Homfly Skein Theory
of
Reversed String Satellites

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Abstract

This thesis aims to use Homfly skein theory to give a geometric interpretation of useful and interesting algebraic objects. We consider tangles and the skein of the annulus. Previous work has generally been restricted to considering strings in tangles or around the annulus travelling in one direction. Our extension allows strings travelling in both directions. We extend many of the existing results into this arena, at the same time as developing some new ideas.

Acknowledgements

Firstly I would like to thank my supervisor and mentor Professor Hugh R. Morton. His support and invaluable advice, not to mention his endless patience, has been essential in the writing of this thesis.

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Thank you to all my family for their support; Mum, Dad, Louise and Gemma. Finally, a very special thank you to Carol, your love is everything to me.

To *Mum* and *Dad* for always being there, and,
To *Carol* for being my soulmate...
I couldn't have done any of this without you all

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Introduction

In this work we aim to extend the understanding of Homfly skein theory, in particular when trying to give a geometric interpretation of useful and interesting algebraic objects. Much work that precedes this thesis has considered the skein theoretic view of algebraic objects such as the Hecke algebra, including [Jon87, MT90, Mor93, Ais96, AM98, Luk01] and many more besides.

Our extension begins with an algebra $H_{n,p}$ in which strings in the geometric viewpoint can be considered in both directions. We now offer the highlights from each chapter.

The overall aim of this work is to develop some new concepts at the same time as bringing together much recent work that has previously only appeared spread across the literature.

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The first chapter gives the necessary ingredients for the remainder of the work. The concept of Homfly skein theory is introduced. The Homfly polynomial is first defined and is then used to give a general definition of a Homfly skein.

Before giving specific examples of Homfly skeins, a description of some useful skein maps is given, followed by a slight diversion into defining the concepts and terminology associated with Young diagrams.

Finally four Homfly skeins are defined. Firstly the skein of a rectangle with n input and n output points. This is known to be isomorphic to the Hecke algebra H_n.

An extension of this is then reintroduced from a geometric viewpoint (initially given by [MW, Had]). This algebra is denoted $H_{n,p}$ and comes from considering the skein of a rectangle as with H_n, but this time it should have n input and p output points on one side and n output and p input points on the opposite side.

We then give two different skeins of the annulus. The first is denoted $\mathcal{C}$ and is broken down into subspaces which are defined by wiring the previous two skeins, H_n and $H_{n,p}$, into the annulus.

The second is a lesser known skein, denoted $\mathcal{A}$. It arises from considering the annulus with an input point specified on the inner boundary component and an output point specified on the outer boundary component. It is isomorphic to a skein used by Kawagoe [Kaw98] with the input and output point on the same boundary component. It has been adapted more recently as it lends itself well to providing elegant proofs through its unexpected algebraic properties. Although it is linearly isomorphic to the skein of Kawagoe, it is this more recent adaptation that has meant it could be considered as an algebra. It is the commutative algebraic properties that make the calculations we rely upon later in Chapter 4 possible. As we shall see, elements of the skein are used in determinants, see also in [Mor02b, Luk01].

Chapter 2 defines the Murphy operators. The original context for such objects was the group algebra $\mathbb{C}[S_n]$ of the symmetric group and is defined in terms of sums of transpositions. This concept was extended to the Hecke algebra H_n by Dipper and James [DJ87]. We offer a survey of some results involving these elements and the centre of H_n, mainly by Ram and Morton. This includes a nice skein theoretic representation of the Murphy operators and some interesting connections between these elements, the centre of H_n and the symmetric functions (see [Mac79] for a complete survey of symmetric functions).

This chapter ends with an introduction of a potential set of Murphy operators for the algebra $H_{n,p}$. We also attempt to connect these to central elements of $H_{n,p}$. Following the precedent of the H_n case, we find there is a path from these elements to a certain type of symmetric function, the so-called supersymmetric functions.

The third chapter describes the results of work by the author with Morton. These results have now been published in [MH02]. The work of Chapter 2 is used to give an understanding of two natural linear maps defined on the skein of the annulus $\mathcal{C}$ encircling it with a loop once.

This work has arisen as a result of a paper by T.-H. Chan [Cha00]. There Chan discusses the Homfly polynomial of reverse string parallels of the Hopf link. In this chapter we see that the calculations made by Chan can be made very readily using our techniques. An essential ingredient to our techniques is showing that these linear maps have a set of distinct eigenvalues, answering a question raised by Chan.

We end this chapter by using our results to calculate the Homfly polynomials of some specific reverse string satellites of the Hopf link. We also observe that this approach is still incomplete due to a minimal knowledge of the elements $Q_{\lambda,\mu} \in \mathcal{C}$.

The intention of Chapter 4 is to fill a gap in the knowledge as noted in Chapter 3. This gap is the minimal knowledge of the elements $Q_{\lambda,\mu}$ in the full skein $\mathcal{C}$. The final goal is to give an explicit formula for $Q_{\lambda,\mu}$ in terms of the determinant of simpler skein elements.

In trying to achieve this goal we are required to take a diversion through the skein $\mathcal{A}$ whilst introducing a new type of matrix whose entries follow a specific pattern and can be manipulated in a very prescribed way.

After much work on these matrices we draw together the techniques learned and results discovered to give the derivation of a matrix whose determinant will yield an explicit formula for the $Q_{\lambda,\mu}$.

The final chapter, Chapter 5, aims to finish this work by giving a brief survey of some work of other authors that relates to the general themes discussed here. Although the overlap between our work and that to be discussed in this chapter has not been fully explored, it is felt by the author that such an exploration has potential for further study.

It is hoped by the author that these avenues may be given some thought and their potential explored.

Chapter 1

Skein Theory

The purpose of this chapter is to introduce the basic constructions that will be central to the majority of the work to follow.

1.1 The Homfly polynomial

The *Homfly polynomial* is a two-variable isotopy invariant of oriented links and, since its discovery, has been the subject of much study. It was first described by several groups; [FYH$^+$85, PT87]. Its discovery followed the construction of a simpler polynomial invariant V, the so-called Jones polynomial [Jon85], found using von Neumann algebras and braid groups.

Various versions of the Homfly polynomial appear in the literature. The framed version to the fore in this work, denoted for a link L, $P(L)$, is determined by the Homfly polynomial skein relations:

$$\begin{aligned} P(L_+) - P(L_-) &= (s - s^{-1})P(L_0) \\ \text{and } P(T_+) &= v^{-1}P(T_0), \end{aligned}$$

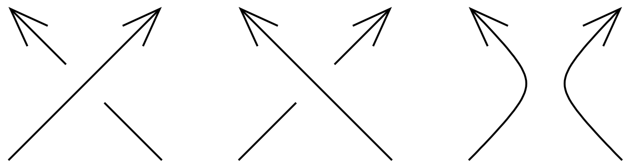
where L_+, L_- and L_0 are oriented links which differ only in a disc as shown in Figure 1.1; and T_+ and T_0 differ only in a disc as shown in Figure 1.2.

The second of the skein relations given above allows one to take account of the writhe of the link.

We normalize the Homfly polynomial by setting $P(\emptyset)$, where $\emptyset$ is the empty link, equal to 1. Also, a direct consequence of the skein relations is that

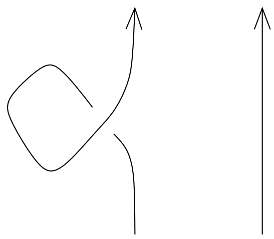
$$P(L \sqcup \bigcirc) = \frac{v^{-1} - v}{s - s^{-1}} P(L)$$

where $L \sqcup \bigcirc$ is the link L with a disjointly embedded null-homotopic oriented loop.



L_+ L_- L_0

Figure 1.1: L_+, L_- and L_0 differ only as shown.



T_+ T_0

Figure 1.2: T_+ and T_0 differ only as shown.

Remark. (i) The Homfly polynomial of the oriented m-component unlink, $\mathcal{U}^m = \sqcup_{i=1}^m \bigcirc$, is $P(\mathcal{U}^m) = \delta^m$, where $\delta = \frac{v^{-1}-v}{s-s^{-1}}$.

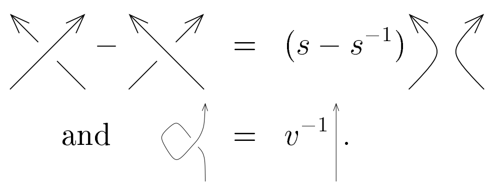
Remark. (ii) If L^* is the reflection of a link L, then

$$P(L^*)(s,v) = P(L)(s^{-1}, v^{-1}).$$

1.2 Homfly skein theory

Skein theory was first introduced by J.H. Conway, a Liverpool born mathematician, [Con70]. Skein theory can be considered from many viewpoints; here we are interested in the skein theory associated to the Homfly polynomial.

Following the description of the Homfly polynomial given above, the Homfly skein relations are



Now let F be a *planar surface* with a fixed (possibly empty) set of input and output points on the boundary. We allow the surface to have holes. We consider *diagrams* in F which consist of oriented arcs joining input points to output points and oriented closed curves, up to Reidemeister moves R_{II} and R_{III} [Rei32] (reminders of all three Reidemeister moves are shown in Figure 1.3).

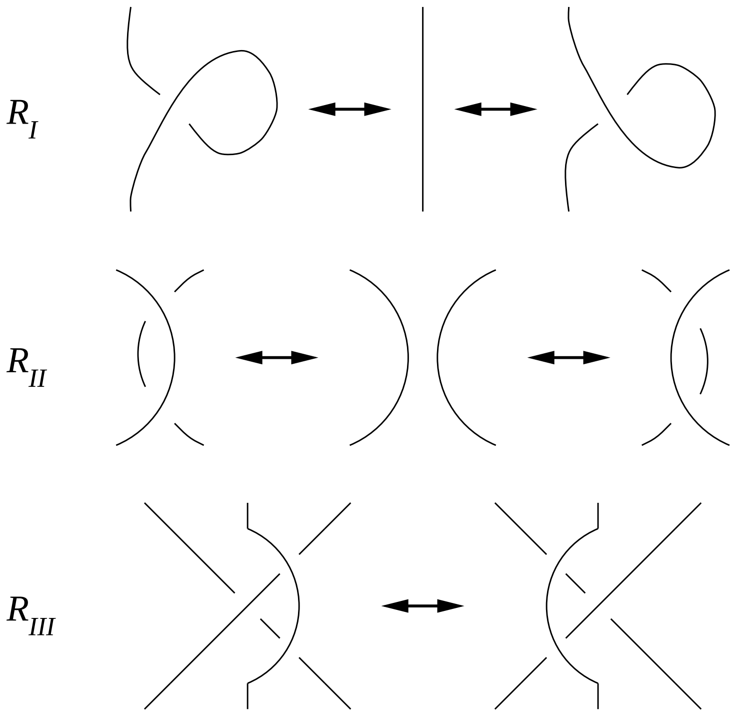


Figure 1.3: Reidemeister moves R_I, R_{II} and R_{III}

Within a diagram in F, the strands at a crossing point are distinguished in the conventional way as an overcrossing and an undercrossing. Clearly, if the surface F is to have input and output points there must be an equal number of each.

Similarly to the Homfly polynomial skein relations, it is a consequence that for a diagram D, $D \sqcup \bigcirc = \frac{v^{-1}-v}{s-s^{-1}} D$.

The *Homfly skein*, $\mathcal{S}(F)$, of a surface F is then defined to be Λ-linear combinations of diagrams in F, modulo the Homfly skein relations given above, for a suitable coefficient ring Λ.

The coefficient ring can be taken as $\Lambda = \mathbb{Z}[v^{\pm 1}, s^{\pm 1}]$ with monomials in $\{s^k - s^{-k} : k \geq 0\}$ admitted as denominators.

We notice the empty diagram is only admitted when F has no boundary points specified. The relation which is given above as a consequence of the Homfly skein relations allows the removal of an oriented nul-homotopic closed curve without crossings, at the expense of multiplication by the scalar $\delta = \frac{v^{-1}-v}{s-s^{-1}}$. This relation is a consequence of the main relations except where the removal of the curve leaves the empty diagram.

1.3 Skein maps

1.3.1 Wiring maps

We can map the skein of a surface, F, into the skein of another, F' say. We do this through a construction called a *wiring*. A wiring w of F into F' is a choice of inclusion of F into F' and a choice of a fixed diagram of curves and arcs in $F' \backslash F$. The boundary of this fixed diagram is the union of the distinguished set of F and F'. Examples of wiring will be essential in some of the work to follow.

1.3.2 A mirror map

We define a *mirror map*,

$$\bar{\ } : \mathcal{S}(F) \to \mathcal{S}(F')$$

induced by switching all crossings in the diagram, coupled with inverting v and s in Λ.

1.3.3 180° rotation

This skein map is induced by a 180° rotation of diagrams in F about the horizontal axis A, as shown in Figure 1.4. This is denoted $* : \mathcal{S}(F) \to \mathcal{S}(F)$. There is no effect on s and v in Λ.

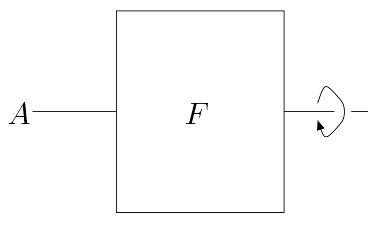


Figure 1.4: The involution $*$ rotates F about the axis A.

1.3.4 An evaluation map

There is also an *evaluation map*,

$$\langle\ \rangle : \mathcal{S}(F) \to \Lambda.$$

This is obtained by wiring F into the plane by some prescribed wiring map, in particular, if F has no boundary points then just "forget" its boundary. Then for an element $X \in \mathcal{S}(F)$, $\langle X \rangle$ is just the framed Homfly polynomial of X after wiring into the plane.

1.3.5 A closure map

Given a surface F with a non-empty set of boundary points, we can wire elements $X \in \mathcal{S}(F)$ into the skein of another surface F' without any boundary points using a *closure map*. Such a map would have arcs in $F' \backslash F$ joining, in some prescribed way, the input points to the output points of F.

1.4 Young diagrams

We now take a temporary diversion from skein theory to discuss the well studied topic of *Young diagrams*. Only a brief description will be given here but a fuller account appears in a great many texts such as [Wey46, FH91, Jon90]. Here we shall concentrate only on the details essential to our studies.

A Young diagram describes both a partition and a graphical representation of the partition. Let λ be a Young diagram representing the integer n. Our λ is then an array of square cells (each of equal size) with l rows. We denote the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_i, \ldots, \lambda_l)$ such that there are λ_i cells in the i^{th} row enumerated from top to bottom, with $\sum_{i=1}^{l} \lambda_i = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_i \geq \lambda_l$.

For $n = 0$ the Young diagram (0) is the empty diagram $\emptyset$.

The number of cells in a Young diagram λ is denoted by $|\lambda|$ and the length $l(\lambda) = l$ is the number of non-zero rows. The *conjugate* of λ is denoted $\lambda^\vee$ and is the transposition of λ such that the rows of λ are the columns of $\lambda^\vee$. In other word, this is equivalent to reflecting in the leading diagonal. We have $(\lambda^\vee)^\vee = \lambda$ for any Young diagram λ.

We also assign a co-ordinate system to each Young diagram. The j^{th} cell in the i^{th} row reading from left-to-right, top-to-bottom, is denoted $(i, j) \in \lambda$, and the *content* $\text{cn}(c)$ of the cell $c = (i, j) \in \lambda$ is defined to be $j - i$. We have that the *hook length* of a cell $(i, j) \in \lambda$ is defined to be $\text{hl}(i, j) = \lambda_i - i + \lambda_j^\vee - j + 1$.

The number of partitions of a natural number n (equivalently, the number of Young diagrams with n cells) shall be denoted $\pi(n)$. (The standard notation used for the number $\pi(n)$ is $p(n)$; our alternative notation has been chosen to avoid a clash with notation required later in this work.) Finally, the *standard tableau* $T(\lambda)$ is a Young diagram for λ with the numbers 1 to n assigned to each cell, such that the numbers increase from left-to-right and from top to bottom.

1.5 The Hecke algebra

The *Hecke algebra*, H_n of type A_{n-1} is a deformed version of the group algebra of the symmetric group S_n. It has been well studied from many different viewpoints, and hence has many different but equivalent incarnations. It will be most conveniently thought of in this context as having explicit presentation

$$H_n = \left\langle \sigma_i : i = 1, \ldots, n-1 \;\middle|\; \begin{array}{ccc} \sigma_i\sigma_j = \sigma_j\sigma_i & : & |i-j| > 1; \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & : & 1 \leq i < n-1; \\ \sigma_i - \sigma_i^{-1} = s - s^{-1}. & & \end{array} \right\rangle.$$

We discuss how to translate from this variant into some of its isomorphic variants at the end of this section.

Now consider the following geometric scenario. Consider a surface $I \times I$, a rectangle, with n input points specified across the bottom and n output points across the top. Denote this surface $F = R_n^n$, as shown diagrammatically in Figure 1.5.

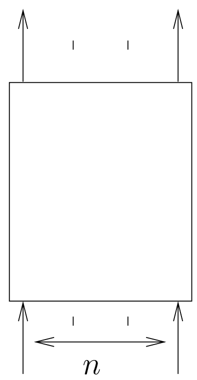
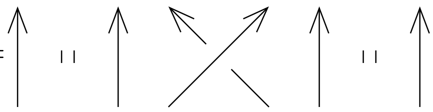


Figure 1.5: The surface R_n^n.

Diagrams in F then consist of oriented arcs joining the inputs to the outputs and oriented closed curves, up to Reidemeister moves II and III. Such diagrams in R_n^n are known as n-tangles.

Now consider the skein $\mathcal{S}(R_n^n)$, Λ-linear combinations of n-tangles in R_n^n, modulo the Homfly skein relations.

Composition of diagrams D_1 and D_2 in R_n^n is achieved by stacking D_2 above D_1. This composition induces a product which makes $\mathcal{S}(R_n^n)$ into an algebra. It has a linear basis of $n!$ elements and its generators are the elementary braids

$\sigma_i =$ 

where the crossing occurs between the i^{th} and $i{+}1^{\text{th}}$ string, for $i = 1, \ldots, n{-}1$.

It is shown in [MT90] that the skein theoretic algebra $\mathcal{S}(R_n^n)$ with coefficient ring extended to include $v^{\pm 1}$, is isomorphic to the Hecke algebra, H_n, of type A_{n-1}. We notice that the variable v does not appear in the presentation of the abstract algebra H_n. It is present when following a geometric route to allow one to reduce general tangles to linear combinations of braids, by means of the Homfly skein relations. The variable v comes into play in dealing with curls using the second Homfly skein relation and in handling disjoint closed curves. In other words it is required to keep track of the framing of the diagrams.

From this point we shall, perhaps rather lazily, consider $\mathcal{S}(R_n^n)$ and H_n synonymously. The juxtaposition of putting tangles $S \in H_n$ to the left of $T \in H_m$ is denoted $S \otimes T$ and is an element of $H_n \otimes H_m \hookrightarrow H_{n+m}$.

In the special case $s - s^{-1} = 0$, the Hecke algebra reduces to $\mathbb{C}[S_n]$ with σ_i becoming the transposition $(i\; i+1)$. In this case there is no possibility of any curls being present hence the v is not required in the presentation.

As said previously, there are different isomorphic variants of the Hecke algebra. We will now describe two others and show how to translate between our standard definition and these variants.

One variant includes an extra variable x whose function it to keep track of the writhe of a diagram. We denote this variant $H_n(x, z)$ and obtain H_n from it by setting $x = 1$ and $z = s - s^{-1}$. The quadratic relation for $H_n(x, z)$ in terms of generators ρ_i is then $x^{-1}\rho_i - x\rho_i^{-1} = z$.

A further variant is seen in many algebraic texts. We shall denote this variant $H_n(q)$ as it is usually seen to include the indeterminate q. The quadratic relation is usually given with roots q and -1. With generators τ_i the quadratic relation is $\tau_i^2 = (q-1)\tau_i + q$.

The three variants of the Hecke algebra given here are all isomorphic, related by the isomorphisms given below:

$$\begin{array}{ccccc} H_n & \cong & H_n(x,z) & \cong & H_n(q) \\ \sigma_i & \mapsto & x^{-1}\rho_i & & \\ & & \rho_i & \mapsto & s^{-1}x\tau_i, \end{array}$$

where q, z and s are related by $z = s - s^{-1}$ and $q = s^2$.

1.5.1 Quasi-idempotent elements in H_n

The group algebra $\mathbb{C}[S_n]$ has idempotent elements which are described by the classical Young symmetrizers. For a Young diagram λ its Young symmetrizer is the product of the sum of permutations which preserve the rows of the standard tableau $T(\lambda)$ and the alternating sum of permutations which preserve the columns.

It is then reasonable to suppose that corresponding elements exist in H_n replacing permutations by suitably weighted positive permutation braids. Jones [Jon87] describes the two idempotents which correspond to the single row and single column Young diagrams, with other authors giving descriptions for general λ, including Gyoja [Gyo86].

Given the Gyoja construction as a starting point, a pleasing skein picture based on the Young diagram λ was given by Aiston and Morton [Ais96, AM98]. With this it was possible to see many pleasing properties for these idempotent elements.

For H_n, we denote these idempotent elements e_λ with $|\lambda| = n$. Before we continue we briefly describe the basic process followed in constructing such elements. However, for a full account of this the interested reader should still refer to [AM98] or [Ais96]. We deliberately avoid any technicalities here to avoid repetition later when we construct single row and single column idempotents in Section 2.3. Instead we shall concentrate on the rather elegant pictorial view of the e_λ and some of the basic properties.

Recall that the quadratic relation for the presentation of the Hecke algebra is

$$\sigma_i - \sigma_i^{-1} = s - s^{-1}.$$

This can be factorised to $(\sigma_i - a)(\sigma_i - b) = 0$ with $a = -s^{-1}$ and $b = s$. Now define

$$a_n = \sum_{\pi \in S_n} (-a)^{-l(\pi)} w_\pi \qquad \text{and} \qquad b_n = \sum_{\pi \in S_n} (-b)^{-l(\pi)} w_\pi,$$

where $l(\pi) = \mathrm{wr}(w_\pi)$, the writhe of the braid w_π.

Now for each $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ we want to define elements e_λ. First we give a three-dimensional picture of the elements, referring to it now as

E_λ. Imagine the strings of the tangle lined up to pass through the centres of templates of the Young diagram λ at its top and bottom. At its input points, the strings are grouped together with linear combinations a_j of braids where the rows have j cells. At the output points, the strings are grouped with linear combinations of b_j of braids where the columns have j cells.

To make this explanation clear we now use an explicit example. Consider the Young diagram $\nu = (4, 3, 1, 1)$. We then have that E_ν is the tangle shown in Figure 1.6.

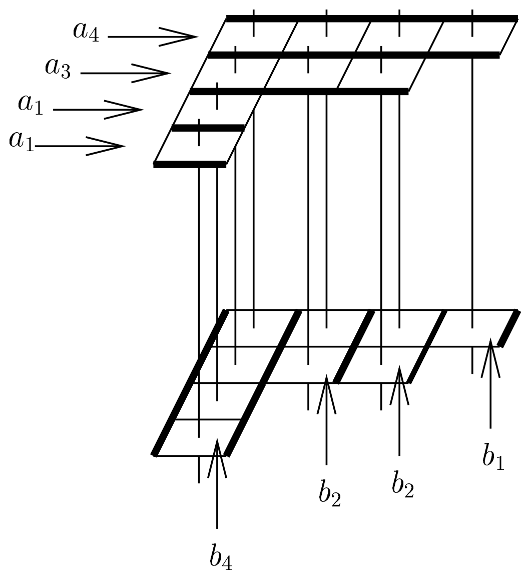


Figure 1.6: The 3-dimensional representation of E_ν with $\nu = (4, 3, 1, 1)$.

Now how do we translate from this three-dimensional picture to our usual flat interpretation of tangles? From this three-dimensional picture we *flatten* it out into two dimensions, ensuring that the resulting crossings that are made are all positive.

A main feature of these elements is captured in the following theorem.

Theorem 1.1 (Aiston-Morton [AM98]). *Let λ and μ be Young diagrams with n cells. Then*

$$\begin{aligned} e_\lambda e_\mu &= 0 \qquad \textit{for } \lambda \neq \mu, \\ e_\lambda^2 &= \alpha_\lambda e_\lambda \qquad \textit{for some scalar } \alpha_\lambda. \end{aligned}$$

Thus distinct Young diagrams determine orthogonal elements, while each e_λ is a quasi-idempotent element of H_n.

More information on these interesting elements will emerge during the course of this work. As a taster, we will be particularly interested in the effect of central elements of H_n on the e_λ. Given elements $c \in Z(H_n)$, we will want to find the values of c_λ where $ce_\lambda = c_\lambda e_\lambda$.

There are clearly $\pi(n)$ of these elements in H_n as they coincide with the number of partitions of n.

1.6 $H_{n,p}$ — A generalized Hecke algebra?

We now consider a family of extended variants of the Hecke algebras discussed previously.

Let us consider a surface $I \times I$, a rectangle, with n input and p output points specified across the top, and matching n output and p input points across the bottom. Denote the surface $F = R^{n,p}_{n,p}$, as shown in Figure 1.7.

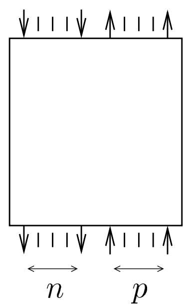


Figure 1.7: The surface $R^{n,p}_{n,p}$.

As before, diagrams in F consist of oriented arcs joining the inputs to the outputs and oriented closed curves, up to Reidemeister moves II and III. Such diagrams in $R^{n,p}_{n,p}$ are to be known as (n,p)-tangles.

Write $H_{n,p}$ for the skein $\mathcal{S}(R^{n,p}_{n,p})$. There is a natural algebra structure on $H_{n,p}$ induced by placing one (n,p)-tangle above the other. When we set $n = 0$ (or $p = 0$), we notice that the resulting algebra is isomorphic to the Hecke algebra H_p (or H_n respectively).

The algebra $H_{n,p}$ has been studied by Kosuda and Murakami, [KM93], in the context of $sl(N)_q$ endomorphisms of the module $V^{\otimes n} \otimes \bar{V}^{\otimes p}$, where V is the fundamental N-dimensional module.

The author of this work has also studied this algebra previously [Had]. This included describing the algebra geometrically as above and finding an explicit skein-theoretic basis for it. We briefly discuss some of the details from [Had], with further details about $H_{n,p}$ being revealed in subsequent chapters of this work as they are required.

Firstly, one should observe that there is a linear isomorphism of $H_{n,p}$ with $H_{(n+p)}$, however this is not in general an algebra isomorphism. This linear isomorphism is a wiring which does nothing to the p positively oriented strings and turns the n negatively oriented strings around into positively oriented strings. Clearly there is an element of choice in this wiring.

The algebra $H_{n,p}$ is generated by the elements σ_i, for $-(n-1) \leq i \leq p-1$, where the skein theoretic representation of the elements $\{\sigma_i : -(n-1) \leq i < 0\}$, σ_0 and $\{\sigma_i : 0 < i \leq p-1\}$ are shown in Figure 1.8 (a), (b) and (c) respectively. Also, $H_{n,p}$ has a linear basis of $(n+p)!$ elements.

(a)

(b)

(c)

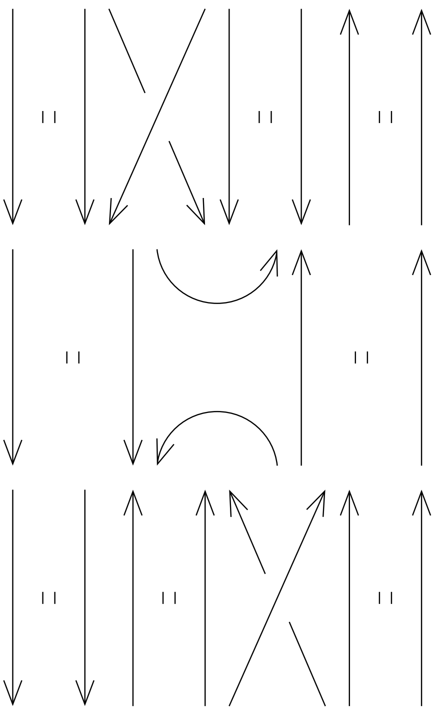


Figure 1.8: (a)$\{\sigma_i : -(n-1) \leq i < 0\}$; (b)$\sigma_0$; (c)$\{\sigma_i : 0 < i \leq p-1\}$.

1.6.1 New elements from old

Using elements of H_n we can immediately find elements of $H_{n,p}$. Consider first the image of H_n under the involution $*$. Clearly then $*(H_n) \otimes H_p \hookrightarrow H_{n,p}$.

Given the Gyoja-Aiston-Morton elements $e_\lambda \in H_n$ described above, we can find an obvious set of idempotent elements in $H_{n,p}$. These elements are to be denoted $e'_{(\lambda,\mu)} := e_\lambda^{(-)} \otimes e_\mu^{(+)}$ formed by the juxtaposition of e_λ and e_μ with appropriate orientations and $|\lambda| = n$ and $|\mu| = p$. There are the $\pi(n) \times \pi(p)$ of these.

1.7 Two skeins of the annulus

In this section we define two skeins of the annulus. The first is very well-known and has received much attention from several authors. The second however has only recently begun to receive the attention it deserves.

1.7.1 The skein $\mathcal{C}$

Let F be the annulus, $F = S^1 \times I$. Then $\mathcal{S}(S^1 \times I)$ is the Homfly skein of the annulus. We denote this by $\mathcal{C}$. This skein is discussed in some detail in [Mor93] and originally in 1988 in the preprint of [Tur97].

We shall represent an element $X \in \mathcal{C}$ diagrammatically as in Figure 1.9.

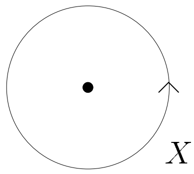


Figure 1.9: An element $X \in \mathcal{C}$.

The skein $\mathcal{C}$ has a product induced by placing one annulus outside another. This defines a bilinear product under which $\mathcal{C}$ becomes an algebra. This algebra is clearly commutative (lift the inner annulus up and stretch it so the outer one will fit inside it).

Turaev [Tur97] showed that $\mathcal{C}$ is freely generated as an algebra by the elements $\{A_m, m \in \mathbb{Z}\}$ where A_m is represented by the skein theoretic element shown in Figure 1.10. The sign of the index m indicates the orientation of the curve. A positive m denotes counterclockwise orientation and a negative m a clockwise orientation. The element A_0 is the identity element, represented by the empty diagram.

Subspaces of $\mathcal{C}$

The algebra $\mathcal{C}$ can be thought of as the product of subalgebras $\mathcal{C}^+$ and $\mathcal{C}^-$ which are generated by $\{A_m : m \in \mathbb{Z}, m \geq 0\}$ and $\{A_m : m \in \mathbb{Z}, m \leq 0\}$ respectively.

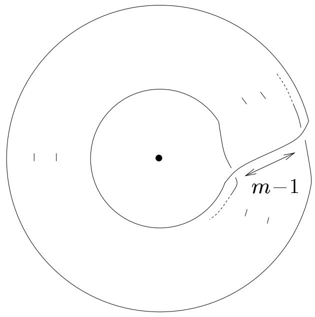


Figure 1.10: An element $A_m \in \mathcal{C}$, for $m \in \mathbb{Z}$.

We now take the surface $F = R_n^n$ and wire it into the annulus, $F' = S^1 \times I$ as shown in Figure 1.11.

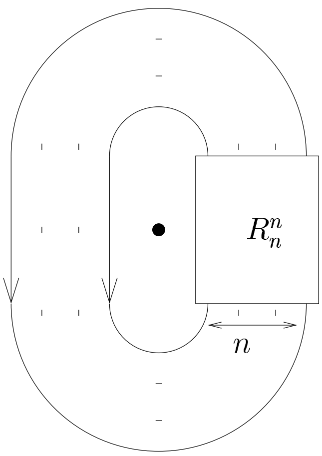


Figure 1.11: R_n^n wired into $S^1 \times I$.

The resulting skein is a linear subspace of $\mathcal{C}^+$ which we shall call $\mathcal{C}^{(n)}$. This subspace can be thought of as the image of H_n under the closure map $\wedge : H_n \to \mathcal{C}^{(n)}$. For an n-tangle $T \in H_n$, we denote its image under this closure map into $\mathcal{C}^{(n)}$ as $\wedge(T)$ or $\hat{T}$.

The subspace $\mathcal{C}^{(n)}$ is then spanned by monomials in $\{A_m\}$, with $m \in \mathbb{Z}^+$, of total weight n, where $\mathrm{wt}(A_m) = m$. It is clear that this spanning set consists of $\pi(n)$ elements, the number of partitions of n. $\mathcal{C}^+$ is then graded as an algebra

$$\mathcal{C}^+ = \bigoplus_{n=0}^{\infty} \mathcal{C}^{(n)}.$$

We can now extend our view of the skein of the annulus to include strings oriented in both directions. We do this through considering the closure of oriented (n,p)-tangles in the annulus. Equivalently, this is achieved through wiring the surface $R^{n,p}_{n,p}$ into the annulus $S^1 \times I$, analagous to the way shown in Figure 1.11.

We denote the algebra formed through considering the image of $H_{n,p}$ under the closure map by $\mathcal{C}^{(n,p)} \subset \mathcal{C}$.

Unlike the case for $\mathcal{C}^{(n)}$ where $\mathcal{C}^{(n)} \cap \mathcal{C}^{(n-1)} = \emptyset$, we have that

$$\mathcal{C}^{(n,p)} \supset \mathcal{C}^{(n-1,p-1)} \supset \mathcal{C}^{(n-2,p-2)} \supset \cdots \supset \begin{cases} \mathcal{C}^{(n-p,0)} & \text{if } \min(n,p) = p, \\ \mathcal{C}^{(0,p-n)} & \text{if } \min(n,p) = n, \end{cases}$$

however, it should be noted that for each $\mathcal{C}^{(i,j)}$ in the sequence above, the difference $i - j$ remains constant throughout. Also

$$\begin{aligned} \mathcal{C}^{(m,0)} &\cong \mathcal{C}^{(m)}_{(-)} \\ \text{and } \mathcal{C}^{(0,m)} &\cong \mathcal{C}^{(m)}_{(+)}, \end{aligned}$$

where the $(-)$ or $(+)$ subscripts indicate the direction of the strings around the centre of the annulus. However, we do have that $\mathcal{C}^{(n_1,p_1)} \cap \mathcal{C}^{(n_2,p_2)} = \emptyset$ if $n_1 - p_1 \neq n_2 - p_2$.

We find that $\mathcal{C}^{(n,p)}$ is spanned by suitably weighted monomials in

$$\{A_{-n}, \ldots, A_{-1}, A_0, A_1, \ldots, A_p\}.$$

We can see that

$$\mathcal{C}^{(n,p)} = \left(\mathcal{C}^{(n)}_{(-)} \times \mathcal{C}^{(p)}_{(+)}\right) + \mathcal{C}^{(n-1,p-1)}.$$

The spanning set of $\mathcal{C}^{(n,p)}$ then consists of $\pi(n,p)$ elements where

$$\begin{aligned} \pi(n,p) &:= \sum_{j=0}^{k} \pi(n-j)\pi(p-j) \\ &(= \pi(n)\pi(p) + \cdots + \pi(n-k)\pi(p-k)), \end{aligned}$$

where $k = \min(n,p)$.

Similar to the grading of $\mathcal{C}^+$ with the $\mathcal{C}^{(n)}$ we can think of the full skein $\mathcal{C}$ in terms of the $\mathcal{C}^{(n,p)}$

$$\mathcal{C} = \bigoplus_{k=-\infty}^{\infty} \left(\bigcup_{n,p \geq 0} \{\mathcal{C}^{(n,p)} : n - p = k\} \right).$$

All that is left for us to do now is to use an example to illustrate what we meant by $\mathcal{C}^{(n,p)}$ being spanned by "suitably weighted" monomials in the range $\{A_i : -n \leq i \leq p\}$.

Example. Consider when $n = 4$ and $p = 2$. The spanning set of $\mathcal{C}^{(4,2)}$ consists of 15 ($= 5 \cdot 2 + 3 \cdot 1 + 2 \cdot 1$) elements, since

$$\mathcal{C}^{(4,2)} = \left(\mathcal{C}^{(4)}_{(-)} \times \mathcal{C}^{(2)}_{(+)}\right) + \left(\mathcal{C}^{(3)}_{(-)} \times \mathcal{C}^{(1)}_{(+)}\right) + \left(\mathcal{C}^{(2)}_{(-)} \times \mathcal{C}^{(0)}_{(+)}\right).$$

The spanning set is therefore

$$\{A_{-4}A_2, A_{-4}A_1^2, A_{-3}A_{-1}A_2, A_{-3}A_{-1}A_1^2, A_{-2}^2A_2, A_{-2}^2A_1^2, A_{-2}A_{-1}^2A_2,$$
$$A_{-2}A_{-1}^2A_1^2, A_{-1}^4A_2, A_{-1}^4A_1^2, A_{-3}A_1, A_{-2}A_{-1}A_1, A_{-1}^3A_1, A_{-2}, A_{-1}^2\}$$

where, for example, the element $A_{-3}A_1$ is obtained from closing an element in $H_{4,2}$ as shown in Figure 1.12.

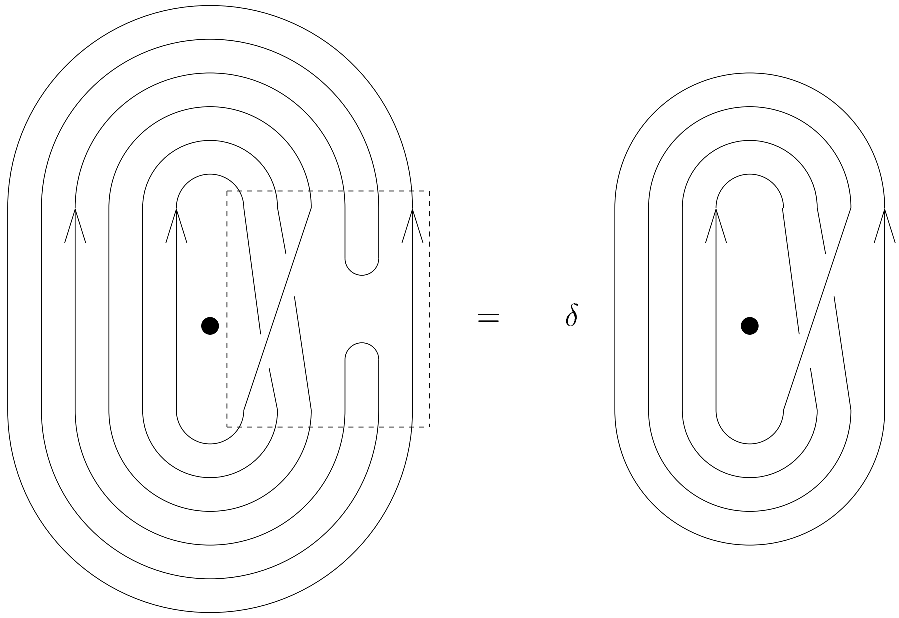


Figure 1.12: The generator $A_{-3}A_1$.

1.7.2 The skein $\mathcal{A}$

Consider again the annulus $S^1 \times I$. Let the outer boundary curve be C_1 and the inner boundary curve C_2. Now pick points $\gamma_1 \in C_1$ and $\gamma_2 \in C_2$ such that γ_1 is an output point and γ_2 is an input point, and denote these by γ_1^{out} and γ_2^{in} respectively.

Let F be the surface $S^1 \times I$ with an associated set of boundary points $\{\gamma_1^{\text{out}}, \gamma_2^{\text{in}}\}$ as described above. Then $\mathcal{S}(F) = \mathcal{S}(S^1 \times I, \{\gamma_1^{\text{out}}, \gamma_2^{\text{in}}\})$ is the

Homfly skein of the surface represented diagramatically in Figure 1.13. We shall denote this skein by $\mathcal{A}$.

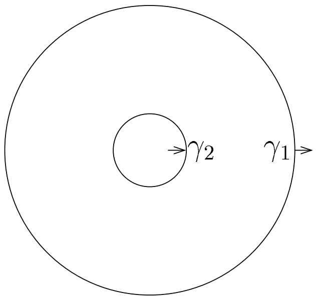


Figure 1.13: The annulus with two boundary points.

Similar to $\mathcal{C}$, the skein $\mathcal{A}$ becomes an algebra under the product induced by placing one annulus outside another. The identity element this time cannot be the empty diagram due to the points specified on the boundary. It is the element $e \in \mathcal{A}$ represented by the diagram shown in Figure 1.14, obtained by joining the two boundary points by a single straight arc.

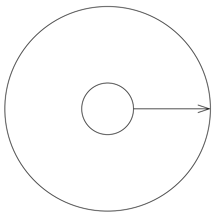


Figure 1.14: $e \in \mathcal{A}$.

A further element of $\mathcal{A}$, also with no crossings, we shall call $a \in \mathcal{A}$ and represent it by the diagram shown in Figure 1.15. From this, powers, a^m for $m \in \mathbb{Z}$, can be constructed, giving for example the elements shown in Figure 1.16.

Another property that $\mathcal{A}$ has in common with $\mathcal{C}$ is

Theorem 1.2 (Morton). *As an algebra, $\mathcal{A}$ is commutative.*

However, unlike the case of $\mathcal{C}$ this is not immediately obvious. After the introduction of a bit more technology, we offer a proof from [Mor02b].

Remark. A skein which is isomorphic to $\mathcal{A}$ is used by Kawagoe [Kaw98] and other authors. Their version is based on the annulus with input and output

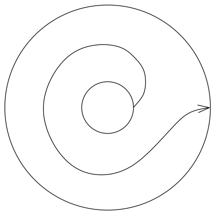


Figure 1.15: $a \in \mathcal{A}$.

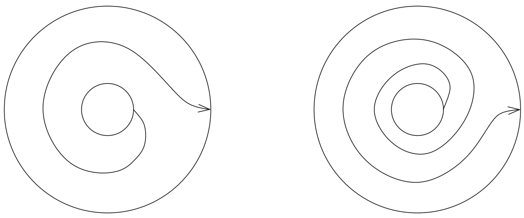
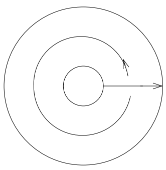
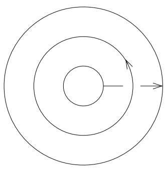


Figure 1.16: a^{-1} and $a^2 \in \mathcal{A}$.

points both on the same boundary component. More recently its use has been adopted by the author as its unexpected algebraic properties allow for some satisfyingly clean proofs. For more work on this interesting skein from this viewpoint, see also [Mor02b], and work by Lukac [Luk01].

We also have two bilinear products which involve the skein $\mathcal{A}$. These are $l : \mathcal{C} \times \mathcal{A} \to \mathcal{A}$ and $r : \mathcal{A} \times \mathcal{C} \to \mathcal{A}$ and are induced by placing an element of $\mathcal{C}$ respectively under or over an element of $\mathcal{A}$. For example, recall that $A_1 \in \mathcal{C}$ is represented by a single counterclockwise loop, so this gives

$l(A_1, e) =$  and $r(e, A_1) =$ .

We now give the proof which was promised above.

Proof of Theorem 1.2 [Mor02b]. Using standard skein theory techniques we can represent any element of $\mathcal{A}$ as a linear combination of tangles consisting of a totally descending arc lying over a number of closed curves. This is achieved through ensuring that on traversing an arc, each time one encircles the centre of the annulus it is passing below the part already traversed, and if not the skein relations can be used to change crossings as required. Each such tangle represents $l(c_m, a^m) = l(c_m, e)a^m$ for some m and some $c_m \in \mathcal{C}$. The general element of $\mathcal{A}$ can then be written as a Laurent polynomial

$$\sum_{m \in \mathbb{Z}} l(c_m, e)a^m$$

in a, with coefficients in the commutative subalgebra $l(\mathcal{C}, e) \subset \mathcal{A}$. Since a commutes with $l(\mathcal{C}, e)$ it follows that any two elements of $\mathcal{A}$ commute. ♠

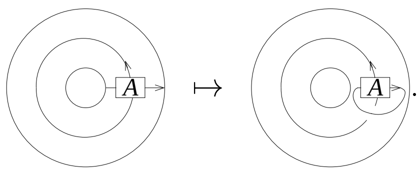
The subalgebras $l(\mathcal{C}, e)$ and $r(e, \mathcal{C})$ are both isomorphic, but they *are not* equal. We can use their difference to define a sort of commutator map

$$[\ , e] : \mathcal{C} \to \mathcal{A}$$

where for $c \in \mathcal{C}$, $[c, e] = l(c, e) - r(e, c)$.

Finally let us define a type of closure map particular to this skein $\mathcal{A}$. Our map will take an element of $\mathcal{A}$ and make it an element of $\mathcal{C}$ by joining the two boundary points over the top of the annulus. We have

$$\diamond : \mathcal{A} \to \mathcal{C}$$



As we alluded to above, we shall not study the skein $\mathcal{A}$ here independently, rather use it as a tool, capitalizing on its unexpected algebraic properties.

Chapter 2

Murphy Operators

Historically, the *Murphy operators* have appeared in various arenas. Initially they were defined independently in the works of Jucys [Juc71] and Murphy [Mur81] as certain sums of transpositions giving elements of the group algebra $\mathbb{C}[S_n]$ of the symmetric group.

Remark. The first reference [Juc71] appears in a then little known Lithuanian journal of theoretical physics. As a result of this it was some time before its content was generally known, hence [Mur81] was published independently by Murphy. As an acknowledgement of this situation we will refer to the algebraic objects of interest as Jucys-Murphy elements.

Let the Jucys-Murphy elements be defined by $m(1) = 0$ and:

$$m(j) = \sum_{i=1}^{j-1} (i\,j) \in \mathbb{C}[S_n], \quad \text{for } j = 2, \ldots, n. \tag{2.1}$$

These elements have two well-known properties; firstly they all commute with one-another, and also every symmetric polynomial in them can be shown to lie in the *centre* of the algebra, $Z(\mathbb{C}[S_n])$.

For example, $m(3) = (1\,3) + (2\,3)$, $m(4) = (1\,4) + (2\,4) + (3\,4)$, and

$$\begin{aligned} m(3)m(4) &= (1\,3)(1\,4) + (1\,3)(2\,4) + (1\,3)(3\,4) \\ &\quad +(2\,3)(1\,4) + (2\,3)(2\,4) + (2\,3)(3\,4) \\ &= (3\,4)(1\,3) + (2\,4)(1\,3) + (1\,4)(1\,3) \\ &\quad +(1\,4)(2\,3) + (3\,4)(2\,3) + (2\,4)(2\,3) \\ &= m(4)m(3). \end{aligned}$$

2.1 Murphy operators in the Hecke algebras

Now given that the Hecke algebra, H_n of type A_{n-1} is a deformation of the group algebra $\mathbb{C}[S_n]$ of the symmetric group, it would be a natural question to ask if there exists a deformed analogue of the Jucys-Murphy elements defined in (2.1).

Such a definition is given by Dipper and James in [DJ87] using a simple deformation of the transpositions. This deformation of the transpositions corresponds geometrically to the positive permutation braid $\omega_{(i\,j)} \in H_n$ for $i < j$ shown in Figure 2.1, where positive permutation braids have all crossings positive.

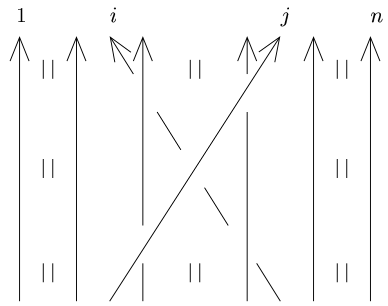


Figure 2.1: The positive transposition braid $\omega_{(i\,j)} \in H_n$.

Remark. Positive permutation braids are first defined by Elrifai and Morton in [EM94]. They subsequently appear in many places such as [Ais96, AM98, Mor02b].

Before we define these elements explicitly, we make the following observations. Again, these elements, denoted $M(j)$, all commute, and also every symmetric polynomial in them lie in the centre of H_n. Moreover, Dipper and James showed that for generic values of the deformation parameter these account for the whole of the centre. This was then extended by Mathas [Mat99] to include the previously omitted non-semisimple case.

Furthermore, Katriel, Abdessalam and Chakrabarti [KAC95] observed the stronger result that in fact any central element can be expressed as a polynomial in just the sum $M = \sum_{j=1}^{n} M(j)$ of the Murphy operators.

Before moving on, we observe that Ram [Ram97] offers generalizations of the Jucys-Murphy elements in other settings. He considers the arbitrary Weyl groups and Hecke algebras of types A_n, B_n, D_n and G_2. He also observes that the Hecke algebras of types F_4, E_6 and E_7 are also within easy reach of the techniques he uses.

Now using the skein model for H_n we find that there are elegant geometric representations of the Murphy operators. The observations that follow in this section are due to the work of Ram [Ram97] and Morton [Mor02b]. This skein theoretic viewpoint immediately facilitates the proofs of the properties stated above.

Definition 1. The *Murphy operator* $M(j) \in H_n$, $j = 1, \ldots, n$ is defined by $M(1) = 0$ and

$$M(j) = \sum_{i=1}^{j-1} \omega_{(i\,j)}. \tag{2.2}$$

These elements clearly project to the Jucys-Murphy elements $m(j) \in \mathbb{C}[S_n]$, therefore (2.2) is the deformed analogue of (2.1).

Proving that these elements possess the properties described above require a bit of algebraic work. As noted above Ram [Ram97] and Morton [Mor02b] found geometric representations of the Murphy operators which are easier to manipulate and indeed make certain properties obvious with no work required. We observe that the sum of the Murphy operators, M, defined above, can be written as:

$$M = \sum_{j=1}^{n} M(j) = \sum_{i<j} \omega_{(i\,j)}.$$

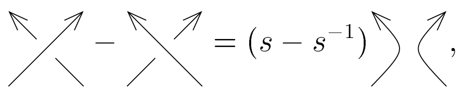
Theorem 2.1 (Ram). *The Murphy operator $M(j)$ can be represented by a single braid $T(j)$, up to linear combination with the identity.*

Theorem 2.2 (Morton). *The sum M of the Murphy operators can be represented in H_n by a single tangle $T^{(n)}$, again up to linear combination with the identity.*

Before embarking on our journey through these elegant proofs, we require one piece of new notation. Let the identity braid on l strings be denoted by $\mathbb{I}_l$ for $l \leq n$ and given a tangle T on $n - l$ strings then we write $T \otimes \mathbb{I}_l \in H_n$ for the juxtaposition of T and the identity.

Proof (of Theorem 2.1 [Ram97]). Let $T(j)$ be the element of H_n represented by the braid shown in Figure 2.2.

Using the framed Homfly skein relation



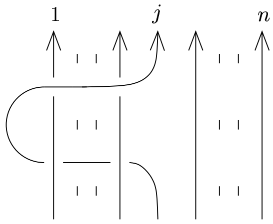
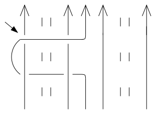
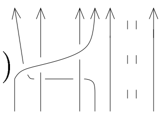
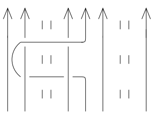


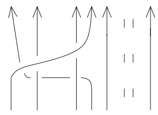
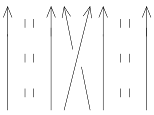
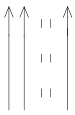
Figure 2.2: $T(j) \in H_n$.

on the crossing indicated we see that

$T(j) =$ 

$= (s - s^{-1})$  $+$ 

$= \ldots$ (repeated applications of the skein relations) $\ldots$

$= (s - s^{-1}) \Big($  $+ \cdots +$  $\Big) +$ 

$$= (s - s^{-1}) \sum_{i=1}^{j-1} \omega_{(i\,j)} + \mathbb{I}_n$$

$$= (s - s^{-1}) M(j) + \mathbb{I}_n.$$

Therefore,

$$M(j) = \frac{T(j) - \mathbb{I}_n}{s - s^{-1}}.$$

♠

Remark. Theorem 2.1 enables us to consider the geometrically more appealing elements $T(j)$ in place of the $M(j)$, provided $s - s^{-1} \neq 0$, or in other words we are away from $\mathbb{C}[S_n]$.

In fact, these elements are not only geometrically more appealing, it is also the case that algebraically they are much easier to work with. Mathas [Mat99] remarks that the original definitions for Murphy operators are quite hard to work with and defines $\mathcal{L}$-Murphy operators which have the same properties as the elements $T(j)$, in particular Theorem 2.1. Results are then proved for the $\mathcal{L}$-Murphy operators.

Remark. It is pictorially clear that the elements $T(j)$ all commute.

Remark. The product of the $T(j)$ is the full curl (often denoted in braid theory by Δ^2), clearly a central element. However, it is not immediately obvious that their sum is central.

Proof (of Theorem 2.2 [Mor02b]). Let $T^{(n)}$ be the element of H_n represented by the tangle $T^{(n)}$, as shown in Figure 2.3.

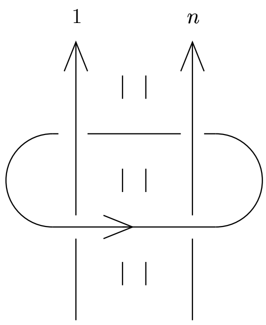
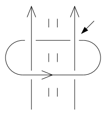
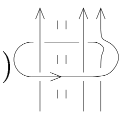
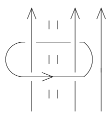


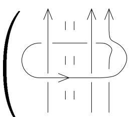
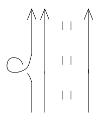
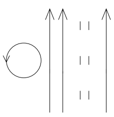
Figure 2.3: $T^{(n)} \in H_n$.

Applying the skein relation to the crossing indicated we have

$T^{(n)} =$ 

$= (s - s^{-1})$  $+$ 

$= \ldots$ (repeated applications of the skein relations) $\ldots$

$= (s - s^{-1}) \Big($  $+ \cdots +$  $\Big) +$ 

$$= (s - s^{-1})v^{-1} \sum_{j=1}^{n} T(j) + T^{(0)} \otimes \mathbb{I}_n.$$

Now since the term $T^{(0)} \otimes \mathbb{I}_n$ is simply a disjoint trivial loop alongside the identity braid, we can remove the loop at the expense of the scalar $\delta = \frac{v^{-1} - v}{s - s^{-1}}$.

Therefore, using the result of Theorem 2.1, we have

$$\begin{aligned} T^{(n)} &= (s-s^{-1})v^{-1}\sum_{j=1}^{n} T(j) + \frac{v^{-1}-v}{s-s^{-1}}\mathbb{I}_n \\ &= (s-s^{-1})v^{-1}\sum_{j=1}^{n}\big((s-s^{-1})M(j)+\mathbb{I}_n\big) + \frac{v^{-1}-v}{s-s^{-1}}\mathbb{I}_n \\ &= (s-s^{-1})^2v^{-1}M + \left((s-s^{-1})v^{-1}n + \frac{v^{-1}-v}{s-s^{-1}}\right)\mathbb{I}_n. \end{aligned}$$

♠

Pictorially it is very clear that the element $T^{(n)}$ is central in H_n, therefore, it is an immediate corollary of Theorem 2.2 that the element M is central.

2.2 The Murphy operators and idempotents of the Hecke algebra

Recall the set of idempotent elements in H_n defined in Section 1.5.1. They are denoted e_λ, one for each partition λ of n, with $\emptyset$ being the unique partition of 0. We now consider the effect of these idempotents on the element $T^{(n)}$. Using skein theoretic techniques it is easy to prove the following corollary of Theorem 19 in [AM98] (see also [Mor02b]),

Corollary (of Theorem 19, [AM98]). *$T^{(n)}e_\lambda = t_\lambda e_\lambda$ where*

$$t_\lambda = (s-s^{-1})v^{-1}\sum_{\substack{c,\ cells\\ in\ \lambda}} s^{2cn(c)} + \delta.$$

Moreover, the scalars t_λ are different for each partition λ.

If we were then to reverse the orientation of the encircling string in $T^{(n)}$ we obtain another central element in H_n. We shall call this element $\bar{T}^{(n)}$. Then, using similar techniques, one can show

Lemma 2.3 ([MH02]). *$\bar{T}^{(n)}e_\lambda = \bar{t}_\lambda e_\lambda$ where*

$$\bar{t}_\lambda = -(s-s^{-1})v\sum_{\substack{c,\ cells\\ in\ \lambda}} s^{-2cn(c)} + \delta.$$

Moreover, the scalars $\bar{t}_\lambda$ are different for each partition λ.

Remark. An alternative proof to this lemma could be made through considering these elements wired into the skein of the annulus combined with the effect of the mirror map. Then it can be shown that the e_λ are invariant under the mirror map and clearly $^{-}(T^{(n)}) = \bar{T}^{(n)}$. We also recall that the mirror map inverts the scalars v and s in the coefficient ring. Applying these facts to the preceding corollary, the result follows immediately.

We remarked above that the product of the Murphy operators is the full curl, Δ^2. This too is a well-known central element. It would therefore be interesting to ask the effect of the idempotent elements e_λ on Δ^2.

For our purposes we choose not to adopt the notation Δ^2, but instead use $F_n \in H_n$ for the full curl on n strings. We have in terms of the Murphy operators the inductive definition

$$F_n = vT(n)(F_{n-1} \otimes \mathbb{I}_1),$$

which gives $F_n = v^n \prod_{j=1}^{n} T(j)$. We then have

Theorem 2.4 (Aiston-Morton). *Let λ be a Young diagram with $|\lambda| = n$. Then $F_n e_\lambda = f_\lambda e_\lambda$, where*

$$\begin{aligned} f_\lambda &= v^{-|\lambda|} s^{n_\lambda} \\ \textit{and} \qquad n_\lambda &= \sum_{(i,j)\in\lambda} 2(j-i). \end{aligned}$$

2.3 Symmetric functions and the skein of the annulus

The theory of symmetric polynomials has been well studied and there are many texts giving a good description with the well-known authority being [Mac79]. In this section we consider elements in the Hecke algebra and their closure in $\mathcal{C}$ within this context of symmetric functions.

Again recall the set of idempotent elements in H_n as described in Section 1.5.1. Here we consider the two simplest, those which correspond to the single row and single column Young diagrams.

Let w_π be the positive permutation braid ([EM94]) corresponding to $\pi \in S_n$. Define two quasi-idempotents by

$$a_n = \sum_{\pi \in S_n} s^{l(\pi)} w_\pi \qquad \text{and} \qquad b_n = \sum_{\pi \in S_n} (-s)^{-l(\pi)} w_\pi,$$

where $l(\pi) = \mathrm{wr}(w_\pi)$, the writhe of the braid w_π. We recall that the writhe of the braid (also known as the algebraic crossing number) is the sum of the signs of the crossings.

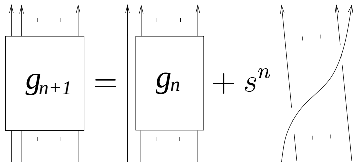
Lemma 2.5.

$$a_n = a_{n-1} g_n,$$

where $g_n = 1 + s\sigma_{n-1} + s^2 \sigma_{n-1}\sigma_{n-2} + \ldots + s^{n-1}\sigma_{n-1} \cdot \ldots \cdot \sigma_1$.

In the above lemma the σ_i correspond to the usual braid group generators, for the braid group B_n.

We have $g_{n+1} = 1 + s\sigma_n g_n$, and also the immediate skein relation



for tangles on $n+1$ strings.

Lemma 2.6. *For any braid* $\beta \in B_n$ *we have* $a_n\beta = \phi_s(\beta)a_n = \beta a_n$, *where* $\phi_s(\beta) = s^{wr(\beta)}$.

Analogous results for b_n hold, replacing s with s^{-1} throughout.

We can then see that the element a_n satisfies

$$a_n^2 = \phi_s(a_n)a_n = \phi_s(a_{n-1})\phi_s(g_n)a_n.$$

Now since $\phi_s(g_n) = 1 + s^2 + \ldots + s^{2n-2} = s^{n-1}[n]$ with $[k] = \frac{s^k - s^{-k}}{s - s^{-1}}$, we have immediately

Corollary. *We can write*

$$s^{n-1}[n]h_n = h_{n-1}g_n,$$

where $h_n = a_n/\phi_s(a_n)$ *is the true idempotent.*

The element h_n constructed above is the idempotent which corresponds to the single row Young diagram with n cells. The single column idempotent, denoted e_n, is constructed in an analogous way from b_n. It can be obtained from h_n by using $-s^{-1}$ in place of s.

With a slight abuse of the notation we write $h_n, e_n \in \mathcal{C}$ for the closures $\wedge(h_n)$, $\wedge(e_n)$ in $\mathcal{C}$.

The skein $\mathcal{C}^+$ when considered as an algebra is spanned by the monomials in $\{h_m : m \geq 0\}$.

Remark. These elements have already been studied by Aiston in [Ais96], however, there the notations Q_{c_n} and Q_{d_n} are used in place of e_n and h_n. Morton adopts this more suggestive notation in [Mor02b] to make it clear that it is the combination of these elements and symmetric function techniques that is being exploited.

Write

$$\begin{aligned} H(t) &= 1 + \sum_{n=1}^{\infty} h_n t^n \\ \text{and} \qquad E(t) &= 1 + \sum_{n=1}^{\infty} e_n t^n \end{aligned}$$

for the generating function of the elements $\{h_n\}$ and $\{e_n\}$ respectively, when considered as formal power series with coefficients in $\mathcal{C}$.

Theorem 2.7 (Aiston).

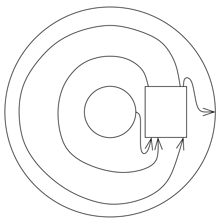
$$E(-t)H(t) = 1$$

as a power series in $\mathcal{C}$.

We shall regard the elements h_n and e_n formally as respectively the n^{th} complete and elementary symmetric functions in a suitably large number N of variables $x_1, \ldots, x_N$, setting

$$\begin{aligned} H(t) &= \prod_{i=1}^{N} \frac{1}{1 - x_i t}, \\ \text{and} \qquad E(t) &= \prod_{i=1}^{N} (1 + x_i t). \end{aligned}$$

Now consider the wiring induced from considering the diagram

$D =$  .

with n strings running around the annulus. Such a wiring is a linear map $W_n : R_{n+1}^{n+1} \to \mathcal{A}$. It is easy to see from the drawing of some simple pictures that given a tangle $T \in H_n$ which is included in H_{n+1} as the element $T \otimes \mathbb{I}_1$ has the property

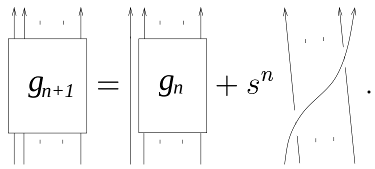
$$W_n(T \otimes \mathbb{I}_1) = W_{n-1}(T)a.$$

This is clear because the final string leaving the top right-hand corner of T passes around the annulus one final time before going to the output point of the annulus, it is this that contributes the a. Also, $W_n(\mathbb{I}_n) = a^n$.

Theorem 2.8 (Morton). *The elements $W_n(h_{n+1})$, $W_n(\mathbb{I}_1 \otimes h_n)$ and $l(h_n, e)$ in $\mathcal{A}$ satisfy the linear relation*

$$[n+1]W_n(h_{n+1}) = s^{-1}[n]W_n(\mathbb{I}_1 \otimes h_n) + l(h_n, e).$$

Proof. [Mor02b] Recall the relation given above,



This immediately gives $W_n(h_n g_{n+1}) = W_n(g_n h_n) + s^n W_n(h_n \sigma_n \cdots \sigma_1)$. Now using the now familiar style of manipulation using the skein relations we can also show that $g_n h_n = s^{n-1}[n]h_n$ and $W_n(h_n \sigma_n \cdots \sigma_1) = l(h_n, e)$. This combined with a previous result that $s^n[n+1]h_{n+1} = h_n g_{n+1}$ the result follows immediately. ♠

Let $Y_n = [n+1]W_n(h_{n+1})$ and use this to define another formal power series

$$Y(t) = \sum_{n=0}^{\infty} Y_n t^n.$$

We then obtain the following corollary

Corollary. *As power series with coefficients in $\mathcal{A}$ we have*

$$l(H(t), e) = (e - s^{-1}at)Y(t). \tag{2.3}$$

Proof. We know that $W_n(h_n) = W_{n-1}(h_n)a$ or equivalently since $\mathcal{A}$ is commutative $W_n(h_n) = aW_{n-1}(h_n)$. We can therefore rewrite the expression for Y_n as

$$Y_n = s^{-1}aY_{n-1} + l(h_n, e).$$

Therefore,

$$Y(t) = s^{-1}atY(t) + l(H(t), e).$$

The result now follows immediately. ♠

Following appropriate use of the mirror map on the skein $\mathcal{A}$ the following result is an immediate consequence of the previous corollary

Proposition 2.9. *As power series with coefficients in $\mathcal{A}$ we have*

$$r(e, H(t)) = (e - sat)Y(t). \tag{2.4}$$

Combining these results gives

$$[H(t), e] = (s - s^{-1})atY(t).$$

This result also appears in the same context in [Luk01].

We finally offer one further result which will be essential in certain subsequent results. The element of $\mathcal{C}$ to appear here is the formal power sum of the variables x_i, $P_m = \sum_{i=1}^{N} x_i^m$.

Theorem 2.10. *For $m \geq 1$ we have $[P_m, e] = (s^m - s^{-m})a^m$.*

Proof. First recall the Newton power series equation

$$\sum_{m=1}^{\infty} \frac{P_m}{m} t^m = \ln H(t).$$

Now, taking logarithms of equations 2.3 and 2.4, then subtracting, we have

$$\begin{aligned} \ln(e - s^{-1}at) - \ln(e - sat) &= \ln(l(H(t), e)) - \ln(r(e, H(t))) \\ &= l(\ln(H(t)), e) - r(e, \ln(H(t))) \\ &= \sum_{m=1}^{\infty} \left[\frac{P_m}{m} t^m, e \right]. \end{aligned}$$

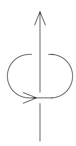
Now $\ln(e - s^{-1}at) = -\sum_{m=1}^{\infty} \frac{s^{-m}a^m t^m}{m}$. Finally, comparing coefficients of t^m, the result follows. ♠

2.4 Symmetric functions of the Murphy operators

The work that appears in this section is intended to summarise the results of Morton in [Mor02b] with a view to extending them later *à la* [Mor02a].

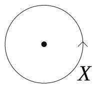
Morton introduces a new relation between the Hecke algebras and the skein of the annulus. This relation is a very natural homomorphism ψ_n from $\mathcal{C}$ to the centre of each algebra H_n.

First take D to be the diagram

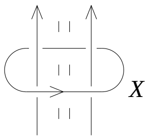
$D =$  .

D then determines a map $\psi_n : \mathcal{C} \to H_n$ which is induced by placing $X \in \mathcal{C}$ around the encircling loop in D and the identity $\mathbb{I}_n \in H_n$ on the arc. We therefore have:

$$\psi_n : \mathcal{C} \to H_n$$



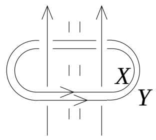
$\mapsto$



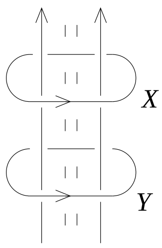
$\in H_n$.

Clearly

$\psi_n(XY) =$



$=$



$= \psi_n(X)\psi_n(Y).$

Therefore, ψ_n defines an algebra homomorphism. Also, it is obvious that the elements $\psi_n(X)$ lies in the centre of H_n for all $X \in \mathcal{C}$.

We shall say that the element $T^{(n)}$ is "almost equal" to the sum $\sum_{j=1}^{n} T(j)$. Denote this by

$$T^{(n)} \approx \sum_{j=1}^{n} T(j).$$

By this we mean that $T^{(n)}$ is equal to a scalar multiple of $\sum_{j=1}^{n} T(j)$ up to a linear combination with the identity as in Theorem 2.2. Also we observe that $T^{(n)} = \psi_n(X_1)$ for $X_1 = A_1 \in \mathcal{C}$. Morton then enquires whether there is an element X_2 such that $\psi_n(X_2) \approx \sum_{j=1}^{n} T(j)^2$, or indeed more generally, whether there are X_m such that $\psi_n(X_m) \approx \sum_{j=1}^{n} T(j)^m$ for any value m.

The surprising part of this result is not that there exist such elements in $\mathcal{C}$, but that there exist elements which are independent of n which have this property.

Theorem 2.11. *For any n we have*

$$\psi_n(P_m) - \psi_0(P_m) = (s^m - s^{-m})v^{-m} \sum_{j=1}^{n} T(j)^m.$$

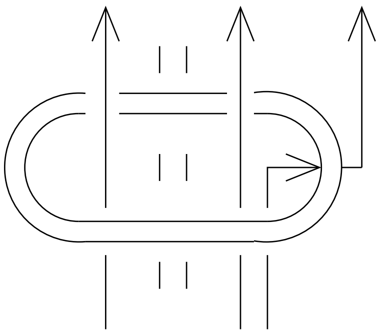


Figure 2.4: The diagram D which induces the wiring V_n.

Proof. First define the wiring $V_n : \mathcal{A} \to H_n$ induced by the diagram D shown in Figure 2.4.

It is clear that for any $X \in \mathcal{C}$, we have

$$\begin{array}{rrcl} & V_n(l(X,e)) & = & \psi_n(X) \\ \text{and} & V_n(r(e,X)) & = & \psi_{n-1}(X) \otimes \mathbb{I}_1. \end{array}$$

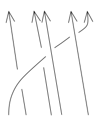
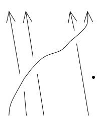
We also observe that $V_n(a) = v^{-1}T(n)$ and hence inductively we have $V_n(a^m) = v^{-m}T(n)^m$.

Therefore $\psi_n(P_m) - \psi_{n-1}(P_m) \otimes \mathbb{I}_1 = (s^m - s^{-m})v^{-m}T(n)^m$, and by induction on n we have

$$\psi_n(P_m) - \psi_0(P_m) \otimes \mathbb{I}_n = (s^m - s^{-m})v^{-m} \sum_{j=1}^{n} T(j)^m,$$

which we abbreviate using the standard inclusion of $H_{n-1} \subset H_n$ to obtain the result. ♠

In [Ais96], Aiston shows that $[m]P_m$ is the sum

$[m]P_m =$  $+ \cdots +$  $+ \cdots +$  .

The proof she gives requires significant knowledge of results about $sl(N)_q$ representations. Morton offers another proof later in [Mor02a] which is purely skein theoretic.

We end this section with one final result.

Theorem 2.12. *The image of ψ_n is the whole centre of H_n.*

Proof. It is shown by Dipper, James and Mathas [DJ87, Mat99] that symmetric polynomials in the Murphy operators account for the whole of the centre of H_n. The power sums P_m are a generating set for the symmetric polynomials. Now by Theorem 2.11 the result follows. ♠

2.5 A set of Murphy operators in $H_{n,p}$

Since the family of algebras $H_{n,p}$ can be thought of as a generalization of the Hecke algebra, an immediate question is whether one can find a set of elements with similar properties in this more general setting.

For some of the results in this section we shall adopt the approach used by Morton in [Mor02b] and [Mor02a] as they have been exhibited above.

We follow an analogous procedure in $H_{n,p}$ as in H_n. Firstly let us consider the elements of $H_{n,p}$ represented by the tangles $T^{(n,p)}$ and $\bar{T}^{(n,p)}$ which are constructed in a similar way to $T^{(n)}$ and $\bar{T}^{(n)}$ respectively. We show $T^{(n,p)}$ diagramatically in Figure 2.5.

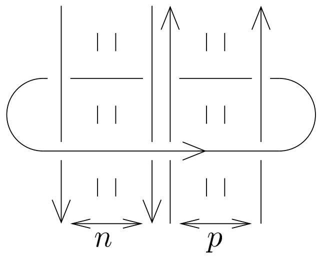


Figure 2.5: The (n,p)-tangle $T^{(n,p)}$.

Definition 2. (see [MW],[Had]) Let $H^{(i)}_{n,p}$ denote the sub-algebra of $H_{n,p}$ spanned by elements with "at least" i pairs of strings turning back.

Remark. An (n,p)-tangle is said to have "at least" l pairs of strings which *turn back* if it can be written as a product T_1T_2 of an $\{(n,p),(n-l,p-l)\}$-tangle T_1 and an $\{(n-l,p-l),(n,p)\}$-tangle T_2 as illustrated in Figure 2.6.

Remark. The $H^{(i)}_{n,p}$ are two-sided ideals and there is a filtration:

$$H_{n,p} \cong H^{(0)}_{n,p} \triangleright H^{(1)}_{n,p} \triangleright \cdots \triangleright H^{(k)}_{n,p},$$

where $k = \min(n,p)$.

We use a similar notation of $\mathbb{I}_{l,m} \in H_{n,p}$ for the identity on l strings down and m strings up, with $l \leq n$ and $m \leq p$.

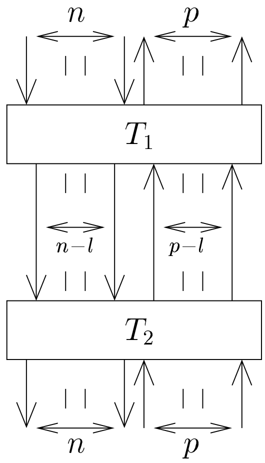


Figure 2.6: A tangle with at least l pairs of strings which *turn back*.

Lemma 2.13.

$$
\begin{aligned}
T^{(n,p)} &= T^{(n,p)\prime} + w, \\
\bar{T}^{(n,p)} &= \bar{T}^{(n,p)\prime} + \bar{w},
\end{aligned}
$$

where

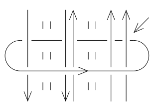
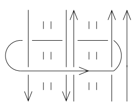
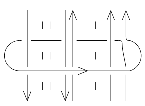
$$
\begin{aligned}
T^{(n,p)\prime} &= T^{(n)}_{(-)} \otimes \mathbb{I}^{(+)}_{p} + \mathbb{I}^{(-)}_{n} \otimes T^{(p)}_{(+)} - \delta\, \mathbb{I}_{n,p}, \\
\bar{T}^{(n,p)\prime} &= \bar{T}^{(n)}_{(-)} \otimes \mathbb{I}^{(p)}_{(+)} + \mathbb{I}^{(n)}_{(-)} \otimes \bar{T}^{(p)}_{(+)} - \delta\, \mathbb{I}^{(n)}_{(-)} \otimes \mathbb{I}^{(p)}_{(+)},
\end{aligned}
$$

and $w, \bar{w} \in H^{(1)}_{n,p}$.

Proof. We prove the result for $T^{(n,p)}$, with the result for $\bar{T}^{(n,p)}$ following in exactly the same way. Throughout this proof, we use a standard notation setting $s - s^{-1} = z$.

We first define some elements in $H_{n,p}$ represented by tangles as shown in Figure 2.7.

Now applying the skein relation once to $T^{(n,p)}$ we obtain:

 $=$  $+\, z$ 

$$
= T^{(n,p-1)} \otimes \mathbb{I}^{(+)}_{1} + z v^{-1} A(p).
$$

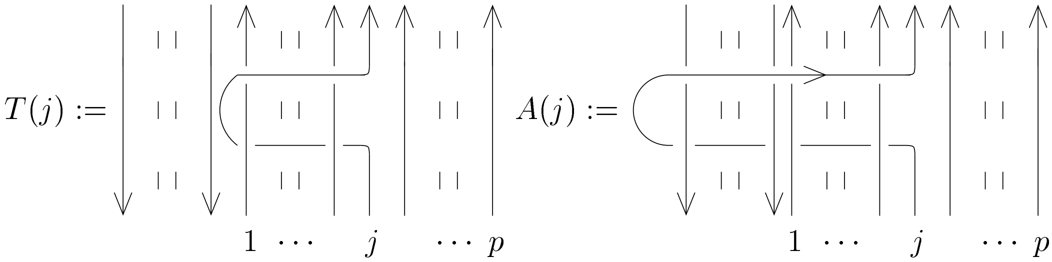
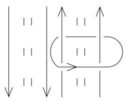
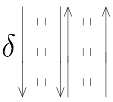


Figure 2.7: The elements $T(j)$ and $A(j)$ for $1 \leq j \leq p$.

Repeated application of the skein relation in this way will clearly yield:

$$
\begin{aligned}
T^{(n,p)} &= T^{(n,0)} \otimes \mathbb{I}_p^{(+)} + zv^{-1} \sum_{j=1}^{p} A(j) \\
&= T_{(-)}^{(n)} \otimes \mathbb{I}_p^{(+)} + zv^{-1} \sum_{j=1}^{p} A(j). \qquad (2.5)
\end{aligned}
$$

Now observe, similar to a result in [Mor02b], we can find:

 $= \delta$  $+ zv^{-1} \sum_{j=1}^{p} T(j)$. (2.6)

Combining equations 2.5 and 2.6, we see that we are only left to show that:

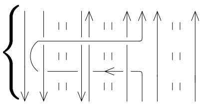
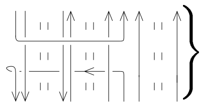
$$zv^{-1} \sum_{j=1}^{p} A(j) = zv^{-1} \sum_{j=1}^{p} T(j) + w,$$

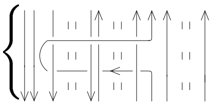
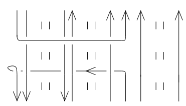
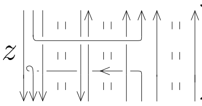
for $w \in H_{n,p}^{(1)}$.

Let $w = \sum_{j=1}^{p} w(j)$. We must now show that for each j, with $1 \leq j \leq p$, there exists a $w(j)$ such that:

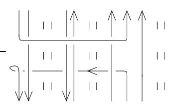
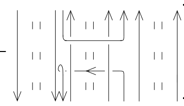
$$zv^{-1} A(j) = zv^{-1} T(j) + w(j).$$

Now,

$zv^{-1}A(j) = zv^{-1}\Big\{$  $- z$  $\Big\}$

$= zv^{-1}\Big\{$  $- z$  $- z$  $\Big\}$

$= \cdots$ (repeating application of the skein relation)

$= zv^{-1}T(j) + z^2v^{-1}\Big\{-$  $- \cdots -$  $\Big\}$

$= zv^{-1}T(j) + w(j).$

with $w(j) \in H^{(1)}_{n,p}$.

The result follows. ♠

We therefore suggest a potential set of Murphy operators in $H_{n,p}$. These are then the elements $T(j)$ defined for the first n strings (as with the H_n case in Figure 2.2 except the strings are obviously oriented in the opposite direction and we then take its inverse). In addition to this set of n elements, we add the $A(j)$ defined in Figure 2.7, defined for the last p strings. These elements are shown in Figure 2.8.

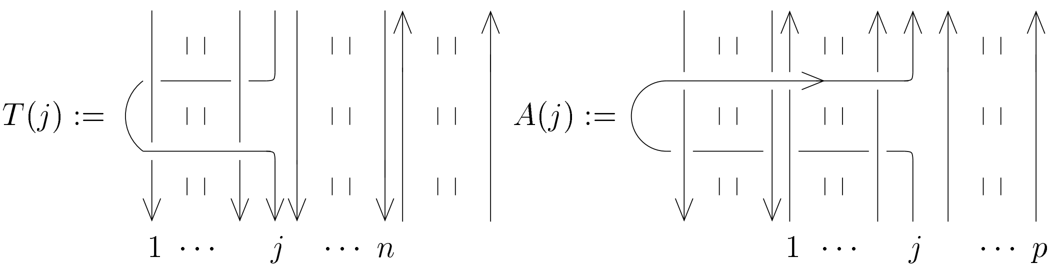


Figure 2.8: The elements $T(j)$ for $1 \leq i \leq n$ and $A(j)$ for $1 \leq j \leq p$.

We find, similar to Theorem 2.2 that

Theorem 2.14. *The sum of these Murphy operators is* almost equal *to* $T^{(n,p)}$.

Proof. It is not difficult to show using the skein relations that

$$T^{(n,p)} = (s - s^{-1})\left(-v\sum_{j=1}^{n} T(j) + v^{-1}\sum_{j=1}^{p} A(j)\right) + \frac{v^{-1} - v}{s - s^{-1}}\mathbb{I}_{n,p}.$$

This is achieved through an analogous method to the one used previously to show that $T^{(n)} = (s - s^{-1})v^{-1}\sum_{j=1}^{n} T(j) + T^{(0)} \otimes \mathbb{I}_n$ in the standard Hecke algebra case, except this time one must pay particular attention to the orietation of the strings. By the definition of *almost equal* the result follows immediately. ♠

2.6 The Murphy operators and idempotents of $H_{n,p}$

We can then use earlier information, combined with Lemma 2.13 to prove the following proposition concerning the elements $e'_{(\lambda,\mu)} = e_\lambda^{(-)} \otimes e_\mu^{(+)}$ as discussed above.

Proposition 2.15.

$$\begin{aligned} T^{(n,p)} e'_{\lambda,\mu} &= t_{\lambda,\mu} e'_{\lambda,\mu} + w e'_{\lambda,\mu} \\ \text{and} \quad \bar{T}^{(n,p)} e'_{\lambda,\mu} &= \bar{t}_{\lambda,\mu} e'_{\lambda,\mu} + \bar{w} e'_{\lambda,\mu}, \end{aligned}$$

where,

$$t_{\lambda,\mu} = (s - s^{-1}) \left(-v \sum_{\substack{cells \\ in\ \lambda}} s^{-2(content)} + v^{-1} \sum_{\substack{cells \\ in\ \mu}} s^{2(content)} \right) + \delta$$

and

$$\bar{t}_{\lambda,\mu} = (s - s^{-1}) \left(v^{-1} \sum_{\substack{cells \\ in\ \lambda}} s^{2(content)} - v \sum_{\substack{cells \\ in\ \mu}} s^{-2(content)} \right) + \delta.$$

Here we had fixed $|\lambda|$ and $|\mu|$ with values n and p respectively. In fact, we find that $t_{\lambda,\mu}$ and $\bar{t}_{\lambda,\mu}$ have the following property:

Lemma 2.16. *As λ and μ vary over all choices of Young diagram, the values of $t_{\lambda,\mu}$ are all distinct; as are the values of $\bar{t}_{\lambda,\mu}$.*

Remark. An equivalent way of stating Lemma 2.16 is that *if* $t_{\lambda,\mu} = t_{\lambda',\mu'}$ *then* $\lambda = \lambda'$ *and* $\mu = \mu'$ (similarly for the $\bar{t}_{\lambda,\mu}$).

Proof. (of Lemma 2.16) We prove the first part of the lemma and note that the second part follows immediately due to the observation that $\bar{t}_{\lambda,\mu} = t_{\mu,\lambda}$.

Given $f(s, v) = t_{\lambda,\mu}$ we now show how to recover the Young diagrams λ and μ.

From the formula for $t_{\lambda,\mu}$ in Lemma 2.15 we see that $f(s,v) - \delta$ is a Laurent polynomial in s and v, and must be of the form:

$$(s - s^{-1})(-vP(s) + v^{-1}Q(s)).$$

Now consider $P(s)$ and $Q(s)$ individually. It is clear that these are also Laurent polynomials, this time only in the variable s. We have

$$\begin{aligned} P(s) &= \sum a_i s^{-2i} \\ \text{and} \quad Q(s) &= \sum b_j s^{2j}, \end{aligned}$$

where a_i is the number of cells in λ with content i, and similarly, b_j is the number of cells in μ with content j. Hence we can uniquely construct λ and μ. ♠

Extending the notion of the full curl into the $H_{n,p}$ setting, we use the notation $F_{n,p} \in H_{n,p}$. Again, $F_{n,p}$ is central in $H_{n,p}$. In terms of our set of Murphy operators we have

$$F_{n,p} = v^{n+p} \prod_{j=1}^{n} T(j)^{-1} \prod_{j=1}^{p} A(j).$$

We now offer without proof a lemma comparable to Lemma 2.13.

Lemma 2.17.

$$F_{n,p} = F_{n,0} \otimes F_{0,p} + u$$

where $u \in H_{n,p}^{(1)}$.

Continuing with this theme we have the following proposition, combining the result of Theorem 2.4 and the techniques of Proposition 2.15.

Proposition 2.18.

$$F_{n,p} e'_{(\lambda,\mu)} = f_{(\lambda,\mu)} e'_{(\lambda,\mu)} + u e'_{(\lambda,\mu)}$$

where $f_{(\lambda,\mu)} = v^{|\lambda|-|\mu|} s^{-n_\lambda + n_\mu}$, *and* $n_\lambda = \sum_{(i,j)\in\lambda} 2(j-i)$ *and* $n_\mu = \sum_{(i,j)\in\mu} 2(j-i)$.

2.7 Supersymmetric polynomials in the Murphy operators

Why should we have chosen this decomposition of $T^{(n,p)}$ to give a set of Murphy operators in $H_{n,p}$? Is there a symmetric function type result in this setting? Well, first we prove the following, a generalization of Theorem 2.11. First we introduce a natural generalisation of the map ψ_n into the $H_{n,p}$ arena and call it $\psi_{n,p}$. Similarly, this defines an algebra homomorphism on $H_{n,p}$, and all elements $\psi_{n,p}$ lie in the centre of $H_{n,p}$.

Theorem 2.19. *The central elements $\psi_{n,p}(P_m)$ of $H_{n,p}$ can be written, up to a linear combination with the identity, as the power sum difference*

$$-v^m \sum_{j=1}^{n} T(j)^m + v^{-m} \sum_{j=1}^{p} A(j)^m.$$

Proof. Using the techniques displayed in Theorem 2.11 and changing the wiring appropriately for the left n strings we find

$$\psi_{n,p}(P_m) - \psi_{0,0}(P_m) \otimes \mathbb{I}_{n,p} = (s^m - s^{-m}) \left(-v^m \sum_{j=1}^{n} T(j)^m + v^{-m} \sum_{j=1}^{p} A(j)^m \right).$$

♠

Now this does not quite resemble the power sum found in the H_n setting for the standard symmetric functions, however, Stembridge discusses supersymmetric polynomials in [Ste84]. Such polynomials appear in terms of two sets of commuting variables $\{x_i\}$ and $\{y_i\}$ say. For a polynomial in these variables to be called supersymmetric they must satisfy the following properties:

1. the polynomial is invariant under permutations of the variables $\{x_i\}$;
2. the polynomial is invariant under permutations of the variables $\{y_i\}$;
3. when the substitution $x_1 = y_1 = t$ is made, the resulting polynomial is independent of t.

Stembridge then continues to prove that the set of supersymmetric polynomials is in fact generated by the power sum difference $\sum x_i^m - \sum y_i^m$, proving a conjecture of Scheunert [Sch84]

We then see that the central elements $\psi_{n,p}(X)$ can be written as such a supersymmetric polynomial in two sets of commuting elements, up to a linear combination with the identity.

Remark. There is an element of choice associated with this set of Murphy operators given here. For example, conjugating all of them by a fixed element will not alter their supersymmetric polynomials.

We end this section, and indeed chapter, with a currently unproved, but morally reasonable conjecture.

Conjecture (Morton). *The image of $\psi_{n,p}$ is the whole centre of $H_{n,p}$.*

Remark. Morton remarks that although it is possible to prove this for the H_n case (see Theorem 2.12), there does not at present exist an immediate skein theory proof for either the H_n and certainly not the more general $H_{n,p}$ case. The information that currently seems to be lacking is an upper bound on the dimension of the centre in the generic case $n, p > 0$.

Chapter 3

The Homfly Polynomial Of Generalized Hopf Links

In this chapter we see how to use some of the results of the previous chapter to calculate the Homfly polynomial of a class of links we shall call *generalized Hopf links*. This work will follow that described in [MH02] by Morton and the author. Although this chapter can be considered as self-contained, it acts very well to whet one's appetite for what is to follow.

3.1 Initial motivation

In [Cha00], T.-H. Chan discusses the Homfly polynomial of reverse string parallels $H(k_1, k_2; n_1, n_2)$ of the Hopf link. Using results described previously, we find that the computations which were more labour intensive in [Cha00] become simplified. A further generalization is then readily available to allow us to calculate the Homfly polynomial of satellites of the Hopf link which consists of a reverse string parallel around one component combined with a completely general reverse string decoration on the other.

3.2 Satellites of Hopf links

The Hopf link is the simplest non-trivial link involving just two unknots linked together. When giving this link orientation, two distinct links are formed. We shall call these H_+ and H_-, as shown in Figure 3.1.

The Homfly polynomial of these links can easily be calculated with the

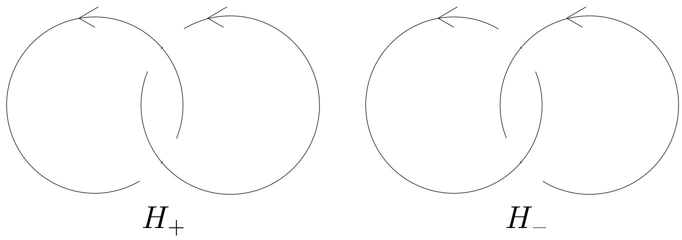


Figure 3.1: The links H_+ and H_-.

Homfly polynomial skein relations. We have that:

$$P(H_+) = \left(\frac{v^{-1} - v}{s - s^{-1}}\right)^2 + v^{-2} - 1;$$
$$\text{and } P(H_-) = \left(\frac{v^{-1} - v}{s - s^{-1}}\right)^2 + v^2 - 1.$$

We now use H_+ and H_- as starting points for the construction of satellite links. We do this by considering the two components of the Hopf links and decorating them. For example, take P_1 and P_2 as diagrams in the annulus. Now starting with H_+, we decorate its two components with P_1 and P_2 respectively, obtaining a new link in the plane which we shall call $H_+(P_1, P_2)$, as shown in Figure 3.2. Now clearly $H_+(P_1, P_2)$ and $H_+(P_2, P_1)$ are equivalent

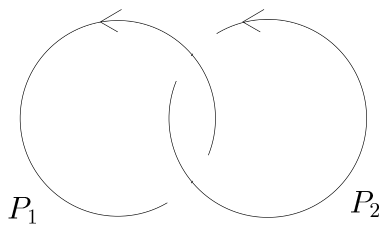


Figure 3.2: The link $H_+(P_1, P_2)$.

links. An analogous construction is now possible for H_-.

With such a construction, it is possible to realise a variety of links. In particular, the generalized Hopf links which are the topic of [Cha00] can be constructed. For example, if we take P_1 and P_2 as shown in Figure 3.3, then $H_+(P_1, P_2)$ is the link Chan refers to as $H(k_1, k_2; n_1, n_2)$. This link is shown in Figure 3.4, somewhat rearranged from how it appears in [Cha00]. This change of view will be seen to be beneficial in our approach.

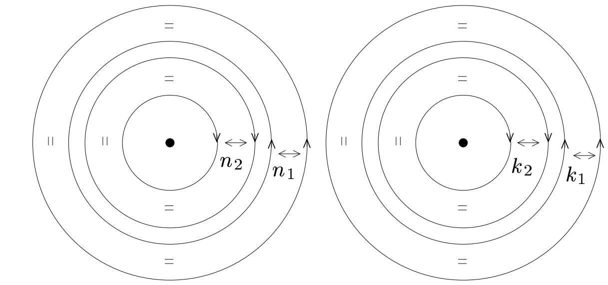


Figure 3.3: The diagrams P_1 and P_2.

With such links in mind, we make the following observation, using the notation that the image of a link H under the involution $*$, described in Section 1.3, shall be denoted H^*

Observation. The links

$$H(k_1, k_2; n_1, n_2), H(n_1, n_2; k_1, k_2), H(k_2, k_1; n_2, n_1), H(n_2, n_1; k_2, k_1),$$

and

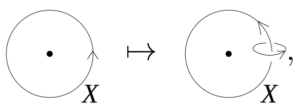
$$H^*(k_2, k_1; n_1, n_2), H^*(n_1, n_2; k_2, k_1), H^*(k_1, k_2; n_2, n_1), H^*(n_2, n_1; k_1, k_2),$$

are all *equivalent links.* For example it is trivial to see that reordering the four groups of strings $H(k_1, k_2; n_1, n_2)$ will give $H(k_2, k_1; n_2, n_1)$.

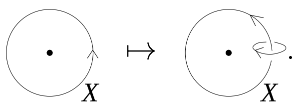
3.3 Maps on the skein of the annulus, $\mathcal{C}$

We now define two natural *linear* maps, φ and $\bar{\varphi}$, on the skein of the annulus in the following way; take an element $X \in \mathcal{C}$ and encircle it once with a single oriented loop. The orientations are opposite for φ and $\bar{\varphi}$. We define these maps pictorially as follows:

$$\varphi : \mathcal{C} \to \mathcal{C}$$



and $\bar{\varphi} : \mathcal{C} \to \mathcal{C}$



Now reconsider the satellites of Hopf links discussed earlier in this chapter, but this time as elements of the skein of the annulus $\mathcal{C}$. We can then use

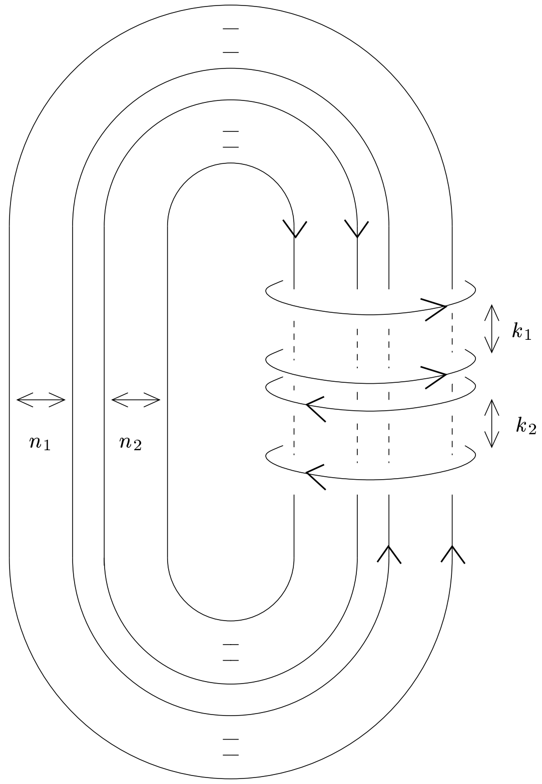


Figure 3.4: The *generalized Hopf link* $H(k_1, k_2; n_1, n_2)$.

compositions of the maps φ and $\bar{\varphi}$ to construct a subset of such links. In particular, for the element $A = A_1^{n_1} A_{-1}^{n_2} \in \mathcal{C}$, we have

$$H(k_1, k_2; n_1, n_2) = \varphi^{k_1} \left(\bar{\varphi}^{k_2}(A) \right).$$

It therefore seems a reasonable proposition that to aid our investigation of the links $H(k_1, k_2; n_1, n_2)$ and their Homfly polynomial, we should look more closely at the maps φ and $\bar{\varphi}$, in particular at their eigenvalues. We shall achieve this during the remainder of this chapter through considering certain already familiar subspaces of $\mathcal{C}$ and the restrictions of the maps φ and $\bar{\varphi}$ to these subspaces.

3.4 Eigenvectors and eigenvalues of the maps φ and $\bar{\varphi}$

We begin with the H_n case. Take an element $S \in H_n$ with $\hat{S} \in \mathcal{C}^{(n)}$ and compose it with $T^{(n)}$. Then $\wedge(ST^{(n)}) = \varphi(\hat{S})$. Similarly $\wedge(S\bar{T}^{(n)}) = \bar{\varphi}(\hat{S})$.

The restrictions $\varphi|_{\mathcal{C}^{(n)}}$ and $\bar{\varphi}|_{\mathcal{C}^{(n)}}$ clearly carry $\mathcal{C}^{(n)}$ to itself.

Theorem 3.1 ([Mor02b]). *The eigenvalues of $\varphi|_{\mathcal{C}^{(n)}}$ are all distinct as are the eigenvalues of $\bar{\varphi}|_{\mathcal{C}^{(n)}}$.*

Proof. We prove the first statement with the second following in exactly the same way.

Set $Q_\lambda = \hat{e}_\lambda \in \mathcal{C}^{(n)}$. Then the closure of $T^{(n)}e_\lambda$ is $\varphi(Q_\lambda)$. However, $T^{(n)}e_\lambda = t_\lambda e_\lambda$, hence $\varphi(Q_\lambda) = t_\lambda Q_\lambda$. The element Q_λ is then an eigenvector of φ with eigenvalue t_λ. There are $\pi(n)$ of these eigenvectors, and the eigenvalues are all distinct by [AM98]. Since $\mathcal{C}^{(n)}$ is spanned by $\pi(n)$ elements we can deduce that the elements Q_λ form a basis for $\mathcal{C}^{(n)}$ and that the eigenspaces are all 1-dimensional. ♠

This proof is quite instructive as it establishes that the Q_λ with $|\lambda| = n$ are a basis for $\mathcal{C}^{(n)}$. Hence any element in $\mathcal{C}^{(n)}$ can be written as a linear combination of the Q_λ with $|\lambda| = n$. It also follows that *any* element of $\mathcal{C}^{(n)}$ which is an eigenvector of φ (and similarly $\bar{\varphi}$) must be a multiple of some Q_λ. Finally, we notice that the eigenvalues of the φ and $\bar{\varphi}$ are the t_λ and $\bar{t}_\lambda$ we found earlier in Chapter 2.

We now extend our view to the $H_{n,p}$ case. First recall the scalars $t_{(\lambda,\mu)}$ and $\bar{t}_{(\lambda,\mu)}$ discussed in Chapter 2. We go straight into some important results about these values.

Theorem 3.2. *The $t_{\lambda,\mu}$ and $\bar{t}_{\lambda,\mu}$ are eigenvalues of $\varphi|_{\mathcal{C}^{(n,p)}}$ and $\bar{\varphi}|_{\mathcal{C}^{(n,p)}}$ respectively. Moreover, they occur with multiplicity* 1.

Proof. We prove the result for the $t_{\lambda,\mu}$ with an identical argument proving the result for the $\bar{t}_{\lambda,\mu}$.

Fix an integer k such that $k = p - n$ and $k \geq 0$ (in other words $p \geq n$ — the case for $p < n$ is identical). Write $\mathcal{C}^{(n,p)}$ as $\mathcal{C}^{(n,k+n)}$ and do induction on n.

For $n = 0$ we have that $\mathcal{C}^{(0,k)} \cong \mathcal{C}^{(k)}$. Now for $|\lambda| = 0$ and $|\mu| = k$ we have that $t_{\lambda,\mu} = t_\mu$. Moreover, in the proof of Theorem 3.1 we saw that the t_μ with $|\mu| = k$ are eigenvalues of $\varphi|_{\mathcal{C}^{(k)}}$. Now since $\mathcal{C}^{(k)} \cong \mathcal{C}^{(0,k)} \subset \mathcal{C}^{(n,k+n)}$ for all n, the t_μ are also eigenvalues of $\varphi|_{\mathcal{C}^{(n,k+n)}}$.

Now assume that for $|\lambda| < n$ and $|\mu| < k + n$ the $t_{\lambda,\mu}$ are eigenvalues of $\varphi|_{\mathcal{C}^{(|\lambda|,|\mu|)}}$. Since $\mathcal{C}^{(|\lambda|,|\mu|)} \subset \mathcal{C}^{(n,k+n)}$ the $t_{\lambda,\mu}$ are also eigenvalues of $\varphi|_{\mathcal{C}^{(n,k+n)}}$.

Consider the $t_{\lambda,\mu}$ with $|\lambda| = n$ and $|\mu| = k+n$. By the inductive hypothesis these $t_{\lambda,\mu}$ are not eigenvalues of $\varphi|_{\mathcal{C}^{(n-1,k+n-1)}}$ since we have $\pi(n-1, k+n-1)$ eigenvalues and $\mathcal{C}^{(n-1,k+n-1)}$ is spanned by $\pi(n-1, k+n-1)$ elements and by Lemma 2.16 we have that if $t_{\lambda,\mu} = t_{\lambda',\mu'}$ then $\lambda = \lambda'$ and $\mu = \mu'$.

Define elements $Q'_{\lambda,\mu} := Q_{\lambda}^{(-)} \cdot Q_{\mu}^{(+)} (= \wedge(e'_{\lambda,\mu}))$ with $|\lambda| = n$ and $|\mu| = k+n$. Clearly $Q'_{\lambda,\mu} \in \mathcal{C}^{(n,k+n)}$.

Now by Lemma 2.15,

$$\varphi|_{\mathcal{C}^{(n,k+n)}}(Q'_{\lambda,\mu}) = t_{\lambda,\mu}Q'_{\lambda,\mu} + w'$$

where $w' \in \mathcal{C}^{(n-1,k+n-1)}$.

We can find a $v \in \mathcal{C}^{(n-1,k+n-1)}$ such that $(\varphi|_{\mathcal{C}^{(n,k+n)}} - t_{\lambda,\mu}I)(v) = w'$.

Now consider $Q'_{\lambda,\mu} - v$. This is clearly non-zero. We find:

$$\begin{aligned}
\varphi|_{\mathcal{C}^{(n,k+n)}}(Q'_{\lambda,\mu} - v) &= \varphi|_{\mathcal{C}^{(n,k+n)}}(Q'_{\lambda,\mu}) - \varphi|_{\mathcal{C}^{(n,k+n)}}(v) + t_{\lambda,\mu}v - t_{\lambda,\mu}v \\
&= \varphi|_{\mathcal{C}^{(n,k+n)}}(Q'_{\lambda,\mu}) - w' - t_{\lambda,\mu}v \\
&= t_{\lambda,\mu}Q'_{\lambda,\mu} + w' - w' - t_{\lambda,\mu}v \\
&= t_{\lambda,\mu}(Q'_{\lambda,\mu} - v).
\end{aligned}$$

Hence such $t_{\lambda,\mu}$ are eigenvalues of $\varphi|_{\mathcal{C}^{(n,k+n)}}$.

Hence by induction, we have that the $t_{\lambda,\mu}$, with $|\lambda| \leq n$, $|\mu| \leq p$ and $|\lambda| - |\mu| = n - p$, are eigenvalues of $\varphi|_{\mathcal{C}^{(n,p)}}$.

Moreover, we have found at least $\pi(n,p)$ eigenvalues for $\varphi|_{\mathcal{C}^{(n,p)}}$. But $\mathcal{C}^{(n,p)}$ is known to be spanned by $\pi(n,p)$ elements, so $\varphi|_{\mathcal{C}^{(n,p)}}$ has at most $\pi(n,p)$ different eigenvalues. Hence it has exactly $\pi(n,p)$ eigenvalues each with multiplicity one. ♠

We now state two useful corollaries.

Corollary. *There is a basis of $\mathcal{C}^{(n,p)}$ given by:*

$$\{Q_{\lambda,\mu} : |\lambda| \leq n, |\mu| \leq p, |\lambda| - |\mu| = n - p\}$$

such that:

$$\varphi(Q_{\lambda,\mu}) = t_{\lambda,\mu}Q_{\lambda,\mu} \quad \text{and} \quad \bar{\varphi}(Q_{\lambda,\mu}) = \bar{t}_{\lambda,\mu}Q_{\lambda,\mu}.$$

Corollary. *Every eigenvector of φ and $\bar{\varphi}$ is a multiple of one such basis element.*

Remark. The eigenvalues $t_{\lambda,\mu}$ and $\bar{t}_{\lambda,\mu}$ correspond to the eigenvalues of the matrix M in equation (1.1) of [Cha00], found there only for $1 \leq k_1 + k_2 \leq 5$ and $k_2 \leq k_1$. Chan uses the Homfly polynomial based on parameters l and m, which are variants of v and z. The numbers $\sqrt{m^2-4}$ in Chan's eigenvalues ρ_i and ρ_i^* correspond to the parameter s here with $z = s - s^{-1}$, which features strongly in our eigenvalues $t_{\lambda,\mu}$ and $\bar{t}_{\lambda,\mu}$. Our use of s is the feature which allows us to give simple formulae for the Gyoja-Aiston elements Q_λ and to extend in principle to $Q_{\lambda,\mu}$.

Unlike the Gyoja-Aiston elements Q_λ which are known and have been well-studied, their generalisations the $Q_{\lambda,\mu}$ described in the above Corollary are not well-understood. We shall show in the following section how they can be found explicitly.

3.5 The Homfly polynomials of some generalized Hopf links

Here we apply the techniques described above to show how computation of the Homfly polynomial ofF some generalized Hopf links is possible.

3.5.1 The Homfly polynomial of $H(k_1, k_2; n, 0)$

Consider $H(k_1, k_2; n, 0)$ in the skein of the annulus. Then we have

$$H(k_1, k_2; n, 0) = \varphi^{k_1}(\bar{\varphi}^{k_2}(A_1^n)).$$

Now since the maps φ and $\bar{\varphi}$ are linear maps, we know that for the Q_λ,

$$\varphi^{k_1}(\bar{\varphi}^{k_2}(Q_\lambda)) = t_\lambda^{k_1}\bar{t}_\lambda^{k_2} Q_\lambda.$$

Also, since the Q_λ are a basis or the skein $\mathcal{C}^{(n)}$, we have

$$A_1^n = \sum_{|\lambda|=n} d_\lambda Q_\lambda$$

for constants d_λ. The d_λ can be calculated by several means, for example by counting the number of standard tableaux of shape λ. Consider the Young diagram $\lambda = (2,2)$, there are two possible standard tableau. The first has the top two cells enumerated 1 and 2 and the bottom two cells 3 and 4, the second has the top two cells enumerated 1 and 3 and the bottom two cells 2 and 4.

Therefore,

$$\begin{aligned} H(k_1, k_2; n, 0) &= \sum_{|\lambda|=n} d_\lambda \varphi^{k_1}(\bar{\varphi}^{k_2}(Q_\lambda)) \\ &= \sum_{|\lambda|=n} d_\lambda t_\lambda^{k_1}\bar{t}_\lambda^{k_2} Q_\lambda. \end{aligned}$$

So evaluating in the plane (using the work of [AM98]), we find

$$P(H(k_1,k_2;n,0)) = \sum_{|\lambda|=n} d_\lambda t_\lambda^{k_1} \bar{t}_\lambda^{k_2} \left(\prod_{(i,j)\in\lambda} \frac{v^{-1}s^{j-i} - vs^{i-j}}{s^{\mathrm{hl}(i,j)} - s^{-\mathrm{hl}(i,j)}} \right),$$

where $\mathrm{hl}(i,j)$ is the hook-length of the cell (i,j), in row i and column j.

3.5.2 The Homfly polynomial of $H(k_1,k_2;n_1,n_2)$

Consider, in a similar way to above, $H(k_1,k_2;n_1,n_2)$ as an element of the skein $\mathcal{C}$. Then we have

$$H(k_1,k_2;n_1,n_2) = \varphi^{k_1}(\bar{\varphi}^{k_2}(A_1^{n_1}A_{-1}^{n_2})).$$

Similar to the restricted case above, we have

$$\varphi^{k_1}(\bar{\varphi}^{k_2}(Q_{\lambda,\mu})) = t_{\lambda,\mu}^{k_1}\bar{t}_{\lambda,\mu}^{k_2}Q_{\lambda,\mu}$$

and

$$A_1^{n_1}A_{-1}^{n_2} = \sum_{\substack{|\lambda|\le n_2 \\ |\mu|\le n_1 \\ |\lambda|-|\mu|=n_2-n_1}} d_{\lambda,\mu}Q_{\lambda,\mu}$$

for constants $d_{\lambda,\mu}$. These constants can be calculated in terms of appropriate d_λ and d_μ (see previous section).

Theorem 3.3 ([Ste87]). *The numbers $d_{\lambda,\mu}$ can be found from the following formula:*

$$d_{\lambda,\mu} = m!\binom{n_2}{m}\binom{n_1}{m}d_\lambda d_\mu,$$

where $|\lambda|\le n_2$, $|\mu|\le n_1$ and $m = n_2-|\lambda| = n_1-|\mu|$.

Therefore,

$$\begin{aligned} H(k_1,k_2;n_1,n_2) &= \sum_{\substack{|\lambda|\le n_2 \\ |\mu|\le n_1 \\ |\lambda|-|\mu|=n_2-n_1}} d_{\lambda,\mu}\varphi^{k_1}(\bar{\varphi}^{k_2}(Q_{\lambda,\mu})) \\ &= \sum_{\substack{|\lambda|\le n_2 \\ |\mu|\le n_1 \\ |\lambda|-|\mu|=n_2-n_1}} d_{\lambda,\mu}t_{\lambda,\mu}^{k_1}\bar{t}_{\lambda,\mu}^{k_2}Q_{\lambda,\mu}. \end{aligned}$$

At present, we do not have a general closed formula for $P(H(k_1,k_2;n_1,n_2))$ due to lack of information about the elements $Q_{\lambda,\mu}$.

We can, however, make explicit calculations in individual cases as illustrated by the following example.

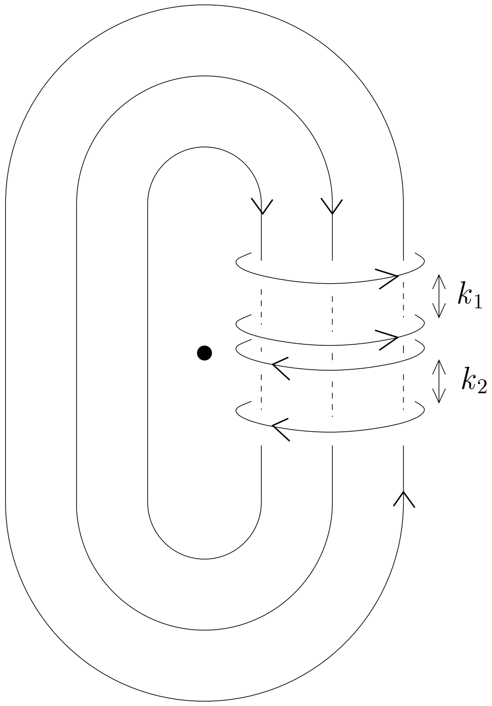


Figure 3.5: The link $H(k_1, k_2; 1, 2)$ in $\mathcal{C}$.

Example. Consider $H(k_1, k_2; 1, 2) \in \mathcal{C}^{(2,1)}$, as shown in Figure 3.5.

Then

$$H(k_1, k_2; 1, 2) = \varphi^{k_1}(\bar{\varphi}^{k_2}(A_1 A_{-1}^2)),$$

where, by Theorem 3.3,

$$A_1 A_{-1}^2 = Q_{\square\square,\square} + 2Q_{\square,\emptyset} + Q_{\substack{\square\\\square},\square}. \tag{3.1}$$

However, we can also find, by using powers of trivial Gyoja-Aiston elements $Q_{\square}$, with appropriate orientation, that since $A_1 = Q_{\square}^{(+)}$ and $A_{-1} = Q_{\square}^{(-)}$ we have

$$A_1 A_{-1}^2 = (Q_{\square}^{(-)})^2 Q_{\square}^{(+)}.$$

Moreover, these elements are known to satisfy the Littlewood-Richardson rule for multiplication of Young diagrams ([Ais96]), so

$$\begin{aligned} A_1 A_{-1}^2 &= (Q_{\square\square}^{(-)} + Q_{\substack{\square\\\square}}^{(-)}) Q_{\square}^{(+)} \\ &= Q_{\square\square}^{(-)} Q_{\square}^{(+)} + Q_{\substack{\square\\\square}}^{(-)} Q_{\square}^{(+)} \\ &= Q'_{\square\square,\square} + Q'_{\substack{\square\\\square},\square}. \end{aligned} \tag{3.2}$$

Now combining equations 3.1 and 3.2 with the observation that

$$Q_{\Box,\emptyset} = Q'_{\Box,\emptyset} = Q^{(-)}_{\Box} Q^{(+)}_{\emptyset}$$

and assuming symmetry under conjugation of Young diagrams, we have

$$\begin{aligned} Q_{\Box\Box,\Box} &= Q'_{\Box\Box,\Box} - Q'_{\Box,\emptyset}, \\ \text{and } Q_{\substack{\Box\\\Box},\Box} &= Q'_{\substack{\Box\\\Box},\Box} - Q'_{\Box,\emptyset}. \end{aligned}$$

Hence, evaluating in the plane, we find,

$$\begin{aligned} P(H(k_1,k_2;1,2)) &= P(\varphi^{k_1}(\bar{\varphi}^{k_2}(A_1 A^2_{-1}))) \\ &= t^{k_1}_{\Box\Box,\Box}\bar{t}^{k_2}_{\Box\Box,\Box} P(Q_{\Box\Box,\Box}) \\ &\quad +2t^{k_1}_{\Box,\emptyset}\bar{t}^{k_2}_{\Box,\emptyset} P(Q_{\Box,\emptyset}) + t^{k_1}_{\substack{\Box\\\Box},\Box}\bar{t}^{k_2}_{\substack{\Box\\\Box},\Box} P(Q_{\substack{\Box\\\Box},\Box}) \\ &= t^{k_1}_{\Box\Box,\Box}\bar{t}^{k_2}_{\Box\Box,\Box}(P(Q'_{\Box\Box,\Box}) - P(Q'_{\Box,\emptyset})) \\ &\quad +2t^{k_1}_{\Box,\emptyset}\bar{t}^{k_2}_{\Box,\emptyset} P(Q'_{\Box,\emptyset}) + t^{k_1}_{\substack{\Box\\\Box},\Box}\bar{t}^{k_2}_{\substack{\Box\\\Box},\Box}(P(Q'_{\substack{\Box\\\Box},\Box} - P(Q'_{\Box,\emptyset})) \\ &= t^{k_1}_{\Box\Box,\Box}\bar{t}^{k_2}_{\Box\Box,\Box} P(Q'_{\Box\Box,\Box}) \\ &\quad +(2t^{k_1}_{\Box,\emptyset}\bar{t}^{k_2}_{\Box,\emptyset} - t^{k_1}_{\Box\Box,\Box}\bar{t}^{k_2}_{\Box\Box,\Box} - t^{k_1}_{\substack{\Box\\\Box},\Box}\bar{t}^{k_2}_{\substack{\Box\\\Box},\Box})P(Q'_{\Box,\emptyset}) \qquad (3.3) \\ &\quad +t^{k_1}_{\substack{\Box\\\Box},\Box}\bar{t}^{k_2}_{\substack{\Box\\\Box},\Box} P(Q'_{\substack{\Box\\\Box},\Box}) \end{aligned}$$

From the definition of the $Q'_{\lambda,\mu}$, we can now use the results in [AM98] to find $P(Q'_{\Box,\emptyset})$, $P(Q'_{\Box\Box,\Box})$ and $P(Q'_{\substack{\Box\\\Box},\Box})$. We have:

$$\begin{aligned} P(Q'_{\Box,\emptyset}) &= \frac{v^{-1}-v}{s-s^{-1}}, \\ P(Q'_{\Box\Box,\Box}) &= \left(\frac{v^{-1}-v}{s^2-s^{-2}}\right)\left(\frac{v^{-1}s-vs^{-1}}{s-s^{-1}}\right)\left(\frac{v^{-1}-v}{s-s^{-1}}\right), \\ \text{and } P(Q'_{\substack{\Box\\\Box},\Box}) &= \left(\frac{v^{-1}-v}{s^2-s^{-2}}\right)\left(\frac{v^{-1}s^{-1}-vs}{s-s^{-1}}\right)\left(\frac{v^{-1}-v}{s-s^{-1}}\right). \end{aligned}$$

Then using Proposition 2.15 we find:

$$\begin{aligned} t_{\Box,\emptyset} &= -v(s-s^{-1}) + \delta, \\ t_{\Box\Box,\Box} &= (s-s^{-1})(-v(1+s^{-2}) + v^{-1}) + \delta, \\ t_{\substack{\Box\\\Box},\Box} &= (s-s^{-1})(-v(1+s^{2}) + v^{-1}) + \delta, \end{aligned}$$

and

$$
\begin{aligned}
\bar{t}_{\square,\emptyset} &= v^{-1}(s - s^{-1}) + \delta, \\
\bar{t}_{\square\square,\square} &= (s - s^{-1})(v^{-1}(1 + s^{2}) - v) + \delta, \\
\bar{t}_{\begin{smallmatrix}\square\\\square\end{smallmatrix},\square} &= (s - s^{-1})(v^{-1}(1 + s^{-2}) - v) + \delta.
\end{aligned}
$$

Substitution of these values into equation 3.3 then gives $P(H(k_1, k_2; 1, 2))$ immediately.

3.6 Some final remarks

We can in principle write any given element of the skein $X \in \mathcal{C}$ as a linear combination of the basis elements $Q_{\lambda,\mu}$. Therefore, one can find $\varphi(X)$ and $\bar{\varphi}(X)$,and hence readily evaluate the Homfly polynomial of $H(k_1, k_2; X) := H_+(X, A_1^{k_1} A_{-1}^{k_2})$. The special case $X = A_1^{n_1} A_{-1}^{n_2}$ gives $H(k_1, k_2; n_1, n_2)$.

In order to be able to write any element of the skein as a linear combination of the basis elements $Q_{\lambda,\mu}$ we must deepen our understanding of these elements. We aim to begin this quest in the next chapter.

Before we embark of this journey we look at some other work related to the findings of the current chapter.

3.6.1 The Homfly polynomial of the decorated Hopf link

Morton and Lukac [Luk01, ML03] show how to calculate the Homfly polynomial of any satellite of the Hopf link, when the decorations are chosen from the more restricted setting of $\mathcal{C}^+$.

This is achieved since the decorations are spanned in the Homfly skein of the annulus by the well-known elements Q_λ. The paper shows that the Homfly polynomial of the Hopf link decorated by Q_λ on one component and Q_μ on the other, denoted $< \lambda, \mu >$, depends on the Schur symmetric function s_μ of an explicit power series depending on λ.

3.6.2 Kauffman polynomials of generalized Hopf links

The techniques developed and used to produce the results of this chapter have been adopted by Zhong and Lu in [ZL02] to investigate the Kauffman polynomials of generalized Hopf links.

They considered the Kauffman skein module of the solid torus which is defined and constructed in an analogous way to the Homfly skein of the annulus, obviously using the unoriented Kauffman skein relations in place of the Homfly skein relations.

Following [MH02], Zhong and Lu define a map φ on the Kauffman skein module and then calculate eigenvalues c_λ. These are then also shown to be distinct for different λ.

Chapter 4

A Basis For The Skein Of The Annulus, $\mathcal{C}$

In the previous chapter we introduced a basis for the full Homfly skein of the annulus. We referred to these skein elements as $Q_{\lambda,\mu}$ where λ and μ are Young diagrams. These basis elements were identified as being eigenvectors of the natural linear skein maps φ and $\bar{\varphi}$ which see the addition of a meridian loop of the annulus.

In this chapter we aim to construct a matrix of simple skein elements whose determinant gives an explicit expression for $Q_{\lambda,\mu}$. Before we can hope to get to that stage we must do some background work. As a taster, we offer some initial observations to the behaviour of the $Q_{\lambda,\mu}$ at a very basic level.

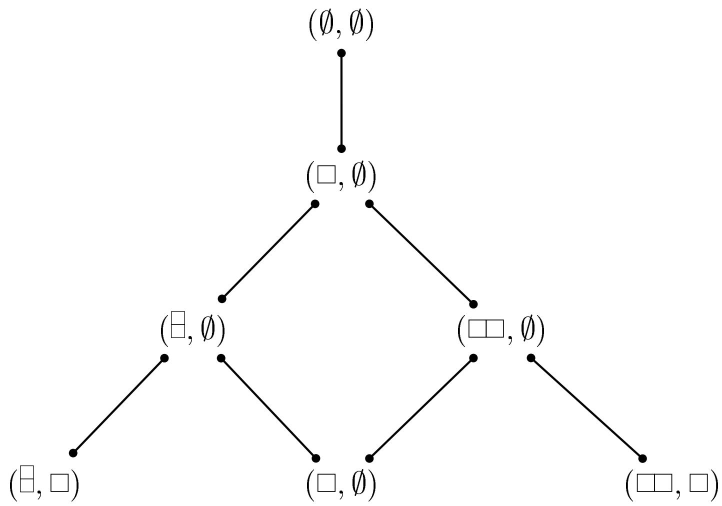
4.1 Basic behaviour of the $Q_{\lambda,\mu}$

It is known that the $Q_{\lambda,\mu} \in \mathcal{C}$ are indexed by pairs of Young diagrams. In this section we ask how these elements behave under multiplication. Since we still have limited knowledge of these elements, we limit ourselves to considering the multiplication by trivial elements, or, in other words

$$Q_{\lambda,\mu} \cdot Q_{\Box,\emptyset} \qquad \text{and} \qquad Q_{\lambda,\mu} \cdot Q_{\emptyset,\Box}.$$

For Young diagrams, such multiplication is illustrated by the Brattelli diagram. For pairs of Young diagrams we can offer an analogue to the Brattelli diagram, it is adapted from a construction offered by Kosuda and Murakami in [KM93]. To illustrate our construction we now build a Brattelli type diagram for the set of Young diagrams relevant to the subspace of $\mathcal{C}$ with $n = 2$

and $p = 1$, $\mathcal{C}^{(2,1)}$. Our diagram is as follows



We notice that to move from one level to the next, we are either multiplying the preceding pairs of Young diagrams by $(\square, \emptyset)$ (the first two steps) or by $(\emptyset, \square)$ (the final step). When multiplying a pair of Young diagrams by $(\square, \emptyset)$ the resulting pairs will either have and extra cell on the *left* Young diagram, or one less cell on the *right* Young diagram. Conversely, when multiplying a pair of Young diagrams by $(\emptyset, \square)$ the resulting pairs will either have and extra cell on the *right* Young diagram, or one less cell on the *left* Young diagram.

Remark. Due to the commutativity in $\mathcal{C}$ we can build up this diagram with identical results even if we were to change the order of the steps.

We use these observations to give the following two rules:

$$\begin{aligned}
Q_{\lambda,\mu} \cdot Q_{\square,\emptyset} (= Q_{\square,\emptyset} \cdot Q_{\lambda,\mu}) &= \sum_{\{(\lambda',\mu):|\lambda'|=|\lambda|+1,\lambda\subset\lambda'\}} Q_{\lambda',\mu} + \sum_{\{(\lambda,\mu'):|\mu'|=|\mu|-1,\mu'\subset\mu\}} Q_{\lambda,\mu'}, \\
Q_{\lambda,\mu} \cdot Q_{\emptyset,\square} (= Q_{\emptyset,\square} \cdot Q_{\lambda,\mu}) &= \sum_{\{(\lambda',\mu):|\lambda'|=|\lambda|-1,\lambda'\subset\lambda\}} Q_{\lambda',\mu} + \sum_{\{(\lambda,\mu'):|\mu'|=|\mu|+1,\mu\subset\mu'\}} Q_{\lambda,\mu'}.
\end{aligned}$$

As a final observation, the number of different paths to a pair of Young diagrams (λ, μ) from top-to-bottom corresponds to the integer $d_{\lambda,\mu}$ given by an explicit formula by Stembridge in Theorem 3.3.

4.2 A spanning set for $\mathcal{C}$

Recall from Chapter 2 the elements of H_n denoted h_n and e_n which correspond respectively to the single row and single column Young diagrams with

n cells. We now consider these elements wired into the annulus, and with a slight abuse of notation we write $h_n, e_n \in \mathcal{C}$ for the closures $\wedge(h_n), \wedge(e_n)$ in $\mathcal{C}$. It can be demonstrated using a symmetric function approach that the skein $\mathcal{C}^+$, when considered as an algebra, is spanned by monomials in the $\{h_m : m \geq 0\}$.

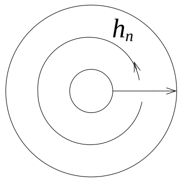
Now consider the image of these elements under the involution $*$. We have $*(h_n) := h_n^*$ and $*(e_n) := e_n^*$. Similarly, the skein $\mathcal{C}^-$ is spanned by monomials in the $\{h_l^* : l \geq 0\}$.

Combining these sets, the whole skein $\mathcal{C}$ is spanned by monomials in $\{h_l^*, h_m : l, m \geq 0\}$.

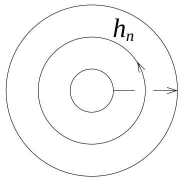
4.3 Some elements of $\mathcal{A}$

Now, if we keep the elements we have just defined in mind, and recall the maps $l : \mathcal{C} \times \mathcal{A} \to \mathcal{A}$ and $r : \mathcal{A} \times \mathcal{C} \to \mathcal{A}$, we can define the following elements of $\mathcal{A}$. Let

$$l_n \quad := \quad l(h_n, e) =$$



and $$r_n \quad := \quad r(e, h_n) =$$



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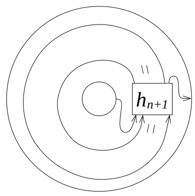
Now given these two elements of $\mathcal{A}$ we define

$$y_n := [n+1] *$$



which satisfies the relation

$$y_n = s^{-1} a y_{n-1} + l_n. \tag{4.1}$$

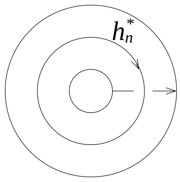
Applying the mirror map, $\bar{}$ to these elements of $\mathcal{A}$ we notice

$$\bar{y}_n = y_n; \quad \bar{a} = a; \quad \bar{l_n} = r_n; \quad s \mapsto s^{-1},$$

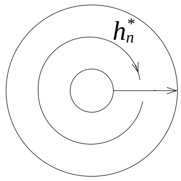
so (4.1) becomes

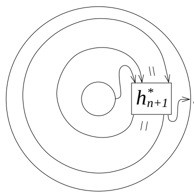
$$y_n = say_{n-1} + r_n. \tag{4.2}$$

We define further elements of $\mathcal{A}$. We have

$$l_n^* \quad = \quad r(e, h_n^*) =$$ 

,

$$r_n^* \quad = \quad l(h_n^*, e) =$$ 

and $\quad y_n^* \quad = \quad [n+1] *$ 

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Similarly we obtain the relation

$$y_n^* = s^{-1}a^{-1}y_{n-1}^* + l_n^* \tag{4.3}$$

and under the mirror map this becomes

$$y_n^* = sa^{-1}y_{n-1}^* + r_n^*. \tag{4.4}$$

We re-write relations (4.4) and (4.3) in order that they are similar in style to (4.1) and (4.2) respectively. We get

$$y_{n-1}^* \quad = \quad s^{-1}ay_n^* + \gamma_{n-1}^* \tag{4.5}$$

$$\text{and} \quad y_{n-1}^* \quad = \quad say_n^* + \rho_{n-1}^*. \tag{4.6}$$

with $\gamma_{n-1}^* = -s^{-1}ar_n^*$ and $\rho_{n-1}^* = -sal_n^*$.

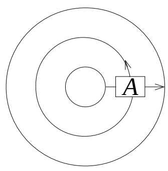
Now solving pairs of equations (4.1,4.2) and (4.5,4.6) we obtain

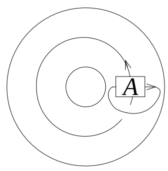
$$(s - s^{-1})y_n \quad = \quad sl_n - s^{-1}r_n \tag{4.7}$$

$$\text{and} \quad (s - s^{-1})y_{n-1}^* \quad = \quad s\gamma_{n-1}^* - s^{-1}\rho_{n-1}^*. \tag{4.8}$$

Finally let us recall the closure map we defined on $\mathcal{A}$. We have

$$\diamond : \mathcal{A} \quad \to \quad \mathcal{C}$$



$\mapsto$


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4.4 Some matrix results

In this section we introduce a system of abbreviations for matrices in order to facilitate the path to our goal. Then using these abbreviations we give some results for determinants of certain matrices of skein elements.

4.4.1 Fixed indexing matrices

Here we describe the idea of a *fixed indexing matrix* (FIM), each of which having associated to it an *indexing vector* (IV). One main feature of the matrices to be considered here is that rows will either contain elements for which all are starred or all are non-starred.

The IV will contain the indices of the elements in the first column of the FIM, the remaining indices then being determined such that the indices of elements in starred rows *decrease* sequentially and the indices of elements in non-starred rows *increase* sequentially.

We shall think of the FIM and the IV as a pair which defines a matrix. We write $M = (A, V)$ for the matrix M represented by the FIM A and the IV V.

Further simplification of notation is possible due to the specific format of the matrices we are interested in. In each FIM we shall only give one row to represent each of the starred and non-starred rows. This will be possible since the elements in any column will be of a similar type, differing only in the indices of its elements. Furthermore, there will be a similarity in elements along rows, with changes occuring in the j^{th} column, for a fixed j. An example will help to clarify this description.

Example. Let A be the 8×8 FIM

$$A = \begin{pmatrix} a^* & \cdots & b^* & b^* & c^* & \cdots \\ \hline a & \cdots & b & b & c & \cdots \end{pmatrix}$$

and V be the IV

$$V = \begin{pmatrix} 3 \\ 4 \\ 3 \\ 5 \\ \hline 1 \\ 2 \\ 3 \\ 1 \end{pmatrix},$$

then taking $j = 4$, we have the matrix M represented by A and V is

$$M = (A, V) = \begin{pmatrix} a_3^* & a_2^* & a_1^* & \overset{\overset{j=4}{\downarrow}}{b_0^*} & b_{-1}^* & c_{-2}^* & c_{-3}^* & c_{-4}^* \\ a_4^* & a_3^* & a_2^* & b_1^* & b_0^* & c_{-1}^* & c_{-2}^* & c_{-3}^* \\ a_3^* & a_2^* & a_1^* & b_0^* & b_{-1}^* & c_{-2}^* & c_{-3}^* & c_{-4}^* \\ a_5^* & a_4^* & a_3^* & b_2^* & b_1^* & c_0^* & c_{-1}^* & c_{-2}^* \\ a_1 & a_2 & a_3 & b_4 & b_5 & c_6 & c_7 & c_8 \\ a_2 & a_3 & a_4 & b_5 & b_6 & c_7 & c_8 & c_9 \\ a_3 & a_4 & a_5 & b_6 & b_7 & c_8 & c_9 & c_{10} \\ a_1 & a_2 & a_3 & b_4 & b_5 & c_6 & c_7 & c_8 \end{pmatrix}.$$

In the forthcoming sections, the matrices we will use will all be of size $(k^* + k) \times (k^* + k)$ with the top k^* rows being starred and the bottom k non-starred. Furthermore, there will be only two different indexing vectors required. We define them now.

Definition 3. Let V_1 and V_2 be the $(k^* + k)$-row IV's

$$V_1 := \begin{pmatrix} i_1 - 1 \\ i_2 - 1 \\ \vdots \\ i_{k^*} - 1 \\ \hline i_{k^*+1} \\ i_{k^*+2} \\ \vdots \\ i_{k^*+k} \end{pmatrix} \quad \text{and} \quad V_2 := V_1 + \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ \hline 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} i_1 \\ i_2 \\ \vdots \\ i_{k^*} \\ \hline i_{k^*+1} \\ i_{k^*+2} \\ \vdots \\ i_{k^*+k} \end{pmatrix}.$$

4.4.2 Matrices of skein elements

Recall the various skein elements constructed and defined in Section 4.3 and the relations between them. We shall now use the notation developed above to state and prove some results for elements of the skein $\mathcal{A}$ which are determinants of matrices of these simple skein elements.

Lemma 4.1. *For all $j \geq 1$ we have*

$$\det\left(\left(\begin{array}{ccccc} & & \overset{j}{\downarrow} & & \\ \gamma^* & \cdots & y^* & \rho^* & \cdots \\ \hline l & \cdots & y & r & \cdots \end{array}\right), V_1 \right) =$$

$$\det\left(\left(\begin{array}{cccccc} & & \overset{j}{\downarrow} & & & \\ \gamma^* & \cdots & y^* & y^* & \rho^* & \cdots \\ \hline l & \cdots & y & y & r & \cdots \end{array}\right), V_1 \right).$$

Proof. Apply column operation

$$c_{j+1} \mapsto sac_j + c_{j+1}$$

using (4.6) on starred rows and (4.2) on non-starred rows. ♠

Corollary. *For all $j \geq 1$ we have*

$$\det\left(\left(\begin{array}{ccccc} & & \overset{j}{\downarrow} & & \\ \gamma^* & \cdots & y^* & \rho^* & \cdots \\ \hline l & \cdots & y & r & \cdots \end{array}\right), V_1 \right) =$$

$$\det\left(\left(\begin{array}{cccc} & & \overset{j}{\downarrow} & \\ \gamma^* & \cdots & y^* & \cdots \\ \hline l & \cdots & y & \cdots \end{array}\right), V_1 \right). \quad (4.9)$$

Lemma 4.2. *For all $j \geq 1$ we have*

$$\det\left(\left(\begin{array}{cccccc} & & \overset{j}{\downarrow} & & & \\ \gamma^* & \cdots & y^* & y^* & \gamma^* & \cdots \\ \hline l & \cdots & y & y & l & \cdots \end{array}\right), V_1 \right) =$$

$$\det\left(\left(\begin{array}{ccccc} & & \overset{j}{\downarrow} & & \\ \gamma^* & \cdots & y^* & \gamma^* & \cdots \\ \hline l & \cdots & y & l & \cdots \end{array}\right), V_1 \right).$$

Proof. Apply column operation

$$c_{j+1} \mapsto -s^{-1}ac_j + c_{j+1}$$

using (4.5) on starred rows and (4.1) on non-starred rows. ♠

Corollary. *For all $j \geq 1$ we have*

$$\det\left(\left(\begin{array}{cccc} \gamma^* & \cdots & \overset{\overset{j}{\downarrow}}{y^*} & \cdots \\ \hline l & \cdots & y & \cdots \end{array} \right), V_1 \right) =$$

$$\det\left(\left(\begin{array}{ccccc} \gamma^* & \cdots & \overset{\overset{j}{\downarrow}}{y^*} & \gamma^* & \cdots \\ \hline l & \cdots & y & l & \cdots \end{array} \right), V_1 \right). \quad (4.10)$$

Lemma 4.3. *For all $j \geq 1$ we have*

$$\det\left(\left(\begin{array}{ccccc} \gamma^* & \cdots & \overset{\overset{j}{\downarrow}}{y^*} & \rho^* & \cdots \\ \hline l & \cdots & y & r & \cdots \end{array} \right), V_1 \right) =$$

$$\det\left(\left(\begin{array}{ccccc} \gamma^* & \cdots & \overset{\overset{j}{\downarrow}}{y^*} & \gamma^* & \cdots \\ \hline l & \cdots & y & l & \cdots \end{array} \right), V_1 \right).$$

Proof. Combine determinantal equations 4.9 and 4.10. ♠

Definition 4.

(a) $\quad \Delta_{k^*+k} := \det\left(\left(\begin{array}{cc} \gamma^* & \cdots \\ \hline l & \cdots \end{array} \right), V_1 \right) \in \mathcal{A};$

(b) $\quad \Delta_0 := \det\left(\left(\begin{array}{cc} \rho^* & \cdots \\ \hline r & \cdots \end{array} \right), V_1 \right) \in \mathcal{A};$

(c) In general, for $j \geq 0$,

$$\Delta_j := \det\left(\left(\begin{array}{ccccc} \gamma^* & \cdots & \overset{\overset{j}{\downarrow}}{\gamma^*} & \rho^* & \cdots \\ \hline l & \cdots & l & r & \cdots \end{array} \right), V_1 \right) \in \mathcal{A}.$$

Lemma 4.4. *For all $j \geq 1$ we have*

$$(s - s^{-1}) \det\left(\left(\begin{array}{ccccc} \gamma^* & \cdots & \overset{\overset{j}{\downarrow}}{y^*} & \rho^* & \cdots \\ \hline l & \cdots & y & r & \cdots \end{array} \right), V_1 \right) = s\Delta_j - s^{-1}\Delta_{j-1}.$$

Proof.

$$(s-s^{-1})\det\left(\begin{pmatrix} \gamma^* & \cdots & \overset{\overset{j}{\downarrow}}{y^*} & \rho^* & \cdots \\ \hline l & \cdots & y & r & \cdots \end{pmatrix}, V_1\right)$$

$$= \det\left(\begin{pmatrix} \gamma^* & \cdots & \overset{\overset{j}{\downarrow}}{(s-s^{-1})y^*} & \rho^* & \cdots \\ \hline l & \cdots & (s-s^{-1})y & r & \cdots \end{pmatrix}, V_1\right)$$

$$= \det\left(\begin{pmatrix} \gamma^* & \cdots & \overset{\overset{j}{\downarrow}}{s\gamma^* - s^{-1}\rho^*} & \rho^* & \cdots \\ \hline l & \cdots & sl - s^{-1}r & r & \cdots \end{pmatrix}, V_1\right)$$

(using equations (4.7) and (4.8))

$$= \det\left(\begin{pmatrix} \gamma^* & \cdots & \overset{\overset{j}{\downarrow}}{s\gamma^*} & \rho^* & \cdots \\ \hline l & \cdots & sl & r & \cdots \end{pmatrix}, V_1\right)$$

$$- \det\left(\begin{pmatrix} \gamma^* & \cdots & \overset{\overset{j}{\downarrow}}{s^{-1}\rho^*} & \rho^* & \cdots \\ \hline l & \cdots & s^{-1}r & r & \cdots \end{pmatrix}, V_1\right)$$

$$= s\Delta_j - s^{-1}\Delta_{j-1}$$

♠

Furthermore,

Lemma 4.5.

$$(s-s^{-1})\sum_{j=1}^{k^*+k} s^{2j-1}\det\left(\begin{pmatrix} \gamma^* & \cdots & \overset{\overset{j}{\downarrow}}{y^*} & \rho^* & \cdots \\ \hline l & \cdots & y & r & \cdots \end{pmatrix}, V_1\right) = s^{2(k^*+k)}\Delta_{k^*+k} - \Delta_0.$$

Proof.

$$
\begin{aligned}
(s-s^{-1})\sum_{j=1}^{k^*+k} s^{2j-1}\det&\left(\left(\begin{array}{cccccc}\gamma^* & \cdots & \overset{\overset{j}{\downarrow}}{y^*} & \rho^* & \cdots \\ \hline l & \cdots & y & r & \cdots\end{array}\right), V_1\right) \\
&= \sum_{j=1}^{k^*+k} s^{2j-1}(s-s^{-1})\det\left(\left(\begin{array}{cccccc}\gamma^* & \cdots & \overset{\overset{j}{\downarrow}}{y^*} & \rho^* & \cdots \\ \hline l & \cdots & y & r & \cdots\end{array}\right), V_1\right) \\
&= \sum_{j=1}^{k^*+k} s^{2j-1}(s\Delta_j - s^{-1}\Delta_{j-1}) \\
&= s^{2(k^*+k)}\Delta_{k^*+k} - \Delta_0.
\end{aligned}
$$

$\spadesuit$

4.5 The final push

In this section we combine all the results of the preceding sections to take us to our final goal, explicitly identifying a basis for the Homfly skein of the annulus $\mathcal{C}$.

Firstly we are required to define some further matrices.

Definition 5.

(a) $$\Delta'_{k^*+k} := \det\left(\left(\begin{array}{cc} r^* & \cdots \\ \hline l & \cdots \end{array}\right), V_2\right) \in \mathcal{A};$$

(b) $$\Delta'_0 := \det\left(\left(\begin{array}{cc} l^* & \cdots \\ \hline r & \cdots \end{array}\right), V_2\right) \in \mathcal{A};$$

Now relating these matrices to those defined in Definition 4 we find

Proposition 4.6. *The following relations hold:*

$$
\begin{aligned}
\textit{(i)}\ \ \Delta'_{k^*+k} &= (-sa^{-1})^{k^*}\Delta_{k^*+k}, \\
\textit{(ii)}\ \ \Delta'_0 &= (-s^{-1}a^{-1})^{k^*}\Delta_0.
\end{aligned}
$$

Proof. To the left hand side of the relations apply the facts that $r_i^* = -sa^{-1}\gamma_{i-1}^*$ and $l_i^* = -s^{-1}a^{-1}\rho_{i-1}^*$ respectively to the top k^* rows. We also allow for the shift in indices on the top k^* rows with the change in indexing vector. The relations follow immediately. $\spadesuit$

The following lemma uses all the results of the previous section to give a relation for elements in $\mathcal{C}$ through application of the closure map $\diamond : \mathcal{A} \to \mathcal{C}$.

Lemma 4.7.

$$
\begin{aligned}
&s^{2k^*}(s^{2k} \diamond (\Delta'_{k^*+k}) - \diamond(\Delta'_0)) = \\
&\qquad \sum_{j=1}^{k^*+k} s^{2j} \left(\delta \det \left(\left(\begin{array}{cc} h^* & \cdots \\ \hline h & \cdots \end{array} \right), V_2 \right) \right. \\
&\qquad\qquad \left. - \det \left(\left(\begin{array}{ccccc} h^* & \cdots & \overset{\overset{j}{\downarrow}}{\bar{\varphi}(h^*)} & h^* & \cdots \\ \hline h & \cdots & s^{-2}\bar{\varphi}(h) & h & \cdots \end{array} \right), V_2 \right) \right)
\end{aligned}
$$

Proof. We begin with the left hand side of the equation:

$$
\begin{aligned}
& s^{2k^*}(s^{2k} \diamond (\Delta'_{k^*+k}) - \diamond(\Delta'_0)) \\
= \;& \diamond(s^{2k^*}(s^{2k}\Delta'_{k^*+k}) - \Delta'_0) \\
& \quad \text{(since } \diamond \text{ is a linear map)} \\
= \;& \diamond(s^{2k^*}(s^{2k}(-sa^{-1})^{k^*}\Delta_{k^*+k} - (-s^{-1}a^{-1})^{k^*}\Delta_0)) \\
& \quad \text{(by Proposition 4.6)} \\
= \;& \diamond((-a^{-1}s)^{k^*}(s^{2(k^*+k)}\Delta_{k^*+k} - \Delta_0)) \\
= \;& \diamond\left((-a^{-1}s)^{k^*}(s - s^{-1}) \sum_{j=1}^{k^*+k} s^{2j-1} \det \left(\left(\begin{array}{ccccc} \gamma^* & \cdots & \overset{\overset{j}{\downarrow}}{y^*} & \rho^* & \cdots \\ \hline l & \cdots & y & r & \cdots \end{array} \right), V_1 \right) \right) \\
& \quad \text{(by Lemma 4.5)} \\
= \;& \diamond\left((-a^{-1}s)^{k^*}(s - s^{-1}) \sum_{j=1}^{k^*+k} s^{2j-1} \det \left(\left(\begin{array}{ccccc} \gamma^* & \cdots & \overset{\overset{j}{\downarrow}}{y^*} & \gamma^* & \cdots \\ \hline l & \cdots & y & l & \cdots \end{array} \right), V_1 \right) \right) \\
& \quad \text{(by Lemma 4.3)} \\
= \;& \diamond\left(\sum_{j=1}^{k^*+k} (-a^{-1}s)^{k^*} s^{2j-1} \det \left(\left(\begin{array}{ccccc} \gamma^* & \cdots & \overset{\overset{j}{\downarrow}}{s\gamma^* - s^{-1}\rho^*} & \gamma^* & \cdots \\ \hline l & \cdots & sl - s^{-1}r & l & \cdots \end{array} \right), V_1 \right) \right) \\
& \quad \text{(after multiplying column } j \text{ by } (s - s^{-1}) \text{ and using relations (4.7,4.8))}
\end{aligned}
$$

$$= \diamond\left(\sum_{j=1}^{k^*+k}(-a^{-1}s)^{k^*}s^{2j}\det\left(\left(\begin{array}{cc}\gamma^* & \cdots \\ \hline l & \cdots\end{array}\right), V_1\right)\right)$$

$$- \diamond\left(\sum_{j=1}^{k^*+k}(-a^{-1}s)^{k^*}s^{2(j-1)}\det\left(\left(\begin{array}{cccccc}\gamma^* & \cdots & \overset{j}{\downarrow}\rho^* & \gamma^* & \cdots \\ \hline l & \cdots & r & l & \cdots\end{array}\right), V_1\right)\right)$$

(by splitting the matrices with the entries in the j^{th} column)

$$= \sum_{j=1}^{k^*+k} s^{2j}\diamond\left(\det\left(\left(\begin{array}{cc}r^* & \cdots \\ \hline l & \cdots\end{array}\right), V_2\right)\right)$$

$$- \sum_{j=1}^{k^*+k} s^{2j}\diamond\left(\det\left(\left(\begin{array}{ccccc}r^* & \cdots & \overset{j}{\downarrow}l^* & r^* & \cdots \\ \hline l & \cdots & s^{-2}r & l & \cdots\end{array}\right), V_2\right)\right)$$

We now apply the closure map to the determinants and recalling the skein map $\bar{\varphi}$ we see the result follows. ♠

We now calculate the values of $\bar{\varphi}(h_n^*)$ and $\bar{\varphi}(h_n)$.

Proposition 4.8.

$$\begin{array}{rrcl} & \bar{\varphi}(h_n^*) & = & (v^{-1}(s^{2n-1}-s^{-1})+\delta)h_n^* \\ \textit{and} & \bar{\varphi}(h_n) & = & (v(s^{-2n+1}-s)+\delta)h_n. \end{array}$$

Proof. We are considering the idempotent closures h_n^* and h_n, therefore we are interested in the single row Young diagram with n cells. We apply the results for the values of t_λ and $\bar{t}_\lambda$ given in Chapter 2 (see also [MH02]), where $\lambda = (n)$ with the content of the cells being (from left-to-right) 0, 1, 2,..., $n-1$. After some cancellation, the results follow. ♠

Corollary (of Lemma 4.7 and Proposition 4.8).

$$s^{2j}\det\left(\left(\begin{array}{cccccc}h^* & \cdots & \overset{j}{\downarrow}\bar{\varphi}(h^*) & h^* & \cdots \\ \hline h & \cdots & s^{-2}\bar{\varphi}(h) & h & \cdots\end{array}\right), V_2\right) =$$

$$\det\left(\left(\begin{array}{ccccc}h^* & \cdots & \overset{j}{\downarrow}(v^{-1}s^{2i_r+1}+s^{2j}(\delta-s^{-1}v^{-1}))h^*_{i_r-j+1} & h^* & \cdots \\ \hline h & \cdots & (vs^{1-2i_r}+s^{2j}(s^{-2}\delta-vs^{-1}))h_{i_r+j-1} & h & \cdots\end{array}\right), V_2\right)$$

where i_r is the value in the r^{th} row of the indexing vector V_2.

Proof. The result is immediate after finding the index for the j^{th} column from the indexing vector. ♠

We notice the following about the second summand of the scalars multiplying the $h^*_{i_r-j+1}$ and the h_{i_r+j-1} in the previous matrix.

Proposition 4.9.

$$\delta - s^{-1}v^{-1} = s^{-2}\delta - s^{-1}v.$$

Proof. Since $\delta = \frac{v^{-1}-v}{s-s^{-1}}$, we have

$$\begin{array}{rrcl} & \delta(s-s^{-1}) & = & v^{-1}-v \\ \Rightarrow & s\delta - v^{-1} & = & s^{-1}\delta - v \\ \Rightarrow & \delta - s^{-1}v^{-1} & = & s^{-2}\delta - s^{-1}v. \end{array}$$

♠

From this point we shall define

$$A_{\lambda,\mu} := \det\left(\begin{pmatrix} h^* & \cdots \\ \hline h & \cdots \end{pmatrix}, V_2\right),$$

and let

$$\begin{array}{rrcl} & \beta^*_{ij} & := & \theta^*_i + \nu_j, \\ \text{and} & \beta_{ij} & := & \theta_i + \nu_j, \end{array}$$

where

$$\begin{array}{rrcl} & \theta^*_i & := & v^{-1}s^{2i_r+1}, \\ & \theta_i & := & vs^{1-2i_r}, \\ \text{and} & \nu_j & := & s^{2j}(\delta - s^{-1}v^{-1}). \end{array}$$

As explained, the entries in V_2 determine the values of the subscripts of the entries in $A_{\lambda,\mu}$. Here, the entries of V_2 will be associated with the number of cells in the Young diagrams λ and μ, although we shall not indicate here how this association is made.

Therefore we have

$$s^{2j}\det\left(\begin{pmatrix} h^* & \cdots & \overset{\overset{j}{\downarrow}}{\bar{\varphi}(h^*)} & h^* & \cdots \\ \hline h & \cdots & s^{-2}\bar{\varphi}(h) & h & \cdots \end{pmatrix}, V_2\right) =$$

$$\det\left(\begin{pmatrix} h^* & \cdots & \overset{\overset{j}{\downarrow}}{\beta^*_{ij}h^*} & h^* & \cdots \\ \hline h & \cdots & \beta_{ij}h & h & \cdots \end{pmatrix}, V_2\right)$$

Lemma 4.10.

$$\sum_{j=1}^{k^*+k} s^{2j}\det\left(\left(\begin{array}{ccccc} h^* & \cdots & \overset{\overset{j}{\downarrow}}{\bar{\varphi}(h^*)} & h^* & \cdots \\ \hline h & \cdots & s^{-2}\bar{\varphi}(h) & h & \cdots \end{array}\right), V_2\right) =$$
$$(\beta^*_{11} + \cdots + \beta^*_{k^*k^*} + \beta_{k^*+1,k^*+1} + \cdots + \beta_{k^*+k,k^*+k})A_{\lambda,\mu}.$$

Proof. We combine the previous statements noting that

$$\sum_{j=1}^{k^*+k} s^{2j}\det\left(\left(\begin{array}{ccccc} h^* & \cdots & \overset{\overset{j}{\downarrow}}{\bar{\varphi}(h^*)} & h^* & \cdots \\ \hline h & \cdots & s^{-2}\bar{\varphi}(h) & h & \cdots \end{array}\right), V_2\right) =$$
$$\sum_{j=1}^{k^*+k} s^{2j}\det\left(\left(\begin{array}{ccccc} h^* & \cdots & \overset{\overset{j}{\downarrow}}{\theta^*_i h^*} & h^* & \cdots \\ \hline h & \cdots & \theta_i h & h & \cdots \end{array}\right), V_2\right)$$
$$+ (\nu_1 + \cdots + \nu_{k^*+k})A_{\lambda,\mu}.$$

Now apply a general formula noted by Lukac in [Luk01] (see also [Luk]) for variables w_{ij} and π_i,

$$\sum_{j=1}^{r}\det\begin{pmatrix} w_{11} & \cdots & w_{1\,j-1} & \pi_1 w_{1j} & w_{1\,j+1} & \cdots & w_{1r} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ w_{r1} & \cdots & w_{r\,j-1} & \pi_r w_{1j} & w_{r\,j+1} & \cdots & w_{rr} \end{pmatrix} = p\det\begin{pmatrix} w_{11} & \cdots & w_{1r} \\ \vdots & & \vdots \\ w_{r1} & \cdots & w_{rr} \end{pmatrix}$$

where $p = \pi_1 + \cdots + \pi_r$. The result follows. ♠

In the following theorem we shall gain a glimpse of the eigenvectors $Q_{\lambda,\mu}$, as required.

Theorem 4.11. *$A_{\lambda,\mu}$ is a scalar multiple of $Q_{\lambda,\mu}$.*

Proof. Recall the statement of Lemma 4.7. The left-hand-side, on calculating the effect of the closure map, is

$$s^{2k^*}(s^{2k} \diamond (\Delta'_{k^*+k}) - \diamond(\Delta'_0)) = s^{2(k^*+k)}\delta A_{\lambda,\mu} - s^{2k^*}\bar{\varphi}(A_{\lambda,\mu}).$$

The right-hand-side, through the preceding manipulation, is

$$\sum_{j=1}^{k^*+k} s^{2j}\left(\delta\det\left(\left(\begin{array}{cc} h^* & \cdots \\ \hline h & \cdots \end{array}\right), V_2\right) - \det\left(\left(\begin{array}{ccccc} h^* & \cdots & \overset{\overset{j}{\downarrow}}{\bar{\varphi}(h^*)} & h^* & \cdots \\ \hline h & \cdots & s^{-2}\bar{\varphi}(h) & h & \cdots \end{array}\right), V_2\right)\right)$$
$$= \left(\left(\sum_{j=1}^{k^*+k} s^{2j}\right)\delta - \left(\sum_{j=1}^{k^*}\beta^*_{jj} + \sum_{j=k^*+1}^{k^*+k}\beta_{jj}\right)\right)A_{\lambda,\mu}.$$

Combining these two statements yields, on re-arranging

$$s^{2k^*}\bar{\varphi}(A_{\lambda,\mu}) = \left(\left(s^{2(k^*+k)} - \sum_{j=1}^{k^*+k} s^{2j}\right)\delta + \sum_{j=1}^{k^*}\beta^*_{jj} + \sum_{j=k^*+1}^{k^*+k}\beta_{jj}\right)A_{\lambda,\mu}.$$

Now, in Chapter 3 we had a result that stated that every eigenvector of $\bar{\varphi}$ is a multiple of one such $Q_{\lambda,\mu}$. We have seen that $A_{\lambda,\mu}$ is an eigenvector of $\bar{\varphi}$, or is zero. We can confirm that it is non-zero by comparing the specialisation of $< A_{\lambda,\mu} >$ (the evaluation of $A_{\lambda,\mu}$ in the plane), when $v = s^N$ with a suitable $< Q_\nu >$, for large enough N. Hence the result follows. ♠

Once we have identified the eigenvalue of $A_{\lambda,\mu}$ as $t_{\lambda,\mu}$, we then know that $A_{\lambda,\mu}$ is a multiple of $Q_{\lambda,\mu}$ and can hence identify the indexing vector appropriate for pairs of Young diagrams (λ, μ).

Chapter 5

A Survey Of Related Work

We end this work with a brief chapter to discuss some recent work of other authors. The work to be discussed here takes a very different approach to the subject with a larger emphasis on algebra and lesser so on the geometric interpretation. It is still however a close relative of what we have been discussing here in the preceding chapters.

5.1 Centralizer algebras of mixed tensor representations

Various parties have discussed a construction similar to the generalized Hecke algebra $H_{n,p}$ discussed here. Its construction however is different from the geometric approach we adopted.

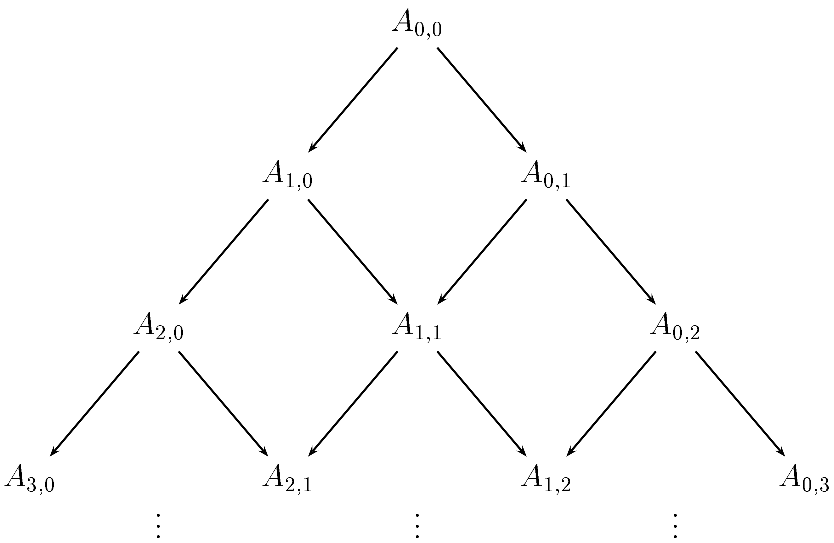
Firstly we consider a one variable algebra. For this algebra, the variable q can be considered in the same context as the variable for the Hecke algebra variant described previously and denoted $H_n(q)$. According to Jimbo [Jim86], the Hecke algebra $H_n(q)$ is the centralizer of the action of the special linear group $H_q(sl_r)$ on $\otimes^n V_q$ where V_q is the natural r-dimensional representation of $U_q(sl_r)$. This is also sometimes called the vector representation.

This is then extended to an algebra denoted $H_{n,p}(q)$. This is then defined to be the centralizer of the action of the general linear group $U_q(gl_r)$ on $(\otimes^n V_q^*) \otimes (\otimes^p V_q)$ where V_q is again the natural r- dimensional representation of $U_q(sl_r)$ and V_q^* its dual.

Such an algebra is considered by Kosuda and Murakami [KM92, KM93] and also by Halverson [Hal96]. The connection is made by these authors between this algebra and the Homfly polynomial of closed (n,p)-tangles.

Perhaps more fitting for our approach, Leduc introduces a two variable algebra in a similar way [Led94], denoted $A_{n,p}(z,q)$ where the q appears as it does above, and the z corresponds to the v we see in the coefficient ring for $H_{n,p}$ and is present to deal with any curls within the tangles. The δ we use is the equivalent to the x used by Leduc.

Leduc offers a convenient way to see how these algebras, shortened now to $A_{n,p}$, display the natural embedding described earlier. We know that $A_{k,l} \subseteq A_{n,p}$ for $0 \leq k \leq n$ and $0 \leq l \leq p$. We then see that the algebras $A_{n,p}$ can be arranged in the form of Pascal's triangle.



In this triangle we may say that an algebra $A_{k,l}$ is a subalgebra of $A_{n,p}$ if and only if there is a path from $A_{k,l}$ to $A_{n,p}$ proceeding from top-to-bottom obeying the directions of the arrows. We also notice that the outer points of the triangle are isomorphic to the Hecke algebra, and the sum $i + j$ for each $A_{i,j}$ is constant at each level.

Remark. Leduc gives a presentation of $A_{n,p}$ in terms of generators and relations (Definition 2.2, [Led94]). The presentation given is isomorphic to the presentation given by the author in the main theorem of [Had] where it is proved to be a presentation for the skein theoretic algebra.

In all these pieces of work, the idea of indexing by pairs of Young diagrams is present, however unlike the approach taken in Chapter 4, they use a concept they describe as staircases.

Leduc ends his thesis [Led94] with a description of the potential connection between this algebra and calculating the Homfly polynomial for closures of tangles.

Barcelo and Ram offer a survey to some of this work and more besides in [BR99]. Their survey is primarily from the point of view of combinatorial representation theory and hence they include much that is beyond the scope of this thesis. They do include a comprehensive list of references.

Remark. In other related work by Kosuda [Kos99], irreducible representations of the Hecke category $\mathcal{H}$ are shown to define isotopy invariants of oriented tangles. The set of oriented tangles (up to isotopy) forms a category denoted $\mathcal{OTA}$. Following Turaev [Tur90], the Hecke category $\mathcal{H}$ is defined as $\mathcal{OTA}$ factored by the Homfly skein relations. This method is then used to compute the Homfly polynomial in [Kos97].

5.2 The Homfly skein module of $S^1 \times S^2$

Gilmer and Zhong discuss the Homfly skein module of $S^1 \times S^2$ in [GZ]. This skein $\mathcal{S}(S^1 \times S^2)$ is described as a certain quotient of $\mathcal{S}(S^1 \times D^2)$, denoted in the preceeding chapters by $\mathcal{C}$. In order to discuss this quotient, the authors first give a basis for the skein $\mathcal{S}(S^1 \times D^2)$ in terms of closures of the Aiston-Morton idempotents of the Hecke algebra. They offer the following proposition, re-written here using our terminology.

Proposition 5.1. *$\mathcal{C}$ has a countable infinite basis given by $Q'_{\lambda,\mu}$ where λ and μ vary over all Young diagrams.*

The space $S^1 \times S^2$ is then considered to be obtained by adding a 2-handle and a 3-handle to the solid torus. The skein of this space is studied via considering another skein, $\mathcal{S}(S^1 \times D^2, A, B)$, the skein of the solid torus with an input point A and an output point B.

Two bases are then given for $\mathcal{S}(S^1 \times D^2, A, B)$. The first is given in terms of the basis of $\mathcal{S}(S^1 \times D^2)$ given by Turaev and denoted here, using our terminology, by the set

$$\{A_m : m \in \mathbb{Z}\}.$$

The second basis is related to the basis of $\mathcal{C}$ described above as $Q'_{\lambda,\mu}$, the closures of two suitably oriented Aiston-Morton idempotent elements.

5.3 Concluding remarks

The author hopes that through this work some interesting questions have been answered. On the one hand it is hoped that the answering of these questions goes a small way in improving our understanding of this area of skein theory; on the other, one hopes that more questions are raised as a result.

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