Generalized Exchangeable Braids

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Abstract

An exchangeable braid is a link with two unknotted components such that each component lies as a closed braid relative to the other as axis. Any such link is constructable from the closure of a certain type of braid known as a Stallings braid, although not all Stallings braids are exchangeable.

In this thesis I study conditions under which a Stallings braid yields an exchangeable braid, including a complete analysis of Stallings braids up to 5 strings. Part of the study involves the consideration of the Alexander polynomial of more general 2-component links, the Alexander polynomial of their cyclic branched covers, and the use of groupoid ideas to then study geometric features of the link.

The setting is then extended to deal with a *generalised exchangeable braid*, where the link may have more than two components, each being a fibered knot.

Contents

Chapter 1

Introduction

In this introductory chapter I will give a brief summary of the ideas and results contained within this thesis, and list the notation used throughout the thesis. The first section of this chapter will be the section on ideas and results used in the thesis, and the second section will be a summary of notation used within this thesis.

1.1 A brief summary of results and ideas.

Traditionally exchangeable braids were defined in the terms of links of two components, each of which are unknotted, and each component having the other component as closed braid relative to it as an axis. A braid, β, is then called exchangeable if its closure together with axis, $\widehat{\beta} \cup A$, form such a link. The essential feature of such a braid is that the closure of the braid, $\widehat{\beta}$, is unknotted and that the axis for the braid A is itself a closed braid relative to $\widehat{\beta}$ as axis.

A Stallings braid on n strings is defined by Morton in [9] to be the product of $n-1$ elementary braids or their inverses which close to a single component, where an elementary braid $\sigma_{i,j} \in B_n$, for $1 \leq i < j \leq n$ is the braid interchanging the i^{th} and j^{th} strings with a single positive crossing, in front of any intermediate strings, and leaving the others alone. Such braids were introduced by Stalling as examples of braids whose closure is the unknot. A spanning surface for the closure of a Stallings braid can be obtained by placing n discs on the braid strings and connecting them by half twisted bands corresponding to the elementary braids in the above product, where a positive half twisted band connecting the i-th and j-th discs corresponds to the elementary braid $\sigma_{i,j}$ and a negative half twisted band connecting the i'-th and j'-th discs corresponds to the inverse of the elementary braid i.e. $\sigma^{-1}_{i',j'}$. Such a disc intersects the axis in only n points, this form of spanning disc for a Stallings braid is known as a *Stallings disc.*

It is shown in [9] that every exchangeable braid is conjugate in the braid group to a Stallings braid, but that some Stallings braids are not exchangeable.

Morton's paper on exchangeable braids then goes on to describe two ways in which a Stallings braid decomposes into simpler Stallings braids, and then proves that the original braid is exchangeable if and only if its decomposition constituents are exchangeable.

The first of these two ways of decomposition is a form of generalized plumbing, called in [9] the *Murasugi sum* of two Stallings braids. The Murasugi sum is a form of plumbing extended to apply to the pairs consisting of the Stallings disc and the axis of the Stallings braid.

Given a Stallings braid $\beta \in B_n$, the braid group on n strings, with Stallings disc D and axis A, following Morton's definition I shall say that the pair (D, A) is the *Murasugi sum* of (D_1, A_1) and (D_2, A_2) if

(i) D_1 and D_2 are subdiscs of D each lying in half of S^3, and meeting only in a disc $D_0 = D_1 \cap D_2$ lying on the sphere, S^2, which separates S^3 into the two halves.

(ii) The axis A meets the separating sphere S^2 in only two points, $d \in D_0$ and $c \notin D_0$.

(iii) The disc D_1 forms a Stallings disc with axis A_1 where A_1 consists of the part of L in the half of S^3 containing D_1 completed by an unknotted arc cd in the other half of S^3, and similarly for the pair (D_2, A_2).

Now that the Murasugi sum of two Stallings braids is defined I will call the decomposition components of the Murasugi sum the Murasugi summands, i.e. the Murasugi summands of $\beta = \partial D$ are $\beta_1 = \partial D_1$ and $\beta_2 = \partial D_2$.

The other possible decomposition of a Stallings braid can be described in terms of the construction of satellite knots and links. In general, to construct a satellite of a link $L = L_1 \cup \ldots \cup L_r$ we need another link $C = C_1 \cup \ldots \cup C_k$, in which one unknotted component, C_k say, is selected. Then one component L_1 say, of the link L is chosen. We then replace a solid torus neighbourhood V of L_1 by the solid torus $W = S^3 - C_k$, the complementary torus to the neighbourhood of C_k. This replacement is by a faithful homeomorphism $h : W \to V$ i.e. one which carries a longitude of W to a longitude of V. The satellite link thus formed consists of $h(C_1) \cup \ldots \cup h(C_{k-1}) \cup L_2 \cup \ldots \cup L_r$, and contains a *splitting torus*, $T = \partial V = h(\partial W)$.

For the case of Stallings braids we have $r = k = 2$ and both $C_1 \cup C_2$ and $L_1 \cup L_2$ consist of closed braid union axis, where C_2 and L_2 are the respective axes. It then follows that $h(C_1) \cup L_2$ is a closed braid with axis L_2 on mn strings when the constituent braids have m and n strings respectively. In the following text I will write the satellite as $(L_1 * C_1) \cup L_2$ to indicate that the component L_1 has been *decorated* by the *pattern* C_1.

For the two preceding decompositions of Stallings braids Morton has proved the following theorems.

Murasugi Sum Theorem *Given that the pair of Stallings braid and axis* $\widehat{\beta} \cup A$ *is the Murasugi sum of* $\widehat{\beta_1} \cup A_1$ *and* $\widehat{\beta_2} \cup A_2$ *, then the Stallings braid* β *is exchangeable if and only if its Murasugi summands* β_1 *and* β_2 *are exchangeable.*

Satellite Theorem *Let the satellite link* $\widehat{\beta_1 * \beta_2} \cup A$ *consist of a closed braid* $\widehat{\beta} = \widehat{\beta_1 * \beta_2}$ *with axis* A*. Then* $\beta = \beta_1 * \beta_2$ *is exchangeable if and only if* β_1 *and* β_2 *are exchangeable.*

Morton also points out that a gap exists between necessary and sufficient geometric conditions for a Stallings braid to be exchangeable.

In Chapter 2 I give a brief survey of Free Differential Calculus results, as a description of why the free calculus can be used to find the Alexander module, and thus the Alexander polynomial.

In Chapter 3 I will investigate the spread in a single variable of the multivariable Alexander polynomial in terms of the genus of the component relating to the variable and the geometric intersection number of the surface and the rest of the link. I will also generalize the k-fold cover theorem on the Alexander polynomial of braids to a similar theorem on tangles. Also I point out that the spread theorem is true in any integer homology three sphere.

In Chapter 4 I will consider the case of exchangeable Stallings braids, as a form of tangle, that is I will investigate the "exchange" of a Stallings braid, and related tangles.

In Chapter 5 I propose to generalize the notion of exchangeable braids to include braids with generalized axis in homology 3-spheres, and using some knowledge of multilinks [3] generalize the Satellite Theorem to a splice decomposition theory of exchangeable braids. In Chapter 5 I will also prove a necessary and sufficient condition on which Stallings braids are exchangeable.

In chapter 6 I will show that the Murasugi sum Theorem may be generalized, that is the Murasugi sum of two generalized braids with generalized axis is exchangeable if and only if the two Murasugi summands are exchangeable. The major part of Chapter 6 will be a discussion of necessary and sufficient conditions on the exchangeability of generalized exchangeable braids, in terms of fibrations of related knots and links. This Chapter contains necessary and sufficient geometric conditions on which Stallings braids are exchangeable. That is a Stallings braid β is Exchangeable if and only if $\widehat{\beta^k}$ is fibered.

The following thesis is dedicated, with many thanks, to my family and friends, and Hugh Morton. I would also like to thank S.E.R.C. for their support.

1.2 Notation used in this thesis.

In this thesis, the space Σ will be a smooth, path connected, orientable 3-manifold,

A **path** in Σ from s_1 to s_2 is a continuous mapping from the interval, I, into Σ

$$p : I \to \Sigma$$

with $p(0) = s_1$ and $p(1) = s_2$.

Write $G(\Sigma, S)$ for the category whose objects are points in S and whose morphisms are paths in Σ with endpoints in S, where S is a non-empty subset of Σ, and write $G(\Sigma, S)(s_1, s_2)$ for the set of all morphisms (paths) from s_1 to s_2 in $G(\Sigma, S)$.

The **inverse** of $p \in G(\Sigma, S)(s_1, s_2)$ is the path $p^{-1} \in G(\Sigma, S)(s_2, s_1)$ defined by $p^{-1}(t) = p(1-t)$.

Write $p_1 \sim p_2$, when $p_1,\ p_2 \in G(\Sigma, S)(s_1, s_2)$ are homotopic, fixing their endpoints. Thus we have ${p_1}^{-1} \circ p_1 \sim id_{s_1}$ and $p_1 \circ {p_1}^{-1} \sim id_{s_2}$. Let us define the homotopy class of p, denoted by $[p] \in G(\Sigma, S)(s_1, s_2)$ as the subset $< p'$: such that $p' \in G(\Sigma, S)(s_1, s_2)$, and $p' \sim p >$.

Write $\pi_1(\Sigma, S)$ for the category whose objects are points in S and whose morphisms are homotopy classes of paths with endpoints in S. This is called the fundamental groupoid of Σ over S.

I shall say that the triple (Σ, F, A) is the *Murasugi sum* of the two triples (Σ_1, F_1, A_1) and (Σ_2, F_2, A_2) if

1. The homology 3-sphere $\Sigma = \Sigma_1 \#_{S^2} \Sigma_2$

2. F_1 and F_2 are subsurfaces of F meeting only in a disc $D_0 = F_1 \cap F_2$ lying on the sphere, S^2, which separates Σ into Σ_1 and Σ_2, such that F_i is in Σ_i, and $F = F_1 \#_{D_0} F_2$.

3. The generalized axis A meets the separating sphere S^2 in only two points, $d \in D_0$ and $c \notin D_0$. Thus A factors into A_1 in Σ_1 and A_2 in Σ_2, $A = A_1 \# A_2$

Let Σ be a smooth, connected orientable, three manifold, and let the *link*, $\mathbf{L}(\Sigma, S_1 \cup \ldots \cup S_N)$, a union of disjoint oriented simple closed curves in Σ (All links in this thesis will be assumed oriented). If $N(S_i)$ denotes a neighbourhood of S_i in Σ then let $\Sigma_0 = ext(\Sigma) = \Sigma - (N(S_1) \cup \ldots \cup N(S_n))$ be the exterior of the link, $\pi : \widetilde{\Sigma_0} \to \Sigma_0$ be the projection from the universal abelian cover to the link exterior, and $\tilde{p} = \pi^{-1}(p)$ be a typical fiber of this map.

Write $G = \pi_1(\Sigma_0)$ be the fundamental group of Σ_0. The group of covering transformations of Σ_0 is $H_1(\Sigma_0)$.

The trivialiser, t, of $\mathbf{Z}(G)$ is defined by

$$t(\Sigma k_i g_i) = (\Sigma k_i)e$$

where $k_i \in \mathbf{Z}$ for all i and $g_i \in G$ with e the identity element of G.

A derivation, D, on the group ring $\mathbf{Z}(G)$ is a map from $\mathbf{Z}(G)$ to itself which satisfies

$$D(u+v) = D(u) + D(v) \tag{1.1}$$

$$D(u.v) = D(u).t(v) + u.D(v) \tag{1.2}$$

where $u, v \in \mathbf{Z}(G)$.

The free group on n generators, $x_1, \ldots, x_n$ is denoted F_n. There is a set of n endomorphisms of $\mathbf{Z}[F_n]$ called the free derivatives. They are determined by

$$\frac{\partial(x_i)}{\partial x_j} = \delta_{ij} \tag{1.3}$$

$$\frac{\partial(u+v)}{\partial x_j} = \frac{\partial u}{\partial x_j} + \frac{\partial v}{\partial x_j} \tag{1.4}$$

$$\frac{\partial(uv)}{\partial x_j} = \frac{\partial u}{\partial x_j} t(v) + u\frac{\partial v}{\partial x_j} \tag{1.5}$$

$$u - t(u) = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}(x_j - 1) \quad \forall u \in \mathbf{Z}(F_n) \tag{1.6}$$

$$w - 1 = \sum_{j=1}^{n} \frac{\partial w}{\partial x_j}(x_j - 1) \quad \forall w \in F_n \tag{1.7}$$

Chapter 2

The Free Differential Calculus.

The purpose of this chapter is to give a description of derivations, in particular the free differential calculus, and to show how to obtain the Alexander module from a finite presentation of the fundamental group.

2.1 The Free Calculus As A Derivation.

When studying the knot group a useful tool is the Free Differential Calculus of Fox in [4]. It is useful in that it converts the study of a non-abelian group into the study of a related abelian group which still retains enough structure to be interesting. To introduce the free calculus it is first necessary to consider the group ring, $\mathbf{Z}(G)$, where an element $\Sigma k_i g_i \in \mathbf{Z}(G)$ is the finite sum of integer multiples of group elements, where $k_i \in \mathbf{Z}$ for all i and all but a finite number of the k_i are zero. Next it is useful to consider the trivialiser t of the group ring. The trivialiser t of $\mathbf{Z}(G) \to \mathbf{Z}(G)$ is defined by

$$t(\Sigma k_i g_i) = (\Sigma k_i)e$$

where $k_i \in \mathbf{Z}$ for all i and $g_i, e \in G$ with e the identity element.

A derivation on the group ring $\mathbf{Z}(G)$ is a map D from $\mathbf{Z}(G)$ to itself which satisfies

$$\begin{aligned} D(u+v) &= D(u) + D(v) && (2.1) \\ D(u.v) &= D(u).t(v) + u.D(v) && (2.2) \end{aligned}$$

where $u, v \in \mathbf{Z}(G)$.

Next we consider derivations on the free group. Write F_n for the free group on n generators, $x_1, \ldots, x_n$ and $\mathbf{Z}[F_n]$ for its integer group ring. As above write

$t : \mathbf{Z}[F_n] \to \mathbf{Z}[F_n]$ for the *trivialiser,* defined by $t(\sum a_i w_i) = (\sum a_i)e$, where $w_i \in F_n$ and e is the identity element of F_n. Let the derivation D be $D = id - t$, where id is the identity mapping on the group ring and t is as before. Thus we have $D(\Sigma k_i g_i) = \Sigma k_i g_i - \Sigma k_i e_{F_n} = \Sigma(k_i g_i - k_i e_{F_n}) = \Sigma k_i D(g_i)$, note that $t(D(g)) = 0$. If $w = \Sigma k_i g_i \in \mathbf{Z}(F_n)$ has $t(w) = 0$ then $w = w - t(w) = D(w)$. Now it can be shown that the $\mathbf{Z}(F_n)$ module defined by $(w \in \mathbf{Z}(F_n) \mid D(w) = 0)$ is a free module with generators $(x_1 - 1), \ldots, (x_n - 1)$. Then,

$$u - t(u) = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}(x_j - 1) \quad \forall u \in \mathbf{Z}(F_n) \tag{2.3}$$

and in particular

$$w - 1 = \sum_{j=1}^{n} \frac{\partial w}{\partial x_j}(x_j - 1) \quad \forall w \in F_n. \tag{2.4}$$

This set of n endomorphisms $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ of $\mathbf{Z}[F_n]$ is called the free differential calculus because they obey

$$\frac{\partial(x_i)}{\partial x_j} = \delta_{ij} \tag{2.5}$$

$$\frac{\partial(u+v)}{\partial x_j} = \frac{\partial u}{\partial x_j} + \frac{\partial v}{\partial x_j} \tag{2.6}$$

$$\frac{\partial(uv)}{\partial x_j} = \frac{\partial u}{\partial x_j} t(v) + u \frac{\partial v}{\partial x_j} \tag{2.7}$$

but (2.6) and (2.7) are the rules to be satisfied for a map to be a derivation.

2.2 Free differential calculus and the Alexander module.

In this section I will present a method for obtaining a presentation of the Alexander module, from a presentation of the fundamental group of the exterior of the link, and thus a method for finding the Alexander polynomial, where it is properly defined. For more complete treatment see the papers of Fox [4, 5, 6].

Let Σ be a smooth, connected orientable, 3-manifold, and let the link $\mathbf{L}(\Sigma, S_1 \cup \ldots \cup S_N)$ in Σ be a proper embedding of simple oriented closed curves, $S_1 \cup \ldots \cup S_N$, into Σ.

Since the aim of this section is to find a presentation of the Alexander module in terms of the exterior of the link, we must first define the exterior of a link in Σ. Let us denote by $N(S_i)$ a neighbourhood of S_i in Σ.

Then the exterior, Σ_0, of the link $\mathbf{L}$ in Σ is $\Sigma_0 = \Sigma - (N(S_1) \cup \ldots \cup N(S_n))$.

Let $\widetilde{\Sigma_0}$ be the universal abelian cover of Σ_0, with associated projection $\pi : \widetilde{\Sigma_0} \to \Sigma_0$ from the universal abelian cover to the link exterior, with $\tilde{p} = \pi^{-1}(p)$ a typical fiber of this map, and let $G = \pi_1(\Sigma_0)$ be the fundamental group of Σ_0. The group of covering transformations of $\widetilde{\Sigma_0}$ is $H_1(\Sigma_0)$.

The aim of this section is to show that a presentation for $H_1(\widetilde{\Sigma}_0, p)$ as a $\mathbf{Z}H_1(\Sigma_0)$ module can be obtained from a presentation of $G = \pi_1(\Sigma_0)$, the fundamental group of Σ_0, because $H_1(\widetilde{\Sigma}_0, p)$ presented as a $\mathbf{Z}H_1(\Sigma_0)$ module is called the Alexander module.

Let P be a presentation of the group G, given by

$$P = (x_1, \ldots, x_n : r_1, \ldots, r_m).$$

This can be considered to be part of a cell decomposition of Σ_0, where the base point p of the fundamental group is the 0-cell, the n 1-cells are the generators of the presentation

$$x_1, \ldots, x_n,$$

and are thus loops in the fundamental group, and the m 2-cells are the discs

$$D_1, \ldots, D_m,$$

with the attaching map of D_i being r_i.

Now let H be some quotient of G, and $\widetilde{\Sigma}_0 \to \Sigma_0$ the regular covering with group of covering transformations isomorphic to H. The cell structure for Σ_0 lifts to a cell structure for $\widetilde{\Sigma}_0$, as follows.

Let $q \in \tilde{p}$ be a specific element of the fiber, then choose $\widetilde{x_i}$ for the unique lift of x_i which starts at q, and $\widetilde{D_i}$ for the unique lift of D_i such that $\partial\widetilde{D_i}$ is the lift $\widetilde{r_i}$ of r_i which starts at q. Then the cell decomposition of the covering space, $C_0(\widetilde{\Sigma_0})$, $C_1(\widetilde{\Sigma_0})$, $C_2(\widetilde{\Sigma_0})$ are free $\mathbf{Z}(H)$ modules.

Now when $H = H_1(\Sigma_0)$ we have the boundary map $\partial_2 : C_2(\widetilde{\Sigma_0}) \to C_1(\widetilde{\Sigma_0})$ is an almost complete description of $H_1(\widetilde{\Sigma_0}, \tilde{p})$ as a $\mathbf{Z}(H_1(\Sigma_0))$ module. (And since $H_1(\widetilde{\Sigma_0}, \tilde{p})$ can be described solely in terms of the group G the result will be independent of the presentation P.)

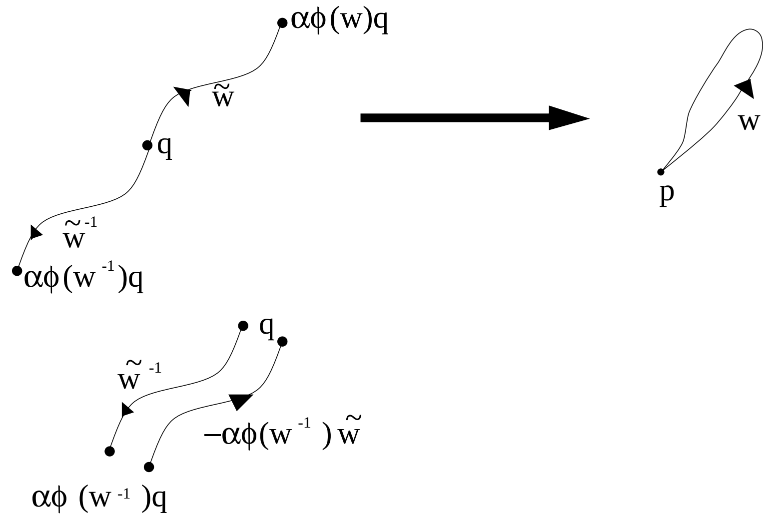
The next step is to let w be any word in the x_j's, regarded as a representative loop in the homotopy class based at p. Then w lifts to a unique path $\tilde{w}$ based at q.

Proposition 2.2.1 *If w is any word in the x_i's then the lift $\tilde{w}$ is the following* 1*-chain in $\widetilde{\Sigma_0}$,*

$$\tilde{w} = \sum_{j=1}^{n} \alpha\phi(\frac{\partial w}{\partial x_j})\widetilde{x_j}$$

where $\frac{\partial w}{\partial x_j}$ is the derivation of the word w with respect to the generator x_j in the free differential calculus.

Proof The proof will be by induction on the word length. When the word length of w is one $w = {x_i}^{-1}$ or $w = x_i$. Firstly we need to see that the end point of $\tilde{w}$ based at q is $\alpha\phi(w)q$, and that $\widetilde{w^{-1}} = \alpha\phi(w^{-1}) \times -(\tilde{w})$. The illustration below shows this to be true.



If $w = x_i$ we have

$$\tilde{w} = \widetilde{x_i} = \sum_{j=1}^{n} \delta_{ij}\widetilde{x_j} \ ,$$

and if $w = {x_i}^{-1}$ we have

$$\tilde{w} = \widetilde{{x_i}^{-1}} = -\alpha\phi({x_i}^{-1})\widetilde{x_i} = \sum_{j=1}^{n} -\delta_{ij}\alpha\phi({x_j}^{-1})\widetilde{x_j} \ .$$

Now we assume that any word of length less than or equal to l has a lift $\tilde{w}$ as required. So we now consider a word, ${x_i}^{\epsilon}w$, of length $l+1$.

Now as above the end point of $\widetilde{{x_i}^{\epsilon}}$ beginning at q is $\alpha\phi({x_i}^{\epsilon})q$, so $\widetilde{{x_i}^{\epsilon}w} = \widetilde{{x_i}^{\epsilon}} + \alpha\phi({x_i}^{\epsilon})\tilde{w}$ thus

$$\widetilde{{x_i}^{\epsilon}w} = \sum_{j=1}^{n} \alpha\phi(\frac{\partial {x_i}^{\epsilon}}{\partial x_j})\widetilde{{x_j}^{\epsilon}} + \alpha\phi({x_i}^{\epsilon}) \sum_{j=1}^{n} \alpha\phi(\frac{\partial w}{\partial x_j})\widetilde{{x_j}^{\epsilon}} \ .$$

Thus by the inductive hypothesis $\tilde{w}$ is as described. $\square$

In particular, with respect to the $\mathbf{Z}(H_1(\widetilde{\Sigma_0}))$ bases of $C_2(\widetilde{\Sigma_0}) = (\widetilde{D_i} : 1 \leq i \leq m)$ and $C_1(\widetilde{\Sigma_0}) = (\widetilde{x_j} : 1 \leq j \leq n)$, we have $\partial_2 : C_2(\widetilde{\Sigma_0}) \to C_1(\widetilde{\Sigma_0})$ given by the $m \times n$ matrix

$$M = \alpha\phi(\frac{\partial r_i}{\partial x_j}) \ .$$

The other boundary map $\partial_1 : C_1(\widetilde{\Sigma_0}) \to C_0(\widetilde{\Sigma_0})$ is given by the following

$$\partial_1(\widetilde{x_j}) = (\alpha\phi(x_j) - 1)q \ .$$

Thus we have the above matrix M as a presentation of $H_1(\widetilde{\Sigma_0}, \tilde{p})$, which leads to a presentation of $H_1(\widetilde{\Sigma_0})$ because $H_1(\widetilde{\Sigma_0}, \tilde{p}) = I \oplus H_1(\widetilde{\Sigma_0})$, where I is the $\mathbf{Z}(H)$ module generated by $(\alpha\phi(x_j) - 1)$, for every j.

The module, $H_1(\widetilde{\Sigma_0})$, presented as a $\mathbf{Z}(H_0(\Sigma_0))$ module is the Alexander module, and the Alexander polynomial can be retrieved from the determinants of the $(n-1) \times (n-1)$ submatrices of M.

Chapter 3

The Alexander polynomial

In this chapter I will show a number of results on the Alexander polynomial, including an upper bound for the spread of each variable in the multivariable Alexander polynomial of a link.

3.1 Spread of the Alexander polynomial

At first we consider the spread in a variable of the multivariable Alexander polynomial in the three sphere.

3.1.1 A bound on a variable in the multi-variable Alexander polynomial.

There is a well-known bound for the degree of the Alexander polynomial $\Delta_K(t)$ of a knot K in terms of the genus of a spanning surface for K.

In this chapter we find similar geometric bounds for the Laurent degree of each variable in the multi-variable Alexander polynomial of a link. We consider a link L with k components $L_1, \ldots, L_k$. The Alexander polynomial, for $k \geq 2$, is a Laurent polynomial $\Delta_L(t_1, \ldots, t_k)$ in k variables $t_1, \ldots, t_k$ corresponding to the meridians of $L_1, \ldots, L_k$ respectively. Write $\Delta_L(t_1, \ldots, t_k)$ in terms of t_i as

$$\Delta_L(t_1, \ldots, t_k) = \sum_{j=m}^{M} t_i^j P_j \ ,$$

where P_j is a Laurent polynomial in the remaining variables with $P_m \neq 0, P_M \neq 0$, and we define the *spread*, or *Laurent degree*, of the variable t_i in $\Delta_L(t_1, \ldots, t_k)$ to be $Spr_{t_i}(\Delta_L) = M - m$. Select one component $K = L_i$ of L, and put $L' = L - K$, and write $t_K = t_i$ for the corresponding variable. Choose a Seifert surface S_K for K, in other words any compact orientable surface whose boundary is K, and arrange that L' intersects S_K transversely. Write $I(S_K, L')$ for the total number of transverse intersections of L' with S_K. We now give an upper bound for the spread, $Spr_{t_K}(\Delta_L)$, of the variable t_K in Δ_L, in terms of $I(S_K, L')$ and the genus $g(S_K)$ of the surface S_K.

Theorem 3.1.1 *In the multi-variable Alexander polynomial $\Delta_L(t_1, \ldots, t_k)$ we have*

$$Spr_{t_K}(\Delta_L) \leq 2g(S_K) + I(S_K, L') - 1.$$

Corollary 3.1.1 *Let $\Delta_L(t_1, \ldots, t_k)$ be as above. Then*

$$Spr_{t_K}(\Delta_L) \leq \min_{S_K}(2g(S_K) + I(S_K, L') - 1).$$

Proof Obvious. □

Corollary 3.1.2 *Let L be a link with an unknotted component U. Then*

$$Spr_{t_U} \leq I(D_U, L') - 1.$$

Proof The disk D_U has genus 0. □

Remark In the case of the closed braid with axis we have a specialisation of Corollary 3.1.2 above. This follows because the Alexander polynomial of an l-braid and axis can be realised as the characteristic polynomial, $\det(xI - B(t))$, of an $(l-1) \times (l-1)$ invertible matrix, known as the reduced Burau matrix of the braid, with x representing the meridian of the unknotted axis U. Then $Spr_U = Spr_x(\det(xI - B(t))) = (l-1)$, while a disk D_U spanning the axis meets the closed braid in at least l points.

3.1.2 Free calculus.

In the proof of theorem 3.1.1. we will make use of Fox's free differential calculus in calculating the multi-variable Alexander polynomial, Δ_L, from a presentation of the group $\pi_1(S^3 - L)$. The following is a brief reprise of chapter 2, for a more complete treatment see [4, 5, 6].

Write F_n for the free group on n generators, $x_1, \ldots, x_n$ and $\mathbf{Z}[F_n]$ for its integer group ring. Write $t : \mathbf{Z}[F_n] \to \mathbf{Z}$ for the *trivialiser*, defined by $t(\sum a_i w_i) = \sum a_i$, where $w_i \in F_n$. The free differential calculus defines a set n endomorphisms $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ of $\mathbf{Z}[F_n]$ with the following properties:

$$\begin{aligned}
\frac{\partial(x_i)}{\partial x_j} &= \delta_{ij} && (3.1)\\
\frac{\partial(u+v)}{\partial x_j} &= \frac{\partial u}{\partial x_j} + \frac{\partial v}{\partial x_j} && (3.2)\\
\frac{\partial(uv)}{\partial x_j} &= \frac{\partial u}{\partial x_j} t(v) + u\frac{\partial v}{\partial x_j} && (3.3)
\end{aligned}$$

Then,

$$u - t(u) = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}(x_j - 1) \quad \forall u \in \mathbf{Z}(F_n) \qquad (3.4)$$

and in particular

$$w - 1 = \sum_{j=1}^{n} \frac{\partial w}{\partial x_j}(x_j - 1) \quad \forall w \in F_n \qquad (3.5)$$

Given a group G with a finite presentation $G = < x_1, \ldots, x_n \mid r_1, \ldots, r_m >$ on n generators $x_1, \ldots, x_n$, write F_n as before for the free group with generators $x_1, \ldots, x_n$ and $\phi : F_n \to G$ for the homomorphism defined by $\phi(x_i) = x_i$. Then $R = \ker \phi$ is the normal subgroup of F_n generated (as a normal subgroup) by $r_1, \ldots, r_m \in F_n$. We may define an $m \times n$ "Jacobian" matrix $M = (\frac{\partial r_i}{\partial x_j})$ with entries in $\mathbf{Z}[F_n]$. This has the property that

$$M \begin{pmatrix} x_1 - 1 \\ \vdots \\ x_n - 1 \end{pmatrix} = \begin{pmatrix} r_1 - 1 \\ \vdots \\ r_m - 1 \end{pmatrix} ,$$

under the natural convention for multiplying matrices with non-commuting entries, since $\sum_{j=1}^{n} \frac{\partial r_i}{\partial x_j}(x_j - 1) = r_i - 1$ by (3.5).

There is a homomorphism $\alpha : G \to G/G'$, called the *abelianiser*, from G to the abelian group G/G', we will also write the group G/G' multiplicatively.

The map $\alpha\phi : F_n \to G/G'$ induces a ring homomorphism $\alpha\phi : \mathbf{Z}[F_n] \to \mathbf{Z}[G/G']$ to the commutative ring $\mathbf{Z}[G/G']$.

Write $y_j = \alpha\phi(x_j)$; the matrix M gives a matrix $\alpha\phi M$ with entries in $\mathbf{Z}[G/G']$, which satisfies

$$\alpha\phi M \begin{pmatrix} (y_1 - 1) \\ \vdots \\ (y_n - 1) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} , \text{ since } \phi(r_i) = 1 \in G.$$

Now let $G = \pi_1(S^3 - L)$, and choose a finite presentation of G. The abelianisation $G/G' \cong H_1(S^3 - L)$ is a free abelian group of rank k, and we can choose generators $t_1, \ldots, t_k$ which are represented by oriented meridians of the components $L_1, \ldots, L_k$. The group ring $\mathbf{Z}[G/G']$ is then the ring $\mathbf{Z}[t_1^{\pm 1}, \ldots, t_k^{\pm 1}]$ of Laurent polynomials in $t_1, \ldots, t_k$.

Using the presentation of G on n generators find the matrix $\alpha\phi M$, called the Alexander matrix of the presentation. The Alexander polynomial $\Delta_L(t_1, \ldots, t_k) \in \mathbf{Z}[t_1^{\pm 1}, \ldots, t_k^{\pm 1}]$ is defined, up to a unit in this ring, as

$$\Delta_L(t_1, \ldots, t_k) = \gcd(\ (n-1) \times (n-1)\,\mathbf{subdet}\ \alpha\phi M\ ).$$

It is shown by Fox in [5] that Δ_L depends only on G and not on the choice of presentation. We now give the proof of Theorem 3.1.1.

3.1.3 Proof of theorem.

Proof of Theorem 3.1.1 We will use Van Kampen's theorem to find a suitable presentation of $G = \pi_1(S^3 - L)$, based on a decomposition of $S^3 - L$ as $U \cup V$, where U, V are open and $U \cap V$ is path-connected.

Let K be the chosen component of L, as above, and $L' = L - K$. Choose a meridian m for K and a solid torus V_m centred on m, such that $(V_m \cap L') = \emptyset$. The spanning surface S_K meets L' transversely in $l = I(S_K, L')$ points. Write $S'_K = (S_K - (S_K \cap L'))$ for S_K with these points removed. Then we may choose our U and V as follows

$$U = (S^3 - L' - S_K),$$

$$V = \operatorname{int}((S'_K \times (-\varepsilon, \varepsilon)) \cup V_m).$$

Then we have

$$U \cup V = (S^3 - L),$$

$$U \cap V = \operatorname{int}((S'_K \times (-\varepsilon, 0)) \cup (S'_K \times (0, \varepsilon)) \cup (V_m - (V_m \cap S'{}_K))).$$

So by Van Kampen's Theorem;

$$\pi_1(S^3 - L) = \pi_1(U) * \pi_1(V) / < (i_U)_*(x) = (i_V)_*(x) \mid \forall x \in \pi_1(U \cap V) >$$

where $(i_U)_*$ and $(i_V)_*$ are the homomorphisms at the level of homotopy induced from the inclusions $i_U : (U \cap V) \to U$ and $i_V : (U \cap V) \to V$.

The fundamental group of any oriented surface with one boundary component and l punctures is a free group on $2g+l$ generators, where g is the genus of the surface.

Now $\mathrm{int}(S'_K \times (-\varepsilon,\varepsilon)) \cap \mathrm{int}(V_m) = D^2$, where D^2 is an open *ball*, which is simply-connected. So we have $\pi_1(V) = \pi_1(S'_K) * \pi_1(V_m) =< \mathbf{h}', \lambda \mid >$, where the orientation of the meridian λ for the component K is chosen so $t_K = \alpha\phi(\lambda)$.

Similarly for $U \cap V$ we have two copies of the punctured Seifert surface joined together by $V_m - (V_m \cap S'_K)$, which gives

$$\pi_1(U \cap V) =< \mathbf{h}'', \mathbf{k}'' \mid >$$

with $(i_V)_*(\mathbf{h}'') = \mathbf{h}'$, and $(i_V)_*(\mathbf{k}'') = \lambda \mathbf{h}' \lambda^{-1}$

Now we build a presentation for $\pi_1(U)$ which is compatible with the generators of $\pi_1(U \cap V)$, in other words we start with generators of $\pi_1(U)$ which exactly match the generators of $\pi_1(U \cap V)$, and then complete the generating set. This gives a presentation of the form

$$\pi_1(U) =< \mathbf{h}, \mathbf{k}, \mathbf{g} \mid \mathbf{r}' > \quad .$$

Van Kampen's theorem then gives the presentation

$$\pi_1(S^3 - L) =< \mathbf{h}', \mathbf{h}, \mathbf{k}, \mathbf{g}, \lambda \mid \mathbf{r}', (i_U)_*(\mathbf{h}'') = (i_V)_*(\mathbf{h}''), (i_U)_*(\mathbf{k}'') = (i_V)_*(\mathbf{k}'') > .$$

Now $(i_U)_*(\mathbf{h}'') = \mathbf{h}$, $(i_U)_*(\mathbf{k}'') = \mathbf{k}$, giving

$$\pi_1(S^3 - L) =< \mathbf{h}', \mathbf{h}, \mathbf{k}, \mathbf{g}, \lambda \mid \mathbf{r}', \mathbf{h} = \mathbf{h}', \mathbf{k} = \lambda \mathbf{h} \lambda^{-1} > .$$

which gives

$$\pi_1(S^3 - L) =< \mathbf{h}, \mathbf{k}, \mathbf{g}, \lambda \mid \mathbf{r}', \mathbf{k} = \lambda \mathbf{h} \lambda^{\mathbf{-1}} >$$

The Alexander matrix of this presentation can be found by doing free calculus on the relations, $r_i = k_i \lambda {h_i}^{-1} \lambda^{-1}$, and r'_j. We have

$$\frac{\partial r_i}{\partial h_j} = -\delta_{ij}(k_i \lambda {h_i}^{-1}) = -\delta_{ij} r_i \lambda,\ \frac{\partial r_i}{\partial k_j} = \delta_{ij},\ \frac{\partial r_i}{\partial g} = 0,\ \frac{\partial r_i}{\partial \lambda} = k_i - r_i$$

and since the relations $\mathbf{r}'$ depend only on $\mathbf{h}$, $\mathbf{k}$ and $\mathbf{g}$ we have $\dfrac{\partial r'_j}{\partial \lambda} = 0$ for all j.

Now the Alexander matrix of the group is found by applying $\alpha\phi$ to the Jacobian matrix M, which can be written in block form as

$$M = \begin{pmatrix} \frac{\partial \mathbf{r}}{\partial \mathbf{h}} & \frac{\partial \mathbf{r}}{\partial \mathbf{k}} & \frac{\partial \mathbf{r}}{\partial \mathbf{g}} & \frac{\partial \mathbf{r}}{\partial \lambda} \\ \frac{\partial \mathbf{r}'}{\partial \mathbf{h}} & \frac{\partial \mathbf{r}'}{\partial \mathbf{k}} & \frac{\partial \mathbf{r}'}{\partial \mathbf{g}} & \frac{\partial \mathbf{r}'}{\partial \lambda} \end{pmatrix} .$$

The calculations above give

$$\alpha\phi M \quad = \quad \begin{pmatrix} I_{2g+l} & -\alpha\phi(\lambda)I_{2g+l} & 0 & \alpha\phi(k_i)-1 \\ \alpha\phi\dfrac{\partial\mathbf{r}'}{\partial\mathbf{h}} & \alpha\phi\dfrac{\partial\mathbf{r}'}{\partial\mathbf{k}} & \alpha\phi\dfrac{\partial\mathbf{r}'}{\partial\mathbf{g}} & 0 \end{pmatrix} \tag{3.6}$$

As all the generators of this presentation, except λ, are homotopic to paths in U we have that the linking numbers of the other generators and K is zero. This implies that the only appearances of $t_K = \alpha\phi(\lambda)$ are the $2g+l$ entries in the (1,2)-block of the matrix.

Now we need the following two lemmas;

Lemma 3.1.1 *Let Λ be a unique factorisation domain, let M be a $n \times (n+1)$ matrix with entries in Λ, let M_j be M with the jth column deleted, and let $D_j = \det M_j$, and suppose that*

$$M \begin{pmatrix} a_1 \\ \vdots \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad ,$$

where $a_j \in \Lambda$ and $\gcd(a_1, \ldots, a_{n+1}) = 1$. Then
$(D_1, \ldots, D_{n+1}) = p(a_1, \ldots, a_{n+1})$ for $p = \gcd(D_1, \ldots, D_{n+1}) \in \Lambda$.

Proof Adjoin a row of zeros to M to give

$$M' = \begin{pmatrix} & M & \\ 0 & \ldots & 0 \end{pmatrix} \quad ,$$

with $\det(M') = 0$.

Then, $\operatorname{adj}(M')M' = 0 \times I_{n+1} = M'\operatorname{adj}(M')$.

Now

$$\operatorname{adj}(M') = \begin{pmatrix} 0 & \ldots & 0 & D_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & D_{n+1} \end{pmatrix}$$

which gives

$$M \begin{pmatrix} D_1 \\ \vdots \\ D_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad .$$

Either $D_j = 0$ for all j or there exists a k such that $D_k \neq 0$. If $D_j = 0$ for all j then $(0, \ldots, 0) = 0(a_1, \ldots, a_{n+1})$. Now if $D_k \neq 0$ we have

$$M\begin{pmatrix} a_kD_1 - D_ka_1 \\ \vdots \\ a_kD_k - D_ka_k \\ \vdots \\ a_kD_{n+1} - D_ka_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

which gives

$$M_k\begin{pmatrix} a_kD_1 - D_ka_1 \\ \vdots \\ a_kD_{n+1} - D_ka_{n+1} \end{pmatrix}_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

where $(v_1, \dots, v_m)_k^T$ is a vector with its kth entry deleted. But as $D_k = \det M_k \neq 0$ we have

$$\begin{pmatrix} a_kD_1 - D_ka_1 \\ \vdots \\ a_kD_{n+1} - D_ka_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Now let $\gcd(a_k, D_k) = d$ and put $a_k/d = q$ and $D_k/d = p$. Then there exists p, q such that $\gcd(p, q) = 1$ and

$$q\begin{pmatrix} D_1 \\ \vdots \\ D_{n+1} \end{pmatrix} = p\begin{pmatrix} a_1 \\ \vdots \\ a_{n+1} \end{pmatrix}$$

This gives $qD_j = pa_j$ for all j. Any prime divisor of q must then divide a_j for all j, but $\gcd(a_1, \dots, a_{n+1}) = 1$, so q has no prime divisors and thus is a unit. Then the vector $(D_1, \dots, D_{n+1}) = p(a_1, \dots, a_{n+1})$ with $\gcd(D_1, \dots, D_{n+1}) = p \times \gcd(a_1, \dots, a_{n+1}) = p \in \Lambda$, as required. $\square$

Let I be an indexing set defined by $I = (i_1, \dots, i_n)$ $(1 \leq i_1 < \dots < i_n \leq r)$. Let M be an $r \times (n+1)$ matrix with $r \geq n$ then define M_j as before to be M with the jth column deleted. Also define M_j^I to be the $n \times n$ submatrix of M_j with the rows determined by I.

Lemma 3.1.2 *Let Λ ,I and $a_1, \dots, a_{n+1}$ be as above and let M be an $r \times (n+1)$ matrix for $r \geq n$ which satisfies*

$$M\begin{pmatrix} a_1 \\ \vdots \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

with $\gcd(a_1,\ldots,a_{n+1}) = 1$. *Let* M_j, *and* M_j^I *be as above then, writing* $d_k = \gcd\limits_I(M_k^I)$, *we have*

$$(d_1,\ldots,d_{n+1}) = \Delta(a_1,\ldots,a_{n+1})$$

with Δ *given by* $\Delta = \gcd\limits_j(d_j) = \gcd\limits_{I,j}(\det(M_j^I))$.

Proof . Write $D_j^I = \det M_j^I$ and set

$$\Delta = \gcd_{I,j}(D_j^I) = \gcd_I(\gcd_j(D_j^I)) \ .$$

By Lemma 3.1.1 applied to the $n \times (n+1)$ matrix M^I we have

$$D_k^I = a_k \times \gcd_j(D_j^I) \ , \text{ for all } k.$$

Hence

$$\gcd_I(D_k^I) = d_k = \Delta a_k$$

and thus $(d_1,\ldots,d_{n+1}) = \Delta(a_1,\ldots,a_{n+1})$ with $d_k = \gcd\limits_I(D_k^I)$. $\square$

Corollary 3.1.3 *Let* Λ, M, M_j, M_j^I, *and* d_k *be as in Lemma 3.1.2,and suppose that*

$$M \begin{pmatrix} b_1 \\ \vdots \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad ,$$

where $b_j \in \Lambda$ *for all* j, *and* $\gcd(b_1,\ldots,b_{n+1}) = b$. *Then*
$b(d_1,\ldots,d_{n+1}) = \Delta(b_1,\ldots,b_{n+1})$ *where* $\Delta = \gcd(d_1,\ldots,d_{n+1}) \in \Lambda$.

Proof Apply Lemma 3.1.2 to $(a_1,\ldots,a_{n+1}) = (b_1/b,\ldots,b_{n+1}/b)$. $\square$

Remark In the case where M is the Alexander matrix of a knot rather than a link of two or more components, the Corollary of Lemma 5.3.2 applies with $(b_1,\ldots,b_{n+1}) = (\alpha\phi((g_1-1),\ldots,\alpha\phi(g_{n+1}-1)) = (t-1)$. Now since $b = \gcd(b_1,\ldots,b_{n+1}) = (t-1)$ we have $(t-1)\Delta_K(t) = (t-1)d_{n+1}$. This implies that the limit on the spread is bound by the spread in d_{n+1} rather than this spread minus one, and this spread is bound by $2g_K$.

Proof of theorem 3.1.1 continued Now apply Lemma 3.1.2 to the Alexander matrix (3.6) of L with

$$\begin{pmatrix} a_1 \\ \vdots \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} \alpha\phi(\mathbf{h}-\mathbf{1}) \\ \alpha\phi(\mathbf{k}-\mathbf{1}) \\ \alpha\phi(\mathbf{g}-\mathbf{1}) \\ \alpha\phi(\lambda-1) \end{pmatrix}.$$

This may be done since $\gcd(a_1, \ldots, a_{n+1}) = 1$. Then we may apply Lemma 3.1.2 to get $(d_1, \ldots, d_{n+1}) = \Delta(a_1, \ldots, a_{n+1})$ where $\Delta = \Delta_L(t_1, \ldots, t_k)$ is the Alexander polynomial of L. So if we find the Alexander polynomial by deleting the $(n+1)$st column, which corresponds to λ, we have $\Delta = d_{n+1}/a_{n+1}$ as $a_{n+1} = t_K - 1 \neq 0$. Thus we have

$$\Delta_L(t_1, \ldots, t_k) = \gcd_{I,j}(\det(\alpha\phi M_j^I)) = \gcd_{I}(\det(\alpha\phi M_{n+1}^I))/(t_K - 1),$$

where

$$M_{n+1} = \begin{pmatrix} I_{2g+l} & -t_K I_{2g+l} & 0 \\ N_1 & N_2 & N_3 \end{pmatrix}$$

and N_1, N_2, and N_3 are matrices that do not involve t_K. Now it is obvious that in the above matrix the maximum power that t_K can attain is $2g + l$. Since $t_K - 1$ divides each of the $n \times n$ subdeterminants of the above matrix, we then must have the following;

$$Spr_{t_K} \leq 2g + l - 1.$$

But $g = genus(S_K)$, and $l = I(S_K, L')$, which gives us the required result. $\square$

Remark This bound is realised in the following example. Let K be a fibered knot of genus g, and let $A \cup \hat{\beta}$ be a closed l-braid with axis A. Now if the braid is chosen so that its closure is a single component then the link $L = K \#_A A \cup \hat{\beta}$ has $I(S_K, L') = l = lk(\hat{\beta}, A) = lk(\hat{\beta}, K)$. A result of Fox gives, for this link,

$$\Delta_L(t_K, 1) = \frac{({t_K}^{lk(\hat{\beta},K)} - 1)}{(t_K - 1)} \Delta_K(t_K),$$

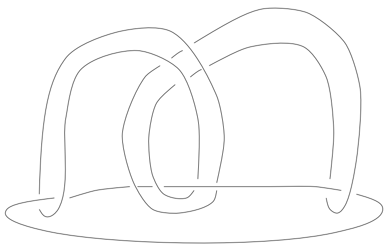
so it follows that

$$Spr_{t_K} \Delta_L(t_K, t_2) \geq Spr_{t_K} \Delta_L(t_K, 1) = Spr_{t_K} \frac{({t_K}^{lk(\hat{\beta},K)} - 1)}{(t_K - 1)} \Delta_K(t_K) = 2g_K + l - 1.$$

Now $Spr_{t_K} \Delta_L(t_K, t_2) \leq 2g_K + l - 1$ by the theorem above, so we have

$$Spr_{t_K} \Delta_L(t_K, t_2) = 2g_K + l - 1.$$

Example The link L shown in the following figure has Alexander polynomial $\Delta_L(a,b) = (1-b)(1-a)(1+a^2)$, giving $Spr_b\Delta_L = 1$ and $Spr_a\Delta_L = 3$. Now by observation there exists a disc spanning the component of L corresponding to b which the other component intersects with only twice. This implies that L may be drawn as a, closed, 2-tangle with b as axis, but if a is chosen as the axis for a tangle representing L then we have by Corollary 3.1.2 that this tangle is at least a 4-tangle. From the picture below a disc spanning a can be seen which meets b in 4 points, so b can indeed be presented as a closed 4-tangle (which it is shown to be again by observation of L).



This shows that the geometric winding number of two components is not symmetrical, and that in general tangles are not exchangeable as I stated in section 1 of chapter 4. This leads to the idea that a sort of error term exists in the linking number of two components.

This error term is the following

$$err(K_1, K_2) = min_{S_{K_1}}(2(g(S_{K_1}) - g_{K_1}) + I(S_{K_1}, K_2) - lk(K_1, K_2)).$$

Notice that this is not necessarily symmetric in K_1 and K_2.

3.2 The Alexander polynomial of the k-fold cyclic cover

The work in this section is motivated by the relationship between the Alexander polynomial of a braid, and the Alexander polynomial of the k-fold cyclic cover of the braid branched over the axis. We shall suppose that $L \cup A \subset S^3$ is a link with a distinguished component A. We may then construct the k-fold cover Σ of S^3 branched over A, with the projection $p_k : \Sigma \to S^3$.

Write $A' = {p_k}^{-1}(A)$ for the inverse image of the branch set and $L^k/A = {p_k}^{-1}(L)$ for the inverse image of the rest of the link. Let μ be a meridian of A' in Σ, and λ a meridian of A in S^3. Then $p_k \mid L^k/A \to L$ is a k-fold cover and $p_k \mid A' \to A$ is a homeomorphism.

Let the induced map from $H_1(\Sigma - p_k^{-1}(L \cup A)) \to H_1(S^3 - (L \cup A)$ be denoted by $H_1(p_k)$.

In this section we shall take A to be unknotted, so that $\Sigma = S^3$ and A' is again unknotted; our results in this case are extended in the next section to the case when Σ is an integer homology 3-sphere.

The aim of this section is to describe the Alexander polynomial of the link ${p_k}^{-1}(L \cup A) = (L^k/B) \cup A'$ in terms of the Alexander polynomial of $L \cup A$.

Using the definition of the Alexander polynomial given in the previous section of this chapter, we have the following theorem.

Theorem 3.2.1 *Let $L \cup A$ be a link in S^3, with A unknotted. Let p_k be the projection as before, with $A' = {p_k}^{-1}(A)$, $L^k/A = {p_k}^{-1}(L)$, and the Alexander polynomial as above, then*

$$H_1(p_k)(\Delta_{(L^k/A)\cup A}(y_1, \ldots, y_t, y_{A'})) = \prod_{i=1}^{k} \Delta_{L\cup A}(x_1, \ldots, x_s, \zeta^i x_A)$$

where ζ is a k-th primitive root of unity, $y_{A'}$ corresponds to the meridian μ, and x_A to the meridian λ.

Remark. This result is easily proved when $L \cup A$ consists of a closed braid $L = \widehat{\beta}$ together with its axis A. Then the cover ${p_k}^{-1}(L)$ is simply the closure of the braid β^k, with A' as its axis. It is known in this case, [9], that the Alexander polynomial $\Delta_{\widehat{\beta}\cup A}$ is the characteristic polynomial $\det(B(t) - xI)$ of the reduced Burau matrix $B(t)$ for the braid β.

Since the reduced Burau matrix determines a representation of the braid group into a matrix group, it follows that the reduced Burau matrix of β^k is $(B(t))^k$, and thus the Alexander polynomial of the cover $p^{-1}(L \cup A) = p^{-1}(L) \cup A'$ can be written as

$$\Delta_{p^{-1}(L)\cup A'}(t, z) = \det((B(t))^k - zI) \ ,$$

where z represents a meridian of A'.

Now if we put $x^k = z$ we have

$$((B(t))^k - zI) = \prod_{i=1}^{k}(B(t) - \zeta^i xI)$$

where ζ is a k-th primitive root of unity.

Thus

$$\det((B(t))^k - zI) = \prod_{i=1}^{k} \det(B(t) - \zeta^i xI)$$

and so, since $H_1(p_k)(z) = x^k$, we have

$$H_1(p_k)\Delta_{p^{-1}(L)\cup A'}(t,z) = \prod_{i=1}^{k} \Delta_{L\cup A}(t, \zeta^i x)$$

with ζ, x and z as above.

The proof of Theorem 3.2.1 will show that the same formula holds in the general case where L is not a closed braid relative to the axis A, and gives a natural extension to the case where L has m components.

We are thus able to give the multivariable Alexander polynomial of ${p_k}^{-1}(L\cup A)$ in terms of the Alexander polynomial of $L\cup A$, at least when ${p_k}^{-1}(L)$ and L have the same number of components. Even in the case that ${p_k}^{-1}(L)$ and L do not have the same number of components we can retrieve some information on the Alexander polynomial of ${p_k}^{-1}(L)\cup A'$.

The final result in this section expresses the Alexander polynomial of ${p_k}^{-1}(L)$ in terms of the Alexander polynomials of $(L\cup A)$ and L.

The following construction is required before the proof of the theorem.

Let L be an embedding of the closed curves into $S^3 - A$, where S^3 is the three sphere and A is an unknot. Define V_A to be solid torus centred on m, with $V_A \cap L = 0$, where m is a meridian of A. Following the construction of the fundamental group in section 1 of this chapter we now choose a surface spanning A, $F(A)$ say, but as A is an unknot we may choose $F(A) = D(A)$ to be a disc spanning A.

Let $L^k/A = p_k^{-1}(L)$ be the k-fold cyclic cover of L, as before, in the k-fold cyclic cover of S^3 branched over A.

Let $\Delta_{L\cup A}(x_1,\dots,x_s,x_A)$, and $\Delta_{L^k/A\cup A'}(y_1,\dots,y_t,y_{A'})$ be the Alexander polynomials of $L\cup A$ and $(L^k/A)\cup A$ respectively, with x_A representing the meridian λ of A and $y_{A'}$ represented by the meridian μ of A', while $x_1,\dots,x_s$ and $y_1,\dots,y_t$ are represented by meridians of the components of L and ${p_k}^{-1}(L)$ respectively.

Proof of theorem 3.2.1. Firstly I will construct the fundamental groups for $S^3 - (L\cup A)$ and $S^3 - ((L^k/A)\cup A)$, using the same method as in section 1 of this chapter. The cover map p_k sends meridians of ${p_k}^{-1}(L)$ homeomorphically to meridians of L, while the meridian μ is a k-fold cover of the meridian λ. Then

$$H_1(p_k)(y_i) = x_j$$

for some x_j and

$$H_1(p_k)(y_{A'}) = {x_A}^k \ .$$

Thus we have $S^3 - (L \cup A) = U \cup V$, where $U = (S^3 - L - int(D(A)))$, $V = (int(D(A) - (D(A) \times (-\epsilon, \epsilon) \cap L) \cup V_A)$, and $U \cap V$ is path connected. By Van Kampen's Theorem

$$\pi_1(S^3 - (L \cup A)) = \pi_1(U) * \pi_1(V) / < (i_U)_*(x) = (i_V)_*(x) \mid \forall x \in \pi_1(U \cap V) >$$

where $(i_U)_*$ and $(i_V)_*$ are the homomorphisms induced from the inclusions
$i_U : (U \cap V) \to U$ and $i_V : (U \cap V) \to V$.

Now

$$\pi_1(V) =< \mathbf{h}', \lambda \mid >$$

and

$$\pi_1(U \cap V) =< \mathbf{h}'', \mathbf{k}'' \mid > \ ,$$

where $(i_V)(\mathbf{h}'') = \mathbf{h}'$ and $(i_V)(\mathbf{k}'') = \lambda \mathbf{h}' \lambda^{-1}$.

Again as in section 1 we build a presentation for $\pi_1(U)$ which is compatible with the generators of $\pi_1(U \cap V)$, that is $\mathbf{h}$, and $\mathbf{k}$, as generators, where $(i_U)(\mathbf{h}'') = \mathbf{h}$ and $(i_U)(\mathbf{k}'') = \mathbf{k}$, together with generators $\mathbf{g}$, to complete a set of generators. Now as $U \sim (D \times I - L)$ the extra generators $\mathbf{g}$ represent local minima of L in $D \times I$, and there is a relation set $\mathbf{r}$, one r_i for each local maximum. The final set of relations in $\pi_1(U)$ is obtained, as $\mathbf{k} = w(\mathbf{h}, \mathbf{g})$, here w is a word in the generators $\mathbf{h}$, and $\mathbf{g}$. Thus

$$\pi_1(U) =< \mathbf{h}, \mathbf{k}, \mathbf{g} \mid \mathbf{r}, \mathbf{k} = w(\mathbf{h}, \mathbf{g}) > \ ,$$

and so,

$$\pi_1(S^3 - L \cup A) =$$
$$< \mathbf{h}', \mathbf{h}, \mathbf{k}, \mathbf{g}, \lambda \mid \mathbf{h}' = \mathbf{h}, \mathbf{k} = \lambda \mathbf{h} \lambda^{-1}, \mathbf{r}, \mathbf{k} = w(\mathbf{h}, \mathbf{g}) > \ ,$$

which is

$$\pi_1(S^3 - L \cup A) =< \mathbf{h}, \mathbf{g}, \lambda \mid w(\mathbf{h}, \mathbf{g}) = \lambda \mathbf{h} \lambda^{-1}, \mathbf{r} > \ .$$

Let $J = (\frac{\partial \mathbf{R}}{\partial \mathbf{G}})$ be the Jacobian matrix of the presentation of $< \mathbf{G} \mid \mathbf{R} >= \pi_1(S^3 - A \cup \widehat{T})$, we have $J = (M - x_A N \mid \mathbf{v})$, where

$$M = \begin{pmatrix} \frac{\partial w(\mathbf{h}, \mathbf{g})}{\partial \mathbf{h}} & \frac{\partial w(\mathbf{h}, \mathbf{g})}{\partial \mathbf{g}} \\ \frac{\partial \mathbf{r}}{\partial \mathbf{h}} & \frac{\partial \mathbf{r}}{\partial \mathbf{g}} \end{pmatrix} \ ,$$

$$N = \begin{pmatrix} \delta_{ij} & 0 \\ 0 & 0 \end{pmatrix} \ ,$$

and $\mathbf{v}$ is a vector given by $\mathbf{v}^T = (\alpha\psi(1-h_1)\ \dots \alpha\psi(1-h_n)\ 0\ \dots\ 0)$, with δ_{ij} the Kronecker delta, and the two matrices M and N having same sized block form. But by section 1 of this chapter, this implies

$$\Delta_{L\cup A}(x_1, \dots, x_s, x_A) = (x_A - 1)^{-1} \det(M - x_A N) \ .$$

We now produce a presentation for $\pi_1(S^3 - ((L^k/A) \cup A'))$, and by chapter 2 section 2, we derive a presentation for $H_1(S^3 - ((\widetilde{L^k/A}) \cup A'), \tilde{p})$. This leads, by section 2.2, to the Alexander module, and thus to the Alexander polynomial.

The fundamental group is given by the fundamental group of the exterior of $\widehat{T^k}$, and $(V_m \cup (D_A - (D_A \cap L)))$, so we need the following lemma.

Lemma 3.2.1 *With (L^k/A) as above*

$$\pi_1(D \times I - (D \times (0) \cup ((L^k/A) \cup A'))) =$$

$$< \mathbf{h^{(i)}}, \mathbf{g^{(i)}} \mid w(\mathbf{h^{(i)}}, \mathbf{g^{(i)}}) = \mathbf{h^{(i+1)}}, \mathbf{r^{(i)}} >$$

for $i = (1, \dots, (k-1))$.

Proof. The proof is by induction on k. When $k = 1$ we have

$$\pi_1(D \times I - (D \times (0) \cup (L \cup A))) =$$

$$< \mathbf{h}, \mathbf{g} \mid w(\mathbf{h}, \mathbf{g}) = \mathbf{h}, \mathbf{r} > ,$$

as required.

Let us assume that the lemma is true for $k = n$, and then consider the case $k = n+1$,

$$\pi_1(D \times I - (D \times (0) \cup ((L^{n+1}/A) \cup A'))) =$$

$$\pi_1(D \times I - (D \times (0) \cup ((L^k/A) \cup A'))) * \pi_1(D \times I - (D \times (0) \cup (L \cup A')))/ < w(\mathbf{h^{(k)}}, \mathbf{g^{(k)}}) = \mathbf{h^{(k+1)}} >$$

which gives

$$\pi_1(D \times I - (D \times (0) \cup ((L^k/A) \cup A'))) =$$

$$< \mathbf{h^{(i)}}, \mathbf{h^{(k+1)}}, \mathbf{g^{(i)}}, \mathbf{g^{(k+1)}} \mid w(\mathbf{h^{(i)}}, \mathbf{g^{(i)}}) = \mathbf{h^{(i+1)}}, w(\mathbf{h^{(k)}}, \mathbf{g^{(k)}}) = \mathbf{h^{(k+1)}}, \mathbf{r^{(k+1)}}, \mathbf{r^{(i)}} > ,$$

for $i = (1, \dots, (k-1))$ as required. $\square$

From the above we thus have, using Van Kamppens theorem

$$\pi_1(S^3 - ((L^k/A) \cup A')) =< \mathbf{G} \mid \mathbf{R} >$$

$$< \mathbf{h^{(i)}}, \mathbf{g^{(i)}}, \mu \mid w(\mathbf{h^{(k)}}, \mathbf{g^{(k)}}) = \mu \mathbf{h^{(1)}} \mu^{-1}, w(\mathbf{h^{(i)}}, \mathbf{g^{(i)}}) = \mathbf{h^{(i+1)}}, \mathbf{r^{(i)}} > \ .$$

Let the link L in $S^3 - A$ be considered to be some closed tangle $\widehat{T^k}$. Let $J = (\frac{\partial \mathbf{R}}{\partial \mathbf{G}})$ be the Jacobian matrix of the presentation of $< \mathbf{G} \mid \mathbf{R} = \pi_1(S^3 - A \cup \widehat{T^k})$, and let $M^{(j)}$ be the matrix M with g_i replaced with $g_i^{(j)}$ and h_i replaced with $h_i^{(j)}$ instead of $\alpha\phi$ we have $\alpha\psi$. It is, of course, not necessary to replace N by $N^{(j)}$ as all entries in N are 0 or 1.

Then we can write $\alpha\psi(J)$ in block form as

$$\alpha\psi(J) = \begin{pmatrix} M^{(1)} & -N & \dots & & \vdots & | & 0 \\ 0 & M^{(2)} & -N & \dots & \vdots & | & 0 \\ \vdots & 0 & \dots & M^{(k-1)} & -N & | & 0 \\ -y_{A'}N & 0 & \dots & 0 & M^{(k)} & | & \mathbf{v} \end{pmatrix}$$

where $\mathbf{v}^T = (\alpha\psi(1 - h_1^{(1)}) \ \dots \alpha\psi(1 - h_n^{(1)}) \ 0 \ \dots \ 0)$.

But by section 1 of this chapter, we have Alexander polynomial given as the following

$$\Delta_{(L^k/A) \cup A'}(y_1, \dots, y_t, y_{A'}) = (y_{A'} - 1)^{-1} \det \begin{pmatrix} M^{(1)} & -N & \dots & & \vdots \\ 0 & M^{(2)} & -N & \dots & \vdots \\ \vdots & 0 & \dots & M^{(k-1)} & -N \\ -y_{A'}N & 0 & \dots & 0 & M^{(k)} \end{pmatrix}.$$

Now $H_1(p_k)\alpha\psi(g_i^{(j)}) = \alpha\phi(g_i)$, $H_1(p_k)\alpha\psi(h_i^{(j)}) = \alpha\phi(h_i)$, and $H_1(p_k)\alpha\psi(\mu) = \alpha\phi(\lambda^k)$ which leads to the matrix

$$H_1(p_k)\alpha\psi(J) = \begin{pmatrix} M & -N & \dots & & \vdots & | & 0 \\ 0 & M & -N & \dots & \vdots & | & 0 \\ \vdots & 0 & \dots & M & -N & | & 0 \\ -{x_A}^k N & 0 & \dots & 0 & M & | & H_1(p_k)(\mathbf{v}) \end{pmatrix}.$$

We thus have

$$H_1(p_k)(\Delta_{(L^k/A) \cup A}(y_1, \dots, y_t, y_{A'})) =$$

$$((x_A)^k - 1)^{-1} \det \begin{pmatrix} M & -N & \dots & & \vdots \\ 0 & M & -N & \dots & \vdots \\ \vdots & 0 & \dots & M & -N \\ -{x_A}^k N & 0 & \dots & 0 & M \end{pmatrix},$$

from which, using the following lemma, we can deduce our result.

Lemma 3.2.2 *Let M and N be $n \times n$ matrices, and J be the $k \times k$ matrix*

$$J = \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & & \vdots \\ \vdots & \dots & \dots & 1 \\ \lambda^k & \dots & \dots & 0 \end{pmatrix} .$$

Then $\det(M \otimes I_k - N \otimes J) = \prod_{i=1}^{k} \det(M - \zeta^i \lambda N)$, *where ζ is a primitive k-th root of unity.*

Proof. The matrix J has eigenvalues $\zeta^i\lambda$, for $i = 1, \dots, k$, and so can be diagonalized whenever $\lambda \neq 0$. So there exists a $k \times k$ matrix P such that $PJP^{-1} = D(\zeta^i\lambda)$, the diagonal matrix with entries $\zeta^i\lambda$. Thus we have

$$((I \otimes P)(M \otimes I - N \otimes J)(I \otimes P^{-1})) = (M \otimes I - N \otimes D(\zeta^i\lambda)) \ .$$

The matrix $(M \otimes I - N \otimes D(\zeta^i\lambda))$ is a direct sum of $M - \zeta^i\lambda N$ for $i = 1, \dots, k$. Now $(\det(I \otimes P))^{-1} = \det(I \otimes P^{-1})$, so

$$\det(M \otimes I - N \otimes J) = \det(M \otimes I - N \otimes D(\zeta^i\lambda)) = \prod_{i=1}^{k} \det(M - \zeta^i\lambda N) \ ,$$

as required. $\square$

Proof of theorem 3.2.1 continued. As $\Delta_{T^k \cup A} = (y_{A'} - 1)^{-1} \det(\alpha\psi J)$ we apply the above lemma with $\lambda = x_A$, so we have

$$H_1(p_k)(\Delta_{T^k \cup A}) = ({x_A}^k - 1)^{-1} \prod_{i=1}^{k} \det(M - \zeta^i x_A N),$$

giving

$$H_1(p_k)(\Delta_{T^k \cup A}(y_1, \dots, y_t, y_{A'})) = \prod_{i=1}^{k} \Delta_{T \cup A}(x_1, \dots, x_s, \zeta^i {x_A}^k) \ .$$

This completes the proof of theorem 3.2.1 $\square$

In Chapter 5 section 3 I give an account of Eisenbud and Neumann's result on the Alexander invariant, this invariant is a modified version of the Alexander polynomial which behaves well with respect to satellite, and splicing operations. Given $L = (\Sigma, S_1 \cup \dots \cup S_n)$, then the Alexander invariant $\Delta_*(L; t_1, \dots, t_n)$ is given by

$$\Delta_*(L; t_1, \dots, t_n) = \begin{cases} \Delta_L(t_1, \dots, t_n) & \text{for } n > 1 \\ (t_1 - 1)^{-1}\Delta_L(t_1) & \text{for } n = 1 \end{cases}$$

where $\Delta(L; t_1, \ldots, t_n)$ is the normal Alexander polynomial.

The Fox Torres formula derives the Alexander polynomial of any sublink L from that of L'', under certain linking conditions. Eisenbud and Neumann use the Alexander invariant in the context of splice decomposition to give a uniform extension to the Fox-Torres formula.

Let $\mathbf{L}'' = (\Sigma, S_0 \cup S_1 \cup \ldots \cup S_n)$, and $\mathbf{L} = (\Sigma, S_1 \cup \ldots \cup S_n)$ with $\mathbf{L}' = (S^3, A)$, where A is an unknot in S^3, then L is the splice of L' and L'', and

$$({t_1}^{l_1} \ldots {t_n}^{l_n} - 1)\Delta_*(L; t_1, \ldots, t_n) = \Delta_*(L''; 1, t_1, \ldots, t_n),$$

where $l_i = lk(S_0, S_i)$ in Σ.

With p_k as above the following proposition gives the Alexander invariant of ${p_k}^{-1}(L)$ in terms of the Alexander invariants of L and $L \cup A$.

Proposition 3.2.1 *Let $\widehat{T} \cup A$ be a link formed as the closure of a tangle T union an unknotted axis A. Then we have the following*

$$\Delta_*(\widehat{T^k}; t_1, \ldots, t_n) = \Delta_*(\widehat{T}; t_1, \ldots, t_n) \prod_{i=1}^{k-1} \Delta_*((A \cup \widehat{T}); \zeta^i, t_1, \ldots, t_n),$$

Proof In this proof we have two cases; either the linking number of each component of the tangle with the axis is zero, or at least one is non-zero.
CASE 1 If there exists a j such that $l_j \neq 0$ then we have the following. With Δ_*, as above, and $H_1(p)$ from above, then

$$\begin{aligned}({t_1}^{l_1} \ldots {t_n}^{l_n} - 1)\Delta_*(\widehat{T^k}; t_1, \ldots, t_n) &= \Delta_*((A \cup \widehat{T^k}); 1, t_1, \ldots, t_n) \\ &= \prod_{i=1}^{k} \Delta_*((A \cup \widehat{T}); \zeta^i, t_1, \ldots, t_n).\end{aligned}$$

Thus

$$({t_1}^{l_1} \ldots {t_n}^{l_n} - 1)\Delta_*(\widehat{T^k}; t_1, \ldots, t_n) =$$

$$({t_1}^{l_1} \ldots {t_n}^{l_n} - 1)\Delta_*(\widehat{T}; t_1, \ldots, t_n) \prod_{i=1}^{k-1} \Delta_*((A \cup \widehat{T}); \zeta^i, t_1, \ldots, t_n),$$

which gives finally

$$\Delta_*(\widehat{T^k}; t_1, \ldots, t_n) = \Delta_*(\widehat{T}; t_1, \ldots, t_n) \prod_{i=1}^{k-1} \Delta_*((A \cup \widehat{T}); \zeta^i, t_1, \ldots, t_n).$$

CASE 2 Now if $l_i = lk(S_i, A) = 0$ for all i the above method fails, but it can be adapted to cope as follows. Let B be an unknot in $S^3 - \widehat{T} \cup A$ such that $lk(A, B) = lk(S_1, B) = 1$ and $lk(S_i, B) = 0$ for i greater than 1. Let $\widehat{T} \cup B = \widehat{T'}$ then if we do the above result on the k-fold cover of $\widehat{T'}$, we have

$$\Delta_*(\widehat{T'^k}; t_1, \ldots, t_n, t_{B^k}) = \Delta_*(\widehat{T'}; t_1, \ldots, t_n, t_B) \prod_{i=1}^{k-1} \Delta_*((A \cup \widehat{T'}); \zeta^i, t_1, \ldots, t_n, t_B).$$

Then as $lk((S_1)^k, B^k) = k$ in the cover space, and with A set equal to ζ^i we have

$$(t_1{}^k - 1)\Delta_*(\widehat{T^k}; t_1, \ldots, t_n) =$$
$$(t_1 - 1)\Delta_*(\widehat{T}; t_1, \ldots, t_n) \prod_{i=1}^{k-1} (\zeta^i t_1 - 1)\Delta_*((A \cup \widehat{T}); \zeta^i, t_1, \ldots, t_n),$$

which gives

$$\Delta_*(\widehat{T^k}; t_1, \ldots, t_n) = \Delta_*(\widehat{T}; t_1, \ldots, t_n) \prod_{i=1}^{k-1} \Delta_*((A \cup \widehat{T}); \zeta^i, t_1, \ldots, t_n),$$

as claimed. □

3.3 The Alexander polynomial in integer homology three spheres.

The two results below have seemingly been restricted to the case of links in S^3, (which is why A was unknotted in the second result.)

Theorem 3.1.1 1 *In the multi-variable Alexander polynomial $\Delta_L(t_1, \ldots, t_k)$ we have*

$$Spr_{t_K}(\Delta_L) \leq 2g(S_K) + I(S_K, L') - 1.$$

Where the notation in the following is that of the previous section.

Theorem 3.2.1 1 *Let $L \cup A$ be a link in S^3, with A unknotted. Let p_k be the projection as before, with $A' = p_k{}^{-1}(A)$, $L^k/A = p_k{}^{-1}(L)$, and the Alexander polynomial as above, then*

$$H_1(p_k)(\Delta_{((L^k/A)\cup A)}(y_1, \ldots, y_t, y_{A'})) = \prod_{i=1}^{k} \Delta_{(L\cup A)}(x_1, \ldots, x_s, \zeta^i x_A)$$

where ζ is a k-th primitive root of unity, $y_{A'}$ corresponds to the meridian μ, and x_A to the meridian λ.

These theorems are equally valid whenever the links considered lie in an integer homology three sphere. That is

Theorem 3.3.1 *In the multi-variable Alexander polynomial,* $\Delta_*(L; t_1, \ldots, t_k)$*, for a link of two or more components* $\mathbf{L} = (\Sigma, L_1 \cup \ldots \cup L_k)$ *we have*

$$Spr_{t_K}(\Delta_*(L; t_1, \ldots, t_k) \leq 2g(S_K) + I(S_K, L') - 1$$

with $K = L_i$ *for some* i *and* $t_K = t_i$.

This is true as none of the results require more that Σ being an integer homology three sphere, rather than S^3. Thus we split $\Sigma - L = (\Sigma - L - F(K)) \cup ((F(K) - (F(K) \cap (L - K))) \times (-\varepsilon, \varepsilon) \cup V_K)$, where $F(K)$ is a surface spanning K, as in section 3.1. Now the Alexander invariant equals the Alexander polynomial for a link of 2 or more components, (see section 5.3), and the Alexander polynomial is still found via the Jacobian, (see section 2.2). But by section 2.2 the Alexander polynomial can be retrieved from the determinants of the $(n-1) \times (n-1)$ sub matrices of the Alexander matrix when Σ is an integer homology three sphere, so the result holds.

In the case of the k-fold cover we require both Σ and Σ^k to be integer homology three spheres. But I conjecture that , even here, it is only necessary for Σ to be an integer homology three sphere, because under the projection $H_1(p)$ the Jacobian of $\pi_1(\Sigma^k - A^k/B - B)$ becomes a module over $H_1(\Sigma - A - B)$, and thus I believe that the same formula holds here also.

Chapter 4

Exchangeable Stallings Braids

In this chapter we consider Stallings braids, and exchangeability, which we now define. Traditionally exchangeable braids were defined as links of two components, each of which is unknotted, and each component has the other component as closed braid relative to it as an axis. A braid β is then called *exchangeable* if its closure together with axis, $\widehat{\beta} \cup A$, form such a link. The essential feature of such a braid is that the closure of the braid, $\widehat{\beta}$, is unknotted and that the axis for the braid, A, is itself a closed braid relative to $\widehat{\beta}$ as axis.

A Stallings braid on n strings is defined by Morton in [9] to be the product of $n-1$ elementary braids or their inverses which close to a single component, where an elementary braid $\sigma_{i,j} \in B_n$, for $1 \leq i < j \leq n$ is the braid interchanging the i^{th} and j^{th} strings with a single positive crossing, in front of any intermediate strings, and leaving the others alone. Such braids were introduced by Stallings as examples of braids whose closure is the unknot. A spanning surface for the closure of a Stallings braid can be obtained by placing n discs on the braid strings and connecting them by half-twisted bands corresponding to the elementary braids in the above product, where a positive half-twisted band connecting the i-th and j-th discs corresponds to the elementary braid $\sigma_{i,j}$ and a negative half-twisted band connecting the i'-th and j'-th discs corresponds to the inverse of the elementary braid i.e. $\sigma^{-1}_{i',j'}$. Such a disc intersects the axis in only n points, this form of spanning disc for a Stallings braid is known as a *Stallings disc*. It is shown in [9] that every exchangeable braid is conjugate in the braid group to a Stallings braid, but that some Stallings braids are not exchangeable.

4.1 Exchangeable braidlike tangles.

In this section we approach the idea of exchangeable braids from the idea of tangles with properties related to those possessed by exchangeable braids. To do this it is necessary to define tangles, both open and closed, and under what conditions an open tangle is closable. Let $(D \times I)$ be the empty *open*-tangle, and let $p : D \times I \to S^3$ be given by

$$p(x, 0) = p(h(x), 1)$$

for all $x \in D$, where $h : D \to D$ is a homeomorphism fixing ∂D.

$$p(y, t) = p(y, 0)$$

for all $y \in \partial D$ and for all t in I, and no other relations. Let $p(\partial D, 0) = A$ then $(S^3 - A) = p(int(D) \times I)$ is the empty *closed*-tangle with axis. In this chapter, we will restrict h to be the identity homotopy $h(D) = id_t(D)$, the identity map.

Define a tangle be a proper embedding of closed curves and arcs, T, into $(int(D) \times I)$, that is $p(\partial(T)) \in int(D) \times \partial(I)$, $p(int(T)) \in int(D) \times int(I)$, and $p(x) = p(y)$ implies $x = y$. Thus the tangle T is a proper embedding of arcs and circles into the cylinder.

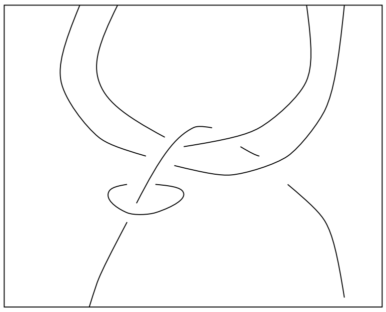
The next definition deals with breaking the set of all open-tangles into manageable sub-sets, i.e. the set of (n,m)-tangles, T^n_m, and leads naturally onto closeability.

Let $T = ((D \times I, id_t), p(C))$ is an open-tangle in $(D \times I, id_t)$, let $I_t(T) = I(D_t, p(C))$ be the intersection number of the disc D_t with the union of curves and arcs $p(C)$, and let $n(T) = I_0(T)$ and $m(T) = I_1(T)$, then $T \in T^n_m$ if $n(T) = n$ and $m(T) = m$.

In the above definition we have a set of arcs and circles in $D \times I$ in general position, but with a known number of points in the two ends of the cylinder. For such a system to close to $S^3 - (A \cup L)$ where A is the axis mentioned above, and L is some link in the complement of A, we need $n = m$ and the positions of the end points must be such that there are no loose ends in the closure.

Let $S_0 = (x \in D \times (0) \ st. \ x \in T)$, and let $S_1 = (x \in D \times (1) \ st. \ x \in T)$, then a tangle T is closable if $n = m$ and $S_0 = S_1$. If a tangle T is closable let $\widehat{T}$ be its closure in $S^3 - A$.

The following is an example of a (4,2)-tangle.

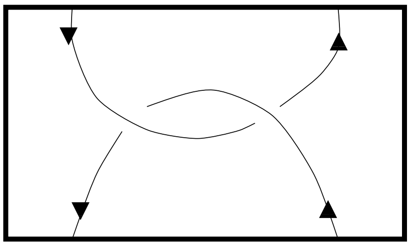


The above set of definitions defines open tangles, so what are closed tangles? A closed tangle is a link in $S^3 - A$. Let L be a link in $S^3 - A$, and D_A some disc spanning A and transverse to $p(C)$, let $T = (((S^3 - A) - D_A, L - (L \cap D_A)), id_t)$, then $S^3 - (A \cup L) = S^3 - (A \cup \widehat{T})$.

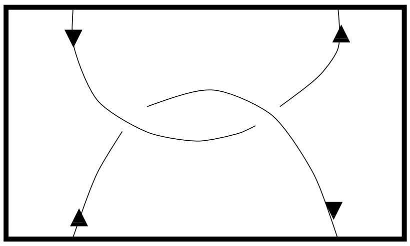
In all of the above the question of orientability was left out, so below is the set of (n,n)-tangles which are already oriented, and can be closed to an oriented link in $S^3 - A$.

Definition.

Let the set of oriented (n,n)-tangles OT_n be $T \in T_n^n$ with orientation consistent with closure. That is the closure of a tangle in OT_n can be oriented consistently, as in the figure below. The figure shows two oriented 2-tangles, one of which is closeable the other not.



Closeable oriented 2-Tangle



Non-Closeable oriented 2_Tangle

Write $\widehat{T}$ for the closure of $T \in OT_n$, this link is an oriented link. Let us write $OT = \bigcup OT_n$ then there is an equivalence relation $\sim$ defined on OT by $T_1 \sim T_2$ if and only if $\widehat{T_1} \cup A_1 \simeq \widehat{T_2} \cup A_2$.

Remark. In the case of braids this equivalence relation $\sim$ is known to be conjugacy, by [10], i.e. if $\beta_1 \sim \beta_2$ then β_1 and β_2 are both n-braids and so β_1 is conjugate to β_2, in the braid group B_n.

We now consider oriented tangles that have unknotted closure, $UT_n \subset OT_n$ where the set $UT_n = \{T : T \in T_n : \widehat{T}\text{is unknotted.}\}$, any such tangles we say belong to the set of unknotted n-tangles. We aim to define exchangeability for the tangles in this set. The example in section 5.3 shows that the natural exchange of a tangle in UT_2 can be a tangle in UT_4. If we then define the set of unknotted tangles to be $UT = \bigcup(UT_n)$ over all n and use the above equivalence relation on the subset of $UT \in OT$ $T_1 \sim T_2$ if and only if $\widehat{T_1} \cup A_1 \simeq \widehat{T_2} \cup A_2$ by component. It is the possible to define the exchange of such a tangle in UT as follows $Exch([T]) = [T']$ where $\widehat{T'} \cup A' \simeq A \cup \hat{T}$ by component.

Define the *tangle number* of $[T]$ to be the smallest n such that $[T'] = [T]$ and $T' \in UT_n$

If a tangle T in the above set were to be an exchangeable braid then it is clear that the tangle number of both $[T]$ and $Exch([T])$ are equal to the linking number of the closure of the tangle and its axis. I propose to call tangles whose closures have linking number with their axis equal to the tangle number braidlike tangles (partially closed braids) denoted BT_n. We can further define exchangeable braidlike tangles ET_n to be tangles $T \in BT_n$ such that $\hat{T}$ is an unknot and $Exch([T]) \in BT_n$.

It is easy to see that if $T \in ET_n$ there exist a pair of discs D_A and $D_{\hat{T}}$ such that $I(D_A, \hat{T}) = I(D_{\hat{T}}, A) = lk(\hat{T}, A)$.

At this point a re-cap of the construction of satellite links is desirable. To construct a satellite of a link $L = L_1 \cup \ldots \cup L_r$ with the pattern link $C = C_1 \cup \ldots \cup C_k$, in which one unknotted component, C_k say, is selected. Then one component of the link L is chosen, L_1 say, we then replace a solid torus neighbourhood V of L_1 by the solid torus $W = S^3 - C_k$, the complementary torus to the neighbourhood of C_k.

In the case of unknotted tangles UT we have $r = k = 2$ and both $\hat{T} \cup A$ and $\widehat{T'} \cup A'$ consist of the closure of an unknotted tangle union axis. It then follows that $h(\hat{T}) \cup A'$ is a the closure of an unknotted tangle. I will write the satellite as follows $\widehat{T' * T} \cup A'$ to indicate that the component $\widehat{T'}$ has been decorated by the pattern $\hat{T}$.

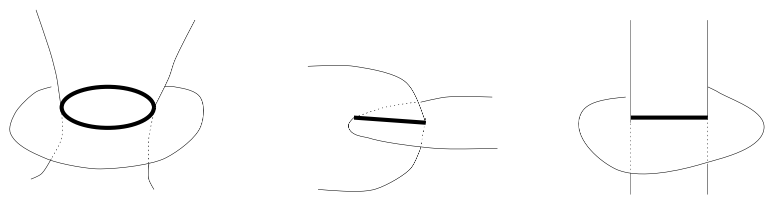
Proposition 4.1.1 *In the general case of tangle exchange above* $Exch([T_1 * T_2]) = Exch([T_2]) * Exch([T_1])$.

Proof Firstly we have $L_1 = \widehat{T_1} \cup A_1$ and $L_2 = \widehat{T_2} \cup A_2$. We then decorate the tangle component of L_1 with the tangle in L_2 this is equivalent to decorating the axis of $A'_1 \cup \widehat{Exch([T_1])}$ with $\widehat{T_1}$. This link is thus $\widehat{T_1} * \widehat{T_2} \cup A_1 \simeq A'_1 * \widehat{T_2} \cup \widehat{Exch([T_1])} \simeq \widehat{T_2} \cup A_2 * \widehat{Exch([T_1])} \simeq A'_2 \cup \widehat{Exch([T_2]) * Exch([T_1])}$. □

Proposition 4.1.2 *If T_1, T_2 are braidlike tangles in UT, then $T = T_1 * T_2$ is an exchangeable tangle if and only if both T_1 and T_2 are exchangeable tangles.*

Proof Let $T_1 \in BT_n$ and $T_2 \in BT_m$ so we have $T \in BT_{nm}$. Firstly we have $Exch([T_1 * T_2]) = Exch([T_2]) * Exch([T_1])$ so if T_1, T_2 are exchangeable tangles the tangle number of $Exch([T_1 * T_2]) =$ the tangle number of $Exch([T_1])$ times the tangle number of $Exch([T_2]) = nm$ so T is exchangeable. Now if either of T_1, T_2 are not exchangeable then the tangle number of its exchange is greater than its own tangle number which give the tangle number of $Exch([T])$ greater than that of T, this imply that if T exchangeable then so are T_1 and T_2. $\square$

It will be useful to understand the way these exchangeable tangles look as disc pairs, but to understand how they look as pairs it is necessary to find out what simple transverse intersections of two surfaces look like. In [2] the simple transverse intersections are shown to be of the following form



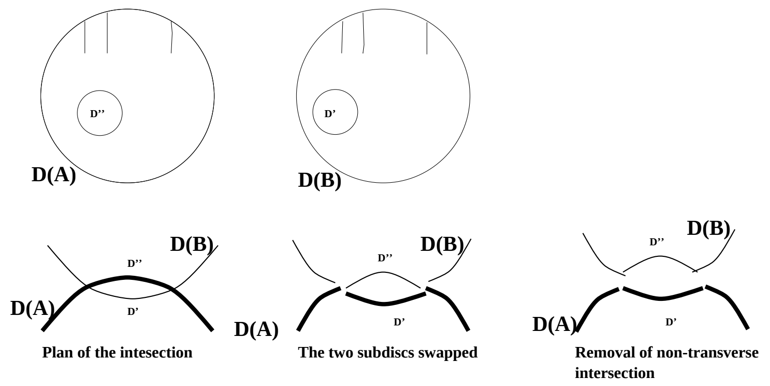
Circle Intersection. Clasp Intersection. Ribbon Intersection.

were the bold lines are the actual intersections.

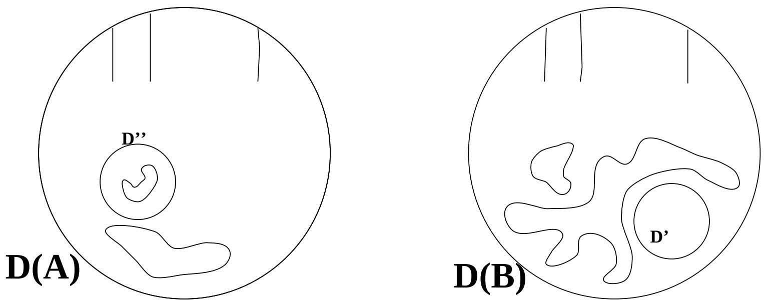
Proposition 4.1.3 *If the tangle $T \in ET_n$ then there exist a pair of discs D_1 and D_2 such that $D_1 \cap D_2$ is n clasps with the same orientation, but no ribbons or circles and $\partial D_1 \cup \partial D_2 \simeq \widehat{T} \cup A$.*

Proof First consider the two discs D_A and $D_{\widehat{T}}$, and consider their intersections, $D_A \cap D_{\widehat{T}}$. These intersections have the properties $D_A \cap \partial(D_{\widehat{T}}) = D_A \cap \widehat{T} = n$ points, and $\partial(D_A) \cap D_{\widehat{T}} = A \cap D_{\widehat{T}} = n$ points, and these intersections have the same orientation. Firstly we must take local isotopies of the discs to these to get rid of any non-transverse intersections, then take isotopies to remove any non-simple intersections. This may be done without affecting the intersections of disc and boundary. Then as the orientation of the n points of intersection, $D_A \cap \partial(D_{\widehat{T}})$, are consistent, $D_A \cap D_{\widehat{T}}$ consists of n clasps, no ribbons plus a number of circles. If the number of circles is zero then we are done, thus the proof is reduced to showing that all circle intersections can be removed.

The proof is by induction on the number of circles. If there is only one circle intersection, then it spans a subdisc of both main discs. Swap the two sub-discs and isotope to remove the non transverse circle intersection.



Next look at the case with $m+1$ discs when it is known that those with m or less discs can have the discs removed. Choose a circle which spans a sub disc in one of the discs that contains no other circle intersections, call this sub disc D',and then recall that it must also span a subdisc of the other disc, which we call D''. The figure below shows a plan of the two discs, with the intersections shown, and the two subdiscs indicated.



The Sub-disc D'' in the disc D(A). The sub-disc D' in the disc D(B).

Remove this circle intersection by swapping the subdiscs over and removing the non transverse intersection thus formed.

The intersection of the two discs is thus n arcs plus one less circle, some of the intersections thus formed may be self intersections of the disc, and as the boundary of the disc is here is an unknot rather than a slice knot, all the self intersections on this sub-disc can be isotoped away, which leads to the following. A self intersection can be removed by the same method of swapping discs when it is realised that any other self intersections remain self intersections and intersections of the two discs remain intersections of the two discs and thus any self intersections can be removed without increasing the number of intersections between the two discs. Thus we have a pair of discs with less than or equal to

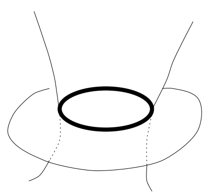
m disc intersections and the induction implies that we may continue this process until all the circles are removed. □

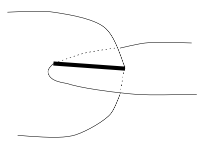
Thus exchangeable tangles are just pairs of discs that intersect in n consistently oriented arcs only. An alternative view of these discs is that we have n discs connected by $n-1$ twisted possibly knotted bands together with a flat disc spanning the axis. Where if the n discs are treated as vertices and the bands as edges the resulting embedded graph is a tree. This view lends itself to the idea of a Murusugi sum of two exchangeable tangles, that is the plumbing of two exchangeable tangles with plumbing disc one of the discs above. The embedded tree has the property that if the plumbing disc is the mth disc then no edge connects from below m to above m, and no edge below m links an edge above m.

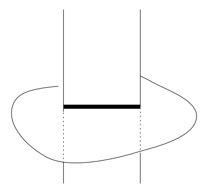
4.2 On the Alexander Polynomial of Exchangeable Tangles.

In this section I will be obtaining the Alexander polynomial of an exchangeable braidlike tangle pair, (see previous section). As these are two component links I use a method for finding the Alexander polynomial of two component links developed in [2]. The reason for using this method rather than another is that when either of two components of the link are unknotted the method simplifies. And when, as is the case for exchangeable braidlike tangles, both components are unknotted the form of the method simplifies still further.

The method of Cooper in [2] for obtaining the Alexander polynomial of a two-component link uses the union of a pair of spanning surfaces for the two-components, with isotopy until the intersections are simple and transverse. Then the intersections of the two surfaces are in the form of clasps, ribbons, and circles. A clasp and a ribbon are intersections that have boundary, while a circle intersection is a closed loop in both surfaces. The clasp has one point of boundary on the boundary of each surface, while a clasp has both ends in one component of the link. These intersections are shown below.


Circle Intersection.


Clasp Intersection.


Ribbon Intersection.

were the bold lines are the actual intersections.

In [2] Cooper calls any pair $S = S(A) \cup S(B)$ a C-complex if the only intersections allowed are clasps. Let $\varepsilon(S)$ be the singularity of S $S(A) \cap S(B)$. Then to use his method the first step is to define two bilinear forms

$$\alpha, \beta : H_1(S) \times H_1(S) \to \mathbf{Z}$$

as follows. Firstly a 1-cycle $u \in S$ is called a loop if whenever u meets $\varepsilon(S)$,it does so at an end point of some component of $\varepsilon(S)$, i.e. the 1-cycle includes the whole of that connected component of the intersection. Any element of the first homology group may be represented by such a loop so given a pair of such loops we may define

$$\begin{array}{rcl} \alpha([u],[v]) & = & lk(u^{--}, v) \\ \beta([u],[v]) & = & lk(u^{-+}, v) \end{array}$$

where u^{-+} is the cycle in S^3 obtained by lifting off the union in the negative normal direction from D_1 and in the positive direction from D_2. The definition of u^{--} is similar i.e. in the negative direction from both.

Now we set about choosing a basis for $H_1(S)$. Firstly a basis $(h_1, \ldots, h_g)$ of the surface $H_1(S(A))$, and then a basis $(h_{g+1}, \ldots, h_{g+k})$ of the surface $H_1(S(B))$. Use these two sets of basis elements to start to define a basis for $H(S)$. Identify the generators of the surfaces by inclusion into the generating set of $H_1(S)$, then extend this set of basis elements into a basis $(h_1, \ldots, h_{g+k+l})$ of $H_1(S)$. Let A and B be the integral matrices of α and β using this basis.

Then for the Alexander polynomial of this two component link we have

$$\Delta(x, y) = (y-1)^{-g}(x-1)^{-k} \det(xyA + A^T - xB - yB^T).$$

This is Cooper's method for finding the Alexander polynomial from a complex of two surfaces, and it is noticeable that if g and k are equal to zero the Alexander polynomial is easier to find. But this is the case when $S(A)$ and $S(B)$ are discs.

As we wish to use the above in the case where S is the union of two discs, with the only intersection types being clasps, the method of choosing a basis for the homology of each surface then extending to a basis for the homology of the union yields the $n-1$ loops formed by the union only. so we have

$$\alpha, \beta : H_1(D_1 \cup D_2) \times H_1(D_1 \cup D_2) \to \mathbf{Z}$$

as follows. let $S = D(A) \cup D(B)$. Let $u \in S$ be a loop as before i.e. the 1-cycle u includes the whole of each connected component, of the intersection of the two discs, that it meets.

Now as previously defined let A and B be the integer matrices of α and β respectively. Using $(h_1, \ldots, h_l)$ as the basis for $H_1(S)$ added for either $D(A)$ or $D(B)$.

Then for the Alexander polynomial of this two component link we have

$$\Delta(x, y) = \det(xyA + A^T - xB - yB^T).$$

Now we are in a position to use this to investigate the Alexander polynomial of exchangeable braidlike tangles.

Let the tangle $T \in ET_n$ and define two single variable polynomials to be $q_T(y) = \det(yA - B)$ and $p_T(x) = \det(xA - B^T)$, which are the polynomials $Max_x(y)$ and $Max_y(x)$ respectively.

Lemma 4.2.1 *If T is an exchangeable Stallings braid then both $p_T(x)$ and $q_T(y)$ are equal to units, that is $p_T(x) = x^n$, and $q_T(y) = y^m$ up to sign.*

Proof. This result follows because as shown in [1] the Alexander polynomial of a braid can be obtained as the characteristic polynomial of an invertible matrix in the coefficient of the braid i.e. $\det(B(x) - yI)$ where the matrix $B(x)$ is the reduced Burau matrix of the braid, and thus the coefficient of the highest order terms in y corresponding to the axis is a unit. As the braid is exchangeable the highest coefficient in x is also a unit. □

Thus given an exchangeable tangle pair if either one of the polynomials is not a unit then the pair is not an exchangeable braid pair.

The following is a construction of the C-complex $S = D(A) \cup D(B)$ for an exchangeable tangle pair. Suppose that the exchangeable tangle pair are n-tangles, then with A the axis and B the tangle, the C-complex S is the union of $D(A)$, for the surface to span A, n discs for each of the n tangle arcs, and $n-1$ twisted bands, to form the rest of $D(B)$. The bands can be treated as the edges of an embedded tree, and the n discs can be treated as vertices.

Given an exchangeable tangle $T \in ET_n$, with D_T and axis A, following Morton's definition I shall say that the pair (D_T, A) is the Murasugi sum of (D_{T_1}, A_1) and (D_{T_2}, A_2) if

(i) D_{T_1} and D_{T_2} are subdiscs of D each lying in half of S^3, and meeting only in a disc $D_0 = D_{T_1} \cap D_{T_2}$ lying on the sphere, S^2, which separates S^3 into the two halves.

(ii) The axis A meets the separating sphere S^2 in only two points, $d \in D_0$ and $c \notin D_0$.

(iii) The disc D_{T_1} forms a disc which the axis A_1 intersects in n_1 points, where A_1 consists of the part of L in the half of S^3 containing D_1 completed by an unknotted arc *cd* in the other half of S^3, and similarly for the pair (D_2, A_2).

Lemma 4.2.2 *If T is the Murasugi sum of T_1 and T_2 then we have $p_T(x) = p_{T_1}(x) \times p_{T_2}(x)$.*

Proof. Let the tangles T_i have matrices A_i and B_i then T has matrices

$$A = \begin{pmatrix} A_1 & 0 \\ M_A & A_2 \end{pmatrix}$$

$$B = \begin{pmatrix} B_1 & M_B \\ 0 & B_2 \end{pmatrix}$$

for some M_A and M_B, but

$$p_T(x) = \det(xA - B^T) = \det(xA_1 - B_1^T) \times \det(xA_2 - B_2^T) = p_{T_1}(x) \times p_{T_2}(x).$$

□

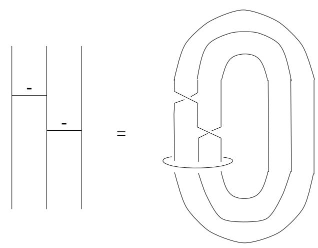
Now as the Alexander polynomial of the satellite of two 2-component links $C_1 \cup C_2$, and $L_1 \cup L_2$ is known to be $\Delta_{L_1 \cup L_2}(x^m, y) \times \Delta_{C_1 \cup C_2}(x, y^n) = \Delta_{(L_1 * C_1) \cup C_2}(x, y)$, where $m = lk(C_1, C_2)$, and $n = lk(L_1, L_2)$. We therefore have, for exchangeable tangles, that the leading term in the Alexander polynomial $p_{T_1 * T_2}(x) = p_{T_1}(x^{n_2}) \times p_{T_2}(x)$ where n_2 is the tangle number of T_2.

4.3 Exchangeability of Stallings braids of small index.

In this section I give a survey of Stallings braids of low braid index, and investigate which are exchangeably braided. A Stallings braid on n strings is defined by Morton in [9] to be the product of $n-1$ elementary braids or their inverses which close to a single component, where an elementary braid $\sigma_{i,j} \in B_n$, for $1 \leq i < j \leq n$ is the braid interchanging the i-th and j-th strings with a single positive crossing, in front of any intermediate strings, and leaving the others alone. Such braids were introduced by Stalling as examples of braids whose closure is the unknot. A spanning surface for the closure of a Stallings braid can be obtained by placing n discs on the braid strings and connecting them by half-twisted bands corresponding to the elementary braids in the above product, where a positive half-twisted band connecting the i-th and j-th discs corresponds to the elementary braid $\sigma_{i,j}$ and a negative half-twisted band connecting the i'-th and j'-th discs corresponds to the inverse of the elementary braid i.e. $\sigma^{-1}_{i',j'}$. Such a disc intersects the axis in only n points, this form of spanning disc for a Stallings braid is known as a *Stallings disc.*

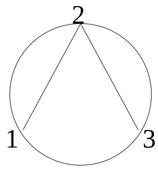
Below we have a nice representation of braids which are presented in the form of a collection of elementary braids.

1. The *fences* represent a braid in the following way, if the number of vertical lines is n then the fence represents an n-string braid and that if a horizontal line connects vertical line i to j with sign ε then in the braid expansion we have the elementary braid ${\sigma_{i,j}}^{\varepsilon}$. I shall call a braid expansion in terms of elementary braid an elementary braid expansion. A spanning surface for the braid can be obtained by placing a disc on each of the braid strings, and for each of the elementary braids a half-twisted band, and the sign on a band determines whether it is a positive or negative half-twisted band. For example,

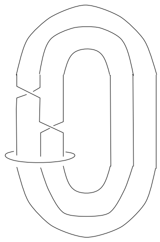


then this braid has three vertical discs and two negative half-twisted bands on its spanning disc, and is thus the Stallings disc of the Stallings braid $\sigma_{1,3}\sigma_{1,2}$. In the disc and twisted band view of the fence of a Stallings braid we have an embedded tree, as follows for each vertical line in a fence, (that is a subdisc of the Stallings disc) we have a vertex of the tree. And each horizontal line of the fence connecting the i-th and j-th strings gives an edge of the embedded tree, connecting the i-th and j-th vertices. That is the tree has n vertices and for each ${\sigma_{i,j}}^{\varepsilon}$ the tree has an edge between the vertices i and j. This embedded tree is not a unique feature, that is the same braid may give many different embedded trees depending on the specific elementary braid expansion. This would seem to limit the use of the embedded tree, but this is not so, as a repeated braid in a survey of stallings braids does not affect the result.

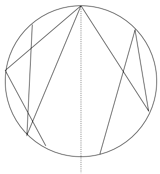
2. The *pattern* of the Stallings braid represents a 'birds-eye view' of the embedded tree described above, that is it shows a ring of vertices with the edges between them. So if the pattern was



then the following 3-string braid has this pattern embedded into its Stallings disc.

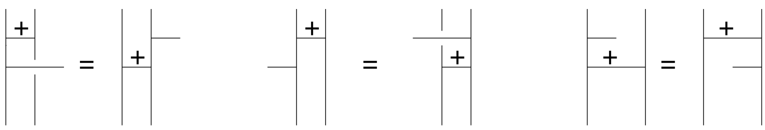


The pattern is of use when deciding whether or not a Stallings braid is a Murasugi sum.

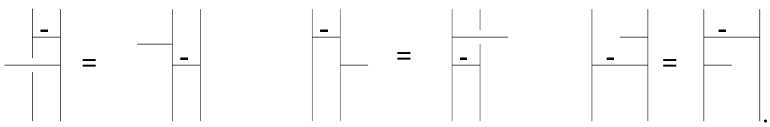


For example, any Stallings braid with the above pattern (embedded tree) will be a Murasugi sum of two smaller Stallings braids, where the dotted line indicates the position of a splitting sphere.

The fences follow braid relation properties, the positively signed horizontal bars satisfy the following relations,



and the negative bars the equivalent properties,



We are thus able to view Stallings braids as *fences* and use the above relations to determine whether or not a given Stallings braid is a Murasugi sum.

We now survey Stallings braids of small index.

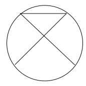
The only one string braid with axis is the Hopf link, and as it has unit Alexander polynomial it follows that $p(x) = 1$.

There are only two 2-string Stallings braids σ_1 and σ_1^{-1}. These are both exchangeable and have $p(x)$ equal to a unit.

All 3-string Stallings braids are the Murasugi sum of two 2-string Stallings braids, so all are exchangeable and all have $p(x)$ equal to a unit, by lemma 2.2.2.

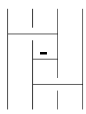
Proposition 4.3.1 *If a* 4*-string Stallings braid is exchangeable then its either a Murasugi sum of two Stallings braids or a satellite of two Stallings braids. Furthermore, if a* 4*-string Stallings braid is not exchangeable then* $p(x)$ *is not a unit.*

Proof The two parts of the proposition can be shown if all Stallings braids with 4-strings that have $p(x)$ a unit can be shown to be composite, and thus either a Murasugi sum or a satellite of Stallings braids that must be exchangeable. However with 4-string Stallings braids we come for the first time to non-exchangeable Stallings braids, so must be careful. All embedded trees except



must produce Stallings braids that are the Murasugi sum of a 3-string Stallings braid and a 2-string one, and so are exchangeable with $p(x)$ equal to a unit.

For the above pattern the Stallings braids are equivalent to the following fence



where the minus sign is required otherwise the braid is equivalent to a Murasugi sum (using the negative bar relations above), and thus equivalent to one treated already. It is worth noting that $\varepsilon_i = \alpha(u_i, u_i)$ is -1 for a positive band and 0 for a negative one, and that $\beta(u_i, u_i) = \alpha(u_i, u_i) + 1$, and that both $\alpha(u_i, u_j)$ and $\beta(u_i, u_j)$ remain unchanged whatever sign the bars take when $i \neq j$.

Using Cooper's method described in section 4.2 to find the Alexander polynomial of a Stallings braid with axis gives

$$A = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ -1 & \varepsilon_2 & 0 \\ -1 & -1 & \varepsilon_3 \end{pmatrix}$$

$$B = \begin{pmatrix} \varepsilon_1 + 1 & 0 & 0 \\ 0 & \varepsilon_2 + 1 & 0 \\ -1 & 0 & \varepsilon_3 + 1 \end{pmatrix}$$

Thus using $p(x) = \det(xA - B^T)$ we have the following cases.

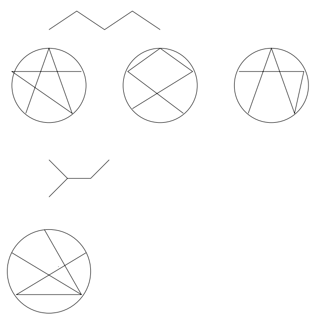
$$\begin{array}{rclrcl}
(\delta_1, \delta_2, \delta_3) & = & (1, -1, 1) & & & \\
(\varepsilon_1, \varepsilon_2, \varepsilon_3) & = & (-1, 0, -1) & p(x) & = & x \\
(\delta_1, \delta_2, \delta_3) & = & (1, -1, -1) & & & \\
(\varepsilon_1, \varepsilon_2, \varepsilon_3) & = & (-1, 0, 0) & p(x) & = & 2x - x^2 \\
(\delta_1, \delta_2, \delta_3) & = & (-1, -1, 1) & & & \\
(\varepsilon_1, \varepsilon_2, \varepsilon_3) & = & (0, 0, -1) & p(x) & = & 2x - x^2 \\
(\delta_1, \delta_2, \delta_3) & = & (-1, -1, -1) & & & \\
 & = & (0, 0, 0) & p(x) & = & 1 + x - x^2
\end{array}$$

All the cases where $p(x)$ are non-unit are not exchangeable, by lemma 4.2.1. The remaining case was shown in [9] to be the satellite $\sigma_1 \widehat{* \sigma_1}{}^{-1}$ which is exchangeable as it decomposes into two exchangeable braids and has $p(x)$ equal to a unit. This shows that the only 4-string Stallings braids having $p(x)$ equal to a unit are the 4-string Stallings braids that are exchangeable, and that the only 4-string exchangeable Stallings braid which is not a Murasugi sum is a satellite.

□

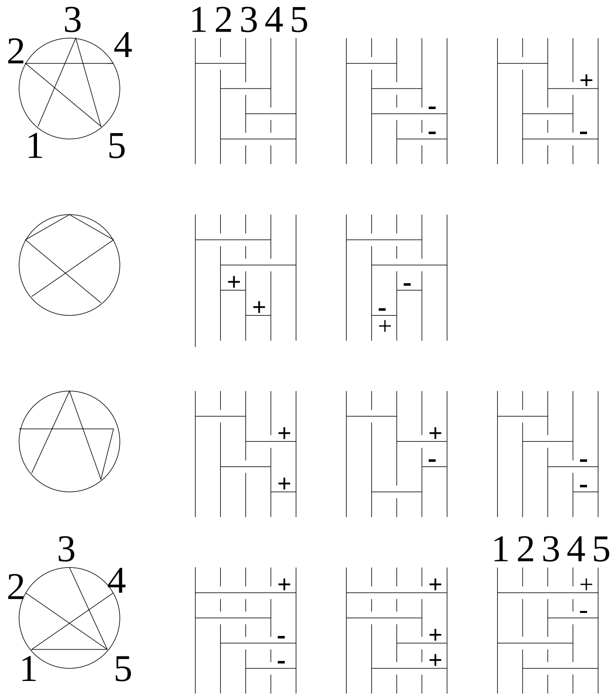
Proposition 4.3.2 *If a 5-string Stallings braid is exchangeable then its a Murasugi sum of two Stallings braids. Furthermore, if a 5-string Stallings braid is not exchangeable then $p(x)$ is not a unit.*

Proof As in proposition 4.3.1 the two parts of the proposition can be shown if all Stallings braids with 5-strings that have $p(x)$ a unit can be shown to be composite, and thus a Murasugi sum of Stallings braids that must be exchangeable. All 5-string Stallings braids except those with the following patterns



are the Murasugi sum of smaller Stallings braids, and these have been treated already, it follows that any 5-string Stallings braid that is a Murasugi sum is exchangeable if and only if $p(x)$ is equal to a unit. This is so because we have shown that for four, three, and two strings a Stallings braid is exchangeable if and only if $p(x)$ is equal to a unit, and thus that the Murasugi sum of two braids to form a five string braid must have $p(x)$ equal to a unit. We must show then that if $p(x)$ is a unit for a Stallings braid, then that Stallings braid is exchangeable.

The above patterns lead to the following braids shown in the form of fences.



In this proof we use Cooper's method, described in section 4.2, to find the Alexander polynomial of a Stallings braid with axis. As in proposition 4.3.1 it is worth noting that $\varepsilon_i = \alpha(u_i, u_i)$ is -1 for a positive band and 0 for a negative one, and that $\beta(u_i, u_i) = \alpha(u_i, u_i) + 1$, and that both $\alpha(u_i, u_j)$ and $\beta(u_i, u_j)$ remain unchanged whatever sign the bars take when $i \neq j$.

In the above fences some of the signs are forced because otherwise the braids are equivalent to Murasugi sums, the other horizontal lines can at first take either sign.

$$A_1 = \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 \\ -1 & \varepsilon_2 & 0 & 0 \\ 0 & -1 & \varepsilon_3 & 0 \\ -1 & -1 & 0 & \varepsilon_4 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} \varepsilon_1+1 & 0 & 0 & 0 \\ -1 & \varepsilon_2+1 & 0 & 0 \\ -1 & -1 & \varepsilon_3+1 & 0 \\ -1 & 0 & 1 & \varepsilon_4+1 \end{pmatrix}$$

For the above matrix if the fourth band is negative i.e. $\delta_4 = -1$ and $\varepsilon_4 = 0$ then the second band must be negative and the third band positive or in either case the braid is equivalent to a Murasugi sum. Also if the third band is negative then the first band must also be negative.

Thus using $p(x) = \det(xA - B^T)$ we have

$$\begin{array}{rclrcl}
(\delta_1,\delta_2,\delta_3,\delta_4) &=& (1,-1,1,-1) & & & \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) &=& (-1,0,-1,0) & p(x) &=& 4x-x^2-2 \\
(\delta_1,\delta_2,\delta_3,\delta_4) &=& (-1,-1,1,-1) & & & \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) &=& (0,0,-1,0) & p(x) &=& 3x^2-x^3-x \\
(\delta_1,\delta_2,\delta_3,\delta_4) &=& (-1,1,-1,1) & & & \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) &=& (0,-1,0,-1) & p(x) &=& 4x^2-2x-x^3 \\
(\delta_1,\delta_2,\delta_3,\delta_4) &=& (-1,-1,-1,1) & & & \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) &=& (0,0,0,-1) & p(x) &=& 3x^2-x^3-x \\
(\delta_1,\delta_2,\delta_3,\delta_4) &=& (1,1,1,1) & & & \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) &=& (-1,-1,-1,-1) & p(x) &=& x^3+x^2-1 \\
(\delta_1,\delta_2,\delta_3,\delta_4) &=& (1,-1,1,1) & & & \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) &=& (-1,0,-1,-1) & p(x) &=& x(x^2-2x+2) \\
(\delta_1,\delta_2,\delta_3,\delta_4) &=& (-1,1,1,1) & & & \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) &=& (0,-1,-1,-1) & p(x) &=& x(x^2+2x-2) \\
(\delta_1,\delta_2,\delta_3,\delta_4) &=& (-1,-1,1,1) & & & \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) &=& (0,0,-1,-1) & p(x) &=& x(3x-x^2-1)
\end{array}$$

thus if a braid has the form of the first fence shown above it is either a Murasugi sum or not exchangeable.

For the second fence we have

$$A_2 = \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 \\ -1 & \varepsilon_2 & 0 & 0 \\ -1 & -1 & \varepsilon_3 & 0 \\ 0 & -1 & -1 & \varepsilon_4 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} \varepsilon_1+1 & 0 & 0 & 0 \\ -1 & \varepsilon_2+1 & 0 & 0 \\ -1 & 0 & \varepsilon_3+1 & 0 \\ -1 & -1 & 0 & \varepsilon_4+1 \end{pmatrix}$$

so the third and fourth bands are negative giving

$$\begin{array}{rclrcl}
(\delta_1,\delta_2,\delta_3,\delta_4) & = & (1,1,-1,-1) & & & \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (-1,-1,0,0) & p(x) & = & x^2-x^3+x \\
(\delta_1,\delta_2,\delta_3,\delta_4) & = & (1,-1,-1,-1) & & & \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (-1,0,0,0) & p(x) & = & 3-2x \\
(\delta_1,\delta_2,\delta_3,\delta_4) & = & (-1,1,-1,-1) & & & \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (0,-1,0,0) & p(x) & = & 3x-2x^2 \\
(\delta_1,\delta_2,\delta_3,\delta_4) & = & (-1,-1,-1,-1) & & & \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (0,0,0,0) & p(x) & = & x^3-4x^2+3x+1
\end{array}$$

as none of these are units we have that Stallings braids with the second fence are not exchangeable if they are not Murasugi sums.

For the third fence we have

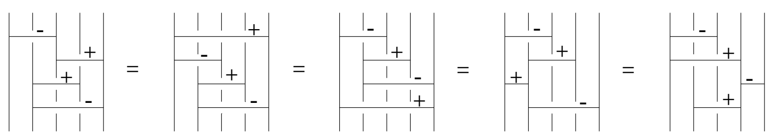
$$A_3 = \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & \varepsilon_2 & 0 & 0 \\ -1 & 1 & \varepsilon_3 & 0 \\ -1 & 0 & -1 & \varepsilon_4 \end{pmatrix}$$

$$B_3 = \begin{pmatrix} \varepsilon_1+1 & 0 & 0 & 0 \\ -1 & \varepsilon_2+1 & 0 & 0 \\ -1 & 1 & \varepsilon_3+1 & 0 \\ -1 & 1 & 0 & \varepsilon_4+1 \end{pmatrix}$$

with second band positive and the fourth band negative giving

$$\begin{array}{rclrcl}
(\delta_1,\delta_2,\delta_3,\delta_4) & = & (1,1,1,-1) & & & \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (-1,-1,-1,0) & p(x) & = & x-2x^2 \\
(\delta_1,\delta_2,\delta_3,\delta_4) & = & (1,1,-1,-1) & & & \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (-1,-1,0,0) & p(x) & = & 5x-2-2x^2 \\
(\delta_1,\delta_2,\delta_3,\delta_4) & = & (-1,1,1,-1) & & & \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (0,-1,-1,0) & p(x) & = & x^2 \\
(\delta_1,\delta_2,\delta_3,\delta_4) & = & (-1,1,-1,-1) & & & \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (0,-1,0,0) & p(x) & = & 2x^2-x^3
\end{array}$$

The third value of $p(x)$ is a unit, but it is easy to show that this braid is equivalent to the Murasugi sum of simpler braids. It can be seen below using the fence relations that the second band may slide over the first and then be transferred to the bottom. It then may slide over the band that was fourth. Then both of these bands slide over the band that was third and the result is a Murasugi sum.



Thus in the case of the third fence, for the Stallings braids to be exchangeable they must be Murasugi sums.

$$A_4 = \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 \\ -1 & \varepsilon_2 & 0 & 0 \\ 0 & 0 & \varepsilon_3 & 0 \\ -1 & 0 & 0 & \varepsilon_4 \end{pmatrix}$$

$$B_4 = \begin{pmatrix} \varepsilon_1+1 & 0 & 0 & 0 \\ -1 & \varepsilon_2+1 & 0 & 0 \\ 0 & 1 & \varepsilon_3+1 & 0 \\ 0 & 0 & -1 & \varepsilon_4+1 \end{pmatrix}$$

The fourth fence has the third and fourth bands positive, which leads to the following determinants

$$\begin{array}{rclrcl} (\delta_1,\delta_2,\delta_3,\delta_4) & = & (1,1,1,1) & & & \\ (\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (-1,-1,-1-1) & p(x) & = & x^3+x^2-x \\ (\delta_1,\delta_2,\delta_3,\delta_4) & = & (1,-1,1,1) & & & \\ (\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (-1,0,-1,-1) & p(x) & = & 2x^3-1 \\ (\delta_1,\delta_2,\delta_3,\delta_4) & = & (-1,1,1,1) & & & \\ (\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (0,-1,-1,-1) & p(x) & = & 2x^3-1 \\ (\delta_1,\delta_2,\delta_3,\delta_4) & = & (-1,-1,1,1) & & & \\ (\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (0,0,-1,-1) & p(x) & = & x^3+x^2-x \end{array}$$

So none of these braids are exchangeable.

$$A_5 = \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 \\ -1 & \varepsilon_2 & 0 & 0 \\ -1 & 0 & \varepsilon_3 & 0 \\ 0 & 0 & 1 & \varepsilon_4 \end{pmatrix}$$

$$B_5 = \begin{pmatrix} \varepsilon_1+1 & 0 & 0 & 0 \\ -1 & \varepsilon_2+1 & 0 & 0 \\ 0 & 0 & \varepsilon_3+1 & 0 \\ 0 & 1 & 0 & \varepsilon_4+1 \end{pmatrix}$$

The fifth fence has the third and fourth bands both negative and positive so that it cannot avoid being a Murasugi sum. So every case here, exchangeable or not, has been treated before.

$$A_6 = \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & \varepsilon_2 & 0 & 0 \\ -1 & 1 & \varepsilon_3 & 0 \\ 0 & -1 & 0 & \varepsilon_4 \end{pmatrix}$$

$$B_6 = \begin{pmatrix} \varepsilon_1+1 & 0 & 0 & 0 \\ -1 & \varepsilon_2+1 & 0 & 0 \\ -1 & 1 & \varepsilon_3+1 & 0 \\ 0 & 0 & -1 & \varepsilon_4+1 \end{pmatrix}$$

Here the second and fourth bands must be positive.

$$\begin{array}{lcllcl}
(\delta_1,\delta_2,\delta_3,\delta_4) & = & (1,1,1,1) & & & \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (-1,-1,-1,-1) & p(x) & = & x^2+2x-2 \\
(\delta_1,\delta_2,\delta_3,\delta_4) & = & (1,1,-1,1) & & & \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (-1,-1,0,-1) & p(x) & = & x^2(3x-2) \\
(\delta_1,\delta_2,\delta_3,\delta_4) & = & (-1,1,1,1) & & & \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (0,-1,-1,-1) & p(x) & = & x(2x^2-1) \\
(\delta_1,\delta_2,\delta_3,\delta_4) & = & (-1,1,-1,1) & & & \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (0,-1,0,-1) & p(x) & = & x(2x-1)
\end{array}$$

So none of these are exchangeable.

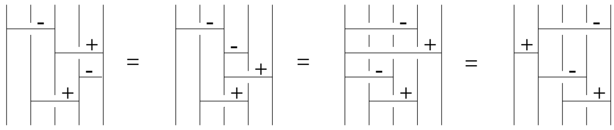
$$A_7 = \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & \varepsilon_2 & 0 & 0 \\ 0 & -1 & \varepsilon_3 & 0 \\ -1 & 1 & 1 & \varepsilon_4 \end{pmatrix}$$

$$B_7 = \begin{pmatrix} \varepsilon_1+1 & 0 & 0 & 0 \\ -1 & \varepsilon_2+1 & 0 & 0 \\ 0 & 0 & \varepsilon_3+1 & 0 \\ -1 & 1 & 0 & \varepsilon_4+1 \end{pmatrix}$$

Here the second is positive and the third negative.

$$\begin{array}{lcllcl}
(\delta_1,\delta_2,\delta_3,\delta_4) & = & (1,1,-1,1) & & & \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (-1,-1,0,-1) & p(x) & = & 2x-1 \\
(\delta_1,\delta_2,\delta_3,\delta_4) & = & (1,1,-1,-1) & & & \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (-1,-1,0,0) & p(x) & = & 3x^2-x^3-1 \\
(\delta_1,\delta_2,\delta_3,\delta_4) & = & (-1,1,-1,1) & & & \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (0,-1,0,-1) & p(x) & = & x \\
(\delta_1,\delta_2,\delta_3,\delta_4) & = & (-1,1,-1,-1) & & & \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (0,-1,0,0) & p(x) & = & 2-x
\end{array}$$

The third value of $p(x)$ above is equal to a unit, so the associated Stallings braid may be exchangeable. Using the fence relations on the third fence, we see that the third band can pass under the second, then these two bands slide over the first, thus the braid is equivalent to a Murasugi sum, shown below.



Thus if a braid with this fence is exchangeable it is a Murasugi sum.

$$A_8 = \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 \\ -1 & \varepsilon_2 & 0 & 0 \\ 0 & -1 & \varepsilon_3 & 0 \\ 0 & 0 & -1 & \varepsilon_4 \end{pmatrix}$$

$$B_8 = \begin{pmatrix} \varepsilon_1+1 & 0 & 0 & 0 \\ -1 & \varepsilon_2+1 & 0 & 0 \\ -1 & -1 & \varepsilon_3+1 & 0 \\ 0 & -1 & 0 & \varepsilon_4+1 \end{pmatrix}$$

Here the third and fourth bands are negative.

$$\begin{array}{rcllcl} (\delta_1,\delta_2,\delta_3,\delta_4) & = & (1,1,-1,-1) & & & \\ (\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (-1,-1,0,-1) & p(x) & = & x^2-x-x^3 \\ (\delta_1,\delta_2,\delta_3,\delta_4) & = & (1,-1,-1,-1) & & & \\ (\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (-1,0,0,0) & p(x) & = & 2x-x^3 \\ (\delta_1,\delta_2,\delta_3,\delta_4) & = & (-1,1,-1,-1) & & & \\ (\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (0,-1,0,0) & p(x) & = & 3x-2x^2 \\ (\delta_1,\delta_2,\delta_3,\delta_4) & = & (-1,-1,-1,-1) & & & \\ (\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (0,0,0,0) & p(x) & = & 2x+1-x^2 \end{array}$$

So none of these fences represents an exchangeable braid.

$$A_9 = \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & \varepsilon_2 & 0 & 0 \\ -1 & -1 & \varepsilon_3 & 0 \\ -1 & -1 & -1 & \varepsilon_4 \end{pmatrix}$$

$$B_9 = \begin{pmatrix} \varepsilon_1+1 & 0 & 0 & 0 \\ 1 & \varepsilon_2+1 & 0 & 0 \\ 0 & -1 & \varepsilon_3+1 & 0 \\ 0 & -1 & 0 & \varepsilon_4+1 \end{pmatrix}$$

In the ninth fence the first band is positive and the third and fourth negative.

$$\begin{array}{rcllcl} (\delta_1,\delta_2,\delta_3,\delta_4) & = & (1,1,-1,-1) & & & \\ (\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (-1,-1,0,0) & p(x) & = & x(4x-x^2-2) \\ (\delta_1,\delta_2,\delta_3,\delta_4) & = & (1,-1,-1,-1) & & & \\ (\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (-1,0,0,0) & p(x) & = & x(3x-x^2-1) \end{array}$$

Thus neither of these fences represents an exchangeable braid.

$$A_{10} = \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & \varepsilon_2 & 0 & 0 \\ -1 & -1 & \varepsilon_3 & 0 \\ -1 & -1 & 0 & \varepsilon_4 \end{pmatrix}$$

$$B_{10} = \begin{pmatrix} \varepsilon_1+1 & 0 & 0 & 0 \\ 1 & \varepsilon_2+1 & 0 & 0 \\ 0 & -1 & \varepsilon_3+1 & 0 \\ 0 & -1 & 1 & \varepsilon_4+1 \end{pmatrix}$$

Here the first third and fourth bands are positive.

$$\begin{array}{lcllcl} (\delta_1,\delta_2,\delta_3,\delta_4) & = & (1,1,1,1) & & & \\ (\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (-1,-1,-1,-1) & p(x) & = & x^3-x+1 \\ (\delta_1,\delta_2,\delta_3,\delta_4) & = & (1,-1,1,1) & & & \\ (\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (-1,0,-1,-1) & p(x) & = & x^2-x+1 \end{array}$$

Thus neither of these fences represents an exchangeable braid.

$$A_{11} = \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 \\ -1 & \varepsilon_2 & 0 & 0 \\ 0 & 1 & \varepsilon_3 & 0 \\ -1 & 0 & -1 & \varepsilon_4 \end{pmatrix}$$

$$B_{11} = \begin{pmatrix} \varepsilon_1+1 & 0 & 0 & 0 \\ 0 & \varepsilon_2+1 & 0 & 0 \\ 1 & 1 & \varepsilon_3+1 & 0 \\ 0 & 1 & -1 & \varepsilon_4+1 \end{pmatrix}$$

Here we must have the first band positive and the second band negative.

$$\begin{array}{lcllcl} (\delta_1,\delta_2,\delta_3,\delta_4) & = & (1,-1,1,1) & & & \\ (\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (-1,0,-1,-1) & p(x) & = & 2x^2-x \\ (\delta_1,\delta_2,\delta_3,\delta_4) & = & (1,-1,1,-1) & & & \\ (\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (-1,0,-1,0) & p(x) & = & 3x-x^2-1 \\ (\delta_1,\delta_2,\delta_3,\delta_4) & = & (1,-1,-1,1) & & & \\ (\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (-1,0,0,-1) & p(x) & = & 3x^2-x^3-x \\ (\delta_1,\delta_2,\delta_3,\delta_4) & = & (1,-1,-1,-1) & & & \\ (\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (-1,0,0,0) & p(x) & = & 2x-x^2 \end{array}$$

So none of these fences represent an exchangeable braid.

The above shows that up to 5-strings a Stallings braid is exchangeable if and only if the polynomial $p(x)$ is equal to a unit, and that if it is not a Murasugi sum then it is a satellite. □

This leads to the following two conjectures.

Guess 1 *Given the above definition of $p(x)$ a Stallings braid is exchangeable if and only if $p(x)$ is equal to a unit.*

Guess 2 *A Stallings braid is exchangeable if and only if it is the Murasugi sum or satellite of two smaller exchangeable braids.*

In Corollary 6.3.2 I show that a Stallings braid is exchangeable if and only if its double cover is fibered as a pluming of hopf band on a disc. This implies that an exchaneable Stallings braid is a Murasugi sum of two exchangeable tangles in the sence of the previous sections of this chapter. The above question is thus are there any exchangeable braids whose splittings into exchangeable tangles contain no exchangeable braid pairs, and are also not satellites.

Chapter 5

Exchangeable braids as fibered multilinks

In this chapter we look at exchangeable braids in the framework of Eisenbud and Neumann's multilinks.

5.1 Splicing Spaces.

Before I begin describing the splice of two spaces, it would be useful to have some definitions. The following definitions will be used, and adapted throughout this section. Let Σ be an integer homology 3-sphere, and S, and S' be closed curves embedded within Σ. Define $N(S)$ to be a solid torus neighbourhood of S within Σ, and let $\partial(N(S))$ be the boundary torus of $N(S)$. Now we can define $lk(S, S')$ to be the homology class of S' in $\Sigma - S$.

In $\partial(N(S))$ we have two distinguished curves, m and l. Define m to be an essential curve which spans a disc in $N(S)$, and take l to lie in the same homology class as S in $N(S)$, with $lk(l, S) = 0$.

Remark. This definition of l is of use only when Σ is an integer homology 3-sphere. I believe that l can be defined in terms of $F(S) \cap \partial(N(S))$, where $F(S)$ is an oriented spanning surface for S when Σ is a smooth, connected, orientable 3-manifold.

The work in this section applies to integer homology 3-spheres, so I will restrict the following to a discussion of these spaces only.

In the days of sailing ships a splice was an operation performed to join two, possibly different, ropes together, with the side effect of reducing the number of end points of the rope. Although this is not quite the mathematical operation of splicing, it does give an idea of why the name *splice* was chosen. The following gives a definition of the splice of two spaces over closed curves within them.

Let Σ', and Σ'' be a pair of integer homology 3-spheres, with ${S'}_0$, and ${S''}_0$ closed curves embedded in Σ', and Σ'' respectively. Following the previous ideas we have $N({S'}_0)$, and $N({S''}_0)$ are neighbourhoods of ${S'}_0$, and ${S''}_0$ in Σ', and Σ'' respectively, with m', m'' as meridians of ${S'}_0$, and ${S''}_0$, and l', l'' be the longitudes.

Definition. Then the *splice* of Σ', and Σ'', over ${S'}_0$, and ${S''}_0$ is the space $(\Sigma' - N({S'}_0)) \cup (\Sigma'' - N({S''}_0))$, with the identification $m' = l''$ and $m'' = l'$.

If we call this space Σ we thus say that Σ is the splice of Σ', and Σ'', over ${S'}_0$, and ${S''}_0$, denoted by $\Sigma = \Sigma' \underset{S' \qquad\qquad S''}{\rule{6em}{0.4pt}} \Sigma''$. It follows that Σ is a smooth, connected, orientable 3-manifold.

Thus we cannot simply say that a space is the splice of two other spaces, but must refer to an embedding of closed curves in spaces, but this is exactly what a knot is. In the above definition of the splice it should be noted that if either, or both of the knots were part of a link, the other link components are now part of a link in Σ. That is to say if ${S'}_0$ is a component of a link ${S'}_0 \cup S_1 \cup \ldots \cup S_k$ in Σ' and ${S''}_0$ is a component of the link ${S''}_0 \cup S_{k+1} \cup \ldots \cup S_n$ in Σ'' then this induces a link $S_1 \cup \ldots \cup S_k \cup S_{k+1} \cup \ldots \cup S_n$ in Σ.

The following definitions show how the splice of two manifolds can lead to the splice of two links.

Let Σ', and Σ'' be, as before, integer homology 3-spheres, with ${S'}_0 \cup S_1 \cup \ldots \cup S_k$ a proper embedding of closed curves in Σ', and ${S''}_0 \cup S_{k+1} \cup \ldots \cup S_n$ a proper embedding of closed curves in Σ''. Let the *link* $\mathbf{L}'$ be the pair $\mathbf{L}' = (\Sigma', {S'}_0 \cup S_1 \cup \ldots \cup S_k)$ denoting the embedding of the the closed curves in the space and similarly let $\mathbf{L}'' = (\Sigma'', {S''}_0 \cup S_{k+1} \cup \ldots \cup S_n)$.

Definition. Let Σ be the *splice* of Σ', and Σ'' over ${S'}_0$ and ${S''}_0$, then the link

$$\mathbf{L} = (\Sigma, S_1 \cup \ldots \cup S_k \cup S_{k+1} \cup \ldots \cup S_n)$$

is called the splice of the link $\mathbf{L}'$ and $\mathbf{L}''$ over ${S'}_0$ and ${S''}_0$.

Now although above we have a definition for the link embedding in the spliced manifolds, we do not yet have a definition on the link exteriors. The following gives a definition of the link exterior, and, following Eisenbud and Neumann [3], a definition for the exterior of the splice of two link exteriors.

Definition.

- As before the link $\mathbf{L} = (\Sigma, K) = (\Sigma, S_1 \cup \ldots \cup S_n)$ is an embedding of the closed curves $K = S_1 \cup \ldots \cup S_n$ in Σ.
- The neighbourhood $N(K)$ of K in Σ is the union of the neighbourhoods of the components of K, that is $N(K) = N(S_1) \cup \ldots \cup N(S_n)$

- The link exterior is defined as $\Sigma - int(N(K))$, where $N(K) = N(S_1) \cup \ldots \cup N(S_m)$ is a neighbourhood of K.
- The meridian and longitude of S_i denoted m_i and l_i are given by the relations $m_i \sim 0$, $l_i \sim S_i$ in $H_1(N(S_i))$ and $lk(m_i, S_i) = 1$, and $lk(l_i, S_i) = 0$.
- Then given the pair of links $\mathbf{L}$, and $\mathbf{L}'$ the splice of the two links, following *Eisenbud and Neumann* [3], is the following link $(\Sigma, (K' - S') \cup (K'' - S''))$, where $\Sigma = (\Sigma' - intN(S')) \cup (\Sigma'' - intN(S''))$, and $M' = L''$, $M'' = L'$. This is denoted as before $L \underset{S \qquad\qquad\quad S'}{\rule[0.5ex]{4cm}{0.4pt}} L'$

In the next section these links are considered together with an integer multiplicity associated by component, this form of link is called a multilink.

5.2 Multilinks

In this section I will give a definition of multilinks, following Eisenbud and Neumann, and then discuss when a multilink is defined on the splice of two links. It should be clear that the splice of two links over S' and S'' can always be taken, but for a splice of multilinks it is not so. That is if a splice of two links has been taken, and we place a multilink over it, this leads to a multilink on each of the splice components. But it is not true the other way round, that is given a splice and a pair of multilinks over the splice components, there may not be a splice multilink of the splice of the two component links that corresponds to the pair of multilinks.

Definition. Given the link $\mathbf{L} = (\Sigma, K)$ with m components, by a multilink $\mathbf{L}(n_1, \ldots, n_m)$ we mean the link together with an integer multiplicity n_i associated with each component S_i. We adopt the convention that S_i with multiplicity n_i means the same as $-S_i$ with $-n_i$ (where $-S_i$ is the component given the opposite orientation).

Then write $(\Sigma, n_1 S_1 \cup \ldots \cup n_m S_m)$ for the multilink $\mathbf{L}(n_1, \ldots n_m)$.

Given a splice $\mathbf{L}$ of two links $\mathbf{L}'$ and $\mathbf{L}''$, the multilink splice can only be defined between the two multilinks $\mathbf{L}'(\underline{n'})$ and $\mathbf{L}''(\underline{n''})$ when

$$\underline{n'}(L') = n'', \text{ and } \underline{n''}(L'') = n'$$

where $\underline{n}(S) = lk(n_1 S_1 + \ldots + n_m S_m, S)$, and L', L'' are the longitudes of S' and S'' respectively (by Eisenbud and Neumann). That is given the splice of the underlying links we may put a multilink on the splice components so that it extends to a multilink on the splice only if

$$\underline{n'}(L') = n'', \text{ and } \underline{n''}(L'') = n' \ .$$

Associated with a multilink $\mathbf{L}(\underline{n})$ we have a group homomorphism $h(\underline{n}) : \pi_1(\Sigma_0) \to C_\infty$ with $h(\underline{n})(x_i) = t^{n_i}$ in the infinite cyclic group C_∞.

A Seifert surface for a multilink is an oriented spanning surface of the link which has intersection with $N(S_i)$ consisting of $|m_i|$ leaves or if $m_i = 0$ it has, on that component, consistently oriented transverse intersection.

A fibered multilink is then a link which has a fibration of the link exterior to the circle, $\Sigma_0 \to S^1$, all of whose fibers are Seifert surfaces of the multilink. A Seifert surface has the following characterising properties;

(1) F is an oriented surface, properly embedded in $\Sigma_0 = \Sigma - int(\mathbf{L})$ (that is $F \cap \partial\Sigma_0 = \partial F$ transversally).

(2) $F \cap \partial N(S_i) = d_i S_i(p_i, q_i)$, a $d_i(p_i, q_i)$-cable on S_i, where d_i, p_i, q_i are determined by:gcd$(p_i, q_i) = 1$, $d_i p_i = n_i$, $d_i q_i = -\sum_{j \neq i} n_j lk(S_j, S - i)$.

The idea of a generalized exchangeable braid can now be formalised as a link which is a fibered multilink for the following sets of integer multiplicities $\underline{d_i} = (\delta_{1i}, \ldots, \delta_{ii}, \ldots, \delta_{mi})$ for i from 1 to m, where δ_{ij} is the usual Kronecker delta.

The following theorem is from Eisenbud and Neumann [3],

Theorem 5.2.1 Eisenbud and Neumann *The multilink $\mathbf{L}(\underline{m})$ is fibered if and only if it is irreducible and each of its multilink splice components are fibered.*

A corollary to this is the following

Corollary 5.2.1 *A link is a generalized exchangeable braid if and only if its splice components are also generalized exchangeable braids.*

Proof If we are given two generalized exchangeable braids

$$\mathbf{L}' = (\Sigma', S'_0 \cup S_1 \cup \ldots \cup S_k),$$

and,

$$\mathbf{L}'' = (\Sigma'', S''_0 \cup S_{k+1} \cup \ldots \cup S_{k+l}),$$

the multilink $\mathbf{L}'(1, 0, \ldots, 0)$ fibered multilink implying that $\mathbf{L}'(n, 0, \ldots, 0)$ is also a fibered multilink, for any n.

Which together with the link $\mathbf{L}''$ being a fibered multilink for the following integer multiplicities $(0, \delta_{(k+1)(k+j)}, \ldots, \delta_{(k+l)(k+j)})$, with j fixed, gives the splice

$$\mathbf{L} = \mathbf{L}' \xrightarrow{\hspace{3cm}} \mathbf{L}''$$

$$S_0 \qquad\qquad S_0'$$

fibered as a multilink with $(0, \ldots, 0, \delta_{(k+1)(k+j)}, \ldots, \delta_{(k+l)(k+j)})$ integer multiplicity. The other half of the required fibrations come by allowing the variation in the integer multiplicities of the first link and noting that the second link is also a $(n, 0, \ldots, 0)$ fibered multilink. Thus if $\mathbf{L}'$ and $\mathbf{L}''$ are exchangeable then so is the splice

$$\mathbf{L} = \mathbf{L}' \xrightarrow{\hspace{3cm}} \mathbf{L}''.$$

$$S_0 \qquad\qquad S_0'$$

If the splice, $\mathbf{L}$, of $\mathbf{L}'$ and $\mathbf{L}''$ is a generalized exchangeable braid then $\mathbf{L}$ is a $\underline{d_i} = (\delta_{1i}, \ldots, \delta_{ii}, \ldots, \delta_{(k+l)i})$ fibered multilink for i from 1 to $k+l$, where δ_{ij} is the usual Kronecker delta. Recall that if the splice of two links is a fibered multilink then the two splice components are fibered multilinks with the same values on each link component of the splice and two related values on the components along which the splice took place, this is because of the separating torus in the splice intersecting the fiber surfaces, see Eisenbud and Neumann for more details.

The splice components of $\mathbf{L}$, $\mathbf{L}'$ and $\mathbf{L}''$ are fibered multilinks for the following sets of integers $(n_0', \delta_{1i}, \ldots, \delta_{ki})$ and $(n_0'', \delta_{(k+1)i}, \ldots, \delta_{(k+l)i})$.

Recall that $\underline{n}(S) = lk(n_1S_1 + \ldots + n_mS_m, S)$, and $\underline{n'}(L') = n''$, and $\underline{n''}(L'') = n'$ where L', and L'' are the longitudes of S' and S'' respectively. Thus

$$\underline{n'}(L') = lk(\delta_{1i}{S'}_1 + \ldots + \delta_{ki}{S'}_k, L') = {n_0}''$$

and

$$\underline{n''}(L'') = lk(\delta_{(k+1)i}{S''}_{k+1} + \ldots + \delta_{(k+l)i}{S''}_{k+l}, L') = {n_0}'.$$

So we have that if $1 \leq i \leq k$ then ${n'}_0 = 0$ and if $k+1 \leq i \leq k+l$ then ${n''}_0 = 0$. We have by (2) above $F(\mathbf{L}'') \cap \partial N(S_0) = d_0S_0(p_0, q_0)$, $d_0q_0 = 0$ so as $\mathbf{gcd}(p_0, q_0) = 1$ we have $p_0 = 1$ and there are d_0 parallel copies of the surface but this fibration can be unified to give a $(1, 0, \ldots, 0)$ fibration, for $F(\mathbf{L}'')$.

To complete this part of the proof recall that the link $\mathbf{L}'$ was fibered with integer multiplicity $(n_0', \delta_{1i}, \ldots, \delta_{ki})$ for each i, but with $n_0' = 0$. Now the process is symmetric in splice components so as well as $\mathbf{L}'$ being a $(0, \ldots, \delta_{mi}, \ldots, 0)$ fibered multilink, and $\mathbf{L}''$ a $(1, 0, \ldots, 0)$ fibered multilink, $\mathbf{L}'$ is also a $(1, 0, \ldots, 0)$ fibered multilink and thus exchangeable. Equally $\mathbf{L}''$ is a $({n''}_0, \delta_{(k+1)i}, \ldots, \delta_{(k+l)i})$ fibered multilink for each i with $n_0'' = 0$, and so is also exchangeable. $\square$

The following uses the ideas of multi-fibration to good effect in the form of satellites. I will first give the construction of satellite knots and links.

In general, to construct a satellite of a link $L = L_1 \cup \ldots \cup L_r$ we need another link $C = C_1 \cup \ldots \cup C_k$, in which one unknotted component, C_k say, is selected. Then one component, L_1 say, of the link L is chosen. We then replace a solid torus neighbourhood V of L_1 by the solid torus $W = S^3 - C_k$, the complementary torus to the neighbourhood of C_k. This replacement is by a faithful homeomorphism $h : W \to V$ i.e. one which carries a longitude of W to a longitude of V. The satellite link thus formed consists of $h(C_1) \cup \ldots \cup h(C_{k-1}) \cup L_2 \cup \ldots \cup L_r$, and contains a *splitting torus*, $T = \partial V = h(\partial W)$.

Corollary 5.2.2 *Given a knot K in S^3 and an exchangeable tangle T, then $K*T$ is fibered if and only if K is fibered and $exch(T)$ is a braid (thus a Stallings braid), in particular, if T is a Stallings braid, T is exchangeable, if and only if $K*T$ is fibered.*

Proof First we assume $K*T$ is a (1)-fibered multilink, and we put $n = (lk(\widehat{T}, A))$. Thus K is an n-fibered multilink and $T \cup A$ is a $(1, 0)$ fibered multilink. The second of these two shows us that the axis is braided with respect to the closure of the tangle. The first together with (2) above shows that K's fibration has the following property $F_\theta(K) \cap \partial N(K) = dK(p, q)$ where $\gcd(p, q) = 1$, $dp = lk(\widehat{T}, A)$ and $dq = 0$. So we have the multifiber surface of K is $lk(\widehat{T}, A)$ parallel surfaces for each θ and thus K is fibered with $lk(\widehat{T}, A)$ parallel copies of its fiber surface at each level of the multifibration $F_\theta(K) = \bigcup F_{\theta,i}$ and so the surfaces $F_{\theta,i}$ form a fibration for K and so K is fibered. Conversely if K is fibered and $exch(T)$ is braided with respect to the closure of T, we have K is a (1)-fibered multilink, and $T \cup A$ is a $(1, 0)$-fibered multilink. Now with n as above, K is also a n-fibered multilink, so the splice can be extended to a multilink, that is $K*T$ is a 1-fibered multilink.

For the case that T is a Stallings braid, we have $exch(T)$ is a braid if and only if $K*T$ is fibered, but T is braided, so T is exchangeable if and only if $K*T$ is fibered. □

Thus if the tangle had been a Stallings braid and the knot a known fibered knot we would have that a Stallings braid is exchangeable if and only if the satellite around a fibered knot is fibered.

5.3 The Alexander polynomial of splices.

In this section I discuss Eisenbud and Neumann's work on the Alexander polynomial of the splice, and show that it can be used to give the Alexander polynomial of the satellite, and that it can induce the result of Fox and Torres on the link with a component excised.

Firstly I will introduce Eisenbud and Neumann's notation for the Alexander invariant. For a link $\mathbf{L} = (\Sigma, S_1 \cup \ldots \cup S_n)$, they define

$$\Delta_*(L; t_1, \ldots, t_n) = \begin{cases} \Delta(L, t_1, \ldots, t_n) & \text{if } m > 1 \\ (t_1 - 1)^{-1}\Delta(L, t_1) & \text{if } m = 1 \end{cases}$$

where $\Delta(L; t_1, \ldots, t_n)$ is the normal Alexander polynomial.

We then have the following

Theorem 5.3.1 Eisenbud and Neumann. *Let $\mathbf{L} = (\Sigma, S_1 \cup \ldots \cup S_n)$ be the result of splicing*

$$\mathbf{L}' = (\Sigma', S_0{}' \cup S_1 \cup \ldots \cup S_k),$$

and,

$$\mathbf{L}'' = (\Sigma'', S_0{}'' \cup S_{k+1} \cup \ldots \cup S_n)$$

along $S_0{}'$, $S_0{}''$ with $0 \leq k < n$. Let $b_i = lk(S_0{}', S_i)$ for $i = 1, \ldots, k$ and $a_j = lk(S_0{}'', S_j)$ for $j = k+1, \ldots, n$. Then, unless $k = a_1 = \ldots = a_n = 0$,

$$\Delta_*(L; t_1, \ldots, t_n) = \Delta_*(L'; T_0{}', t_1, \ldots, t_k).\Delta_*(L''; T_0{}'', t_{k+1}, \ldots, t_n),$$

with

$$T_0{}' = t_{k+1}{}^{a_{k+1}} \ldots t_n{}^{a_n},$$

and

$$T_0{}'' = t_1{}^{b_1} \ldots t_k{}^{b_k}.$$

If $k = a_1 = \ldots = a_n = 0$, then $\Delta_*(L) = \Delta_*(L_0{}'')$, where $L_0{}'' = L''$ with $S_0{}''$ deleted.

This theorem leads to the corollary

Corollary 5.3.1 *Let $\mathbf{L}'' = (\Sigma, S_0 \cup S_1 \cup \ldots \cup S_n)$, and $\mathbf{L} = (\Sigma, S_1 \cup \ldots \cup S_n)$ with $\mathbf{L}' = (S^3, A)$, where A is an unknot in S^3. Then*

$$(t_1{}^{l_1} \ldots t_n{}^{l_n} - 1)\Delta_*(L; t_1, \ldots, t_n) = \Delta_*(L''; 1, t_1, \ldots, t_n),$$

where $l_i = lk(S_0, S_i)$ in Σ.

Proof. If each of these linking numbers are zero then we have $0 = 0$ for the above equation, any other case is just a rewriting of the above theorem. $\square$

This is the result of Fox and Torres mentioned at the start of this section.

Chapter 6

Cyclic covers of fibered links

In this chapter I consider cyclic covers of spaces. We shall suppose that $A \cup B \subset \Sigma$ is a link with a distinguished component B. We may then construct the k-fold cover Σ^k/B of Σ branched over A, with the projection $\phi_k : (\Sigma^k/B) \to \Sigma$.

Write $B' = {\phi_k}^{-1}(B)$ for the inverse image of the branch set and $A^k/B = {\phi_k}^{-1}(A)$ for the inverse image of the rest of the link. Then $\phi_k \mid A^k/B \to L$ is a k-fold cover and $\phi_k \mid B' \to B$ is a homeomorphism.

The following proposition shows that the cyclic cover of an axis A over a closed braid B, A^k/B is fibered. The question the rest of the chapter deals with is, does the k-fold cyclic cover A^k/B fibered imply B is a braid with axis A.

Proposition 6.0.1 *Let $\widehat{\beta} \cup A$ be a closed braid with axis A in S^3, let Σ be the k-fold cover of S^3 branched over $\widehat{\beta}$, and K the k-fold cover of A branched over $\widehat{\beta}$. Then K is fibered in Σ.*

Proof Let D_t be a fibration of $S^3 - A$. Let p be the projection map from $\Sigma - K$ to $S^3 - A$, then $p^{-1}(D_t)$ is a fibration of $\Sigma - K$. $\square$

Here we have a tool that can show that the link $A \cup B$ in S^3 is not equivalent to $\widehat{\beta} \cup A$ in S^3, that is if the k-fold cyclic cover of an unknot branched over a knot is not fibered, then it is not a closed braid with axis. However this tool is not easy to use as it is not always easy to tell when a knot is fibered.

6.1 Some elements of groupoid theory.

In the k-fold cover of a space the lift of a loop may be a path with end points not equal, thus the fundamental group is of limited use. In this section I will use some elementary groupoid theory to replace the function of the fundamental group with the fundamental groupoid. The idea is to find a groupoid in the cover space which maps onto the fundamental group of the original space, and thus information on the cover space can be transferred to the original.

Let Σ be a path connected smooth oriented 3-manifold. A **path** in Σ from s_1 to s_2 is a continuous mapping from the interval, I, into Σ

$$p : I \to \Sigma$$

with $p(0) = s_1$ and $p(1) = s_2$.

Write $G(\Sigma, S)$ for the category whose objects are points in S and whose morphisms are paths in Σ with endpoints in S, where S is a non-empty subset of Σ, and write $G(\Sigma, S)(s_1, s_2)$ for the set of all morphisms (paths) from s_1 to s_2 in $G(\Sigma, S)$.

The **inverse** of $p \in G(\Sigma, S)(s_1, s_2)$ is the path $p^{-1} \in G(\Sigma, S)(s_2, s_1)$ defined by $p^{-1}(t) = p(1-t)$.

Write $p_1 \sim p_2$, when p_1, $p_2 \in G(\Sigma, S)(s_1, s_2)$ are homotopic, fixing their endpoints. Thus we have ${p_1}^{-1} \circ p_1 \sim id_{s_1}$ and $p_1 \circ {p_1}^{-1} \sim id_{s_2}$. Let us define the homotopy class of p, denoted by $[p] \in G(\Sigma, S)(s_1, s_2)$ as the subset $< p'$: such that $p' \in G(\Sigma, S)(s_1, s_2)$, and $p' \sim p >$.

Write $\pi_1(\Sigma, S)$ for the category whose objects are points in S and whose morphisms are homotopy classes of paths with endpoints in S. This is called the *fundamental groupoid* of Σ over S.

Conjunction of paths when defined, is defined as usual by splitting the interval into two parts giving

$$p_2 \circ p_1(t) = \begin{cases} p_1(2t) & \text{for } t \in [0, 0.5] \\ p_2(2t-1) & \text{for } t \in [0.5, 1] \end{cases}$$

Proposition 6.1.1 *Let Σ be a path connected space. If S contains finitely many points and $\pi_1(\Sigma)$ is finitely generated, then $\pi_1(\Sigma, S)$ is finitely generated.*

Proof Firstly number the points in S by $s_1, \ldots, s_n$. As Σ is path connected it is possible to define a set of paths $p_2, \ldots, p_n$, where $p_i \in G(\Sigma, S)(s_1, s_i)$ for each i. Consider a path $p \in G(\Sigma, S)(s_i, s_j)$, now $p \sim (p_j {p_j}^{-1}) p (p_i {p_i}^{-1}) \sim p_j ({p_j}^{-1} p p_i) {p_i}^{-1}$, but ${p_j}^{-1} p p_i$ is a loop in $\pi_1(\Sigma, s_1)$, so the generating set of the groupoid is the generating set of the group plus the set of paths above and is thus finitely generated. $\square$

The above result will be useful as the inverse image of a single point in a k-fold cyclic cover is either a single point or k points.

Let $\phi : \Sigma' \to \Sigma$ be a continuous onto projection with the additional property that any path in Σ has a path as a lift in Σ'. Let S be a subset of Σ as before, and let T be a subset of Σ' defined by $T =< t : t \in \Sigma' \ \phi(t) \in S >$. Then we have the following result.

Proposition 6.1.2 *With Σ', Σ, S, T and ϕ as above the map $\pi_1(\phi) : \pi_1(\Sigma', T) \to \pi_1(\Sigma, S)$ is onto.*

Proof Let $[p] \in \pi_1(\Sigma, S)$ be the homotopy class of p. By the path lifting property of ϕ, there exists a lift q in Σ. The end points of p are in S thus by definition of T the end points of q are in T. So $q \in G(\Sigma', T)$, and thus $[q] \in \pi_1(\Sigma', T)$ and $\pi_1([q]) = [p]$. □

6.2 Cyclic covers.

In this section I consider the question of k-fold cyclic covers and fibration. Recall that Σ is a smooth path connected 3-manifold. We shall suppose that $A \cup B \subset \Sigma$ is a link with a distinguished component B. We may then construct the k-fold cover M of Σ branched over A, with the projection $\phi_k : M \to \Sigma$.

Let $\Sigma^k/B = M$ be our notation for the k-fold cover of Σ branched over B.

Write $B_k = {\phi_k}^{-1}(B)$ for the inverse image of the branch set and $A^k/B = {\phi_k}^{-1}(A)$ for the inverse image of the rest of the link. Then $\phi_k \mid A^k/B \to L$ is a k-fold cover and $\phi_k \mid B_k \to B$ is a homoeomorphism.

Let (Σ, A) denote the embedding of the knot A into the space Σ. Let us denote by $(\Sigma, A)^k/B = (\Sigma^k/B, A^k/B)$, the k-fold cover of the pair.

Lemma 6.2.1 *With the above notation,*

$$(\Sigma, A)^{k_1 k_2}/B = (\Sigma^{k_1}/B, A^{k_1}/B)^{k_2}/B_{k_1} \ .$$

Proof Let $k = k_1 k_2$, choose a spanning surface for B, $F(B)$ along which Σ is cut in the construction of Σ^k/B then

$$(\Sigma, A)^{k_1 k_2}/B - F(B_{k_1 k_2}) = \bigcup_{j=1}^{k_1 k_2} ((\Sigma, A) - F(B)), = \bigcup_{i=1}^{k_2} \bigcup_{j=1}^{k_1} ((\Sigma, A) - F(B)),$$

which gives the required result when $F(B_{k_1 k_2})$ is replaced. □

The following notation is concerned with the infinite cyclic covers over the knot A in Σ and A^k/B in Σ^k/B.

Notation. Recall that Σ is a smooth path connected 3-manifold, with $A \cup B \subset \Sigma$ a link. We may construct the infinite cover $\overline{(\Sigma - A)}$ of Σ branched over A, with the projection $\pi_\infty : \overline{(\Sigma - A)} \to (\Sigma - A)$. We have the induced group homomorphism $h : \pi_1(\Sigma_0) \to C_\infty$ with $h(x_A) = t$ in the infinite cyclic group C_∞.

Write $\tilde{B} = {\pi_\infty}^{-1}(B)$ for the inverse image of the link component B in the infinite cover set.

Let $\overline{((\Sigma - B), A)} = \overline{(\Sigma - B) - A}$ be the infinite cyclic cover $\overline{(\Sigma - A)} - \tilde{B}$.
Let π be the projection of $\overline{(\Sigma, A)}$ on to (Σ, A)

Proposition 6.2.1 *With the above notation,*

$$\overline{(\Sigma^k/B, A^k/B)} = \overline{(\Sigma, A)}^k/\tilde{B}.$$

Proof Let $F(B)$ be a spanning surface for B in (Σ, A), and $F(A)$ a spanning surface for A. Now as ${\phi_k}^{-1}((\Sigma, A) - F(B)$ equals k disjoint copies of $((\Sigma, A) - F(B))$, and $\overline{((\Sigma - F(B)), A)} - \pi^{-1}(F(A))$ equals an infinite number of disjoint copies of $((\Sigma - F(B)), A) - F(A)$. The process of obtaining the infinite cyclic cover $\overline{(\Sigma^k/B, A^k/B)}$ is thus a gluing of

$$\bigcup_{i=1}^{k} \bigcup_{j=-\infty}^{\infty} (((\Sigma - F(B)), A) - F(A))$$

which equals

$$\bigcup_{i=1}^{k} \overline{((\Sigma - F(B)), A)} = \bigcup_{i=1}^{k} \overline{(\Sigma, A)} - \pi_{\infty}^{-1}(F(B)),$$

and so the proposition follows. □

Proposition 6.2.2 *With the notation above, if A^k/B is fibered in Σ^k/B then A is fibered in Σ.*

Proof Let $S = (s_1)$ be a single point subset of $\overline{(\Sigma, A)}$, and let $T = {\phi_k}^{-1}(s_1)$. We thus have, using the groupoid result proposition 6.1.2, that

$$\pi_1(\overline{\phi}) : \pi_1(\overline{((\Sigma, A)^k/B)}, T) \to \pi_1(\overline{(\Sigma, A)}, S)$$

is an onto morphism, and since A^k/B is fibered in Σ^k/B we know

$$\pi_1(\overline{((\Sigma, A)^k/B)})$$

is finitely generated, and T is a finite set of points. So by prop 6.1.1

$$\pi_1(\overline{((\Sigma, A)^k/B)}, T)$$

is finitely generated as a groupoid. Then $\pi_1(\overline{(\Sigma, A)}, S)$ is finitely generated and thus A is fibered in Σ. □

Proposition 6.2.3 *With the notation above, if B is braided in the fibration of A^k/B in Σ^k/B then B is braided in the fibration of A in Σ.*

Proof Let $S = (s_1)$ be a single point subset of $\overline{(\Sigma - B, A)}$, and let $T = {\phi_k}^{-1}(s_1)$. We thus have, using the groupoid result proposition 6.1.2,

$$\pi_1(\overline{\phi}) : \pi_1(\overline{((\Sigma - B, A)^k/B)}, T) \to \pi_1(\overline{(\Sigma_B, A)}, S)$$

is an onto morphism. Since B is braided relative to the fiberation of A^k/Bs inside Σ^k/B we know that

$$\pi_1(\overline{((\Sigma - B, A)^k/B)})$$

is finitely generated. Now T is a finite set of points, so by prop 6.1.1

$$\pi_1(\overline{((\Sigma - B, A)^k/B)}, T)$$

is finitely generated as a groupoid. Then $\pi_1(\overline{(\Sigma - B, A)}, S)$ is finitely generated and thus B is braided relative to the fibration of A inside Σ. □

We are now ready to begin to prove the main result of this section, the following.

Theorem 6.2.1 *If $F(A^k/B)$ covers $F(A)$ with both A^k/B and A fibered then B is braided with respect to A^k/B and thus to A.*

The proof will consist of the rest of this section, but first I must define a generalized tangle, with respect to a fibration. Let $(F \times I)$ be the empty *open*-tangle, and let $p : F \times I \to \Sigma$ be given by

$$p(x, 0) = p(h(x), 1)$$

for all $x \in F$, where $h : F \to F$ is a homeomorphism fixing ∂F.

$$p(y, t) = p(y, 0)$$

for all $y \in \partial F$ and for all t in I, and no other relations. Let $p(\partial F, 0) = A$ then $(\Sigma - A) = p(int(F) \times I)$ is the empty *closed*-tangle with axis. Define a tangle be a proper embedding of closed curves and arcs, T, into $(int(F) \times I)$, that is $p(\partial(T)) \in int(F) \times \partial(I)$, $p(int(T)) \in int(F) \times int(I)$, and $p(x) = p(y)$ implies $x = y$. For such a system to close to $\Sigma - (A \cup L)$ where A is the axis mentioned above, and L is some link in the complement of A, we need $n = m$ and the positions of the end points must be such that there are no loose ends in the closure.

Let $S_0 = (x \in F \times (0)\ st.\ x \in T)$, and let $S_1 = (x \in F \times (1)\ st.\ x \in T)$, then a tangle T is closable if $n = m$ and $h(S_0) = S_1$. If a tangle $T(F, h)$ is closable let $T\widehat{(F, h)}$ be its closure in $S^3 - A$.

Lemma 6.2.2 *Given a fibered link A to act as generalised axis to the link $\widehat{T_A}$ then the link $\widehat{T_A}$ is a closed braid with respect to A if and only if there exists a fiber surface $F(A)$ such that the open tangle T_A is an open braid.*

Proof If $T(\widehat{F,h})$ is a closed braid then the multilink definition any $(1,0)$-fiber surface has the required property. Conversely if $T(F,h)$ is an open braid there exists a fibration for which $I(F_i(A), T_A) = n$ for $i \in [0,1]$. Use h to glue the fibration together to form a closed braid, since the boundary surfaces are unmoved. □

It is now useful to consider a theorem of Meeks and Scott [7]

Theorem (Meeks and Scott) 1 *If F is a compact surface not S^2 or P^2 and if G is a finite group acting smoothly on $F \times I$ so as to preserve $F \times \partial I$, then the action of G is conjugate to an action which preserves the product structure.*

We are now in a position to prove theorem 6.2.1.

Proof of theorem 6.2.1 The link $A^k/B \cup B$ is a closed braid with axis in the manifold M if and only if B is an open braid in $M - F(A^k/B)$. Now in this space we have a finite group action $G =< t : t^k = 1 >$ but the surface has boundaries so it is necessary to adjoin some disc cross the interval to the boundary components. So the group action G on $F \times I$ extends to an action on $F \times I \cup D_i \times I$ where the action on the D_i is defined by the action on the boundary components of F. Then by the Meeks and Scott theorem above the action G is conjugate to an action preserving the product structure, i.e. $\phi t \phi^{-1} = \tau \times id$. Now by the nature of surfaces there exists a map s which can restore the discs to there original position i.e. $s\tau s^{-1}(D_i) = id(D_i)$ so $(s \times id)\phi(s^{-1} \times id)$ is an isotopy of $(F \cup D_i) \times I$ fixing $D_i \times I$. So now we have $s \times id\phi$ is an isotopy of $F \times I$ which takes the fixed point set of the G action to points cross the interval, this is the condition for a set of curves to form an open braid in the fibration, but the fixed point set of the action is just B. And so B is braided with respect to the fibration of A^k/B, and thus to the fibration of A. □

6.3 How this affects Stallings braids.

In this section I will show that if the k-fold cover of a Stallings braid is fibered, then it is an exchangeable braid.

Lemma 6.3.1 *If the k-fold cover of a Stallings braid is fibered and k_1 is a prime divisor of k, then the k_1-fold cover of the Stallings braid is fibered.*

Proof Firstly we have $(S^3)^k/A = S^3$, and so $(S^3, \widehat{\beta})^k/A = (S^3, \widehat{\beta^{k_1}})^{k/k_1}/A$, so by prop 6.2.2 we have $\widehat{\beta^{k_1}}$ is fibered. $\square$

The following is a result from number theory.

Lemma 6.3.2 *Let $p(x) \in \mathbf{Z}[x]$, $p(1) \neq 0$, let k_1 be a prime number, and ζ a k_1-th primitive root of unity. Then if $p(\zeta^i) = 0$ for any i then k_1 divides $p(1)$.*

Proof If $p(\zeta^i) = 0$, then $p(\zeta^j) = 0$ for $j = 1 \ldots (k_1 - 1)$ and thus $p(x) = (1 + \ldots + x^{k_1-1})q(x)$ for some $q(x) \in \mathbf{Z}(x)$, and so $p(1) = k_1 q(1)$. $\square$

Corollary 6.3.1 *A Stallings braid, $\beta \in B_n$, is exchangeable if and only if the closure of its k-th power, $\widehat{\beta^k}$, is fibered for any $k \geq 2$.*

Proof By theorem 6.2.1 if $\widehat{\beta^k}$ is fibered, and if the fiber surface $F(\widehat{\beta^k}) = \phi^{-1}(F(\widehat{\beta}))$, then the axis A is braided with respect to $\widehat{\beta}$. So by lemma 6.3.1, it is enough to consider the case when k is prime. Now the Alexander polynomial of $\widehat{\beta^k} \cup A$ is given by the following,

$$\Delta_{\widehat{\beta^k} \cup A}(x, t^k) = \prod_{i=1}^{k} \Delta_{\widehat{\beta} \cup A}(x, \zeta^i t)$$

where ζ is a k-th primitive root of unity. Torres shows in [13] that

$$\Delta_{\widehat{\beta^k} \cup A}(x, 1) = (x^n - 1)/(x-1)\Delta_{\widehat{\beta^k}}(x)$$

when $\widehat{\beta^k}$ is a knot, and

$$\Delta_{\widehat{\beta^k} \cup A}(x, 1) = (x^n - 1)\Delta_{\widehat{\beta^k}}(x)$$

whenever $\widehat{\beta^k}$ is a link of two or more components. Now the spread of x in $\Delta_{\widehat{\beta} \cup A}(x,t) = p_M(t)x^M + \ldots + p_m(t)x^m$ is equal to $n-1$ so if spread in $\Delta_{\widehat{\beta} \cup A}(x, \zeta^i t)$ is less than $n-1$, one of $p_M(\zeta^i) = 0$ or $p_m(\zeta^i) = 0$. By the lemma 6.3.2 this means one of $p_M(1)$ or $p_m(1)$ would not be equal to plus or minus one. But we know that both $p_M(1) = \pm 1$ and $p_m(1) = \pm 1$, since $\Delta_{\widehat{\beta} \cup A}(x, 1) = 1 + \ldots + x^k$.

Now

$$\Delta_*(\widehat{\beta^k}; x) = \Delta_*(\widehat{\beta}; x) \prod_{i=1}^{k-1} \Delta_*((A \cup \widehat{\beta}); \zeta^i, x),$$

by Proposition 3.2.1. Let $\Delta(x,y) = p_m(y)x^m + \ldots + p_M(y)x^M$, with lemma 6.2.3 applied to p_m and p_M.

Then in the case when $\widehat{\beta^k}$ is a knot and by the spread in x in the Alexander polynomial is $(k-1)(n-1)$. As $\Delta_*(\widehat{\beta}; x) = (x^n - 1)/(x-1)^{-1}$, and $\Delta_*(\widehat{\beta^k}; x) =$

$(x-1)^{-1}\Delta_{\widehat{\beta^k}}(x)$. So the rank of the first homology group of the fiber surface of $\widehat{\beta^k}$ is $(k-1)(n-1)$. Now we have $rk(H_1(D^k/A)) = (k-1)(n-1)$ so D^k/A is a minimal genus surface, and thus a fiber surface of $\partial(D^k/A)$.

When $\widehat{\beta^k}$ is a link of two or more components, we have $\Delta_*(\widehat{\beta};x) = (x^n - 1)/(x-1)^{-1}$, and $\Delta_*(\widehat{\beta^k};x) = \Delta_{\widehat{\beta^k}}(x)$. So the spread in x in the Alexander polynomial is $(n-1)(k-1)-1$ and thus in a link of two or more components we have the rank of the first homology group of the fiber surface of $\widehat{\beta^k}$ is $(k-1)(n-1)$. But we know that the rank of the first homology group of D^k/A is $(k-1)(n-1)$, and since the link is fibered this is a fiber surface.

Thus if $\widehat{\beta^k}$ is fibered then $\widehat{\beta^{k'}}$ is fibered where k' is a prime divisor of k. Hence by theorem 6.2.1 if $\widehat{\beta^k}$ is fibered then $\widehat{\beta}\cup A$ are an exchangeable braid pair. □

Remark. Let $\widehat{\beta}\cup A$, and $\widehat{\beta'}\cup A$ be a pair of links and axes, and let $\widehat{\beta}$ differ from $\widehat{\beta'}$ by the addition of a string, and a band $\sigma_{i(n+1)}$, a positive Markov move. Using the above notation let $K = A^2/\widehat{\beta}$, and $K' = A^2/\widehat{\beta'}$. In [8] Morton and Montesinos show that the double cover of the axis branched over two braids that differ only by a positive Markov move, adding a band differ by the plumbing of a Hopf band. That is K' is K with a Hopf band plumbed on.
In particular we have

Corollary 6.3.2 *A Stallings braid, β, is exchangeable if and only if $\widehat{\beta^2}$ is fibered as the plumbing of Hopf bands on a disc.*

Proof. The last corollary shows us that $\widehat{\beta^2}$ is fibered if and only if β is exchangeable, but if β is exchangeable then $\widehat{\beta^2} \simeq A^2/Exch(\beta)$ implies that $\widehat{\beta^2}$ is a plumbing of Hopf bands on a disc. □

6.4 Generalized Murasugi sum.

A generalized Murasugi sum is a form of plumbing applied to triples, (Σ, F, A).

Definition. I shall say that the triple (Σ, F, A) is the *Murasugi sum* of the two triples (Σ_1, F_1, A_1) and (Σ_2, F_2, A_2) if

1. The homology 3-sphere Σ decomposes as $\Sigma = \Sigma_1 \#_{S^2} \Sigma_2$
2. F_1 and F_2 are subsurfaces of F meeting only in a disc $D_0 = F_1 \cap F_2$ lying on the separating sphere, S^2, of Σ, such that F_i is in Σ_i, and $F = F_1 \#_{D_0} F_2$.
3. The generalized axis A meets the separating sphere S^2 in only two points, $d \in D_0$ and $c \notin D_0$. Thus A factors as $A = A_1 \# A_2$, with A_1 in Σ_1 and A_2 in Σ_2.

For the above definition to be the Murasugi sum of braids, we have the further property that $\partial(F_i)$ is braided relative to A_i in Σ_i.

Theorem 4.1.1 has the following corollary,

Corollary 6.4.1 *The Murasugi sum of two generalized braids with generalized axes is a generalized exchangeable braid if and only if the Murasugi summands are exchangeable.*

Proof If each of the Murasugi summands are exchangeable then the k-fold cover of the Murasugi sum is the plumbing of the k-fold covers of the summands and thus is fibered, this surface covers the surface of the Murasugi sum and thus is exchangeable. Conversely if the Murasugi sum is exchangeable then the k-fold cover of the braid is fibered. Now this braid is a Murasugi sum, and so the k-fold cover is a plumbing of the k-fold covers of the summands and these are therefore fibered. These k-fold cover fiber surfaces are the inverse images of the Murasugi summand surfaces. $\square$

Chapter 7

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