the target is the group obtained by quotienting the monoid by its minimal well-balanced submonoid. Here duals get sent to inverses. The first map factors through the second but it is *not* the case that

$$\pi_n(X) \cong Q\psi_n(X)$$

in general.

For example, take $\psi_2(\mathbb{S}^1)$, which by Proposition 5.12, we will see can be represented by isotopy classes of framed, possibly nested, circles in the interior of $[0,1]^2$. If we allow cancelation of adjacent circles, (nested or side by side), with opposite framing, i.e. circles that are dual to each other, we obtain the group

$$Q\psi_2(\mathbb{S}^1) \ncong \pi_2(\mathbb{S}^1) \cong 0,$$

since $Q\psi_2(\mathbb{S}^1)$ has non-trivial classes, for instance the class represented by a single framed circle. In fact $Q\psi_2(\mathbb{S}^1)$ is not even finitely generated, as it has generators, n nested framed circles, for all $n \in \mathbb{Z}$.

5.2 Transversal Homotopy Monoids of Spheres

The original motivation for the Pontryagin-Thom construction, part of which we will now use, was to give a geometric interpretation of the homotopy groups of spheres. The following general result about the transversal homotopy monoids of spheres gives us a geometric interpretation of what have so far been described as purely algebraic objects.

Definition 5.11. let N and M be manifolds and g and h be embeddings of N in M. A continuous map

$$F : M \times [0,1] \to M$$

is defined to be an *ambient isotopy* taking g to h if F_0 is the identity map, each map F_t is a homeomorphism from M to itself, and $F_1 \circ g = h$.

Proposition 5.12. [26] There is an isomorphism,

$$\psi_k(\mathbb{S}^n) \cong \left\{ \begin{array}{c} \text{Isotopy classes of framed} \\ \text{codimension } n \text{ submanifolds of } S^k \end{array} \right\}.$$

Proof. Any transversal $S^k \xrightarrow{f} \mathbb{S}^n$ gives a framed submanifold $f^{-1}(p) \subset S^k$. If f is homotopic to g through transversal maps then $f^{-1}(p)$ is ambient isotopic to

$g^{-1}(p)$ (we will show this in detail in Proposition 5.13). We therefore have a well-defined map

$$\psi_k(\mathbb{S}^n) \to \left\{ \begin{array}{c} \text{Isotopy classes of framed} \\ \text{codimension } n \text{ submanifolds of } S^k \end{array} \right\}.$$

We now use the notion of a collapse map, taken from the Pontryagin-Thom construction, to show this map is surjective. Given a framed codimension n submanifold A of $[0,1]^k$ and its tubular neighbourhood U, we have

$$U \cong NA \cong A \times \mathbb{R}^k.$$

Define a collapse map on A by mapping all of A to the deepest stratum and on U by extending to a map

$$\begin{array}{ccccc} A & \hookrightarrow & U & \xrightarrow{\cong} & A \times \mathbb{R}^k \\ \downarrow & & \downarrow & & \downarrow \\ p \in \mathbb{S}^n & \hookrightarrow & \mathbb{S}^n & \hookleftarrow & N_s \cong \mathbb{R}^k \end{array}$$

Extension to the whole of $[0,1]^k$ is defined by sending the rest to the base-point and then smoothing as in Theorem 4.11, in the classical Pontryagin-Thom construction.

Finally, suppose $f^{-1}(p)$ and $g^{-1}(p)$ are ambient isotopic in S^k i.e. there exists

$$h : S^k \times [0,1] \to S^k$$

with $h_0 = id$ and h_1 is a diffeomorphism such that $h_1^{-1}(f^{-1}(p)) = g^{-1}(p)$. Then

$$S^k \times [0,1] \xrightarrow{h} S^k \xrightarrow{f} \mathbb{S}^n$$

so $h_1 \circ f$ and g have the same Pontryagin manifold and so by Lemma 4.12,

$$h_1 \circ f \simeq g.$$

We can check that the explicit construction of the homotopy in the proof of Lemma 4.12 gives a homotopy through transversal maps. □

We will now list some examples:

- The 0th transversal homotopy monoid of any Whitney stratified space is simply the set of open strata and so $\psi_0(\mathbb{S}^n) = 1$, i.e. the one element set, for all $n \in \mathbb{N}$.

- The 1st transversal homotopy monoid of the stratified circle is generated by a loop a crossing the 0-stratum and the dual a^* of such a loop, i.e. the loop traversed in the opposite direction. This gives us

$$\psi_1(\mathbb{S}^1) = \langle a, a^\dagger \rangle,$$

the free dagger monoid on one generator (equivalently the free monoid on two generators). By Proposition 5.12 we can also interpret $\psi_1(\mathbb{S}^1)$ as the set of isotopy classes of framed codimension 1 submanifolds of S^1. The generator a corresponds to the class of the map $a : [0,1] \to \mathbb{S}^1 : t \mapsto e^{2\pi i t}$.

- Since any based loop is contractible on the n-sphere for $n > 1$ and cannot intersect the 0-stratum without failing to be transversal, $\psi_1(\mathbb{S}^n) = 1$ for $n > 1$.

- If we look at the pullback stratification of I^2 given by a transversal map into $\mathbb{S}^1$ we get isotopy classes of (possibly nested) framed circles in the interior of I^2, with framing pulled back from the framing on the 0-stratum of the sphere. The monoid structure on $\psi_2(\mathbb{S}^1)$ is then given by stacking squares and reparametrising.

- The 2nd transversal homotopy monoid of $\mathbb{S}^2$ is given by transversal homotopy classes of based transversal maps $S^2 \to \mathbb{S}^2$ and so

$$\psi_2(\mathbb{S}^2) = \{\text{framed points in the interior of the square}\},$$

up to isotopy. We can represent the monoid structure by sticking squares together and then reparametrising. There are no points on the boundary of the square so in fact we can stick them together in both directions, i.e. top-to-bottom and side-by-side, but again the usual Eckmann-Hilton in Remark 5.2 argument can be used to show these structures are the same and so we have a commutative monoid.

- Again we have a trivial monoid for $\psi_2(\mathbb{S}^3)$ and in fact, as in ordinary homotopy, if $i < n$,

$$\psi_i(\mathbb{S}^n) = 1.$$

5.3 Transversal Homotopy Monoids of Complex Projective Space

In this section we will again use a variant of the Pontryagin-Thom construction to give a geometric description of $\psi_n(\mathbb{CP}^k)$, the nth transversal homotopy monoid of k-dimensional complex projective space. $\mathbb{CP}^k$ has a natural stratification, induced from the filtration $\mathbb{CP}^0 \subset \mathbb{CP}^1 \subset \ldots \subset \mathbb{CP}^k$. Also, it generalises S^2 since $\mathbb{CP}^1 \cong S^2$. To give this geometric description we answer the question: "What are the necessary and sufficient conditions for a stratification of S^n to be the pullback stratification of a transversal map $f : S^n \to \mathbb{CP}^k$?"

A class in $\psi_n(\mathbb{CP}^k)$ is represented by a transversal map $f : S^n \to \mathbb{CP}^k$. The pre-image stratification arises from a filtration of the form $X_0 \subset \cdots \subset X_k = S^n$ where

$$X_i = f^{-1}(\mathbb{CP}^i)$$

and the strata are the differences $X_i - X_{i+1}$.

What are the necessary conditions on such a filtration to arise this way? We know $f^{-1}(\mathbb{CP}^{i-1})$ must be a manifold of (real) codimension 2 inside $f^{-1}(\mathbb{CP}^i)$ from Theorem 2.13, where we take the submanifold to be $\mathbb{CP}^{i-1} \subset \mathbb{CP}^i$.

Also, since the normal bundle to $\mathbb{CP}^i$ inside $\mathbb{CP}^{i+1}$ is a complex line bundle, we can classify it (up to isomorphism) by its first Chern class. For an oriented submanifold $A^i \subset M^n$ we write $[A]$ for the Poincaré dual class to A in $H^{n-i}(M;\mathbb{Z})$.

Using naturality to pull back the Chern class along f, and then the fact that f is a transversal map we have,

$$\begin{aligned} f^* c_1(N_{\mathbb{CP}^{i+1}}\mathbb{CP}^i) &= c_1(f^* N_{\mathbb{CP}^{i+1}}\mathbb{CP}^i) \\ &= c_1(N_{f^{-1}\mathbb{CP}^{i+1}} f^{-1}\mathbb{CP}^i) \\ &= c_1(N_{X_{i+1}} X_i). \end{aligned}$$

But it is well-known that $c_1(N_{\mathbb{CP}^{i+1}}\mathbb{CP}^i) = [\mathbb{CP}^{i-1}] \in H^2(\mathbb{CP}^i;\mathbb{Z})$ so,

$$\begin{aligned} f^* c_1(N_{\mathbb{CP}^{i+1}}\mathbb{CP}^i) &= f^*[\mathbb{CP}^{i-1}] \\ &= [f^{-1}\mathbb{CP}^{i-1}] \\ &= [X_{i-1}] \in H^2(f^{-1}\mathbb{CP}^i;\mathbb{Z}) = H^2(X_i;\mathbb{Z}) \end{aligned}$$

where again we have used the fact that f is transversal to identify $f^*[\mathbb{CP}^{i-1}]$ with $[f^{-1}\mathbb{CP}^{i-1}]$. So we see that for each X_i it is necessary that $c_1(N_{X_{i+1}} X_i) = [X_{i-1}] \in H^2(X_i;\mathbb{Z})$ and that X_i must be of (real) codimension 2 inside X_{i+1}.

Lemma 5.13. If two maps $S^n \to \mathbb{CP}^k$ are homotopic through transversal maps then the induced filtrations of S^n are ambiently isotopic.

Proof. Let $h : S^n \times [0,1] \to \mathbb{CP}^k$ be such a homotopy, then the Transversality Theorem (Theorem 2.17) tells us that projecting onto the second factor

$$S^n \times [0,1] \xrightarrow{\pi_2} [0,1]$$

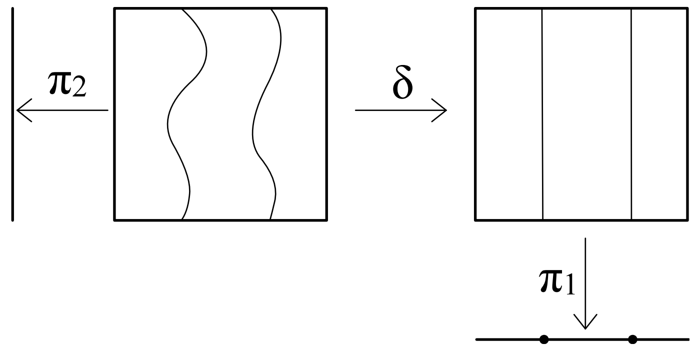


Figure 5.1: Here the square represents $S^n \times [0,1]$. By the Transversality Theorem the projection map π_2 is a stratified submersion. We can then implement our extension of Thom's First Isotopy Lemma to obtain a diffeomorphism δ with which we can compose the projection map π_1 to produce an ambient isotopy, i.e. a map $S^n \times [0,1] \to S^n$ that is a diffeomorphism on each slice, $S^n \times t$.

is a stratified submersion, where $S^n \times [0,1]$ has the preimage stratification from h. So by the extension of Thom's First Isotopy Lemma (Proposition 5.7) there exists a level preserving diffeomorphism

$$S^n \times [0,1] \xrightarrow{\delta} S^n \times [0,1]$$

from the stratification induced from h to the product of the stratification induced from $h_0 = f$ on S^n and the standard one on $[0,1]$. Composing with the projection map onto the first factor we obtain the desired ambient isotopy

$$\pi_1 \circ \delta,$$

see Figure 5.1. □

For the remainder of this section we will use the fact that the first Chern class c_1, of a complex line bundle over a manifold can be identified with the Euler class e of its underlying oriented real bundle.

Proposition 5.14. Suppose we have a filtration by submanifolds

$$X_0 \subset \cdots \subset X_k = S^n,$$

such that each X_j has codimension $2k-2j$ and $e(N_{X_{j+1}}X_j) = [X_{j-1}] \in H^2(X_j;\mathbb{Z})$. Then there exists a transversal map $f : S^n \to \mathbb{CP}^k$ inducing the filtration.

Proof. We will use an inductive version of the collapse map from the Pontryagin-Thom construction.

Let $k = 0$ and consider the map $f|_{X_0} : X_0 \to \mathbb{CP}^0$ sending all of X_0 to $\mathbb{CP}^0$, i.e a point. It is clearly well-defined and transversal as a map to $\mathbb{CP}^0$.

Now suppose $f|_{X_i} : X_i \to \mathbb{CP}^i$ is defined and transversal, with the property that the pullback stratification is obtained from the filtration

$$X_0 \subset \cdots \subset X_i.$$

By assumption, X_j is a codimension $2i - 2j$ submanifold and $e(N_{X_{i+1}} X_i) = [X_{i-1}] \in H^2(X_i; \mathbb{Z})$. Then we can extend to a transversal map

$$f|_{X_{i+1}} : X_{i+1} \to \mathbb{CP}^{i+1}$$

as follows.

We define the collapse map on X_{i+1} by first choosing a tubular neighbourhood U_{X_i} of X_i inside X_{i+1} i.e an isomorphism,

$$U_{X_i} \cong N_{X_{i+1}} X_i.$$

Next since

$$\begin{aligned} e(N_{X_{i+1}} X_i) &= [X_{i-1}] \\ &= [f^{-1}\mathbb{CP}^{i-1}] \\ &= f^*[\mathbb{CP}^{i-1}] \in H^2(\mathbb{CP}^i; \mathbb{Z}) \\ &= f^* e(N_{\mathbb{CP}^{i+1}} \mathbb{CP}^i) \\ &= e(f^* N_{\mathbb{CP}^{i+1}} \mathbb{CP}^i) \end{aligned}$$

we can choose an isomorphism,

$$N_{X_{i+1}} X_i \cong f^* N_{\mathbb{CP}^{i+1}} \mathbb{CP}^i.$$

We can now extend $f|_{X_i}$ to an isomorphism $U_{X_i} \to N_{\mathbb{CP}^{i+1}} \mathbb{C}_i$, by defining

$$f|_{U_{X_i}} : (x, w) \mapsto (f(x), w)$$

where w is in the fibre of $f^* N_{\mathbb{CP}^{i+1}} \mathbb{CP}^i$ at x, which by the definition of a pullback bundle is canonically the same as the fibre of $N_{\mathbb{CP}^{i+1}} \mathbb{CP}^i$ at $f(x)$. From this we can complete

$$U_{X_i} \cong N_{X_{i+1}} X_i \cong f^* N_{\mathbb{CP}^{i+1}} \mathbb{CP}^i \xrightarrow{f|_{U_{X_i}}} N_{\mathbb{CP}^{i+1}} \mathbb{CP}^i \cong U_{\mathbb{CP}^i} \subset \mathbb{CP}^{i+1}$$

where $U_{\mathbb{CP}^i}$ is a chosen tubular neighbourhood in $\mathbb{CP}^i$. Finally to extend to the rest of X_{i+1}, let $D_{X_i} \subset U_{X_i}$ be the image of a closed disk bundle inside $N_{X_{i+1}}X_i$. Consider $f|_{\partial D_{X_i}} : \partial D_{X_i} \to \mathbb{CP}^{i+1}\backslash\mathbb{CP}^i \cong \mathbb{C}^{i+1}$. Since $\mathbb{C}^{i+1}$ is contractible we can extend this to a continuous map

$$f : X_{i+1}\backslash \mathrm{int} D_{X_i} \to \mathbb{CP}^{i+1}\backslash\mathbb{CP}^i.$$

Moreover, such extensions are unique up to homotopy. This map is smooth in a neighbourhood of X_i and continuous outside so by the Whitney Approximation Theorem it is homotopic relative to U_{X_i} to a smooth map. □

Remark 5.15. It was necessary to make some choices in the construction. We needed to choose a tubular neighbourhood of X_i inside X_{i+1} but any two choices of tubular neighbourhood are connected by a family of tubular neighbourhoods so our choice is unique up to homotopy. We made a similar choice of tubular neighbourhood of $\mathbb{CP}^i$ in $\mathbb{CP}^{i+1}$ but again this choice was unique up to homotopy using the same reasoning. We needed to choose an isomorphism $N_{X_i+1}X_i \cong f^*N_{\mathbb{CP}^{i+1}}\mathbb{CP}^i$ but so long as we fix the homotopy class, the resulting extension is unique up to homotopy through transversal maps. We will refer to these choices as the extension data.

In summary, at each step of the induction the extension to $f|_{X_i}$ is unique up to homotopy through transversal maps, i.e. there is a homotopy

$$h : X_i \times [0,1] \to \mathbb{CP}^i$$

inducing the same filtration on X_i at each h_t.

Proposition 5.16. A transversal map $f : S^n \to \mathbb{CP}^k$ inducing a given filtration is unique up to homotopy through transversal maps.

Proof. Given a homotopy $g_i : X_i \times [0,1] \to \mathbb{CP}^i$ through transversal maps such that g_t induces the same stratification for all t, and homotopic choices of extension data for $g_i(-,0)$ and $g_i(-,1)$, we construct extension data for the homotopy, and then use the same argument as in Proposition 5.14 to extend the homotopy to $g_{i+1} : X_{i+1} \times [0,1] \to \mathbb{CP}^{i+1}$, a homotopy between respective extensions of $g_i(-,0)$ and $g_i(-,1)$.

To extend g we need to make a choice of tubular neighbourhood of $X_i \times [0,1]$ in $X_{i+1} \times [0,1]$ and we need to choose an isomorphism

$$N_{X_{i+1}\times[0,1]}(X_i \times [0,1]) \cong g^*N_{\mathbb{CP}^{i+1}}\mathbb{CP}^i.$$

We now explain how to obtain these data. Given two choices of tubular neighbourhoods of X_i in X_{i+1} and a homotopy

$$\tau : N_{X_{i+1}}X_i \times [0,1] \to X_{i+1}$$

connecting them, we can construct a tubular neighbourhood to $X_{i+1} \times [0,1]$, i.e an embedding

$$N_{X_{i+1}\times[0,1]}(X_i \times [0,1]) \cong N_{X_{i+1}}X_i \times [0,1] \to X_{i+1} \times [0,1],$$

$$(a,t) \mapsto (\tau(a,t),t).$$

Similarly, given two choices of isomorphism $N_{X_{i+1}}X_i \cong f^* N_{\mathbb{CP}^{i+1}}\mathbb{CP}^i$ connected by a family of isomorphisms

$$N_{X_{i+1}}X \times [0,1] \to f^* N_{\mathbb{CP}^{i+1}}\mathbb{CP}^i$$

$$(a,t) \mapsto \phi(a,t)$$

we can extend to an isomorphism

$$N_{X_{i+1}}X \times [0,1] \to f^* N_{\mathbb{CP}^{i+1}}\mathbb{CP}^i \times [0,1]. \tag{5.1}$$

$$(a,t) \mapsto (\phi(a,t),t).$$

Now, consider the homotopy

$$(X_i \times [0,1]) \times [0,1] \to \mathbb{CP}^i \times [0,1]$$

$$((x,t),s) \mapsto g(x,st)$$

which is $g_0 = f \circ \pi_1$ when $s = 0$ and g when $s = 1$. This homotopy induces an isomorphism of bundles $f^* N_{\mathbb{CP}^{i+1}}\mathbb{CP}^i \times [0,1] \cong g^* N_{\mathbb{CP}^{i+1}}\mathbb{CP}^i$, [5, Theorem 6.8, page 57]. Combining this with (5.1) we have the required isomorphism $N_{X_{i+1}\times[0,1]}(X_i \times [0,1]) \cong g^* N_{\mathbb{CP}^{i+1}}\mathbb{CP}^i$.

As in Proposition 5.14 we can use the extension data for g_i, constructed above, to show that we can extend to a homotopy through transversal maps connecting extensions of $g_i(-,0)$ and $g_i(-,1)$. Proceeding inductively we have shown that $f : S^n \to \mathbb{CP}^k$ is unique up to homotopy through transversal maps. □

Proposition 5.17. There is a bijection

$$\left\{ \begin{array}{c} \text{Transversal homotopy classes} \\ \text{of transversal maps} \\ S^n \to \mathbb{CP}^k \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Isotopy classes} \\ \text{of stratifications of the} \\ \text{right kind} \end{array} \right\}$$

where by ‘right kind’ we mean stratifications satisfying the conditions on codimension and Euler class described in Proposition 5.14.

Proof. We have so far shown that given a homotopy class of transversal maps $S^n \to \mathbb{CP}^k$, there is an isotopy class (with conditions) of stratifications of S^n. Also if given a fixed filtration we can produce a homotopy class of transversal maps. What is left is to show that if we vary such a filtration by ambient isotopy we get a homotopy through transversal maps of corresponding collapse maps.

Given an isotopy $X \times [0,1] \xrightarrow{i} X$ where the stratification on $X \times [0,1]$ varies with t and the stratification on X is fixed, with i_t being a stratified diffeomorphism for all t and i_0 being the identity, consider

$$X \times [0,1] \to X \times [0,1],$$

$$(x,t) \mapsto (i(x,t),t).$$

Applying the inductive argument from Proposition 5.14 to the right hand side which is $X \times [0,1]$ with the product stratification, gives a homotopy through transversal maps between a collapse map for the stratification of $X \times \{0\}$ (on the left hand side) and $X \times \{1\}$ (on the left hand side). □

Proposition 5.18. If we have two filtrations of S^n as in Proposition 5.14, then any two collapse maps for these are homotopic through transversal maps.

Proof. Consider two filtrations of the right kind, and an isotopy

$$S^n \times [0,1] \xrightarrow{\phi} S^n$$

with $\phi(-,0) = \mathbf{id}$ and $\phi(-,t)$ a diffeomorphism for all t. We take the pullback filtration on $S^n \times [0,1]$ and construct a collapse map for it. This gives a homotopy between collapse maps for filtrations of $S^n \times 0$ and $S^n \times 1$. But any two collapse maps for the filtration of $S^n \times 0$ (respectively $S^n \times 1$) are homotopic by Proposition 5.16. □

Finally we show that the choice of homotopy class of the isomorphism $N_{X_i+1}X_i \cong f^*N_{\mathbb{CP}^{i+1}}\mathbb{CP}^i$ is equivalent to choosing a framing of $X_i \times \mathbb{R}^2$. This is analogous to the choice of framing required in the Pontryagin–Thom construction.

Lemma 5.19. The set of isomorphisms (up to homotopy)

$$N_{X_i+1}X_i \cong f^*N_{\mathbb{CP}^{i+1}}\mathbb{CP}^i$$

is (non-canonically) the set of framings of $X_i \times \mathbb{R}^2$, i.e. bundle isomorphisms of $X_i \times \mathbb{R}^2$.

Proof. Choose one isomorphism $\phi : N_{X_i+1}X_i \to f^*N_{\mathbb{CP}^{i+1}}\mathbb{CP}^i$ and let L be a complex line bundle on X_i, where $c_1(L) = -c_1(f^*N_{\mathbb{CP}^{i+1}}\mathbb{CP}^i)$. There are bijections,

$$
\begin{array}{ccc}
\theta & \in & \mathrm{Isom}(N_{X_i+1}X_i, f^*N_{\mathbb{CP}^{i+1}}\mathbb{CP}^i) \\
\big\downarrow & & \Big\Updownarrow \cong \\
\theta \circ \phi^{-1} & \in & \mathrm{Isom}(f^*N_{\mathbb{CP}^{i+1}}\mathbb{CP}^i, N_{X_i+1}X_i) \\
 & & \Big\Updownarrow \\
 & & \mathrm{Isom}(f^*N_{\mathbb{CP}^{i+1}}\mathbb{CP}^i \otimes L, f^*N_{\mathbb{CP}^{i+1}}\mathbb{CP}^i \otimes L) \\
 & & \Big\Updownarrow \\
 & & \mathrm{Isom}(X_i \times \mathbb{C}, X_i \times \mathbb{C})
\end{array}
$$

This identification respects homotopies. □

Chapter 6

Transversal Homotopy Categories

There is a well known categorical extension of the fundamental group of a space to the fundamental groupoid. We will develop an analogous extension to transversal homotopy theory.

Definition 6.1. [6] Given a topological space X we associate to it a category, $\Pi_1(X)$, the *fundamental groupoid of* X, whose set of objects is X. Morphisms are paths in X up to homotopy relative to their endpoints. Composition of morphisms is defined via concatenation of paths. The inverse of a morphism, represented by a path, is the homotopy class of the 'reverse' path.

In this chapter we describe a similar extension of transversal homotopy monoids.

6.1 Transversal Homotopy Categories

Definition 6.2. Given a based Whitney stratified manifold $(M, \star)$ with basepoint in the open stratum, the *nth transversal homotopy category*, $\Psi_{n,n+1}(M)$, is the category whose objects are based transversal maps $(I^n, \partial I^n) \to (X, \star)$ and whose morphisms are represented by based transversal maps

$$f : (I^n \times [0,1], \partial I^n \times [0,1]) \to (M, \star)$$

such that

$$f(p,t) = \begin{cases} f(p,0) & t \in [0,\epsilon] \\ f(p,1) & t \in [1-\epsilon,1] \end{cases}$$

for some $\epsilon > 0$. Two morphisms are equivalent if there is a homotopy through transversal maps between them.

Composition is given by $[f] \circ [g] = [f \cdot g]$ where

$$f \cdot g(p,t) = \begin{cases} f(p, 2t) & t \in [0, \frac{1}{2}] \\ g(p, 2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

If $s : X \to Y$ is a stratified normal submersion between Whitney stratified manifolds then there is a functor

$$\Psi_{n,n+1}(s) : \Psi_{n,n+1}(X) \to \Psi_{n,n+1}(Y),$$

given by post composition, i.e. $f \mapsto s \circ f$ where $f : S^n \to X$. If s and s' are homotopic through transversal maps then there is a natural isomorphism $\Psi_{n,n+1}(s) \cong \Psi_{n,n+1}(s')$.

6.2 Transversal Homotopy Categories of Spheres

Definition 6.3. The category $Tang^{fr}$, of framed tangles has objects, framed codimension 2 submanifolds in the interior of $[0,1]^2$, i.e framed points in the square. The morphisms are then isotopy classes of closed, compact, framed, codimension 2 submanifolds of $[0,1]^3$ with possible boundary in $[0,1]^2 \times \{0,1\}$ i.e framed cobordisms between objects. We call these morphisms tangles. They must also be of the form $X \times [0,\epsilon)$ and $Y \times (1-\epsilon, 1]$ in a neighbourhood of $[0,1]^2 \times \{0\}, \{1\}$, where X and Y are objects. Composition is defined by stacking cubes and reparametrising. We see that the above boundary conditions mean that composition is smooth.

There are two monoidal structures, given by glueing squares in either direction and reparametrising, where the unit object for both is the empty square. These structures are both equivalent by the usual Eckmann-Hilton argument, see [13, §1.4] and [12] for details of how Eckmann-Hilton applies to categories with two monoidal structures. Similarly, since submanifolds can only have boundary in the 'top' and 'bottom' of the cube the two monoid structures are given by sticking cubes together in the remaining two directions. The identity object is the empty tangle.

The dual X^* of an object X is given by a square with the same points in the interior but with opposite framings. The morphisms $1 \to X \otimes X^*$ and $X \otimes X^* \to 1$ are represented by cubes with framed 'cups' and 'caps' connecting corresponding, oppositely framed points. The framing on the cups and caps matches the framing of the points at their endpoints.

Morphisms have dagger duals given by reversing the direction of the tangle, i.e. reparametrising by $(x, y, z) \mapsto (x, y, 1 - z)$. It should be clear that when a morphism is composed with its dual the resulting morphism is the identity morphism.

We can generalise this definition as follows:

Definition 6.4. The category $nTang_k^{fr}$, of framed, codimension k tangles in dimensions n and $n+1$, has objects represented by framed, codimension k, closed submanifolds of $[0,1]^n - \partial[0,1]^n$ and morphisms by isotopy classes of codimension k, closed submanifolds of $[0,1]^{n+1}$. We insist on the same boundary condition as in Definition 6.3, i.e. the tangle must look like a product in the neighbourhood of $[0,1]^{n+1} \times \{0,1\}$.

Using Pontryagin-Thom we again discover an equivalence:

Proposition 6.5. [26] The nth transversal homotopy category of the stratified sphere is equivalent to the category of framed codimension k tangles in dimensions n and $n+1$, i.e.

$$\Psi_{n,n+1}(\mathbb{S}^k) \cong nTang_k^{fr}.$$

Sketch of Proof. We refer the reader to [26, §4.1] for full details. A transversal map $f : S^n \to \mathbb{S}^k$ induces a pullback stratification on $[0,1]^n$ by framed codimension k submanifolds. We are using the correspondence between the n-sphere and the n-cube with boundary collapsed to a point. This sets up a functor

$$\Psi_{n,n+1}(\mathbb{S}^k) \to nTang_k^{fr}.$$

The functor is well defined on morphisms by Remark 5.8.

We then use a version of the Pontryagin-Thom collapse map on each framed submanifold and bordism, to produce a functor

$$nTang_k^{fr} \to \Psi_{n,n+1}(\mathbb{S}^k).$$

We can make these choices compatibly so that the collapse map for a bordism agrees with the chosen ones for the boundaries. These functors define an equivalence. □

We can now use the previous theorem to describe the homotopy categories of the stratified sphere $\Psi_{n,n+1}(\mathbb{S}^k)$, at least for low dimension.

Choose a basepoint $q \in \mathbb{S}^k$ in the open stratum.

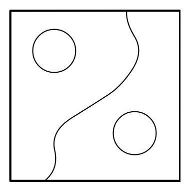
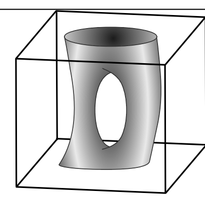
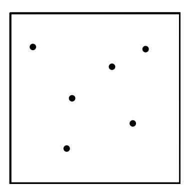
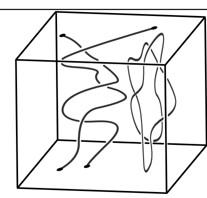
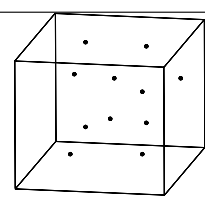
Periodic table of $\Psi_{n,n+1}(\mathbb{S}^k)$				
	$n = 0$	$n = 1$	$n = 2$	$n = 3$ ...
$k = 0$				
$k = 1$				knotting and nesting
$k = 2$				higher links and tangles
$k = 3$				no knotting

Table 6.1: A table of the geometric categories equivalent to the transversal homotopy categories, $\Psi_{n,n+1}(\mathbb{S}^k)$ via the description in Section 4.1 in [26]. Each category is indicated by a picture of a typical representative of a morphism (without framing). A morphism is an isotopy class of such. The greyed boxes indicate that the category contained therein is equivalent to the trivial category. The reader should note that the indexing in this table differs from that of the periodic table in [1].

The first example is the nth transversal homotopy category of the 0-sphere:

- Since all maps are based, $\Psi_{n,n+1}(\mathbb{S}^0)$ is the category with one object, the unique map $S^n \mapsto q$ and one morphism, the unique homotopy class of $S^n \times [0,1] \mapsto q$.

The second example is the 0th transversal homotopy category of the k-sphere:

- $\Psi_{0,1}(\mathbb{S}^k)$ is the category whose objects are based transversal maps $(S^0, \star) \to (\mathbb{S}^k, q)$ and whose morphisms are represented by based transversal maps

  $$f : (S^0, \star) \times [0,1] \to (\mathbb{S}^k, q)$$

  with boundary conditions. That is the category whose set of objects is $\mathbb{S}^k - \{q\}$ and whose morphisms are paths in $\mathbb{S}^k$ that miss the 0-stratum or they would fail to be transversal paths. So for $k \geqslant 0$,

  $$\Psi_{0,1}(\mathbb{S}^k) \cong \Pi_{0,1}(\mathbb{R}^k).$$

Next we give the some examples of transversal homotopy categories of the circle:

- The first transversal homotopy category of the circle $\Psi_{1,2}(\mathbb{S}^1)$, this is the category whose objects are based transversal maps of $S^1 \to \mathbb{S}^1$, and whose morphisms are based transversal maps $S^1 \times [0,1] \to \mathbb{S}^1$, up to homotopy through such maps, relative to the basepoint, with boundary conditions as in Definition 6.2. An object in $\Psi_{1,2}(\mathbb{S}^1)$ can be represented by an interval, whose endpoints get mapped to the basepoint, with points marked with $+/-$. These are the preimages of the zero-stratum in $\mathbb{S}^1$ and the $+/-$ labelling is pulled back from a choosen framing on the zero-straum. A morphism can be represented by a square with marked points on the top and bottom edge and 1-dimensional strata connecting them. Also since these maps need not be level-wise transversal there may also be closed codimension 1 submanifolds, i.e. circles, in the interior of the square. A neighbourhood of the vertical edges maps to the basepoint. A framing is 'pulled back' from one chosen on the zero-straum of $\mathbb{S}^1$. Two such squares are equivalent if the submanifolds within, are ambient isotopic to one another, relative to the boundary of the square.

- The second transversal homotopy category of the circle $\Psi_{2,3}(\mathbb{S}^1)$, has objects represented by squares whose edges map to the basepoint, with framed (possibly nested) circles in their interior. Morphisms are framed cobordisms in the square between two such objects, up to ambient isotopy relative to the boundary of the cube.

Moving to the 2-sphere we have:

- The first transversal homotopy category of the 2-sphere $\Psi_{1,2}(\mathbb{S}^2)$, objects of which are based loops in the 2-sphere that miss the 0-stratum. Morphisms are squares whose vertical edges map to the basepoint and have points (exclusively) in their interior with framings pulled back from the framing on the 0-stratum. Such squares are equivalent up to ambient isotopy relative to the boundary.
- The second transversal homotopy category of the 2-sphere $\Psi_{2,3}(\mathbb{S}^2)$, has objects represented by points in the square, with framing induced from the framing of the 0-stratum in $\mathbb{S}^2$. Morphisms are framed cobordisms between such objects up to ambient isotopy relative to the boundary. This is in fact a description of framed tangles in $\mathbb{S}^3$.
- Objects in $\Psi_{1,2}(\mathbb{S}^3)$ are based loops in the 3-sphere that miss the 0-stratum. Morphisms are squares with such loops as top and bottom edges. In this case the squares cannot intersect the 0-stratum so the induced stratification is trivial.

Part III

Whitney Categories

Chapter 7

Whitney n-categories

We now propose a new definition of n-category with duals. This notion and the other concepts in this thesis were jointly worked out from suggestions proposed by the first author in [23], inspired in turn by ideas of Baez and Dolan, and Morrison and Walker's definition of n-category with duals. This definition may strike the reader as somewhat different to the usual notion of an n-category but in later chapters we will show that it matches the accepted definitions, for small n at least.

7.1 A Grothendieck Topology on $Strat_n$

We remind the reader that when we talk about stratified spaces we are in fact referring to stable germs of Whitney stratified spaces, as described in Defintion 2.6.

Definition 7.1. Let $Strat_n$ be the category whose objects are compact cellular stratified spaces with strata of dimension $\leqslant n$. By cellular we mean that each stratum is homeomorphic to an open ball. The morphisms are stratified maps up to homotopy relative to all strata of dimension $< n$. In particular $Strat_\infty$ is the category of all such stratified spaces and stratified maps between them.

Definition 7.2. Let $Prestrat_n$ be the category whose objects are the same as those of $Strat_n$ but in which we allow a less restrictive choice of morphism, namely *prestratified* maps up to homotopy relative to all strata of dimension $< n$. A map is *prestratified* if it becomes stratified after a refinement of the stratification of the source, i.e. we increase the number of strata.

Definition 7.3. [16, Page 37, Chapter 1, §4] Let X be an object in some category $\mathscr{C}$. A *sieve* $\mathcal{S}$ on X is a collection of morphisms with target X that is closed under pre-composition with arbitrary morphisms, i.e. given $f : Y \to X$ with $f \in \mathcal{S}$ and any $g : Y' \to Y$ then $f \circ g$ must also be in $\mathcal{S}$. We say a sieve is generated by a morphism f if it is the set of all morphisms that factor through f.

Definition 7.4. A *local stratum L of X at x*, is a connected component of $S \cap U$, where S is a stratum of X whose closure $\overline{S}$ contains x, and U is a sufficiently small open neighbourhood of x. (By sufficiently small we mean that $L \cap V$ is connected for smaller $V \subset U$.)

Definition 7.5. A stratified map $X' \xrightarrow{f} X$, of stratified spaces, *trivially covers a local stratum L at x* if there exist open subsets $V \subset X$ with $x \in V$ and $V' \subset X'$ such that

$$(f^{-1}\overline{L}) \cap V' \to \overline{L} \cap V$$

is an isomorphism of stratified spaces (i.e. an isomorphism of stable germs of the underlying spaces).

Definition 7.6. [16, Page 70, Chapter 2, §2] A sieve $\mathcal{S} = \{X_i \xrightarrow{f_i} X\}$ is *covering* if for every $x \in X$, each local stratum L at x is trivially covered by some f_i. These covering sieves are the analogues of open covers in the defintion of a sheaf on a topological space.

Examples 7.7. (1) The sieve generated by $\mathbb{I} \to \mathbb{S}^1 : t \mapsto e^{2\pi it}$ is covering since a local stratum in S^1 is isomorphic to an open interval in the interior of $\mathbb{I}$, isomorphic to an open interval containing an endpoint of $\mathbb{I}$.

(2) The sieve generated by $\{\overline{S} \hookrightarrow X\}_{S \subset X}$ is covering since any local stratum contained in a stratum $S \subset X$ is covered by $\overline{S}$.

We will need the following:

Lemma 7.8. [23] Suppose $f : X \to Z \leftarrow Y : g$ are stratified maps. Then $X \times_Z Y$ can be stratified by the fibre products of the strata of X and Y so that

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & Y \\ \downarrow & & \downarrow g \\ X & \xrightarrow[f]{} & Z \end{array} \tag{7.1}$$

is a commuting diagram of stratified maps. Moreover, if X, Y and Z are cellular then this stratification is cellular.

Proof. Consider $X \times_Z Y = \{(x, y) \in X \times Y \ : \ f(x) = g(y)\} \subset X \times Y$. We equip this with the germ along this subset of the product of the germs of X and Y. It is decomposed into the subsets $A \times_{f(A)=g(B)} B$ where $A \subset X$ and $B \subset Y$ are strata. Each of these is a manifold because $f|_A$ and $g|_B$ are transversal. This decomposition satisfies the Whitney B condition: Suppose $(a_i, b_i) \in A \times_Z B$ and $(s_i, t_i) \in S \times_Z T$ are sequences in $X \times_Z Y$ with the same limit $(a, b) \in A \times_Z B$. The product stratification of $X \times Y$ satisfies the Whitney B condition. Hence (when the limits exist in the ambient tangent space)

$$\lim \ell_i \in \lim T_{(s_i,t_i)}(S \times T)$$

where ℓ_i is the secant line between (a_i, b_i) and (s_i, t_i). In fact since these pairs lie in the fibre product, the limiting secant line lies in the subspace

$$U = \{(v, w) \in \lim T_{(s_i,t_i)}(S \times T) \ : \ df(v) = dg(w)\}.$$

Clearly $U \supset \lim T_{(s_i,t_i)}(S \times_Z T)$; in fact they are equal. For suppose $(v_i, w_i) \in T_{(s_i,t_i)}(S \times T)$ is a sequence with limit (v, w). Then

$$df(v_i) - dg(w_i) \to df(v) - dg(w) = 0.$$

Since f is submersive onto the tangent space of $f(S) = g(T)$ we can find $v_i' \in T_{s_i}S$ with $df(v_i') = df(v_i) - dg(w_i)$ and $v_i' \to 0$. Then

$$(v_i - v_i', w_i) \in T_{(s_i,t_i)}(S \times_Z T)$$

and $(v_i - v_i', w_i) \to (v, w)$. Hence $U \subset \lim T_{(s_i,t_i)}(S \times_Z T)$ as claimed. Therefore the given decomposition of the fibre product satisfies the Whitney B condition, and the fibre product becomes a stratified space. It is easy to check that the maps in (7.1) are stratified.

Suppose that X, Y and Z are cellular. Then by considering the long exact sequences of homotopy groups induced respectively from the fibrations
$F \hookrightarrow T \to g(T)$ and $F \hookrightarrow S \times_Z T \to S$ and using the fact that S, T and $g(T)$ are all contractible, we see that $S \times_Z T$ is weakly contractible. Since it is a smooth manifold it is homotopy equivalent to a CW complex, and so by Whitehead's Theorem it is contractible. Hence $X \times_Z Y$ is cellular. □

Definition 7.9. [16, Page 110, Chapter 3, §4] A Grothendieck topology is a structure on category which makes the objects act like open sets in a topological space. This then allows us to define sheaves on categories.

Proposition 7.10. The covering sieves in Definition 7.1 define a Grothendieck topology on $Strat_n$. We then refer to $Strat_n$ as the stratified site.

Proof. We verify the axioms for a Grothendieck topology on a category $\mathscr{C}$. These are

1. Suppose $\mathcal{S} = \{X_i \xrightarrow{f_i} X\}$ is a covering sieve, and $g : Y \to X$ is any stratified map. Then the *pullback sieve*
$$g^*\mathcal{S} = \{Y_i \xrightarrow{g_i} Y | gg_i \in \mathcal{S}\}$$
is a covering sieve on Y.

2. Let S be a covering sieve on X, and let T be any sieve on X. Suppose that for each object Y of C and each arrow $f : Y \to X$ in $S(Y)$, the pullback sieve f^*T is a covering sieve on Y. Then T is a covering sieve on X.

3. The maximal sieve, $\mathrm{Hom}(-, X)$, is a covering sieve on X for any object X in $\mathscr{C}$.

To show the first axiom holds, pick $y \in Y$ a local stratum L at gy, and suppose $f_i \in \mathcal{S}$ trivially covers L at gy. We have the fibre product diagram

$$\begin{array}{ccc} Y \times_X X_i & \longrightarrow & X_i \\ {\scriptstyle g_i}\downarrow & & \downarrow{\scriptstyle f_i} \\ Y & \xrightarrow[g]{} & X \end{array}$$

and by construction $g_i \in g^*\mathcal{S}$ and g_i trivially covers any local stratum of Y at y contained within $g^{-1}L$. Since any local stratum is in $g^{-1}L$ for some L we are done.

For the second axiom, suppose $\mathcal{S} = \{X_i \xrightarrow{f_i} X\}$ is covering and $\mathcal{T} = \{Y_j \xrightarrow{g_j} X\}$ is a sieve such that $f_i^*\mathcal{T}$ is covering for X_i then we need to show $\mathcal{T}$ is covering for X. This follows because if f_i trivially covers a local stratum L at x, and $h_j \in f_i^*\mathcal{T}$ trivially covers the 'preimage' local stratum in X_i, then
$$f_i \circ h_j \in \mathcal{T}$$
trivially covers L at x.

Finally, it is clear that the maximal sieve is covering. □

Definition 7.11. [16, Page 121, Chapter 3, §4] We define $\mathscr{A}$ to be a sheaf on a site if, given a covering sieve $\mathcal{S} = \{X_i \xrightarrow{f_i} X\}$ and elements $a_{f_i} \in \mathscr{A}(X_i)$ that match in the sense that for any $g : Y \to X_i$ the induced map gives $g^*a_{f_i} = a_{f_i g}$, then there is a unique amalgamation $a \in \mathscr{A}(X)$ such that $f_i^*a = a_{f_i}$ for all i.

Remark 7.12. Since $\{\overline{S} \hookrightarrow X\}_{S \subset X}$ generates a covering sieve,

$$\mathscr{A}(X) = \lim_{S \subset X} \mathscr{A}(\overline{S}) \tag{7.2}$$

for any sheaf $\mathscr{A}$ in the topology. Here we take the limit, in the categorical sense, over the diagram containing the sets $\mathscr{A}(\overline{S})$ and the maps induced by inclusions $\overline{S} \hookrightarrow \overline{T}$ of closures of strata. We refer to this correspondence as *the gluing property.* Consider the fact that elements of $\mathscr{A}(X)$ correspond to matching families over spaces in the sieve. Any such gives a matching family over the subset of these spaces consisting of the $\overline{S}$, i.e. an element of $\lim_{S \subset X} \mathscr{A}(\overline{S})$. So there is map

$$\mathscr{A}(X) \to \lim_{S \subset X} \mathscr{A}(\overline{S}).$$

It is injective because the sieve is generated by the $\overline{S}$, if two elements restrict to the same in each $\mathscr{A}(\overline{S})$ then they restrict to the same in $\mathscr{A}(Y)$ for any Y in the sieve and hence are the same in $\mathscr{A}(X)$. To see that it is surjective we need to check that any compatible family of elements in the $\mathscr{A}(\overline{S})$ extends to a compatible family of elements in the $A(Y)$ for all Y in the sieve generated by the inclusions of the $\overline{S}$. This follows from the fact that any $Y \to X$ which factors through $\overline{S}$ and $\overline{T}$ must perforce factor through $\overline{S} \cap \overline{T}$.

While it may be true we cannot use this type of proof to show the converse, i.e. that $\mathscr{A}$ satisfying the gluing property means that it is a sheaf.

To see this consider the example in Figure 7.1. The two squares generate a covering sieve for the triangle and contain the closure of each 0 and 1 stratum but there is no stratified map (with 2 dimensional image) from the triangle (the closure of the entire space) to the square.

Proposition 7.13. Suppose $\{X_i \xrightarrow{f_i} X\}$ is covering and we have compatible continuous maps $X_i \xrightarrow{g_i} Y$ for each i. Then there is a continuous map $X \xrightarrow{g} Y$ such that $g_i = g \circ f_i$, for all i. (By compatible we mean that for any stratified map $X_i \xrightarrow{h} X_j$ with $f_j \circ h = f_i$, $g_j \circ h = g_i$.)

Proof. Let L be a local stratum at x and suppose f_i trivially covers L at x. Then we define g on $\overline{L} \cap U$ by

$$\begin{array}{ccc} f_i^{-1}\overline{L} \cap U' & \xrightarrow{g_i} & \\ \downarrow \cong & & Y \\ \overline{L} \cap U & \xrightarrow{g} & \end{array}$$

.

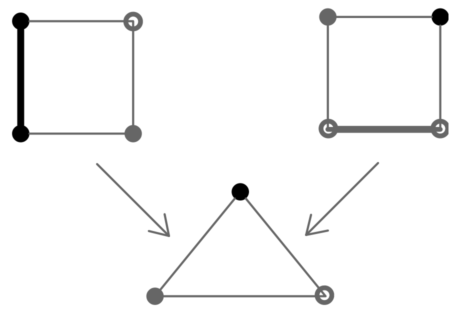
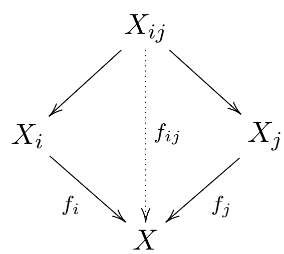


Figure 7.1: The covering sieve over the triangle generated by the two squares does not contain the closure of the 2 dimensional stratum, i.e. the triangle itself. The 0 strata are shaded to indicate where they map to, and the fattened edges indicate that they collapse onto a 0 stratum.

We claim that the existence of fibre products ensures that this is independent of the choice of f_i trivially covering L at x. Consider two local covers f_i, f_j of L at x.



By definition $(f_i^{-1}\overline{L}) \cap V_i \cong \overline{L} \cap V \cong (f_j^{-1}\overline{L}) \cap V_j$ for suitable open V_i and V_j. So the fibre product gives a trivial cover, i.e. $(f_{ij}^{-1}\overline{L}) \cap V_{ij} \cong \overline{L} \cap V$ for some open V_{ij} in X_{ij}. We can always choose this as our trivial cover, thus proving the claim.

Moreover, if L and K are local strata at x with $L \subset \overline{K}$ then in the above claim if f_j trivially covers K, it certainly trivially covers L. So for the same reason the definitions of g on $\overline{L}$ and $\overline{K}$ agree near x. It follows that g is well-defined in a neighbourhood of x. Since x is arbitrary, g is well-defined on X. □

Remark 7.14. Any local property of the g_i is inherited by g. To be more precise:

1) If all the g_i are smooth then g is smooth.

2) If all the g_i are weakly stratified then g is weakly stratified.

3) If all the g_i are stratified then g is stratified.

4) If all the g_i are prestratified then g is prestratified.

5) If all the g_i are transversal then g is transversal.

7.2 Whitney n-categories

We are now ready to define a Whitney n-category.

Definition 7.15. Fix $n \in \mathbb{N} \cup \{\infty\}$. A *Whitney n-category* $\mathscr{A}$ is a presheaf

$$Prestrat_n \to Set,$$

whose restriction to the subcategory $Strat_n$ is a sheaf. We refer to the elements of $\mathscr{A}(X)$ as the X-shaped morphisms or X-morphisms of $\mathscr{A}$. In particular we refer to the set $\mathscr{A}(pt)$ associated to the point as the objects of $\mathscr{A}$.

Recall that a category with one object is a monoid, that a bicategory with one object is a monoidal category and that a bicategory with one object and one morphism is a commutative monoid. By analogy we define the following:

Definition 7.16. A *k-tuply monoidal Whitney n-category* $\mathscr{A}$ is Whitney $(n+k)$-category where $\mathscr{A}(X) = 1$ is a one element set whenever $\dim(X) < k$.

We give three classes of examples of Whitney n-categories:

Example 7.17 (Representable Whitney Categories)**.** For any stratified space $Y \in Prestrat_n$ there is a Whitney n-category given by the presheaf

$$Rep(Y) = Prestrat_n(-, Y),$$

where $Prestrat_n(-, Y) = Hom_{Prestrat_n}(, Y)$.

Proposition 7.13 and Remark 7.14 immediately tell us that given a covering sieve $X_i \xrightarrow{f_i} X$ and compatible family of stratified maps $g_i : X_i \to Y$ there is a unique amalgamation, i.e. a prestratified map $g : X \to Y$. Thus, $Rep(Y)$ is a sheaf when restricted to $Strat_n$.

Example 7.18 (Transversal Homotopy Whitney Categories)**.** In the discussion [3], John Baez asked the following question. Can one assign to any Whitney stratified manifold a fundamental n-category with duals? In answer to this we propose the following. Let M be a Whitney stratified manifold with a basepoint

p lying in some open stratum. We associate to M the Whitney $(n+k)$-category $\Psi_{k,n+k}(M)$, where given $X \in Strat_{n+k}$,

$$\Psi_{k,n+k}(M)(X) = \left\{ \begin{array}{c} \text{Germs of transversal maps } g : X \to M \\ \text{such that whenever} \\ S \subset X \text{ and } \dim S < k, \text{ then } S \subset g^{-1}(p) \end{array} \right\} / \sim .$$

Here $\sim$ is the equivalence relation given by homotopy through transversal maps relative to all strata $S \subset X$ with $\dim S < n+k$.

Given prestratified $f : X \to Y$ we define $f^*[g] = [g \circ f]$. Then $g \circ f$ is transversal to all strata of M and $[g \circ f]$ depends only on the morphism in $Prestrat_{n+k}$ represented by f. The verification that this restricts to a sheaf on $Strat_{n+k}$ is similar to that for representable presheaves, *mutatis mutandis*.

The condition that $g(S) = p$ whenever $\dim S < k$ means that this is a k-tuply monoidal Whitney n-category.

We call $\Psi_{k,k+n}(M)$ the $(k, n+k)$-transversal homotopy Whitney category of M. This notation clashes somewhat with that for 'ordinary' transversal homotopy categories in Chapter 6, but we will resolve this later by showing the two are equivalent, at least in some cases.

These categories are functorial for sufficiently nice maps between Whitney stratified manifolds. Specifically, they are functorial for *weakly stratified normal submersions* $h : M \to N$, i.e. weakly stratified maps such that the induced mappings $N_pS \to N_{h(p)}h(S)$ of normal spaces to strata are always surjective. Whenever $h : M \to N$ is a weakly stratified normal submersion and $g : X \to M$ is transversal then the composite $h \circ g : X \to N$ is transversal. So we can define a map

$$\Psi_{k,n+k}M(X) \to \Psi_{k,n+k}N(X) : [g] \mapsto [h \circ g].$$

Since composition on the left and right commute this specifies a natural transformation of presheaves and so there is a functor $\Psi_{k,n+k}M \to \Psi_{k,n+k}N$ of Whitney categories.

Example 7.19 (The Whitney Category of Framed Tangles)**.** Given $k, n \in \mathbb{N}$, we define the Whitney $(n+k)$-category $nTang_k^{fr}$ of framed tangles, where given $X \in Strat_{n+k}$ and its ambient manifold M, we define a tangle to be represented by a codimension k framed, closed submanifold of a neighbourhood of X in M, which intersects each stratum of X transversally. Tangles are equivalent if they agree in some open neighbourhood of X. For short we will say,

$$nTang_k^{fr}(X) = \left\{ \begin{array}{c} \text{Germs } T \text{ of } M \text{ at transversal intersections of } X \\ \text{with codimension } k \text{ framed submanifolds of } M. \end{array} \right\} / \sim .$$

The equivalence relation $\sim$ is given by ambient isotopy of T in X, relative to all strata of dimension strictly less than $n+k$. A tangle T is codimension k in X with a framing given by the restriction of the framing of the codimension k submanifold of M.

Given a prestratified map $f: Y \to X$ we define

$$f^* : nTang_k^{fr}(X) \to nTang_k^{fr}(Y).$$

If T is the germ of M at $X \cap S$, where $S \subset M$ is a framed codimension k submanifold of M, then $T_f = f^*T$ is the germ of N at $Y \cap f^{-1}S$ (where N is the ambient manifold of Y).

Finally to see that $nTang_k^{fr}$ is a sheaf, fix a covering sieve $\mathcal{S} = \{X_i \xrightarrow{f_i} X\}$, a compatible family of tangles $\{T_{f_i} \in \mathscr{A}(X_i)\}$ and take $X_i \xrightarrow{f_i} X$ which trivially covers a local stratum L at $x \in X$. We define the amalgamation in $\overline{L} \cap U$ via the isomorphism with $f_i^{-1} \cap U'$. Again the existence of fibre products ensures this is well-defined, i.e. is independent of the choice of f_i trivially covering L at x, and if L and K are local strata at x with $L \subset \overline{K}$ then the amalgamation is well-defined for the same reason. (See Proposition 7.12 for details.) Here again we have used the fact that if we know what a tangle is like locally we can piece it together to get a well-defined global tangle.

7.3 Equivalence of Whitney Categories

Definition 7.20. Recall that $\mathbb{I}$ is the interval $[0,1]$ stratified by its endpoints and their complement. Given a Whitney n-category $\mathscr{A}$, fix objects $a, a' \in \mathscr{A}(pt)$. The assignment

$$X \mapsto \{\alpha \in \mathscr{A}(X \times \mathbb{I}) : \iota_0^*\alpha = p^*a,\ \iota_1^*\alpha = p^*a'\}$$

where $\iota_t : X \times t \hookrightarrow X \times \mathbb{I}$ is the inclusion and $p : X \to pt$ the map to a point, defines a Whitney $(n-1)$-category $\mathscr{A}(a, a')$ which we refer to as *the Whitney category of morphisms from a to a'*. The objects of $\mathscr{A}(a,a')(pt)$ are the elements $a \in \mathscr{A}(\mathbb{I})$ with the property that $i_0^*\alpha = a$ and $i_1^*\alpha = a'$.

See Corollary 3.7 in [23] for proof.

We model the notion of equivalence of Whitney n-categories on the symmetric version of equivalence of ordinary small categories, i.e. an equivalence of A and B is given by a span

$$A \xleftarrow{F} C \xrightarrow{G} B$$

in which the functors F and G are fully-faithful and surjective (not merely essentially surjective) on objects. We use this and the notion of a morphism category to make the inductive definition:

Definition 7.21. A 0-equivalence of Whitney 0-categories is a span

$$A \xleftarrow{F} C \xrightarrow{G} B$$

of functors such that $F(pt)$ and $G(pt)$ are bijections. For $n > 0$ an *n-equivalence* is a span

$$A \xleftarrow{F} C \xrightarrow{G} B$$

of functors which are surjective on objects, i.e. the maps

$$F(pt) : C(pt) \to A(pt) \quad \text{and} \quad G(pt) : C(pt) \to B(pt)$$

are surjective, and which induces $(n-1)$-equivalences for

$$A(Fc, Fc') \xleftarrow{F} C(c, c') \xrightarrow{G} B(Gc, Gc')$$

for each pair $c, c' \in C(pt)$.

We will see explicit examples of such an equivalence in Chapters 9 and 10.

Chapter 8

Whitney 1-Categories

If our definition of a Whitney n-category is to be a reasonable notion of an n-category with duals, then it should at least agree with some accepted definition for small n. The notion of 1-category with duals we will use is a dagger category, from [21], and for a one-object 2-category with duals we use the notion of a dagger pivotal category, see [20]. In what follows we will first show the equivalence of Whitney 1-categories with dagger categories and then in the sequel show how one-object Whitney 2-categories and dagger pivotal categories are closely related.

Recall the category $Strat_1$ whose objects are stratified spaces X of dimension $\leqslant 1$ and whose morphisms are stratified maps, of these spaces up to homotopy relative to strata of dimension < 1. Similarly recall the category $Prestrat_1$ whose objects are stratified spaces X of dimension $\leqslant 1$ and whose morphisms are prestratified maps, of these spaces up to homotopy relative to strata of dimension < 1.

The classification of spaces in $Strat_1$ and $Prestrat_1$ is relatively simple because our spaces are one dimensional. Objects are finite unions of the following connected spaces: the point, the space diffeomorphic to the interval stratified by its endpoints and a finite number of point strata in the interior, and the circle with at least one point stratum. The circle can be understood as an interval with endpoints identified. For convenience we will sometimes use

$$pt = \text{a point},\ \mathbb{I} = [0,1],\ \mathbb{I}_2 = [0,2],\ \cdots$$

to represent objects in $Strat_1$. Here each interval is stratified by the integer points and the connected components of their complement.

Remark 8.1. In $Strat_1$ and $Prestrat_1$ complicated spaces can be defined by glueing together simpler spaces such as points and intervals on their endpoints so, for example, $\mathbb{I}_2$ is two copies of $\mathbb{I}$ glued together at a boundary point. The gluing property from Remark 7.12 ensures that given a Whitney 1-category $\mathscr{A}$,

$$\mathscr{A}(\mathbb{I}_2) \cong \mathscr{A}(\mathbb{I}) \times_{\mathscr{A}(pt)} \mathscr{A}(\mathbb{I}),$$

$$\mathscr{A}(\mathbb{I}_3) \cong \mathscr{A}(\mathbb{I}) \times_{\mathscr{A}(pt)} \mathscr{A}(\mathbb{I}) \times_{\mathscr{A}(pt)} \mathscr{A}(\mathbb{I}),$$

and so on, where

$$\mathscr{A}(\mathbb{I}) \times_{\mathscr{A}(pt)} \mathscr{A}(\mathbb{I}) = \lim\left(\mathscr{A}(\mathbb{I}) \to \mathscr{A}(pt) \leftarrow \mathscr{A}(\mathbb{I}).\right)$$

Here we are taking the categorical limit over the diagram on the right.

More generally $\mathscr{A}(X)$ is always a limit over an analogous diagram with one copy of $\mathscr{A}(pt)$ for each 0-dimensional stratum and one copy of $\mathscr{A}(\mathbb{I})$ for each 1-dimensional stratum. In particular $\mathscr{A}(X)$ is completely determined by $\mathscr{A}(pt)$, $\mathscr{A}(\mathbb{I})$ and the maps

$$\mathscr{A}(\mathbb{I}) \underset{i^*}{\overset{h^*}{\rightrightarrows}} \mathscr{A}(pt)$$

induced from the inclusions $\{0\} \overset{i}{\hookrightarrow} \mathbb{I} \overset{h}{\hookleftarrow} \{1\}$.

8.1 Constructing a Dagger Category from a Whitney 1-Category

Recall, a Whitney 1-category is a presheaf $\mathscr{A} : Prestrat_1 \to Sets$ which restricts to a sheaf on $Strat_1$.

Definition 8.2. We fix a Whitney 1-category $\mathscr{A}$, from which we define a dagger category $C(\mathscr{A})$ with

$$Obj_{C(\mathscr{A})} = \mathscr{A}(pt) \quad \text{and}$$

$$Mor_{C(\mathscr{A})} = \mathscr{A}(\mathbb{I}).$$

- Since $\mathscr{A}$ is a Whitney 1-category, that is a contravariant functor, the inclusion map $i : \{0\} \to \mathbb{I}$ induces a map $i^* : \mathscr{A}(\mathbb{I}) \to \mathscr{A}(pt)$, we define the source map $\mathbf{s} := i^*$.
- Similarly take $h : \{1\} \to \mathbb{I}$ and define the target map $\mathbf{t} := h^*$. In fact, i and h are the only prestratified maps from $pt \to \mathbb{I}$ and so we have unique choices of source and target maps.

- The prestratified map $j : \mathbb{I} \to \mathbb{I}_2$ where $j(t) = 2t$, induces a map

$$j^* = \mathscr{A}(j) : \mathscr{A}(\mathbb{I}_2) \to \mathscr{A}(\mathbb{I})$$

and we define the composition map $\mathbf{c} = j^*$. The map j represents the unique homotopy class of maps $\mathbb{I} \to \mathbb{I}_2$ where $0 \mapsto 0$ and $1 \mapsto 2$.

- The unique map $k : \mathbb{I} \to pt$ induces a map $k^* : \mathscr{A}(pt) \to \mathscr{A}(\mathbb{I})$, we define the map $\mathbf{id} = k^*$.

- The map $l : \mathbb{I} \to \mathbb{I}$, where $l(t) = 1 - t$, induces an involutive map $l^* = \mathscr{A}(l) : \mathscr{A}(\mathbb{I}) \to \mathscr{A}(\mathbb{I})$ we define $\dagger = l^*$.

Lemma 8.3. The induced maps $\mathbf{s}$, $\mathbf{t}$, $\mathbf{id}$, $\mathbf{c}$ and $\dagger$ satisfy the conditions in Definition 3.16, and so we obtain a dagger category with objects $\mathscr{A}(pt)$ and morphisms $\mathscr{A}(\mathbb{I})$.

Proof. • We check that $\mathbf{s}(\mathbf{c}(f, g)) = \mathbf{s}(f)$ and $\mathbf{t}(\mathbf{c}(f, g)) = \mathbf{t}(g)$. Figure 8.1

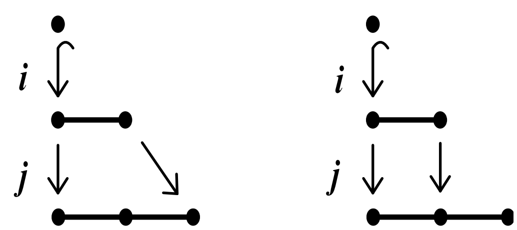


Figure 8.1: Diagrammatic representation of maps inducing $\mathbf{s}(\mathbf{c}(f, g))$ and $\mathbf{s}(f)$.

shows two mappings of the point to $\mathbb{I}_2$ which are clearly the same and therefore induce the same map of sets. Similarly for $\mathbf{t}(\mathbf{c}(f, g)) = \mathbf{t}(g)$.

- Next we check that $\mathbf{c}(\mathbf{id}_x, f) = f = \mathbf{c}(f, \mathbf{id}_y)$, where $(f : x \to y) \in Mor_{\mathscr{C}(\mathscr{A})}$. The maps in Figure 8.2 are homotopic via $G : \mathbb{I} \times \mathbb{I} \to \mathbb{I}$ where

$$G(t, s) = \begin{cases} 2t(1 - s) + st & \text{if } t \in [0, \frac{1}{2}] \\ 1 - s + st & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

and therefore induce the same map of sets.

- To see that the map $\mathbf{c}$ is an associative binary operation, take the maps $m, n : \mathbb{I}_2 \to \mathbb{I}_3$ where

$$m(t) = \begin{cases} t & \text{if } t \in [0, 1] \\ 2t - 1 & \text{if } t \in [1, 2] \end{cases}$$

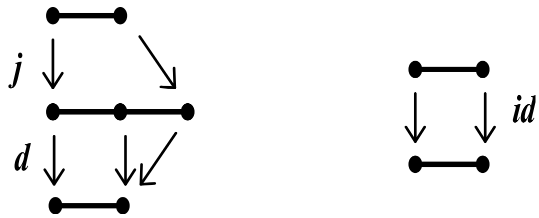


Figure 8.2: Diagrammatic representations of $d \circ j$, with $d : \mathbb{I}_2 \to \mathbb{I}$ given by $d(t) = \min\{t, 1\}$ on the left, and the identity on $\mathbb{I}$ on the right. These induce $\mathbf{c}(id_x, -)$ and the identity respectively.

and

$$n(t) = \begin{cases} 2t & \text{if } t \in [0,1] \\ t+1 & \text{if } t \in [1,2] \end{cases}$$

and consider the composites $m \circ j,\ n \circ j : \mathbb{I} \to \mathbb{I}_3$ where

$$(m \circ j)(t) = \begin{cases} 2t & \text{if } t \in [0, \frac{1}{2}] \\ 4t-1 & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

and

$$(n \circ j)(t) = \begin{cases} 4t & \text{if } t \in [0, \frac{1}{2}] \\ 2t+1 & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

(see Figure 8.3).

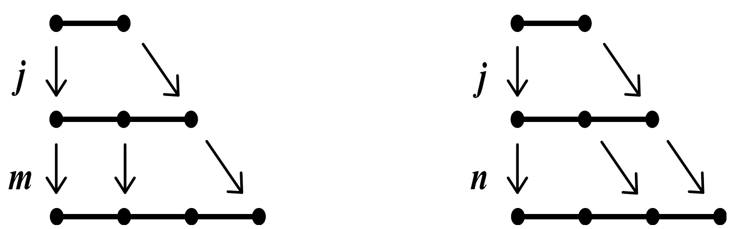


Figure 8.3: Two homotopic composites $m \circ j,\ n \circ j : \mathbb{I} \to \mathbb{I}_3$.

If we apply the functor we get

$$\begin{aligned} \mathscr{A}(m \circ j) &= \mathscr{A}(j) \circ \mathscr{A}(m) \\ &= \mathscr{A}(j) \circ (\mathscr{A}(j) \times \mathbf{1}) : [\mathscr{A}(\mathbb{I}) \times_{\mathscr{A}(pt)} \mathscr{A}(\mathbb{I})] \times_{\mathscr{A}(pt)} \mathscr{A}(\mathbb{I}) \to \mathscr{A}(\mathbb{I}), \end{aligned}$$

and

$$\begin{aligned} \mathscr{A}(n \circ j) &= \mathscr{A}(j) \circ \mathscr{A}(n) \\ &= \mathscr{A}(j) \circ (\mathbf{1} \times \mathscr{A}(j)) : \mathscr{A}(\mathbb{I}) \times_{\mathscr{A}(pt)} [\mathscr{A}(\mathbb{I}) \times_{\mathscr{A}(pt)} \mathscr{A}(\mathbb{I})] \to \mathscr{A}(\mathbb{I}). \end{aligned}$$

Here

$$\mathscr{A}(j) \circ (\mathscr{A}(j) \times \mathbf{1})(f, g, h) = \mathscr{A}(j)(\mathscr{A}(j)(f, g), h) = \mathbf{c}(\mathbf{c}(f, g), h)$$

and

$$\mathscr{A}(j) \circ (\mathbf{1} \times \mathscr{A}(j))(f, g, h) = \mathscr{A}(j)(f, \mathscr{A}(j)(g, h)) = \mathbf{c}(f, \mathbf{c}(g, h)).$$

To see that $\mathscr{A}(m) = \mathscr{A}(j) \times \mathbf{1}$ apply $\mathscr{A}$ to the following commutative diagram

$$\begin{array}{ccccc} \mathbb{I} & \hookrightarrow & \mathbb{I}_2 & \hookleftarrow & [1,2] \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle +1} \\ \mathbb{I}_2 & \hookrightarrow & \mathbb{I}_3 & \hookleftarrow & [2,3] \end{array}$$

to get a commutative diagram

$$\begin{array}{ccccc} \mathscr{A}(\mathbb{I}) & \longleftarrow & \mathscr{A}(\mathbb{I}) \times_{\mathscr{A}(pt)} \mathscr{A}(\mathbb{I}) & \longrightarrow & \mathscr{A}(\mathbb{I}) \\ \uparrow{\scriptstyle \mathscr{A}(j)} & & \uparrow{\scriptstyle \mathscr{A}(m)} & & \uparrow{\scriptstyle \mathscr{A}(+1)=\mathbf{1}} \\ \mathscr{A}(\mathbb{I}) \times_{\mathscr{A}(pt)} \mathscr{A}(\mathbb{I}) & \longleftarrow & \mathscr{A}(\mathbb{I}) \times_{\mathscr{A}(pt)} \mathscr{A}(\mathbb{I}) \times_{\mathscr{A}(pt)} \mathscr{A}(\mathbb{I}) & \longrightarrow & \mathscr{A}(\mathbb{I}) \end{array}$$

for which the claimed equality follows.

The composites $m \circ j,\ n \circ j : \mathbb{I} \to \mathbb{I}_3$ are homotopic via $H : \mathbb{I} \times \mathbb{I} \to \mathbb{I}_3$ where

$$H(t, s) = \begin{cases} 2t + 2st & \text{if } t \in [0, \frac{1}{2}] \\ 4t + 2s(1-t) - 1 & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

and therefore induce the same map of sets, thus $\mathscr{A}(m \circ j) = \mathscr{A}(n \circ j)$ and so $\mathbf{c}(\mathbf{c}(f, g), h) = \mathbf{c}(f, \mathbf{c}(g, h))$.

- Finally to see that $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$ consider the maps in Figure 8.4. These maps induce $(\mathbf{c}(f, g))^\dagger$ and $\mathbf{c}(f^\dagger, g^\dagger)$. On the left of the figure $t \mapsto 1 - t \mapsto 2(1-t)$, on the right $t \mapsto 2t \mapsto 2 - 2t$. And so, the maps are not only homotopic but in fact equal. They therefore induce the same map of sets. Similarly it can be easily shown that $\mathbf{id}_x = \mathbf{id}_x^\dagger$ and $f^{\dagger\dagger} = f$.

□

Remark 8.4. In general we can understand the map $\mathscr{A}(f)$ induced by $f : X \to Y$ in terms of the fibre product description of $\mathscr{A}(X)$ and $\mathscr{A}(Y)$, interval by interval, to reduce to the case $X = \mathbb{I}$. If f is stratified then for each 1-dimensional stratum, there is a commutative diagram,

Figure 8.4: Homotopic maps inducing $(g, f) \mapsto (g \circ f)^\dagger$ and $(g, f) \mapsto f^\dagger \circ g^\dagger$.

$$\begin{array}{ccc} \mathbb{I} & \hookrightarrow & X \\ \downarrow & & \downarrow f \\ f(\mathbb{I}) & \hookrightarrow & Y \end{array}$$

where the top arrow is the inclusion of the stratum. Applying $\mathscr{A}$ we can understand the component of $\mathscr{A}(f)$ in the factor corresponding to the stratum in the fibre product. Informally, all the information in the maps $\mathscr{A}(f)$ for $f : X \to Y$ is actually contained in the $\mathscr{A}(f)$ where $X = \mathbb{I}$ (in which case we may easily reduce to the case in which Y is either a subdivided interval, or a subdivided interval with the endpoints glued). Therefore, although it seems that the construction of $C(\mathscr{A})$ (which only makes use of $\mathscr{A}(f)$ for maps of the above special form) uses only a small part of the data of the Whitney 1-category $\mathscr{A}$, in fact it really uses all of the relevant information.

8.2 Constructing a Whitney 1-Category from a Dagger Category

Lemma 8.5. Given a dagger category $\mathscr{C}$, there is a functor $W(\mathscr{C}) : Prestrat_1 \to Sets$ such that

$$W(\mathscr{C})(pt) = Obj_{\mathscr{C}},$$

$$W(\mathscr{C})(\mathbb{I}) = Mor_{\mathscr{C}}$$

$$W(\mathscr{C})(i) \text{ is the source map,}$$

$$W(\mathscr{C})(h) \text{ is the target map,}$$

$$W(\mathscr{C})(j) \text{ is the composition map,}$$

$$W(\mathscr{C})(k) \text{ is the identity map,}$$

$$W(\mathscr{C})(l) \text{ is the dagger dual.}$$

Proof. Such a functor assigns to each object $X \in Prestrat_1$, a set $W(\mathscr{C})(X)$ and to each morphism a map of sets. We define an element of $W(\mathscr{C})(X)$ to be an equivalence class of labellings of $X \in Prestrat_1$ by $\mathscr{C}$. To assign such a labelling consider X as a graph, choose an orientation for each edge, assign to each vertex an element of $Obj_{\mathscr{C}}$ and to each edge assign an element of $Mor_{\mathscr{C}}$, compatibly chosen to match the source and targets. We define an equivalence relation on labellings, generated by the following: an oriented edge labelled by f is equivalent to one with opposite orientation labelled by $f^{\dagger}$, see Figure 8.5. By construction $W(\mathscr{C})(pt) = Obj(\mathscr{C})$ and $W(\mathscr{C})(\mathbb{I}) \cong Mor_{\mathscr{C}}$, via the map

Figure 8.5: Equivalent labellings of oppositely oriented edges.

taking the morphism f to $\mathbb{I}$ with standard orientation, labelled by f. Given $X, Y \in Prestrat_1$ and a prestratified map $X \to Y$, we must have an induced map $W(\mathscr{C})(Y) \to W(\mathscr{C})(X)$, that is a labelling in $W(\mathscr{C})(Y)$ defines a labelling in $W(\mathscr{C})(X)$. Since prestratified maps send vertices to vertices we 'pull back' the labelling from the target vertex to the source vertex.

Next, how would we label the edges of X?

Remark 8.6. Fix an edge in X. This gives us a characteristic map $\mathbb{I} \xrightarrow{\chi} X$ which is degree 1 onto the edge, where $\mathbb{I}$ has canonical orientation. Then

$$\mathbb{I} \xrightarrow{\chi} X \xrightarrow{\alpha} Y$$

is prestratified and so can be factorised as follows,

$$\begin{array}{ccc} \mathbb{I} & \xrightarrow{\alpha\circ\chi} & Y \\ & {\scriptstyle \times n}\searrow \quad \nearrow{\scriptstyle \text{stratified}} & \\ & \mathbb{I}_n & \end{array}$$

where $n \in \mathbb{N}$.

Given a factorization as in Remark 8.6 we can assign a label to the unique edge in $\mathbb{I}$ for any stratified map $\mathbb{I} \to Y$, in the following ways:

- If $\mathbb{I}$ maps to a vertex labelled a we would assign the label 1_a.

- If $\mathbb{I}$ maps onto an edge with the same orientation, then simply pull back the labelling from the target edge. If it maps to an edge with opposite orientation then 'pull back' the dagger of the label.
- The final possibility is subdivision, the basic example being $\mathbb{I} \to \mathbb{I}_2$. First we use equivalence to orient the edges in the same direction, then, erasing the subdivision equates to composing the labels. If for instance $\mathbb{I}_2$ was labeled f on [0,1] and g on $[1, 2]$ we 'pull back' fg to label $\mathbb{I}$. This can be iterated for any prestratified $\mathbb{I} \to \mathbb{I}_n$.

Given these rules on individual edges we can assign labellings to any stratified map $\mathbb{I}_n \to Y$ and to any prestratified map $\mathbb{I}_1 \to \mathbb{I}_n \to Y$. Clearly, by Remark 8.6 and the second rule, this suffices to assign the label to any edge in X. Thus we have a map

$$W(\mathscr{C})(f) : W(\mathscr{C})(Y) \to W(\mathscr{C})(X).$$

Clearly $W(\mathscr{C})(id) = id$ and we can also verify that $W(\mathscr{C})(f \circ g) = W(\mathscr{C})(f) \circ W(\mathscr{C})(g)$ so that we have defined a functor $W(\mathscr{C})$. By construction $W(\mathscr{C})$ has the stated properties. □

Proposition 8.7. *$W(\mathscr{C})$ is a Whitney 1-category.*

Proof. By construction,

$$W(\mathscr{C})(X) = \lim_{S \subset X} W(\mathscr{C})(\overline{S}).$$

And, by Remark 7.12, for $n = 1$ this implies that $W(\mathscr{C})$ is a sheaf on $Strat_1$. □

8.3 The Relationship Between the Constructions

Whitney 1-categories and Whitney functors, i.e. natural transformations, between them form the category $1Whit$. Dagger categories and dagger functors between them form a category Dag. In fact Dag is a 2-category but we will restrict our attention, at least in this thesis, to the 0-morphisms and 1-morphisms.

Theorem 8.8. *We have functors*

$$1Whit \underset{W}{\overset{C}{\rightleftarrows}} Dag.$$

which are inverse to each other and so have an equivalence of categories, i.e. given a Whitney functor $\eta : \mathscr{A} \to \mathscr{A}'$ of Whitney 1-categories there is a functor

$C(\eta) : C(\mathscr{A}) \to C(\mathscr{A}')$ between the corresponding dagger categories. And in the opposite direction, given a functor $F : \mathscr{C} \to \mathscr{D}$ between dagger categories, there is a natural transformation $W(F) : W(\mathscr{C}) \to W(\mathscr{D})$ between the corresponding Whitney 1-categories.

Proof. Take a Whitney functor, i.e. natural transformation of Whitney 1-categories $\eta : \mathscr{A} \to \mathscr{A}'$. This tells us that given an object $X \in Prestrat_1$ there exists a morphism $\eta_X : \mathscr{A}(X) \to \mathscr{A}'(X)$ in $Sets$ such that for every prestratified map $\alpha : X \to Y \in Prestrat_1$ the square

$$\begin{array}{ccc} \mathscr{A}(X) & \xleftarrow{\mathscr{A}(\alpha)} & \mathscr{A}(Y) \\ \downarrow \eta_X & & \downarrow \eta_Y \\ \mathscr{A}'(X) & \xleftarrow{\mathscr{A}'(\alpha)} & \mathscr{A}'(Y) \end{array}$$

commutes. To define a functor we must have a map $Obj_{C(\mathscr{A})} \to Obj_{C(\mathscr{A}')}$. The natural transformation gives us such a map, $\eta_{pt} : \mathscr{A}(pt) \to \mathscr{A}'(pt)$. Similarly there should be a mapping from the morphisms in $C(\mathscr{A})$ to those in $C(\mathscr{A}')$, the obvious candidate for this is $\eta_{\mathbb{I}} : \mathscr{A}(\mathbb{I}) \to \mathscr{A}(\mathbb{I})$. Taking the above square and choosing $X = \mathbb{I}$ and $Y = \mathbb{I}_2$, and setting $\mathscr{A}(\alpha) = \mathbf{c}$ and $\mathscr{A}'(\alpha) = \mathbf{c}'$, each an example of the composition map previously defined, we get the commuting square,

$$\begin{array}{ccc} \mathscr{A}(\mathbb{I}) & \xleftarrow{\mathbf{c}} & \mathscr{A}(\mathbb{I}_2) \\ \downarrow \eta_{\mathbb{I}} & & \downarrow \eta_{\mathbb{I}_2} \\ \mathscr{A}'(\mathbb{I}) & \xleftarrow{\mathbf{c}'} & \mathscr{A}'(\mathbb{I}_2) \end{array}$$

From this we see that

$$\eta_{\mathbb{I}}(\mathbf{c}(f,g)) = \mathbf{c}'(\eta_{\mathbb{I}_2}(f,g)),$$

i.e.

$$\eta_{\mathbb{I}}(fg) = \mathbf{c}'(\eta_{\mathbb{I}}(f), \eta_{\mathbb{I}}(g)) = \eta_{\mathbb{I}}(f)\eta_{\mathbb{I}}(g),$$

and so composition is preserved. Here $\eta_{\mathbb{I}_2} = \eta_{\mathbb{I}} \times \eta_{\mathbb{I}}$. Similarly consider

$$\begin{array}{ccc} \mathscr{A}(\mathbb{I}) & \xleftarrow{\mathbf{id}} & \mathscr{A}(pt) \\ \downarrow \eta_{\mathbb{I}} & & \downarrow \eta_{pt} \\ \mathscr{A}'(\mathbb{I}) & \xleftarrow{\mathbf{id}'} & \mathscr{A}'(pt) \end{array} \qquad \begin{array}{ccc} \mathscr{A}(pt) & \xleftarrow{\mathbf{s}} & \mathscr{A}(\mathbb{I}) \\ \downarrow \eta_{pt} & & \downarrow \eta_{\mathbb{I}} \\ \mathscr{A}'(pt) & \xleftarrow{\mathbf{s}'} & \mathscr{A}'(\mathbb{I}) \end{array}$$

These commuting squares show

$$\eta_{pt}(\mathbf{id}_x) = \mathbf{id}(\eta_{\mathbb{I}}(x))$$

$$\text{and} \qquad \mathbf{s}(\eta_{pt}(f)) = \eta_{\mathbb{I}}(\mathbf{s}(f))$$

and similarly for t. Hence we have constructed a functor

$$C(\eta) : C(\mathscr{A}) \to C(\mathscr{A}').$$

What about the other direction? Given a functor $F : \mathscr{C} \to \mathscr{D}$ between dagger categories can we define a functor $W(\mathscr{C}) \to W(\mathscr{D})$ between the corresponding Whitney 1-categories? We need a natural transformation assigning to each $X \in Strat_1$, a map of sets $W(\mathscr{C})(X) \to W(\mathscr{D})(X)$, mapping the equivalence classes of labellings of X induced by $\mathscr{C}$, to equivalence classes of labellings induced by $\mathscr{D}$. The fact that we have a functor $\mathscr{C} \to \mathscr{D}$ means we have maps, i.e. $Obj_{\mathscr{C}} \to Obj_{\mathscr{D}}$ and $Mor_{\mathscr{C}} \to Mor_{\mathscr{D}}$. Consider the following diagram:

$$\begin{array}{ccc}
W(\mathscr{C})(X) & \xrightarrow{W(F)(X)} & W(\mathscr{D})(X) \\
\uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} \\
W(\mathscr{C})(Y) & \xrightarrow[W(F)(Y)]{} & W(\mathscr{D})(Y)
\end{array}$$

We define $W(F)(X)$ by applying F to all labels of X by $\mathscr{C}$ to obtain a labelling by $\mathscr{D}$. This square commutes because F is a functor and so preserves composition and identities. Therefore $W(\mathscr{C})(F)$ is a natural transformation.

The constructions

$$C : 1Whit \to Dag$$
$$W : Dag \to 1Whit$$

are inverse to each other in the sense that there is a natural isomorphism of dagger categories

$$\mathscr{C} \to C(W(\mathscr{C})),$$

and a natural isomorphism of Whitney 1-categories

$$\mathscr{A} \to W(C(\mathscr{A})).$$

To see this fix a dagger category $\mathscr{C}$, from it construct a Whitney 1-category $W(\mathscr{C})$ and then use this to define a new dagger category $C(W(\mathscr{C}))$. As we have previously defined, to construct a dagger category from a Whitney 1-category we assign $W(\mathscr{C})(pt) = Obj_{C(W(\mathscr{C}))}$. Now, an element of $W(\mathscr{C})(pt)$ is a labelling of the point, we use the objects of $\mathscr{C}$ to label vertices and so $W(\mathscr{C})(pt) = Obj_{\mathscr{C}}$. And so

$$Obj_{\mathscr{C}} = Obj_{C(W(\mathscr{C}))}.$$

Similarly we get the bijection,

$$Mor_{C(W(\mathscr{C}))} = W(\mathbb{I}) = \{\text{labellings of an edge}\}/_{\sim} \cong Mor_{\mathscr{C}}.$$

This identification respects sources and targets (by the definition of labelling) and composition. Thus we have defined a dagger functor

$$\mathscr{C} \quad \to \quad C(W(\mathscr{C})), \tag{8.1}$$

which is an isomorphism of categories. To see this isomorphism is natural consider the commuting square

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\quad F \quad} & \mathscr{C}' \\ \downarrow & & \downarrow \\ C(W(\mathscr{C})) & \xrightarrow[C(W(F)]{} & C(W(\mathscr{C}')). \end{array}$$

We can either apply the functor F to $\mathscr{C}$ and then construct $C(W(\mathscr{C}')$ or equivalently construct $C(W(\mathscr{C})$ and then apply the functor to the labels to get $C(W(\mathscr{C}')$, i.e the isomorphism is natural.

Next we need to define a functor between Whitney 1-categories, i.e. a natural transformation

$$\mathscr{A} \to W(C(\mathscr{A})). \tag{8.2}$$

We do this by compatibly defining,

$$\mathscr{A}(X) \to W(C(\mathscr{A}))(X) \tag{8.3}$$

for all $X \in Strat_1$. Here $C(\mathscr{A})$ is the dagger category constructed from $\mathscr{A}$ and $W(C(\mathscr{A}))$ is the functor which assigns an equivalence class of labellings from $C(\mathscr{A})$ to each $X \in Strat_1$.

First choose an orientation of the edges of X. Given $a \in \mathscr{A}(X)$, we want to assign an element of $\mathscr{A}(pt)$ to each vertex and an element of $\mathscr{A}(\mathbb{I})$ to each (oriented) edge in a compatible way. The characteristic map for a vertex, $pt \xrightarrow{u} X$, gives $\mathscr{A}(X) \xrightarrow{u^*} \mathscr{A}(pt)$. We therefore label this vertex by u^*a.

Since we have choosen an orientation there is a characteristic map, $\mathbb{I} \xrightarrow{v} X$, onto an oriented edge, giving $\mathscr{A}(X) \xrightarrow{v^*} \mathscr{A}(\mathbb{I})$ and so we label this edge by v^*a. This gives a labelling of X by $C(\mathscr{A})$, i.e. an element of $W(C(\mathscr{A}))(X)$.

As $\mathscr{A}$ is a sheaf, we have

$$\mathscr{A}(X) = \lim_{S \subset X} \mathscr{A}(\overline{S}).$$

It follows from this that (8.3) is a bijection. To see that (8.2) is a natural isomorphism fix an object X and consider the commuting square,

$$\begin{array}{ccc}
\mathscr{A}(X) & \xrightarrow{\eta_X} & \mathscr{A}'(X) \\
\downarrow & & \downarrow \\
W(C(\mathscr{A}))(X) & \xrightarrow[W(C(\eta_X))]{} & W(C(\mathscr{A}'))(X).
\end{array}$$

Given $a \in \mathscr{A}(X)$ we apply η_X to get an element in $\mathscr{A}'(X)$ and then apply the construction to get an element of $W(C(\mathscr{A}'))(X)$. Equivalently, given $a \in \mathscr{A}(X)$ we first apply the construction to get an element of $W(C(\mathscr{A}))(X)$, i.e. an equivalence class of labellings of X by $C(\mathscr{A})$, and then apply η_X to all the labels to get an equivalence class of labellings of X by elements of $C(\mathscr{A}')$, i.e. an element of $W(C(\mathscr{A}'))(X)$. $\square$

Chapter 9

One-Object Whitney 2-Categories

We now direct our attention to one-object Whitney 2-categories. Recall a Whitney 2-category is a presheaf $\mathscr{A} : Prestrat_2 \to Sets$ which is a sheaf when restricted to $Strat_2$. In this section we assume there to be only one object, i.e. $\mathscr{A}(pt) = \mathbf{1}$. We will give a construction similar to the Whitney 1-category case which from a one-object Whitney 2-category produces a dagger pivotal category. This is the notion of a one-object 2-category with duals that we adopt. In the other direction we will give constructions that produce one-object Whitney 2-categories from dagger pivotal categories.

9.1 Constructing a Dagger Pivotal Category from a One-Object Whitney 2-Category

In this section we will construct a functor between the category of dagger pivotal categories and the category of one-object Whitney 2-categories,

$$C : 2Whit_1 \to DagPiv.$$

Definition 9.1. From $\mathscr{A}$ we define a category $C(\mathscr{A})$. We set

$$Obj_{C(\mathscr{A})} = \mathscr{A}(\mathbb{I}),$$

and

$$Mor_{C(\mathscr{A})} = \{\alpha \in \mathscr{A}(\mathbb{I}^2) \mid \alpha|_{0\times\mathbb{I}} = \mathbf{1} = \alpha|_{1\times\mathbb{I}}\}$$

where $\alpha|_{t\times\mathbb{I}} = \mathscr{A}(f)(\alpha)$ for $f : \{t\} \times \mathbb{I} \hookrightarrow \mathbb{I}_2$. We also use $\mathbf{1} \in \mathscr{A}(\mathbb{I})$ for image of the the unique element in $\mathscr{A}(pt)$, induced from the unique map of $\mathbb{I}$ to a point. We have four maps:

- The source and target maps, $\mathbf{s}, \mathbf{t} : \mathscr{A}(\mathbb{I}^2) \to \mathscr{A}(\mathbb{I})$, are induced from the inclusions $i_0 : \mathbb{I} \to \mathbb{I}^2$ where $t \mapsto (t, 0)$ for all t and $i_1 : \mathbb{I} \to \mathbb{I}^2$ where $t \mapsto (t, 1)$ for all t, respectively.
- The map $\mathbf{id} : \mathscr{A}(\mathbb{I}) \to \mathscr{A}(\mathbb{I}\times\mathbb{I})$ is induced from the projection of $\mathbb{I}^2$ onto its first factor. This map takes an object and returns the identity morphism on that object.
- The composition map $\mathbf{c} : \mathscr{A}(\mathbb{I}^2) \times_{\mathscr{A}(\mathbb{I})} \mathscr{A}(\mathbb{I}^2) \to \mathscr{A}(\mathbb{I}^2)$ is induced from the map $\mathbb{I}^2 \to \mathbb{I} \times \mathbb{I}_2 : (x, y) \mapsto (x, 2y)$. Recall that the sheaf property tells us

$$\mathscr{A}(\mathbb{I} \times \mathbb{I}_2) = \mathscr{A}(\mathbb{I}^2) \times_{\mathscr{A}(\mathbb{I})} \mathscr{A}(\mathbb{I}^2).$$

Lemma 9.2. The maps in Definition 9.1 satisfy:

- Source and target of composites: $\mathbf{s}(\mathbf{c}(f, g)) = \mathbf{s}(f)$ and $\mathbf{t}(\mathbf{c}(f, g)) = \mathbf{t}(g)$.
- Associativity of composition: $\mathbf{c}(\mathbf{c}(f, g), h) = \mathbf{c}(f, \mathbf{c}(g, h))$.
- Identities act as units for composition: $\mathbf{c}(\mathbf{id}_a, f) = f = \mathbf{c}(f, \mathbf{id}_b)$, where $(f : a \to b) \in Mor_{C(\mathscr{A})}$.

Proof. The proof of this lemma is the same as that for Lemma 8.3 in the Whitney 1-category ($n = 1$) case, but here we carry along an extra factor. For example, the equality in the second property comes from the homotopy

$$H(x, y, t) = \begin{cases} (x, 2y + 2ty) & \text{if } y \in [0, \frac{1}{2}] \\ (x, 4y + 2t(1 - y) - 1) & \text{if } y \in [\frac{1}{2}, 1] \end{cases}$$

which should be compared with the homotopy H in Lemma 8.3. □

Remark 9.3. Since we have defined $\mathscr{A}(pt) = \mathbf{1}$ we have

$$\begin{aligned} \mathscr{A}(\mathbb{I}_2) &= \mathscr{A}(\mathbb{I}) \times_{\mathscr{A}(pt)} \mathscr{A}(\mathbb{I}) \\ &= \mathscr{A}(\mathbb{I}) \times \mathscr{A}(\mathbb{I}) \end{aligned}$$

Lemma 9.4. The category $C(\mathscr{A})$ has a specific monoidal structure.

Proof. We define a bifunctor $\otimes : C(\mathscr{A})\times C(\mathscr{A}) \to C(\mathscr{A})$. On objects this functor is the map

$$\otimes : \mathscr{A}(\mathbb{I}) \times \mathscr{A}(\mathbb{I}) \to \mathscr{A}(\mathbb{I}) : (A, B) \mapsto A \otimes B$$

induced by $\mathbb{I} \to \mathbb{I}_2 : x \mapsto 2x$.

On morphisms it is the map

$$\otimes : \mathscr{A}(\mathbb{I}^2) \times_{\mathscr{A}(\mathbb{I})} \mathscr{A}(\mathbb{I}^2) \to \mathscr{A}(\mathbb{I}^2) : (f,g) \mapsto f \otimes g$$

induced by the prestratified map $\mathbb{I}^2 \to \mathbb{I}_2 \times \mathbb{I} : (x,y) \mapsto (2x,y)$, see Figure 9.1.

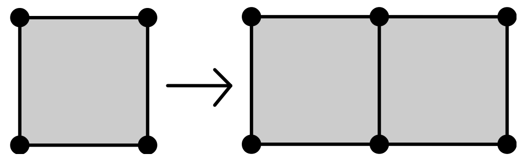


Figure 9.1: The map $(x,y) \mapsto (2x,y)$, inducing the tensor product on $C(\mathscr{A})$

The fact that the functor respects composition is illustrated in Figure **??**. More explicitly composing then tensoring is induced from the composite

$$(x,y) \mapsto (x,2y) \mapsto (2x,2y)$$

and tensoring then composing is induced from the composite

$$(x,y) \mapsto (2x,y) \mapsto (2x,2y).$$

These are different factorisations of the same map and so induce the same map of sets, giving the 'exchange law'

$$(f \otimes g) \circ (f' \otimes g') = (f \circ f') \otimes (g \circ g').$$

There is an associator, i.e. a natural isomorphism α with components,

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

induced from the homotopy $H : \mathbb{I} \times \mathbb{I} \to \mathbb{I}$

$$H(x,t) = \begin{cases} x(t+1) & \text{if } x \in [0, \frac{1}{4}] \\ x + \frac{t}{4} & \text{if } x \in [\frac{1}{4}, \frac{1}{2}] \\ x + \frac{t}{2}(1-x) & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

The inverse corresponds to replacing t by $1-t$ in the homotopy.

The unit object $\mathbf{1} \in \mathscr{A}(\mathbb{I})$ is defined as $p^*\mathbf{1}$ from the unique map $\mathbb{I} \xrightarrow{p} pt$ and the unique element $\mathbf{1} \in \mathscr{A}(pt)$. $\mathbf{1}$ acts as left and right identity, i.e. there are two natural isomorphisms λ^* and ρ^*, with components $\lambda^*_A : \mathbf{1} \otimes A \cong A$ and

$\rho^*_A : A \otimes \mathbf{1} \cong A$. We abuse notation, denoting each unit element $\mathbf{1}$ to avoid overuse of subscripts. We define $\rho^*_A : A \otimes \mathbf{1} \cong A$ as the morphism induced from

$$\rho : \mathbb{I}^2 \to \mathbb{I} : (x, t) \mapsto \max\{(1 + t)x, 1\}$$

which is a homotopy from the identity to $\max\{2x, 1\} : \mathbb{I} \to \mathbb{I}$. We define λ^*_A similarly.

□

Lemma 9.5. The maps $\mathscr{A}(\mathbb{I}) \xrightarrow{*} \mathscr{A}(\mathbb{I})$ induced by $x \mapsto 1-x$ and $\mathscr{A}(\mathbb{I}^2) \xrightarrow{*} \mathscr{A}(\mathbb{I}^2)$ induced by $(x, y) \mapsto (1 - x, y)$ define a dual for each object and morphism respectively.

Proof. To prove this lemma we must construct for each object A, a unit $\eta_A : \mathbf{1} \to A \otimes A^*$, a counit $\epsilon_A : A^* \otimes A \to \mathbf{1}$, and we must verify the triangle identities, i.e. verify that the composites

$$A \xrightarrow{\eta_A \otimes A} A \otimes A^* \otimes A \xrightarrow{A \otimes \epsilon_A} A$$

and

$$A^* \xrightarrow{A^* \otimes \eta_A} A^* \otimes A \otimes A^* \xrightarrow{\epsilon_A \otimes A^*} A^*$$

are identities. Given $A \in \mathscr{A}(\mathbb{I})$ we want an element of $\mathscr{A}(\mathbb{I}^2)$ such that the restriction to the edges gives elements of $\mathscr{A}(\mathbb{I})$ corresponding to the labelling in Figure 9.2. This will be the unit corresponding to A, i.e. η_A.

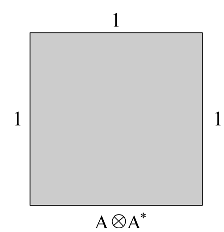


Figure 9.2: Schematic diagram of the boundary conditions on the object in $Strat_2$ which corresponds to the unit element η_A, labelled with the elements of $\mathscr{A}(\mathbb{I})$ obtained by restricting $\mathscr{A}$ to those edges. Here and in what follows $y = 0$ is at the top of the diagram.

Take the map $F : \mathbb{I}_2 \to \mathbb{I}$ given by

$$F(x, y) = \begin{cases} 1 - 2(\sqrt{(x - \frac{1}{2})^2 + (y - 1)^2}) & \text{if } \sqrt{(x - \frac{1}{2})^2 + (y - 1)^2} \leqslant \frac{1}{2} \\ 0 & \text{if } \sqrt{(x - \frac{1}{2})^2 + (y - 1)^2)} \geqslant \frac{1}{2}, \end{cases}$$

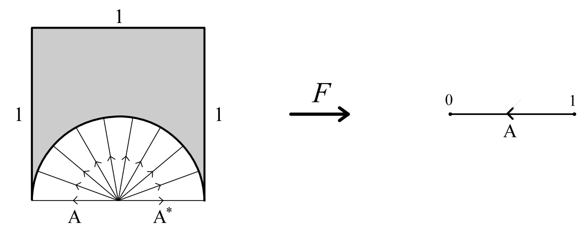


Figure 9.3: We mark a half-disk. F maps each radial line segment on to A, with the arrows showing orientation. The boundary of the disk, the remaining edges of the square and the shaded region all get mapped to 0.

see Figure 9.3. This map is continuous and, by the Whitney Approximation Theorem [14, §10], it is homotopic to a smooth map $\tilde{F} : \mathbb{I}_2 \to \mathbb{I}$. Since the map F is smooth everywhere except the point $(\frac{1}{2}, 1)$ and the set

$$\{(x, y) | \sqrt{(x - \frac{1}{2})^2 + (y - 1)^2} = \frac{1}{2}\},$$

we can choose a homotopy relative to the closed complement of an open neighbourhoods of these sets. We can choose a smoothing so that $\tilde{F}$ is prestratified and is symmetric about the line $x = \frac{1}{2}$. Moreover one can choose the homotopy so that it is through prestratified maps. After the smoothing process we have an induced map

$$\widetilde{F}^* : \mathscr{A}(I) \to \mathscr{A}(I^2) : A \mapsto (1 \to \widetilde{A \otimes A^*}).$$

But,

$$\widetilde{A \otimes A^*} = \tilde{A} \otimes (\tilde{A})^* \cong A \otimes A^*. \tag{9.1}$$

The first equality comes from the fact that we chose a symmetric smoothing and the isomorphism (of objects, in the usual categorical sense) from restricting the homotopy from F to $\tilde{F}$. We define

$$\widetilde{F}^* A = \widetilde{\eta_A} : 1 \to \widetilde{A \otimes A^*}$$

and compose with the isomorphism in (9.1) to get η_A.

Next we verify the triangle identity for η_A (ϵ can be constructed similarly). To do this we need a homotopy between two maps $\mathbb{I}^2 \to \mathbb{I}$. Rather than write down an explicit homotopy we direct the reader to Figure 9.4. In it morphisms are composed via stacking. We can choose a homotopy that 'straightens' the picture on the left. We construct the counit $\epsilon_A : A * \otimes A \to \mathbf{1}$ in the same way and use the same procedure to show the composite

$$A^* \xrightarrow{id_A * \otimes \eta_A} A^* \otimes (A \otimes A^*) \xrightarrow{\alpha} (A^* \otimes A) \otimes A^* \xrightarrow{\epsilon_x \otimes id_x} A^*$$

is the identity. □

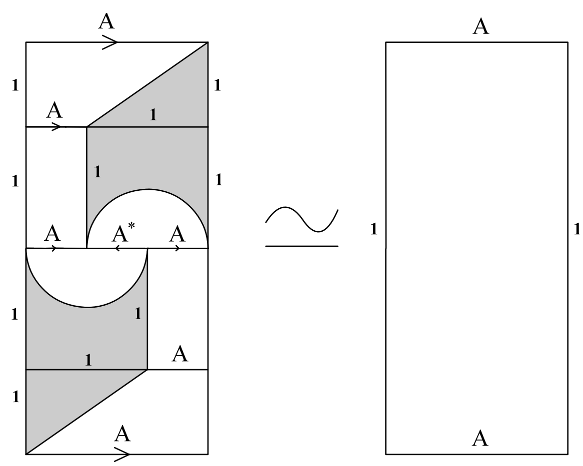


Figure 9.4: Two morphisms, i.e. elements of $\mathscr{A}(\mathbb{I}^2)$, are described diagrammatically. The composite $A \otimes 1 \xrightarrow{id_A \otimes \eta_A} A \otimes (A^* \otimes A) \xrightarrow{\alpha} (A \otimes A^*) \otimes A \xrightarrow{\epsilon_A \otimes id_A} 1 \otimes A$ is represented on the left the identity morphism on A is represented on the right. Each one is obtained by pulling back $A \in \mathscr{A}(\mathbb{I})$ along a prestratified map $\mathbb{I}^2 \to \mathbb{I}$. By inspection these maps are homotopic and so the resulting pullbacks in $\mathscr{A}(\mathbb{I}^2)$ are the same.

Lemma 9.6. The map † induced from $(x, y) \mapsto (x, 1 - y)$ gives the dagger structure on the category $C(\mathscr{A})$.

Proof. The proof is the same as in Lemma 8.3 in the $n = 1$ case but with an extra factor. □

Proposition 9.7. The above construction is functorial. That is, given a natural transformation $\mu : \mathscr{A} \to \mathscr{A}'$ of Whitney 2-categories (with one object) there is a functor $C(\mu) : C(\mathscr{A}) \to C(\mathscr{A}')$ preserving the monoidal, dagger and dual structures. More explicitly,

- $C(\mu)(A \otimes B) \cong C(\mu)(A) \otimes C(\mu)(B)$,
- $C(\mu)(f^\dagger) = C(\mu)(f)^\dagger$,
- $C(\mu)(A^*) \cong C(\mu)(A)^*$.

Proof. The proof is similar to that of Theorem 8.8, i.e. the $n = 1$ case. We therefore omit some details, but sketch the general idea. All structures are induced by maps $X \xrightarrow{f} Y$, for suitable choices of X, Y and f, and so we always have a commuting square

$$\begin{array}{ccc} \mathscr{A}(Y) & \longleftarrow & \mathscr{A}(X) \\ \downarrow{\scriptstyle \mu_Y} & & \downarrow{\scriptstyle \mu_X} \\ \mathscr{A}'(Y) & \longleftarrow & \mathscr{A}'(X). \end{array}$$

By the definition of μ we have an object map $\mu_{\mathbb{I}} : \mathscr{A}(\mathbb{I}) \to \mathscr{A}'(\mathbb{I})$ and a map of morphisms $\mu_{\mathbb{I}\times\mathbb{I}} : \mathscr{A}(\mathbb{I} \times \mathbb{I}) \to \mathscr{A}'(\mathbb{I} \times \mathbb{I})$. The commutativity of the following diagrams show that composition is preserved as well as monoidal, dagger and rigid structure all due to the naturality of μ :

$$\begin{array}{ccc} \mathscr{A}(\mathbb{I} \times \mathbb{I})^2 & \xrightarrow{\mathbf{c}} & \mathscr{A}(\mathbb{I} \times \mathbb{I}) \\ \downarrow{\scriptstyle (\mu_{\mathbb{I}\times\mathbb{I}})^2} & & \downarrow{\scriptstyle \mu_{\mathbb{I}\times\mathbb{I}}} \\ \mathscr{A}'(\mathbb{I} \times \mathbb{I})^2 & \xrightarrow{\mathbf{c}} & \mathscr{A}'(\mathbb{I} \times \mathbb{I}) \end{array} \qquad \begin{array}{ccc} \mathscr{A}(\mathbb{I} \times \mathbb{I}) & \xrightarrow{\otimes} & \mathscr{A}(\mathbb{I}) \\ \downarrow{\scriptstyle \mu_{\mathbb{I}\times\mathbb{I}}} & & \downarrow{\scriptstyle \mu_{\mathbb{I}}} \\ \mathscr{A}'(\mathbb{I} \times \mathbb{I}) & \xrightarrow{\otimes} & \mathscr{A}'(\mathbb{I}) \end{array}$$

$$\begin{array}{ccc} \mathscr{A}(\mathbb{I} \times \mathbb{I}) & \xleftarrow{\dagger} & \mathscr{A}(\mathbb{I} \times \mathbb{I}) \\ \downarrow{\scriptstyle \mu_{\mathbb{I}\times\mathbb{I}}} & & \downarrow{\scriptstyle \mu_{\mathbb{I}\times\mathbb{I}}} \\ \mathscr{A}'(\mathbb{I} \times \mathbb{I}) & \xleftarrow{\dagger} & \mathscr{A}'(\mathbb{I} \times \mathbb{I}) \end{array} \qquad \begin{array}{ccc} \mathscr{A}(\mathbb{I}) & \xleftarrow{*} & \mathscr{A}(\mathbb{I}) \\ \downarrow{\scriptstyle \mu_{\mathbb{I}}} & & \downarrow{\scriptstyle \mu_{\mathbb{I}}} \\ \mathscr{A}'(\mathbb{I}) & \xleftarrow{*} & \mathscr{A}'(\mathbb{I}). \end{array}$$

□

9.2 The Labelling Construction W.

The aim of this and the following two sections is to construct a one-object Whitney 2-category from dagger pivotal category $\mathscr{D}$. Each construction has its own virtues. This section describes the labelling construction which is a simple generalisation of the Whitney 1-category construction. The equivalence relation we will define on labellings of a polygon relates explicitly to the algebraic structure of a dagger pivotal category, see Proposition 9.9. We will define this construction only for polygons and for $X \in Prestrat_2$ such that the closure of each 2-dimensional stratum is polygonal. While we strongly believe it forms a sheaf on such spaces, we find it technically very difficult to show, so we give a sketch and in the next section take a different approach. Nevertheless we include the construction as it clearly shows how the algebraic structure appears.

Definition 9.8. Let $X \in Strat_2$ be strictly cellular and $\mathscr{D}$ a dagger pivotal category. We define an element of $W(\mathscr{D})(X)$ to be an equivalence class of labellings of X by the objects and morphisms of $\mathscr{D}$. By strictly cellular we mean that the closure of each stratum is cellular.

If X is 0 or 1-dimensional then a labelling is similar to the Whitney 1-category case. The difference being we label each vertex with the monoidal identity and edges get objects as labels. Also if we change the orientation of an edge, we relabel with the dual of an object as opposed to the dagger.

Next we treat the case where X is an n-gon and define a labelling using the following procedure:

- First label each vertex with the monoidal identity.
- Choose an orientation for each edge and then label it with an object V. This is equivalent to labelling by V^* if the opposite orientation is chosen.
- Choose an orientation for the face.
- Subdivide the boundary of X by choosing a connected chain of edges as the source and the chain of remaining edges as the target. (We may choose the entire boundary as the source and any vertex as the target or vice versa.) The choice of orientation on the face gives a well defined beginning and end for the chains of edges in the source and target. We use the equivalence on edge labellings to orient all edges of the source in the same direction, changing the label to the dual object where appropriate. The edges in the target should be labelled with the opposite orientation. We call this particular set of choices the standard orientation. Finally label the face with a morphism from $\mathscr{D}$ with compatible source and target. See Figure 9.5.

We define *a labelling* to be a set of the above choices. We then define an equivalence relation on the set of such labellings of an n-gon, generated by the following list of elementary equivalences. In what follows we abuse notation by denoting an object and the identity morphism on that object by the same symbol. The unit of an object V^* is η_{V^*}, and ϵ_{V^*} is the counit.

Take a face labelled f:

1) If we keep everything fixed but reverse the orientation, we can relabel it with $(f^*)^\dagger$.
2) Again keeping everything fixed, if we instead reverse the choice of source and target we can relabel with f^*.

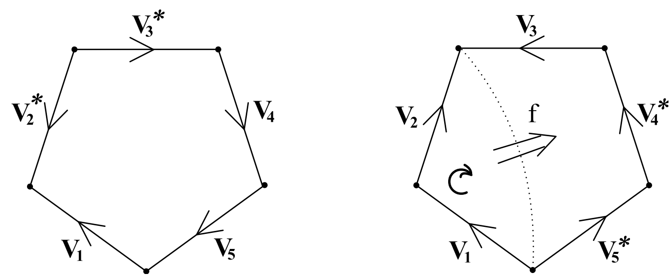


Figure 9.5: On the left, a space with oriented and labelled edges. On the right the face has been oriented, a choice of source and target made, the orientation of edges has been standardised and finally the face is labelled with the morphism $f : V_1 \otimes V_2 \to V_5^* \otimes V_4^* \otimes V_3$

3) Subdivide the boundary of X, fix a standard orientation and label the face f. If we were to keep everything fixed but specify a new subdivision that differs only by moving the edge V from the end of the source to the end of the target (relabelling the edge V^* to preserve the standard orientation, see Figure 9.6 for the local picture) and label the face $g : A \to B \otimes V^*$, then the two labellings are considered equivalent if $g = (f \otimes V^*) \circ (A \otimes \eta_V)$.

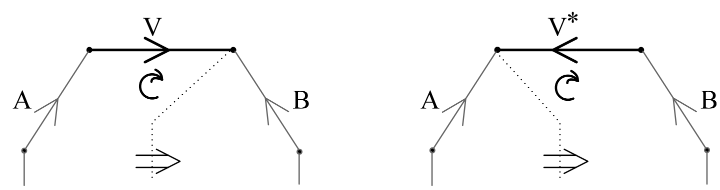


Figure 9.6: On the left a local picture of a face labelled with the morphism $f : A \otimes V \to B$, where A and B represent labels on chains of edges. On the right, the edge V has been moved from the source to the target and the face has been relabelled $g : A \to B \otimes V^*$. These two labellings are equivalent if $g = (f \otimes V^*) \circ (A \otimes \eta_V)$.

4) We can also move an edge from the beginning of the source to the beginning of the target, see Figure 9.7.

5) We can reverse both of the above procedures using the counit in place of the unit.

Proposition 9.9. Each equivalence class contains a unique labelling for a given choice of orientation and subdivision. More explicitly we show that:

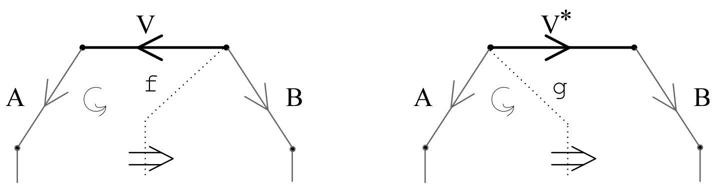


Figure 9.7: On the left a local picture of a face labelled with the morphism $f : V \otimes A \to B$, where A and B represent labels on chains of edges. On the right, the edge V has been moved from the source to the target and the face has been relabelled $g : A \to V^* \otimes B$. These two labellings are equivalent if $g = (V^* \otimes f) \circ (\eta_V^* \otimes A)$.

(i) Changing the subdivision and then changing it back results in the same labelling.

(ii) Changing the subdivision by 180° results in the same labelling as reversing the source and target and relabelling with the dual.

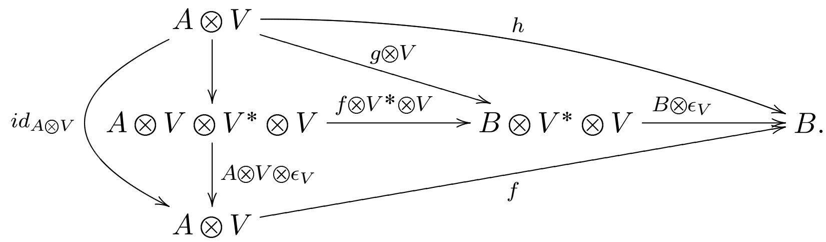
(iii) Changing the subdivision and then reversing the orientation of the face results in the same labelling as reversing the orientation and then changing the subdivision.

(iv) Swapping the source and target and then changing the subdivision results in the same labelling as changing the subdivision and then swapping the source and target.

Proof. (i) Take a face labelled with the morphism $f : A \otimes V \to B$, move the edge labelled V from the end of the source to the end of the target, relabelling with the morphism g, and then reverse the procedure, relabelling with h. By definition of the equivalence relation the following diagrams commute:

$$\begin{array}{ccc} A & \xrightarrow{g} & B \otimes V^* \\ {\scriptstyle A\otimes\eta_V}\downarrow & \nearrow{\scriptstyle f\otimes V^*} & \\ A \otimes V \otimes V^* & & \end{array} \qquad \begin{array}{ccc} A \otimes V & \xrightarrow{h} & B \\ {\scriptstyle g\otimes V}\downarrow & \nearrow{\scriptstyle B\otimes\epsilon_V} & \\ B \otimes V^* \otimes V & & \end{array} .$$

Now consider



The top middle triangle commutes from the definition of g and the top right triangle from the definition of h. To $A \otimes V \otimes V^* \otimes V$ we can apply first f and then ϵ or we can apply ϵ and then f, giving the same result, and so the bottom triangle of the diagram commutes. The vertical composite on the left is the triangle identity, see Definition 3.18. Thus we have $f = h$.

(ii) We take a face labelled by the morphism $f : A \to B$ where A and B represent the entire source and target respectively. Changing the subdivision by 180° corresponds to relabelling by the composite $g = (A^* \otimes \epsilon_{B^*}) \circ (A^* \otimes f \otimes B^*) \circ (\eta_{A^*} \otimes B^*)$. This is precisely the composite used in Remark 3.21 to define the dual of a morphism.

(iii) We change a subdivision and relabel by $(f \otimes V^*) \circ (A \otimes \eta_V)$ as in Figure 9.6, reverse the orientation and change the label to

$$\begin{aligned}[(f \otimes V^*) \circ (A \otimes \eta_V)]^{*\dagger} &= [(A \otimes \eta_V)^* \circ (f \otimes V^*)^*]^\dagger \\ &= [(\epsilon_{V^*} \otimes A^*) \circ (V \otimes f^*)]^\dagger \\ &= (V \otimes f^{*\dagger}) \circ (\eta_V \otimes A^*)\end{aligned}$$

Here we have used the fact from Remark 3.19, that $(\eta_V)^* = \epsilon_{V^*}$. If we had first changed the orientation of the face, changing the label to $f^{*\dagger}$ and then changed the subdivision in the same way (which now means moving the edge from the beginning of the source to the beginning of the target), we would get the same label

$$(V \otimes f^{*\dagger}) \circ (\eta_V \otimes A^*)$$

by applying Part 4) of the definition of the equivalence relation (see Figure 9.7) to

$$V^* \otimes A^* \xrightarrow{f^{*\dagger}} B^*.$$

Similar arguments apply to the elementary ways in which a labelling can be changed.

(iv) Swapping the source and target and then changing the subdivision by moving an edge from the beginning of the source to the beginning of the target, for instance, we obtain the new label

$$(\epsilon_{V^*} \otimes A^*) \circ (V \otimes f^*)$$

because starting with $A \otimes V \xrightarrow{f} B$ we change the label to

$$B^* \xrightarrow{f^*} V^* \otimes A^*$$

and then to the composite

$$\begin{array}{ccc} V \otimes B^* & \longrightarrow & A^* \\ {\scriptstyle V \otimes f^*} \downarrow & \nearrow {\scriptstyle \epsilon_{V^*} \otimes A^*} & \\ V \otimes V^* \otimes A^* & & \end{array} .$$

This results in the same labelling as first changing the subdivision and then swapping the source and target.

$$\begin{aligned} [(f \otimes V^*) \circ (A \otimes \eta_V)]^* &= (A \otimes \eta_V)^* \circ (f \otimes V^*)^* \\ &= (\epsilon_{V^*} \otimes A^*) \circ (V \otimes f^*). \end{aligned} \tag{9.2}$$

Again similar arguments apply to the other elementary changes of subdivision.
□

So we see that given choices of orientation and subdivision we obtain a bijection

$$\left\{ \begin{array}{c} \text{Labellings on } X \\ \text{with with source labelled by } A \\ \text{and target labelled by } B \end{array} \right\} \cong \left\{ \text{ Morphisms } A \xrightarrow{f} B \text{ in } \mathscr{D} \right\}.$$

If we change our choices, the equivalence relation gives a bijection between the labellings with respect to the old and new choices, and the structure of $\mathscr{D}$ gives a bijection between sets of morphism. By construction of the equivalence relation (where we chose the relabelling using the latter property of $\mathscr{D}$) these bijections are compatible with the identifications of labellings and morphisms.

Definition 9.10. Given a strictly cellular $X \in Strat_2$, there is a strictly cellular cover

$$\begin{array}{c} \sum C_i \\ \downarrow \\ X \end{array}$$

where each cell C_i is either a point, an interval or an n-gon. For strictly cellular X we define

$$W(\mathscr{D})(X) = \lim W(\mathscr{D})(C_i).$$

Lemma 9.11. Given a prestratifed map $\alpha : X \to Y$ between strictly cellular objects in $Strat_2$, a labelling on Y defines a labelling on X.

Proof. We only sketch the proof because the details are technically very difficult. As in the $n = 1$ case all prestratified maps $X \to Y$ can be factorised as the composite of a subdivision and a stratified map. Here a subdivision is a refinement of the stratification of X. As in the $n = 1$ case, a labelling of the subdivided space gives a labelling on the original space by taking each n-gon in turn, standardising the orientations, tensoring objects and composing or tensoring morphisms in the category to obtain the pullback, see Figure 9.8. To verify that this composite is independent of the order in which we construct it is tricky and so we do not attempt it here.

Next we must consider what labelling gets 'pulled back' across a stratified map. Firstly 0-cells must map to 0-cells so their labellings are pulled back identically, satisfying the condition that all 0-cells must be labelled with the monoidal identity. Edges that map to a 0-cell are labelled with $\mathbf{1}$. If they map with degree 1 to another edge they get labelled with the object on the target edge or if they map with degree -1 they get labelled with the dual object. So far it is almost identical to the Whitney 1-category case. Faces may map constantly to a vertex, in which case we label with the morphism $\mathbf{1} \to \mathbf{1}$, or they may map to an edge, where we label by the identity morphism on the object labelling that edge. Finally, since stratified maps are submersive, faces that map to a face must do so diffeomorphically, and so we pull back the same morphism if the map is degree 1, or the dagger if the map is degree -1. □

This defines the pullback labelling. While we believe that in principle one can verify that this is well-defined and makes $W(\mathscr{D})$ into a presheaf (for the full subcategory of $Prestrat_2$ on the strictly cellular spaces X for which $W(\mathscr{D})(X)$ is defined) we were unable to prove it satisfactorily and so we take a different approach.

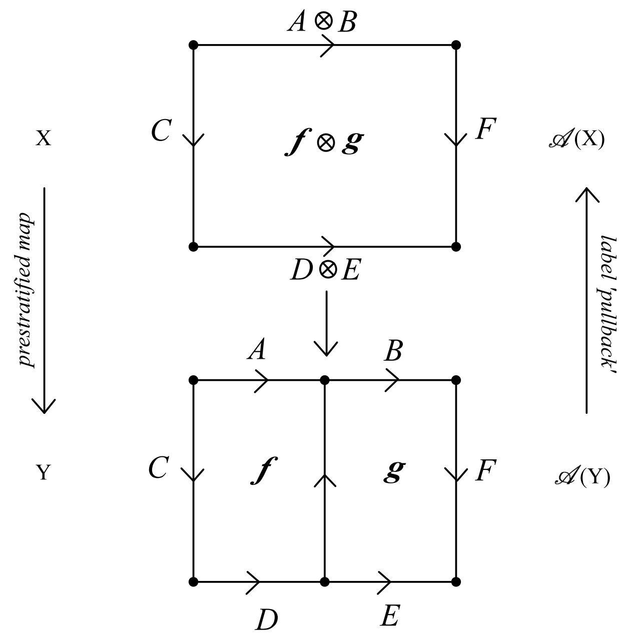


Figure 9.8: An example of the way in which a prestratified map between spaces induces a 'pullback' of labellings.

Conjecture 9.12. There is a unique Whitney 2-category whose restriction to strictly cellular X is $W(\mathscr{D})$.

9.3 The Graph Construction $\widetilde{G}$.

In this section we describe the graph construction which is a functor

$$\widetilde{G} : DagPiv \to 2Whit_1.$$

This construction is more geometrically natural, it generalise tangles, is the simplest to define and it is easily seen to give Whitney 2-categories. It is also integral to defining the ribbon construction $\widetilde{R}$ in the next section.

Definition 9.13. Given $X \in Prestrat_2$, a *graph* Γ in X is a codimension 1 Whitney stratified subspace, of the ambient manifold M of X, which intersects X transversally. A *transversal graph* on X is the germ of the ambient manifold at this intersection, $\Gamma \cap X$. The intersection of codimension 1 and 2 strata with X are the edges and vertices of the transversal graph, respectively, see Figure 9.9. Each vertex is given a framing and each edge is framed such that there is a limiting tangent and framing vector which match (up to sign) the framing of the vertex they are incident to, see Figure 9.10. Edges are then labelled with objects of $\mathscr{D}$ and vertices are labelled with morphisms of $\mathscr{D}$ chosen to be compatible with the labels on edges incident to them, where the framing specifies the input and output objects.

There is an equivalence on the choice of objects as follows: Labelling a framed edge with an object A is equivalent to labelling the oppositely framed edge with A^*. If we choose a different framing on a vertex labelled f we can equivalently label by f^* or $f^{*\dagger}$ depending on whether we reflect corresponding to the tangent direction or the framing direction of the incident edges, see Figure 9.11.

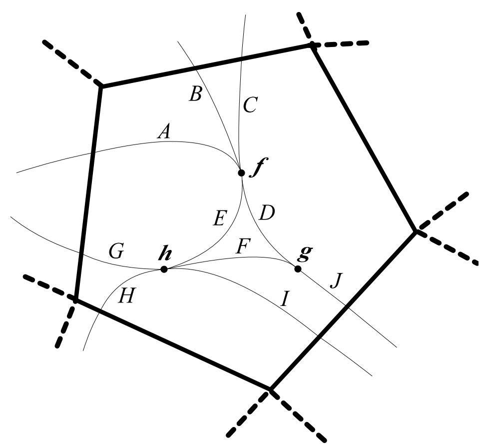


Figure 9.9: A transversal graph in a space X. The strata of X are shown in bold, the edges of the transversal graph are labelled by objects and vertices labelled by morphisms of the category $\mathscr{D}$.

Definition 9.14. Let $\widetilde{G}(\mathscr{D})(X)$ be the set of $\mathscr{D}$-labelled transversal graphs on X up to isotopy relative to their intersection with the 1-skeleton of X.

Lemma 9.15. Given a prestratified map $f : X \to Y$, a $\mathscr{D}$-labelled transversal graph Γ on Y gives a well-defined $\mathscr{D}$-labelled transversal graph on X, i.e. there is a map $f^* : \widetilde{G}(\mathscr{D})(Y) \to \widetilde{G}(\mathscr{D})(X)$.

Proof. First we will show that the preimage of a $\mathscr{D}$-labelled transversal graph in X is a transversal graph in Y. As already mentioned any prestratified map factors as the composite of a subdivision and stratified map. A subdivision is a refinement of the stratification so if a graph is transversal on Y it would certainly be transversal on the same space with fewer strata. We now consider a stratified map. Let M and N be the ambient manifolds to X and Y respectively. Since f

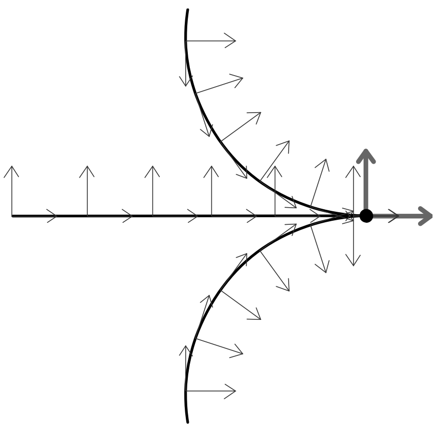


Figure 9.10: A local picture of three edges meeting at a vertex. The limiting tangent vectors are required to be the same (up to sign), and the limiting normal vectors defining the framing are also required to agree (up to sign). (In particular these limits must exist). Moreover the limiting tangent and normal vectors (in that order) should coincide with the framing of the vertex.

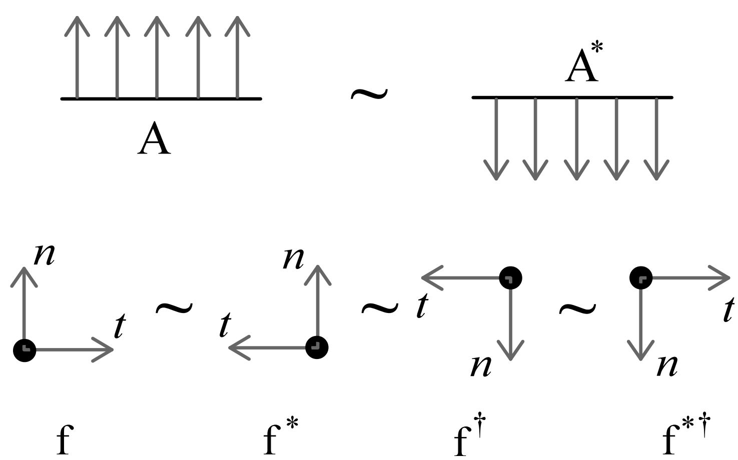


Figure 9.11: On top, the labelling equivalence on edges of a transversal graph. On the bottom, the labelling equivalence on vertices generated by reflecting in the tangent or framing direction.

is stratified, given a stratum $A \subset X$ there is some stratum $B = f(A) \subset Y$. By definition f is the germ of a smooth map $M \to N$ (which we will also call f). Consider the following diagram:

$$\begin{array}{ccccccc} A & \hookrightarrow & X & \hookrightarrow & M & \hookleftarrow & f^{-1}(S) \\ \downarrow{\scriptstyle f|_A} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow \\ B & \hookrightarrow & Y & \hookrightarrow & N & \hookleftarrow & S. \end{array}$$

where S is a codimension 1 stratum of Γ, and $f(a) \in B \cap S$ for some $a \in A$. Then since by definition S intersects B transversally,

$$T_{f(a)}B + T_{f(a)}S = T_{f(a)}N.$$

Since $f : X \to Y$ is a stratified map and in particular $f|_A : A \to B$ is submersive, $df(T_aA) = T_{f(a)}B$. Therefore

$$df(T_aA) + T_{f(a)}S = T_{f(a)}N$$

which implies that $f : M \to N$ is transversal to $S \hookrightarrow N$. By Theorem 2.13, $f^{-1}(S)$ is a codimension 1 submanifold of M, and therefore either transversal to A or $T_aA \subset T_af^{-1}(S)$. The latter is impossible since that would imply $T_{f(a)}B = df(T_aA) \subset df(T_af^{-1}S) \subset T_{f(a)}S$ which is a contradiction since S is transversal to B. The edges of the required transversal graph on X are the union of the germs of intersections of such preimages with X.

Since the restriction of a stratified map to a face is a diffeomorphism, a neighbourhood of a vertex of a transversal graph in Y is diffeomorphic to its preimage. The union of all such preimages (some may be empty) along with the edges already defined constitute a transversal graph in X. Labels are pulled back in the obvious way. □

Remark 9.16. Given a functor $\mathscr{D} \xrightarrow{F} \mathscr{D}'$ we can apply F to the labels of a $\mathscr{D}$-labelled transversal graph to obtain a $\mathscr{D}'$-labelled transversal graph. This defines

$$\widetilde{G}(F) : \widetilde{G}(\mathscr{D}) \to \widetilde{G}(\mathscr{D}').$$

Proposition 9.17. $\widetilde{G}(\mathscr{D})$ is a Whitney 2-category.

Proof. Lemma 9.15 and Remark 9.16 show that $\widetilde{G}(\mathscr{D})$ is a presheaf. The fact that it is a sheaf follows intuitively because to know a graph locally, everywhere, is to know it globally. More precisely it follows by an argument very similar to that for tangles (see Example 7.19). □

9.4 The Ribbon Constructions $\widetilde{R}$.

Next we define the ribbon construction which gives us a functor

$$\widetilde{R} : DagPiv \to 2Whit_1$$

which is left adjoint to the functor $C : 2Whit_1 \to DagPiv$ defined in Section 8.2.

A ribbon graph is a 'fattening' of a transversal graph in X. We will give a detailed description of a fattening but for a more visually intuitive description see Figure 9.12. Given a transversal graph on X :

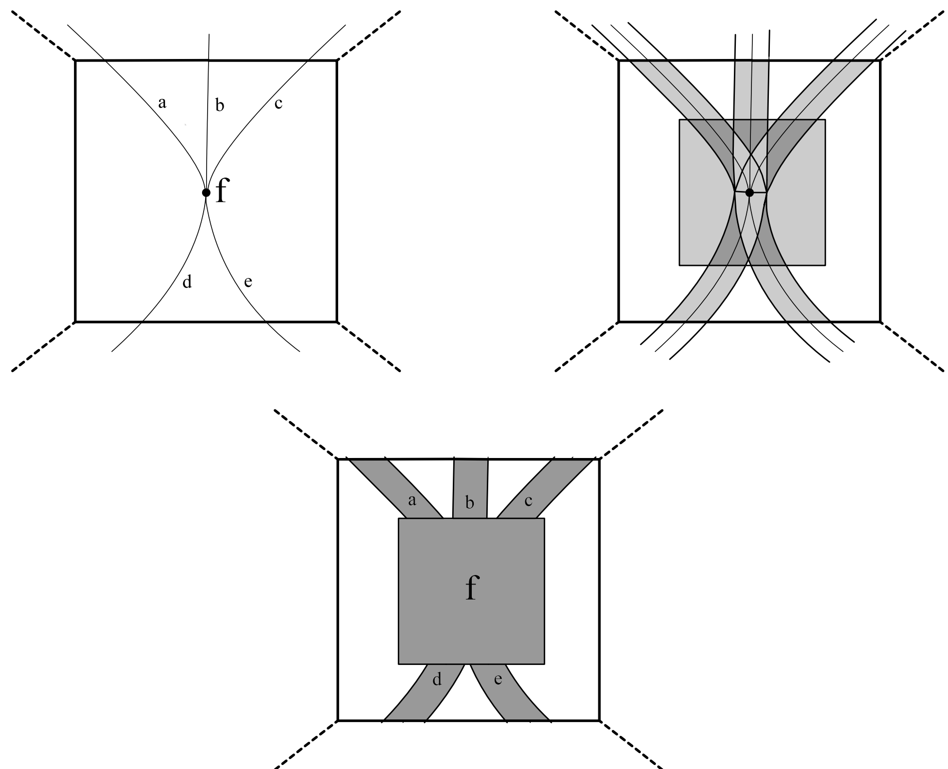


Figure 9.12: Starting on the top left and moving clockwise we have (a) part of a transversal graph on a face of some $X \in Strat_2$, (b) tubular neighbourhoods of the edges and vertices and (c) a refinement of the stratification of X to obtain a ribbon graph.

- Let E be an edge and take a tubular neighbourhood $E \times [0,1] \subset X$, and stratify it by its boundary and interior.

- Let V be a vertex and take a tubular neighbourhood $V \times [0,1]^2 \subset X$, and stratify it by its boundary and interior. These will be known as boxes.

The above 'fattenings' must satisfy:

(i) The interiors of a fattened edge only cross 1-strata which the underlying edge crosses.

(ii) The interior of boxes are entirely inside the 2-stratum which contains the underlying vertex.

(iii) The following form a refinement of the stratification of X:

a) the interior of a box.,

b) a stratum of $E \times [0,1] \cap$ a stratum of $X \cap (X - \bigcup \text{ boxes})$,

c) a stratum of a boundary of a box $\cap$ a stratum of $X \cap (X - \bigcup E \times [0,1])$,

d) a stratum of a boundary of a box $\cap$ a stratum of $X \cap$ a stratum of $E \times [0,1]$,

e) a stratum of $X \cap (X - (\bigcup E \times [0,1] \cup \bigcup \text{boxes}))$.

Moreover an open stratum S as described in b), must be the image of $F \times (0,1)$ for some $F \subset E$ where

$$\begin{array}{ccc} F \times (0,1) & \longrightarrow & S \\ \big\downarrow & & \big\downarrow \\ E \times [0,1] & \longrightarrow & X \end{array}$$

The closure of such an open stratum, i.e. the image of $F \times [0,1]$, is known as a ribbon. We identify $E = E \times \{\frac{1}{2}\} \subset E \times [0,1]$ and $V = (\frac{1}{2}, \frac{1}{2}) \in V \times [0,1]^2 \cong [0,1]^2$.

We refer to the underlying transversal graph as *the skeleton* of the ribbon graph.

A ribbon is then labelled with the object that labels the underlying edge and a box is labelled with the same morphism as the underlying vertex. If a 1-dimensional stratum in the boundary of a ribbon or box intersects an edge of the underlying graph then it is labelled with the same object as that edge. All other 1-dimensional strata get labelled by **1**. We have the same equivalence on labellings of ribbons as that on labellings of edges, see Definition 9.13.

Definition 9.18. Given a dagger pivotal category $\mathscr{D}$, define $\widetilde{R}(\mathscr{D})(X)$ to be the set of $\mathscr{D}$-labelled ribbon graphs on $X \in Strat_n$, up to isotopy relative to the 1-dimensional strata of X.

Lemma 9.19. Given a map $f : X \to Y$, there is an induced map of ribbon graphs $\widetilde{R}(\mathscr{D})(f) : \widetilde{R}(\mathscr{D})(Y) \to \widetilde{R}(\mathscr{D})(X)$.

Proof. We have constructed a ribbon graph as a fattening of a skeleton, i.e. an underlying transversal graph. So to define the induced map we simply pullback this underlying transversal graph as in Lemma 9.15 and then fatten it using the above procedure. □

Remark 9.20. Given a functor $\mathscr{D} \xrightarrow{F} \mathscr{D}'$ we can apply F to the labels of a $\mathscr{D}$-labelled ribbon graph to obtain a $\mathscr{D}'$-labelled ribbon graph. This defines

$$\widetilde{R}(F) : \widetilde{R}(\mathscr{D}) \to \widetilde{R}(\mathscr{D}').$$

Lemma 9.21. $\widetilde{R}(\mathscr{D})$ is a Whitney 2-category.

Proof. Given a prestratified map $f : X \to Y$ and a ribbon graph in $\widetilde{R}(\mathscr{D})(Y)$ we define the pullback ribbon graph in $\widetilde{R}(\mathscr{D})(X)$ by taking the pullback of the underlying transversal graph and fattening as above. Thus $\widetilde{R}(\mathscr{D})$ is a presheaf on $Prestrat_2$.

The argument that $\widetilde{R}(\mathscr{D})(X)$ is a sheaf on $Strat_2$ is very similar to that for tangles (see Example 7.19). If we know a ribbon graph locally we can piece it together to get a global ribbon graph.

□

Definition 9.22. A *maximal ribbon graph* on the interval is a ribbon graph with a single ribbon and no box. The ribbon fills the entire interval, hence maximal. There is the obvious characteristic map to the underlying interval. Similarly on the square we have a single box that is maximal, i.e. it fills the entire square. The ribbons incident to this box are maximal ribbons on the top and bottom of the square.

Theorem 9.23. The functor $\widetilde{R} : DagPiv \to 2Whit_1$ is left adjoint to $C : 2Whit_1 \to DagPiv$, i.e. there exist two natural transformations

$$\begin{aligned} \eta &: 1_{DagPiv} \to C\widetilde{R} \\ \epsilon &: \widetilde{R}C \to 1_{2Whit_1}, \end{aligned}$$

such that the diagrams

$$\begin{array}{ccc} \widetilde{R} & \xrightarrow{\widetilde{R}\eta} & \widetilde{R}C\widetilde{R} \\ & \overset{id_{\widetilde{R}}}{\searrow} & \downarrow{\scriptstyle \epsilon\widetilde{R}} \\ & & \widetilde{R} \end{array} \qquad\qquad \begin{array}{ccc} C & \xrightarrow{\eta C} & C\widetilde{R}C \\ & \overset{id_{C}}{\searrow} & \downarrow{\scriptstyle C\epsilon} \\ & & C. \end{array} \tag{9.3}$$

commute.

Proof. We define the natural transformation η component-wise as

$$\eta_{\mathscr{D}} : \mathscr{D} \to C\widetilde{R}(\mathscr{D})$$

$$d \mapsto \text{the maximal ribbon graph on the interval, labelled by } d,$$

$$\delta \mapsto \text{the maximal ribbon graph on the square, labelled by } \delta,$$

where d is an object of $\mathscr{D}$ and δ is a morphism of $\mathscr{D}$. Given a functor $\mathscr{F} : \mathscr{D} \to \mathscr{D}'$ we get an induced functor $C\widetilde{R}(\mathscr{D}) \to C\widetilde{R}(\mathscr{D}')$ by changing the labels from d to $\mathscr{F}(d)$ and from δ to $\mathscr{F}(\delta)$ in each ribbon graph. Thus

$$\eta_{\mathscr{D}'} \circ \mathscr{F} = C\widetilde{R}(\mathscr{F}) \circ \eta_{\mathscr{D}},$$

and so η is a natural transformation, i.e. the following diagram

$$\begin{array}{ccc} \mathscr{D} & \xrightarrow{\mathscr{F}} & \mathscr{D}' \\ {\scriptstyle \eta_{\mathscr{D}}}\downarrow & & \downarrow{\scriptstyle \eta_{\mathscr{D}'}} \\ C\widetilde{R}(\mathscr{D}) & \xrightarrow[C\widetilde{R}(\mathscr{F})]{} & C\widetilde{R}(\mathscr{D}') \end{array}$$

commutes for a given $\mathscr{D}$.

Next, we define the component $\epsilon_{\mathscr{A}} : \widetilde{R}C(\mathscr{A}) \to \mathscr{A}$.

Given a $C(\mathscr{A})$-labelled ribbon graph in X, we will define an element of $\mathscr{A}(X)$ by considering a ribbon graph to be a refinement X', of the stratification of X. We can then use the characteristic maps of ribbons and boxes together with the labels in $\mathscr{A}(\mathbb{I})$ and $\mathscr{A}(\mathbb{I}^2)$, to define an element of $\mathscr{A}(X')$. Finally we compose using $\mathscr{A}(X') \to \mathscr{A}(X)$.

More precisely, for a ribbon $S = F \times [0,1]$ labelled with $a \in \mathscr{A}(\mathbb{I})$, we have the characteristic map

$$\chi_S : F \times [0,1] \to [0,1],$$

i.e. projection onto the second factor and therefore an induced map

$$\chi_S^* : \mathscr{A}(\mathbb{I}) \to \mathscr{A}(S)$$

which determines a distinguished element $a^* = \chi_S^*(a) \in \mathscr{A}(S)$, with which we label S.

The characteristic map on a box is more complicated, (see Figure 9.13) but we still obtain a distinguished element in $\mathscr{A}(B)$ with which we label B. We do this for all ribbons and boxes. These labels are compatible by construction so we have an element of $\mathscr{A}(X')$. Finally we can compose using the map induced from the refinement $X \to X'$ to obtain an element of $\mathscr{A}(X)$.

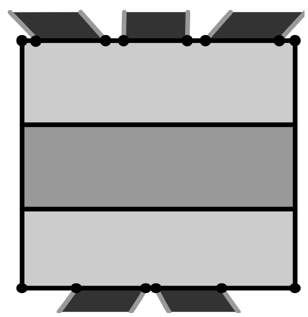


Figure 9.13: To define the characteristic map for a box we divide it into thirds. The middle third maps to $\mathbb{I}^2$. The top third maps to the top edge which is homeomorphic to $\mathbb{I}_{2k+1}$ where k is the number of ribbons incident to the top of the box. Similarly, the bottom third maps to $\mathbb{I}_{2l+1}$ where l is the number of ribbons incident to the bottom of the box. Then given $\alpha \in \mathscr{A}(\mathbb{I}^2)$ labelling the box, and the labels in $\mathscr{A}(\mathbb{I})$ for the incident ribbons, we obtain $a \in \mathscr{A}(\mathbb{I}_{2k+1})$ and $a' \in \mathscr{A}(\mathbb{I}_{2l+1})$. Pulling back across the characteristic maps to $\mathbb{I}^2, \mathbb{I}_{2k+1}$ and $\mathbb{I}_{2l+1}$ respectively allow us obtain an element in $\mathscr{A}(B)$, where B is the box stratified by edges and thirds.

Now suppose we have a morphism between one-object Whitney 2-categories, i.e a natural transformation $\mu : \mathscr{A} \to \mathscr{A}'$. By definition

$$\begin{array}{ccc} \mathscr{A}(\mathbb{I}) & \xrightarrow{\chi_S^*} & \mathscr{A}(S) \\ \downarrow{\scriptstyle \mu_{\mathbb{I}}} & & \downarrow{\scriptstyle \mu_S} \\ \mathscr{A}(\mathbb{I}) & \xrightarrow{\chi_S^*} & \mathscr{A}(S) \end{array}$$

commutes, and a similar statement holds for characteristic maps of boxes. It follows that

$$\begin{array}{ccc} \mathscr{C}\widetilde{R}(\mathscr{A}) & \xrightarrow{\epsilon_{\mathscr{A}}} & \mathscr{A} \\ \downarrow{\scriptstyle \mathscr{C}\widetilde{R}(\mu)} & & \downarrow{\scriptstyle \mu} \\ \mathscr{C}\widetilde{R}(\mathscr{A}') & \xrightarrow{\epsilon_{\mathscr{A}'}} & \mathscr{A}' \end{array}$$

commutes, i.e. that ϵ is natural.

It is left is to show that the diagrams (9.3) commute.

Consider $\widetilde{R}\eta : \widetilde{R}(\mathscr{D})(X) \to \widetilde{R}C\widetilde{R}(\mathscr{D})(X)$. Here, given a $\mathscr{D}$-labelled ribbon graph on X, we relabel to get a $C\widetilde{R}(\mathscr{D})$-labelled graph, i.e. if a ribbon S is labelled with $d \in Obj_{\mathscr{D}}$ we relabel with the maximal ribbon graph on the interval labelled by d. This is an element of $Obj_{C\widetilde{R}(\mathscr{D})} = \widetilde{R}(\mathscr{D})(\mathbb{I})$.

Now take a ribbon S, labelled with a maximal ribbon graph on the interval labelled with d. The characteristic map induces a map $\widetilde{R}(\mathscr{D})(\mathbb{I}) \to \widetilde{R}(\mathscr{D})(S)$ taking the maximal ribbon graph on the interval to a ribbon graph on S and since the characteristic map is projection on to the second factor this ribbon graph on S is labelled with the identity morphism 1_d. (A similar proceedure applies for boxes). We do this for all strata of the ribbon graph and finally we compose to get the original ribbon graph in $\widetilde{R}(\mathscr{D})(X)$.

Finally $C_{\epsilon_{\mathscr{A}}} : C(\mathscr{A}) \to C\widetilde{R}C(\mathscr{A})$ sends $a \in \mathscr{A}(\mathbb{I})$ to the maximal ribbon graph on the interval labelled by a and $f \in \mathscr{A}(\mathbb{I}^2)$ to the maximal ribbon graph on the square labelled by f. The characteristic maps in this case are identities and so the original objects and morphisms are returned. Thus the right-hand diagram commutes. □

9.5 Reduced Constructions $G(\mathscr{D})$, $R(\mathscr{D})$.

Remark 9.24. Recall the description of a graphical language of categories. Consider embedding a copy of $[0,1]^2$ into the interior of X, 'boxing off' a part of the graph in such a way that edges of the graph meeting the interior of the box, only intersect $\{0,1\} \times [0,1]$ transversally and the projection of the graph onto the first coordinate of $[0,1]^2$ is Morse (except at vertices), see Figure 9.14. We can always choose an isotopy of the graph relative to the complement of the box to satisfy these conditions. The Coherence Theorem, (Lemma 3.26) tells us that such an isotopy gives a well-formed equation between morphisms in the category $\mathscr{D}$. Therefore, we can erase the perhaps complicated graph in the boxed off area and replace it with a single vertex labelled by a morphism in $\mathscr{D}$. We then proceed inductively to eventually replace the entire graph in each face of X by one with a single vertex.

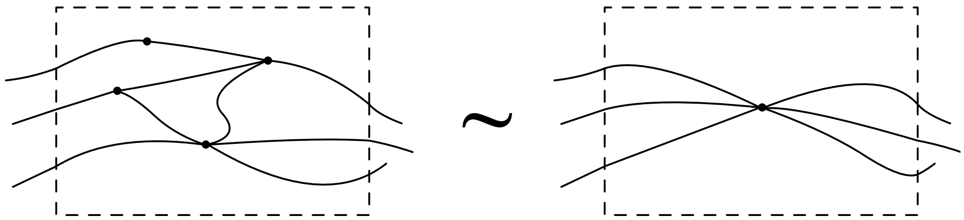


Figure 9.14: Part of the graph is 'boxed off' and replaced with a single vertex. Notice the part of the graph outside the boxed off area remains unchanged.

Definition 9.25. We define an equivalence relation on $\widetilde{G}(\mathscr{D})(X)$ generated by the boxing off relation illustrated in Figure 9.14. Using this equivalence we define

$$G(\mathscr{D})(X) = \widetilde{G}(\mathscr{D})(X)/\sim .$$

Proposition 9.26. $G(\mathscr{D})(X)$ is a Whitney 2-category.

Proof. Since prestratified maps when restricted to the preimage of a 2-dimensional stratum are diffeomorphisms and the equivalence is trivial on higher codimension strata, it should be clear that the equivalence is compatible with $\widetilde{G}(\mathscr{D})(Y) \xrightarrow{f^*} \widetilde{G}(\mathscr{D})(X)$, induced from $f : X \to Y$. We therefore have a presheaf $G(\mathscr{D})(X)$. Moreover, using a similar argument as for $\widetilde{G}(\mathscr{D})$, it is a sheaf. □

Definition 9.27. Using the same procedure as in Remark 9.24, we can replace a general ribbon graph by one with a single box in each 2-dimensional stratum. We define an equivalence relation on $\widetilde{R}(\mathscr{D})(X)$ generated again by the boxing off relation. Using this equivalence we define

$$R(\mathscr{D})(X) = \widetilde{R}(\mathscr{D})(X)/\sim .$$

This equivalence relation is compatible with

$$\widetilde{R}(\mathscr{D})(Y) \xrightarrow{f^*} \widetilde{R}(\mathscr{D})(X)$$

induced from a prestratified $f : X \to Y$ using the same argument as in Proposition 9.26 which also shows that $\widetilde{R}(\mathscr{D})(X)$ is a sheaf.

Proposition 9.28. Given a dagger pivotal category $\mathscr{D}$, the one-object Whitney 2-categories, $W(\mathscr{D})$ (assuming it exists), $G(\mathscr{D})$ and $R(\mathscr{D})$ are all equivalent as Whitney 2-categories.

The following explicit descriptions will be helpful.

Recall that $C(a,b)(Y) = \{f \in C(Y \times \mathbb{I}) \mid f|_{Y\times 0} = a,\ f|_{Y\times 1} = b\}$, where C is a Whitney category, Y is an object in $Prestrat_2$ and $a, b \in Obj_C$. We will use this construction repeatedly.

- $R(\mathscr{D})(pt)$ is the set of equivalence classes of ribbon graphs on a point. There is only one element. We will call this ribbon graph $\mathbf{1}$.
- $W(\mathscr{D})(pt)$ is the set of equivalence classes of labellings of a point. Again there is only one element since a point can only be labelled with $\mathbf{1}$.
- $G(\mathscr{D})(pt)$ is the set of equivalence classes of transversal graphs in a point. This has only one element, i.e. the empty graph, which for convenience we will also call $\mathbf{1}$.
- $R(\mathscr{D})(\mathbf{1},\mathbf{1})(pt)$ is the equivalence class of ribbon graphs on the interval. The equivalence generated by the coherence theorem is trivial on the interval so this is the set of finite subdivisions of the interval labelled with objects of $\mathscr{D}$.
- $W(\mathscr{D})(\mathbf{1},\mathbf{1})(pt)$ is the set of equivalence classes of labellings of the interval. This is in bijection with the set of objects in $\mathscr{D}$.
- $G(\mathscr{D})(\mathbf{1},\mathbf{1})(pt)$ is the set of equivalence classes of transversal graphs in the interval. Again the coherence equivalence on the interval is trivial so this is the set of finite sets of framed points in the interval labelled with objects of $\mathscr{D}$.
- $R(\mathscr{D})(\mathbf{1},\mathbf{1})(d,d')(pt)$ is the set of equivalence classes of ribbon graphs on $[0,1]^2$, where $\{0\} \times [0,1]$ has label $d \in R(\mathscr{D})(\mathbf{1},\mathbf{1})(pt)$ and $\{1\} \times [0,1]$ has label $d' \in R(\mathscr{D})(\mathbf{1},\mathbf{1})(pt)$. Any such ribbon graph is equivalent to one with a single box whose interior is the interior of $[0,1]^2$. The only remaining data in the equivalence class is the morphism used to label this box so this is in bijection with the set $\mathrm{Hom}_{\mathscr{D}}(d,d')$.
- $W(\mathscr{D})(\mathbf{1},\mathbf{1})(d,d')(pt)$ is the set of equivalence classes of labellings of $[0,1]^2$, where $\{0\} \times [0,1]$ is labelled with $d \in W(\mathscr{D})(\mathbf{1},\mathbf{1})(pt)$ and $\{1\} \times [0,1]$ is labelled with $d' \in W(\mathscr{D})(\mathbf{1},\mathbf{1})(pt)$. This is in bijection with the set $\mathrm{Hom}_{\mathscr{D}}(d,d')$.
- $G(\mathscr{D})(\mathbf{1},\mathbf{1})(d,d')(pt)$ is the set of equivalence classes of transversal graphs in $[0,1]^2$, where $\{0\} \times [0,1]$ is labelled with $d \in G(\mathscr{D})(\mathbf{1},\mathbf{1})(pt)$ and $\{1\} \times [0,1]$ is labelled with $d' \in G(\mathscr{D})(\mathbf{1},\mathbf{1})(pt)$. Each such graph is equivalent to one with a single vertex labelled by a morphism so this is in bijection with the set $\mathrm{Hom}_{\mathscr{D}}(d,d')$.

Proof of Proposition 9.28. We define the functor $R(\mathscr{D}) \xrightarrow{M} W(\mathscr{D})$ that given a ribbon graph on a strictly cellular X gives a labelling on X. We first choose an orientation for each stratum of X. Each ribbon has a characteristic map (see proof of Theorem 9.23) so it makes sense to ask if the orientation of a ribbon matches the orientation, as described in the labelling construction, of a 1-dimensional stratum. Taking all the object-labels of the ribbons incident to a particular 1-dimensional stratum we label that 1-dimensional stratum by the tensor product of the labels, changing to their dual if the orientations don't match. If there are no incident ribbons we label by $\mathbf{1}$.

Recall a labelling is a set of choices for each polygon in X. We need a choice of source and target, a chain of edges representing each and an orientation for each face.

There is a characteristic map on the middle third of each box, as described in 9.13. This allows us to talk about an orientation on boxes, i.e. top, bottom, left and right. We choose this as the orientation for each face. In particular we can also specify the leftmost ribbon entering the top of a box. We trace this ribbon back to the edge it intersects, and choose this edge to be the first in the chain representing the source. We continue to add edges in sequence to this chain but stop before adding the edge intersecting the rightmost ribbon incident to the bottom of the box. The remaining chain of edges, with orientations and labels changed appropriately defines the target. Finally we label with the morphism with compatible source and target.

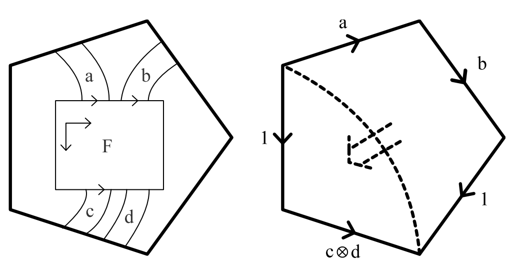


Figure 9.15: From a $\mathscr{D}$-labelled ribbon graph on X we construct a $\mathscr{D}$-labelling. In this figure we have a local picture. The face is labelled with the morphism $a \otimes b \otimes \mathbf{1} \to \mathbf{1} \otimes c \otimes d$.

We next define the functor $R(\mathscr{D}) \xrightarrow{N} G(\mathscr{D})$ that given an equivalence class of ribbon graphs in X gives an equivalence class of transversal graphs on X. We achieve this by simply taking the skeleton of the ribbon graph, which by definition must be a transversal graph.

Consider the span

$$W(\mathscr{D}) \xleftarrow{M} R(\mathscr{D}) \xrightarrow{N} G(\mathscr{D}).$$

- This span of functors is surjective on objects, i.e. $M(pt) : R(\mathscr{D})(pt) \to W(\mathscr{D})(pt)$ and $N(pt) : R(\mathscr{D})(pt) \to G(\mathscr{D})(pt)$ are surjective since they are maps between one-element sets.

- The induced span

$$W(\mathscr{D})(\mathbf{1},\mathbf{1}) \xleftarrow{M} R(\mathscr{D})(\mathbf{1},\mathbf{1}) \xrightarrow{N} G(\mathscr{D})(\mathbf{1},\mathbf{1})$$

is surjective on objects also, where $\mathbf{1}$ is the only object in $R(\mathscr{D})(pt)$. To see this consider $R(\mathscr{D})(\mathbf{1},\mathbf{1})(pt) \to W(\mathscr{D})(\mathbf{1},\mathbf{1})(pt)$. Here we relabel the interval with the tensor product of the labels on the subdivisions so this is clearly surjective. Next given a $\mathscr{D}$-labelled ribbon graph on the interval we take the skeleton to obtain a transversal graph. Each ribbon graph is a fattening of a transersal graph and each transversal graph has many fattenings so $R(\mathscr{D})(\mathbf{1},\mathbf{1})(pt) \to G(\mathscr{D})(\mathbf{1},\mathbf{1})(pt)$ is surjective.

- Finally the maps of sets

$$W(\mathscr{D})(\mathbf{1},\mathbf{1})(f,g) \xleftarrow{M} R(\mathscr{D})(\mathbf{1},\mathbf{1})(f,g) \xrightarrow{N} G(\mathscr{D})(\mathbf{1},\mathbf{1})(f,g)$$

give a 0-equivalence, i.e. $R(\mathscr{D})(\mathbf{1},\mathbf{1})(f,g)(pt) \to W(\mathscr{D})(\mathbf{1},\mathbf{1})(f,g)(pt)$ and $R(\mathscr{D})(\mathbf{1},\mathbf{1})(f,g)(pt) \to G(\mathscr{D})(\mathbf{1},\mathbf{1})(f,g)(pt)$ are bijections. This follows from explicit description of these in the above list which show that all are in bijection with $\mathrm{Hom}(f,g)$.

□

Corollary 9.29. There is an equivalence of dagger pivotal categories,

$$CW(\mathscr{D}) \cong CG(\mathscr{D}) \cong CR(\mathscr{D}).$$

Proof. This follows from a more general fact. Suppose

$$\begin{array}{ccc} & A'' & \\ \swarrow & & \searrow \\ A & & A' \end{array}$$

is an equivalence of one-object Whitney 2-categories, see Definiton 7.3 . Applying C we obtain the span

$$\begin{array}{ccc} & CA'' & \\ \swarrow & & \searrow \\ CA & & CA' \end{array}$$

of dagger pivotal categories. Moreover $CA'' \to CA$ and $CA'' \to CA$ are surjective on objects and fully faithful. Hence we obtain an equivalence of the sort defined in 7.3, of CA and CA'. □

Lemma 9.30. Let $\mathscr{D}$ be a one-object Whitney 2-category. Then

$$\mathscr{D} \xrightarrow{\eta_{\mathscr{D}}} C\widetilde{R}(\mathscr{D}) \xrightarrow{CQ} CR(\mathscr{D})$$

is an isomorphism of categories, where η is the unit of the adjunction and Q is the quotient $\widetilde{R} \to R$.

Proof. It suffices to show that $Q \circ \eta_{\mathscr{D}}$ is a bijection on objects and morphisms of $\mathscr{D}$.

This follows since an object d maps bijectively to the maximal ribbon graph on the interval labelled by d. The coherence equivalence is trivial on the interval so objects in $C\widetilde{R}(\mathscr{D})$ are in bijection with those in $CR(\mathscr{D})$. Similarly a morphism δ maps bijectively to the maximal ribbon graph on the square labelled by δ which in turn maps bijectively to the class of the ribbon graph with a single box labelled by δ. □

It follows that R is faithful and C is full.

Corollary 9.31. R is a fully faithful embedding of $DagPiv$ into $2Whit_1$.

So one-object Whitney 2-categories can be though of as generalisations of dagger pivotal categories.

9.6 Relationship to Previously Defined Concepts

We would hope to recover the 'ordinary' transversal homotopy categories from Section 6.1 (at least for $n = 1, 2$) by applying our construction to transversal homotopy Whitney n-categories.

9.6.1 Transversal Homotopy Theory

Recall in Section 7.2 we defined the transversal homotopy Whitney $(n+k)$-category $\Psi_{k,n+k}(M)$, of a Whitney stratified manifold M, where given $X \in Strat_{n+k}$,

$$\Psi_{k,n+k}(M)(X) = \left\{ \begin{array}{c} \text{Germs of transversal maps } g : X \to M \\ \text{such that whenever} \\ S \subset X \text{ and } \dim S < k, \text{ then } S \subset g^{-1}(p) \end{array} \right\} / \sim .$$

Here $\sim$ is the equivalence relation given by homotopy through transversal maps relative to all strata $S \subset X$ with $\dim S < n + k$.

Example 9.32. In particular, we have the transversal homotopy Whitney 1-category $\Psi_{0,1}(M)$ which to $X \in Strat_1$ associates,

$$\Psi_{0,1}(M)(X) = \left\{ \begin{array}{c} \text{Germs of transversal maps } g : X \to M \text{ such that} \\ \text{whenever } S \subset X \text{ and } \dim S < 0 \text{ then } S \subset g^{-1}(p) \end{array} \right\} / \sim$$

where $\sim$ is the equivalence relation given by homotopy through transversal maps relative to all strata $S \subset X$ with $\dim S < 1$. Since there are obviously no strata of dimension less than 0 the basepoint condition is vacuous and so we are left with the set of transversal maps of X into M up to homotopy through transversal maps, relative to all 0-strata.

Applying the construction C from Section 8.1 we obtain a dagger category $C(\Psi_{0,1}(M)(X))$ with objects

$$\Psi_{0,1}(M)(pt) = \left\{ \text{ Germs of transversal } g : pt \to M \ \right\}$$

and morphisms

$$\Psi_{0,1}(M)(\mathbb{I}) = \left\{ \text{ Germs of transversal } g : \mathbb{I} \to M \ \right\} / \sim$$

up to homotopy through transversal maps fixing the endpoints. This is equivalent to $\Psi_{0,1}(M)$ the 0th transversal homotopy category of M.

Example 9.33. Next take the Whitney 2-category $\Psi_{1,2}(M)$ which to $X \in Strat_2$ associates the set

$$\Psi_{1,2}(M)(X) = \left\{ \begin{array}{c} \text{Germs of transversal maps } g : X \to M \text{ such that} \\ \text{whenever } S \subset X \text{ and } \dim S < 1 \text{ then } S \subset g^{-1}(p) \end{array} \right\} / \sim .$$

Here $\sim$ is the equivalence relation given by homotopy through transversal maps relative to all strata $S \subset X$ with $\dim S < 2$. Because of the basepoint condition any map from the point into M must have image the basepoint, also the homotopy condition is trivial, so $\Psi_{1,2}(M)(pt) = \mathbf{1}$. We thus have a one-object Whitney 2-category, to which we apply the construction from Section 9.1 to obtain the dagger pivotal category $C(\Psi_{1,2}(M))$ with objects,

$$\Psi_{1,2}(M)(\mathbb{I}) = \left\{ \begin{array}{c} \text{Germs of transversal } g : \mathbb{I} \to M \text{ where} \\ \partial\mathbb{I} \mapsto p \end{array} \right\}$$

This is the set of based loops in M, transversal to all strata. Here the homotopy equivalence relation is trivial. The morphisms are

$$\Psi_{1,2}(M)(\mathbb{I}^2) = \left\{ \begin{array}{c} \text{Germs of transversal } g : \mathbb{I}^2 \to M \text{ such that} \\ g(0,y) = g(1,y) = p \end{array} \right\} / \sim .$$

Recall, in Section 6.2 we defined the nth transversal homotopy category of a based Whitney stratified manifold as $\Psi_{n,n+1}(M)$ to be the category whose objects are based transversal maps $(\mathbb{I}^n, \partial\mathbb{I}^n) \to (X, \star)$ and whose morphisms are represented by based transversal maps

$$f : (\mathbb{I}^n \times [0,1], \partial\mathbb{I}^n \times [0,1]) \to (M, \star)$$

with the boundary condition,

$$f(p,t) = \left\{ \begin{array}{ll} f(p,0) & t \in [0,\epsilon] \\ f(p,1) & t \in [1-\epsilon, 1] \end{array} \right.$$

for some $\epsilon > 0$. Two morphisms are equivalent if there is a homotopy through transversal maps between them. We claim the two notions on transversal category are equivalent, at least for $n = 0$ and $n = 1$.

Proposition 9.34. There is a functor $F : \Psi_{1,2}(M) \to C(\Psi_{1,2}(M))$, from the first transversal homotopy category of M, to the dagger pivotal category $C(\Psi_{1,2}(M))$, which is an equivalence.

Proof. Given an object of $\Psi_{1,2}(M)$, i.e. a based transversal map $(\mathbb{I}, \partial\mathbb{I}) \to (X, \star)$, which is constant in some neighbourhood of the boundary, we can extend it to a map $\mathbb{R} \to M$ which is constant outside of $\mathbb{I}$. If we then take the germ of $\mathbb{R}$ at $\mathbb{I}$, we get an element of $\Psi_{1,2}(M)(\mathbb{I})$, i.e. an object of $C(\Psi_{1,2}(M))$.

Given a morphism, i.e. a based transversal map

$$f : (\mathbb{I} \times [0,1], \partial\mathbb{I} \times [0,1]) \to (M, \star),$$

which is constant in some neighbourhood of $\mathbb{I} \times \{0,1\}$, we can extend to a map $\mathbb{R}^2 \to M$ as in Figure 9.16. Again we take the germ of $\mathbb{R}^2$ at $\mathbb{I}^2$ to obtain an element of $\Psi_{1,2}(M)(\mathbb{I}^2)$, i.e. a morphism of $C(\Psi_{1,2}(M))$. We claim this is an equivalence, i.e. F is essentially surjective and fully faithful. Given any object in $C(\Psi_{1,2}(M))$, i.e. the germ of a based transversal map $g : \mathbb{I} \to M$ there is an object in $\Psi_{1,2}(M)$ which is isomorphic to it and so F is fully faithful. We represent this isomorphism by the square in Figure 9.17. Any map representing a morphism in $C(\Psi_{1,2}(M))$ is homotopic to some extension of a map representing

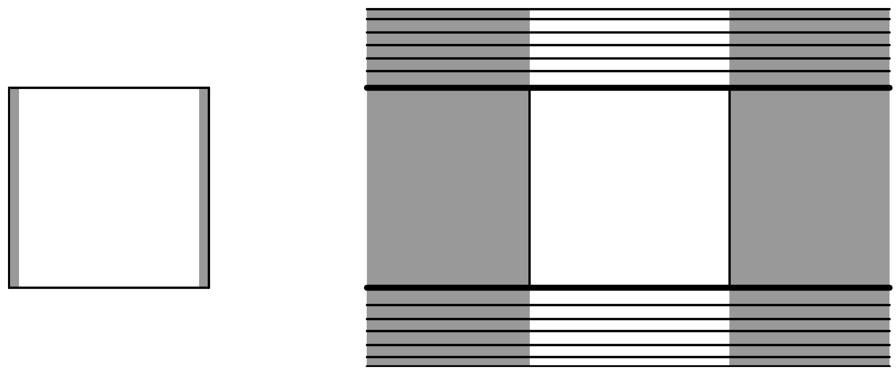


Figure 9.16: On the left, $\mathbb{I} \times [0,1]$ where the shaded area maps to the basepoint. On the right the map is extended to $\mathbb{R}^2$. The endpoints of each slice, i.e. $\{0,1\} \times \{t\}$, map to the basepoint allowing us to extend to $(-\infty, 0) \cup (1, \infty) \mapsto$ basepoint. The slices above and below the square are defined as having the same image as the extended versions of the top and bottom slices of the square.

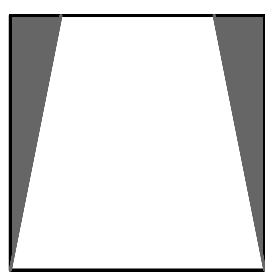


Figure 9.17: Representation of the isomorphism that is a transversal map $f : \mathbb{I} \to M$ on the top slice and a map $g : \mathbb{I} \to M$ on the bottom slice.

a morphism in $\Psi_{1,2}(M)$). And since homotopic maps are considered the same in these categories, F is surjective on morphisms.

Finally, given an isomorphism between two extensions, represented by a cube with one extension on the front and the other on the back, there is a homotopy between the map this represents and the cube which is the same extension on each slice. Thus F is injective and so fully faithful. $\square$

9.6.2 Tangles

In Section 7.2 we defined the Whitney $(n+k)$-category $nTang_k^{fr}$ of framed tangles, where given $X \in Strat_{n+k}$ (with ambient manifold M),

$$nTang_k^{fr}(X) = \left\{ \begin{array}{c} \text{Germs } T \text{ of } M \text{ at transversal intersections of } X \\ \text{with codimension } k \text{ framed submanifolds of } M. \end{array} \right\} / \sim .$$

The equivalence relation $\sim$ is given by ambient isotopy of T in X, relative to all strata of dimension strictly less than $n+k$. A tangle T is codimension k in X with a framing given by the restriction of the framing of the codimension k submanifold of M.

Example 9.35. The Whitney 1-category $0Tang_1^{fr}$ of framed tangles associates to $X \in Strat_1$ the set

$$0Tang_1^{fr}(X) = \left\{ \begin{array}{c} \text{Germs } T \text{ of } M \text{ at transversal intersections of } X \\ \text{with codimension 0 framed submanifolds of } M \end{array} \right\} / \sim .$$

where M is the ambient manifold of X. This is a one element set containing the germ of X.

Example 9.36. Next consider the Whitney 2-category $1Tang_1^{fr}$ with

$$1Tang_1^{fr}(X) = \left\{ \begin{array}{c} \text{Germs } T \text{ of } M \text{ at transversal intersections of } X \\ \text{with codimension 1 framed submanifolds of } M \end{array} \right\} / \sim .$$

where M is the ambient manifold of X. The equivalence relation $\sim$ is given by ambient isotopy, relative to all strata of dimension strictly less than 2. The set of tangles in a point is trivial so ${}_1Tang_1^{fr}(pt) = \mathbf{1}$. We can again apply the construction from Section 9.1 to get a dagger pivotal category with objects:

$${}_1Tang_1^{fr}(\mathbb{I}) = \left\{ \begin{array}{c} \text{Germs } T \text{ of } M \text{ at transversal intersections of } \mathbb{I} \\ \text{with codimension 1 framed submanifolds of } M. \end{array} \right\} / \sim .$$

This is the set of framed points on the interval. The morphisms are:

$$ {}_1Tang_1^{fr}(\mathbb{I}^2) = \left\{ \begin{array}{c} \text{Germs } T \text{ of } M \text{ at transversal intersections of } \mathbb{I}^2 \\ \text{with codimension 1 framed submanifolds of } M. \end{array} \right\} / \sim . $$

By a similar argument as for the transversal homotopy categories, this is equivalent to our earlier definition of tangles.

Chapter 10

The Tangle Hypothesis

In terms of Whitney n-categories the Tangle Hypothesis states:

The Whitney category $nTang_k^{fr}$ of framed tangles is equivalent to the free k-tuply monoidal Whitney n-category on one $\mathbb{S}^k$-morphism.

We will show that there are equivalences,

$$\Psi_{k,n+k}(\mathbb{S}^k) \longrightarrow nTang_k^{fr} \longleftarrow Rep(\mathbb{S}^k)$$

of Whitney n-categories, and that $Rep(\mathbb{S}^k)$ is the free k-tuply monoidal Whitney n-category on one $\mathbb{S}^k$ morphism, in the sense that given any other such k-tuply monoidal Whitney n-category on one $\mathbb{S}^k$ morphism $\mathscr{A}$ and an $\mathbb{S}^k$-morphism $a \in \mathscr{A}(\mathbb{S}^k)$, there is a unique functor up to isomorphism,

$$Rep(\mathbb{S}^k) \to \mathscr{A}$$

$$id_{\mathbb{S}^k} \mapsto a.$$

To show the above we will consider

$$\begin{array}{ccccc} \Psi_{k,n+k}(\mathbb{S}^k)(X) & \longrightarrow & nTang_k^{fr}(X) & \longleftarrow & Rep(\mathbb{S}^k)(X) \\ & & f^{-1}(p) & \longleftarrow\!\shortmid & (X \xrightarrow{f} \mathbb{S}^k) \\ (X \xrightarrow{g} \mathbb{S}^k) & \longmapsto & g^{-1}(0) & & \end{array}$$

where p is a basepoint of $\mathbb{S}^k$ in an open stratum and 0 is the point-stratum of $\mathbb{S}^k$ and the maps f and g are transversal and prestratified respectively.

This connects the three interpretations of Whitney n-categories, i.e. the homotopy theoretic $\Psi_{k,n+k}(M)$, the geometric $nTang_k^{fr}$ and the algebraic $Rep(X)$.

Lemma 10.1. The Whitney $(n+k)$-category $Rep(\mathbb{S}^k)$ is the free k-tuply monoidal Whitney n-category on one $\mathbb{S}^k$-morphism.

Proof. This follows from the Yoneda lemma, see for example [16, Page 26, Chaper 1, §1] which states that functors
$Rep(\mathbb{S}^k) \to \mathscr{A}$ are in bijection with $\mathscr{A}(\mathbb{S}^k)$. Given a k-tuply monoidal Whitney n-category $\mathscr{A}$ and an $\mathbb{S}^k$-morphism $a \in \mathscr{A}(\mathbb{S}^k)$ there is a unique functor of Whitney $(n+k)$-categories, i.e. a natural transformation

$$Rep(\mathbb{S}^k) \to \mathscr{A} : [f : X \to \mathbb{S}^k] \mapsto f^*a$$

mapping the distinguished element $id_{\mathbb{S}^k}$ to a. □

Proposition 10.2. The k-tuply monoidal Whitney n-categories $nTang_k^{fr}$ and $Rep(\mathbb{S}^k)$ are equivalent.

Proof. Fix a basepoint p in the open stratum of $\mathbb{S}^k$. Using the Pontryagin-Thom construction we define a functor

$$F_p : Rep(\mathbb{S}^k) \to nTang_k^{fr}.$$

Take a prestratified map $f : X \to \mathbb{S}^k$. This by definition is a germ of a smooth map of the ambient manifolds, which must also be submersive onto the open stratum when $f^{-1}(p) \neq \varnothing$. Then $f^{-1}(p)$ is a codimension k framed submanifold of M which, when intersected with X, is transversal to all strata. Moreover, if f and g are homotopic through prestratified maps relative to all strata of dimension $< (n+k)$ then the preimages $f^{-1}(p)$ and $g^{-1}(p)$ will be ambient isotopic relative to all strata of dimension $< (n+k)$. To see this note that f and g are homotopic through maps transversal to p and apply Lemma 5.13. That we can perform the homotopy relative to strata of dimension $< (n+k)$ follows from the fact that the homotopy is relative to strata of dimension $< (n+k)$. Define F_p by the assignment $[f] \mapsto [f^{-1}(p)]$.

We claim that F_p is an $(n+k)$-equivalence. Recall for this to be true, $F_p(pt)$ must be surjective and the induced functor

$$Rep(\mathbb{S}^k)([f],[g]) \to nTang_k^{fr}([f^{-1}(p)],[g^{-1}(p)])$$

must be an $(n+k-1)$-equivalence.

Consider an element of $nTang_k^{fr}(X)$, i.e. a framed codimension k tangle in X. We can choose a neighbourhood of T in the ambient manifold M (such that its boundary intersects all strata transversally, and only intersects strata which T does). Moreover, we can choose an isomorphism of NT, the normal bundle to

T in M, with an open neighbourhood so that it gives a tubular neighbourhood of $S \cap T$ in S for each stratum $S \subset X$. We construct a prestratified 'collapse map' $f : X \to \mathbb{S}^k$ so that $T = f^{-1}(p)$, a tubular neighbourhood of T fibres over the open stratum, and the complement of this neighbourhood maps to the 0-stratum. It follows that F_p is surjective for any X and in particular for $X = pt$.

Next we check that we have an induced $(n+k-1)$-equivalence. We now consider prestratified maps $X \times \mathbb{I} \to \mathbb{S}^k$ with boundary conditions given by f and g, and framed tangles in $X \times \mathbb{I}$ with boundary tangles given by $f^{-1}(p)$ and $g^{-1}(p)$. Since we can always extend the collapse maps defined on the boundaries to all of $X \times \mathbb{I}$, the induced functor is surjective on objects. And so we reduce to checking if we have an $(n+k-2)$-equivalence.

Proceeding inductively we reach the base cases:

These concern the map of sets $[f] \mapsto [f^{-1}(p)]$ from homotopy classes of transversal maps $f : [0,1]^{n+k} \to \mathbb{S}^k$, where the restriction of f to the boundary is some fixed map, φ say, to the set of isotopy classes of framed codimension k tangles in $[0,1]^{n+k}$, where the boundary tangle is $\varphi^{-1}(p)$. We can always construct a prestratified collapse map for such a tangle T, extending the given map on the boundary. Indeed, the collapse map is unique up to homotopy through prestratified maps. Moreover, given an isotopy $h_t : [0,1]^{n+k} \to [0,1]^{n+k}$ relative to the boundary, and such that $h_t(T)$ is transversal to all strata for each $t \in [0,1]$, we can construct a family of collapse maps for the tangles $h_t(T)$ yielding a homotopy between a collapse map for $T = h_0(T)$ and a collapse map for $h_1(T)$. Hence in the base case there is a bijection between isotopy classes of framed tangles and homotopy classes of prestratified collapse maps (each with appropriate boundary conditions). It follows that F_p is an $(n+k)$-equivalence. □

Proposition 10.3. The Whitney $(n+k)$-categories, $\Psi_{k,n+k}(\mathbb{S}^k)$ and $nTang_k^{fr}$ are $(n+k)$-equivalent.

Proof. Given a transversal map $f : X \to \mathbb{S}^k$, taking the pre-image of the point-stratum induces a functor

$$\Psi_{k,n+k}(\mathbb{S}^k) \to nTang_k^{fr} : [f] \mapsto [f^{-1}(0)].$$

Notice the differences to the functor F_p. First f must be transversal not to the basepoint but to the point-stratum and secondly it need not be submersive to the entire open stratum but only at the point-stratum.

The proof that F an $(n+k)$-equivalence is almost word-for-word the same as that of Proposition 10.2, but with 0 replacing p, and with transversal maps to S^k replacing prestratified maps. □

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