Fibred knots with a given Alexander polynomial

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1. Introduction. In this paper I shall show that, given any fibred knot k in S^3 of genus > 1, there are infinitely many fibred knots with the same Alexander polynomial as k. These knots all have the same Seifert form, (not necessarily that of k).

The study of fibred knots was given its impetus by the work of Neuwirth and Stallings, [N1], [N2], [S1], where the geometric condition that the complement of a knot $k \subset S^3$ should fibre over S^1 was shown to be equivalent to an algebraic condition on the group $\pi_1(S^3 - k)$. This condition places a restriction on the Alexander polynomial of a fibred knot, beyond the general restrictions on the Alexander polynomial of any knot. Burde, [B], constructed a sequence of fibred knots realising all possible Alexander polynomials, subject only to this one extra restriction.

Apart from Burde's examples, links of algebraic plane curve singularities, and certain alternating knots, [Mu2], there remained a very limited known repertoire of fibred knots. Indeed, if a fibred knot has Alexander polynomial 1, $1 - t + t^2$ or $1 - 3t + t^2$, (the only possibilities of degree $\leqslant 2$), then the knot must be trivial, the trefoil or the figure-eight respectively. This is in sharp contrast to the general case, where infinitely many knots with Alexander polynomial 1, and hence any other Alexander polynomial, can be found.

A brief survey of the known examples led Burde and Neuwirth to make the appealing conjecture, [N3], that there were only finitely many fibred knots with a given Alexander polynomial. This conjecture crumbled when I showed, [M], that an infinite sequence of satellites, each fibred and with the same Alexander polynomial and Seifert form, could be found for any fibred knot.

Quach Cam Van attacked the problem further, [QW] , [Q] , with a series of examples giving infinitely many fibred knots for each possible Alexander polynomial with a repeated root, although in her examples the Seifert forms vary.

The present result, Theorem 4, demolishes the remains of the conjecture by constructing an infinite sequence of fibred knots for each possible polynomial, except the three listed of degree $\leqslant 2$, for which only a single fibred knot can be found.

The constructions start from Burde's examples. Since most of the proofs showing that given knots are fibred rely on the more or less algebraic techniques developed by Stallings, [S2] , I feel that it is worth redressing the balance, and providing in the next two sections a geometric view of the fibrations which arise naturally in many constructions of new fibred knots from old. A knot which is built up in a suitable way from fibred pieces will then be shown to be fibred by a fairly explicit exhibition of the fibres and the monodromy. Burde's examples will be seen in this context using a redrawing of them which Paul Melvin and I developed in the course of trying to understand their fibration geometrically, (fig. 2). From this redrawing it is also easy to calculate the Conway - Kauffman version of their Alexander polynomial. As in Burde's own calculations, the number of twists prescribed in various places relates readily to the coefficients of this polynomial, and a brief description is given in the final section.

The main theorem is presented in section 4. The construction used, twisting s times about a suitable unknotted curve lying in a fibre, is another of the known devices for generating new fibred knots from old, [H] , [S2] . It remains, then, to find such curves in Burde's examples, check the effect on the Alexander polynomial, and finally to distinguish between the resulting knots for infinitely many different choices of s .

The Seifert form will not be at hand in this case to distinguish the knots, but use of Thurston's work on hyperbolic structures allows me to show that either the knots themselves, or some companion, have a hyperbolic structure on their complement, with finite volume which varies with s . Since this volume is a topological invariant, the proof will be complete.

2. Fibred knots and plumbing

An oriented knot or link $k \subset S^3$ is said to be <u>fibred with fibre F</u> if $F \overset{i}{\subset} S^3$ is an orientable surface, with $k = \partial F$, and there is an 'open-book decomposition' of S^3 with binding k and leaf F .

Such a decomposition is equivalent to a fibration of the exterior of k (the complement of an open tubular neighbourhood of k in S^3) over S^1 with fibre F . It can best be described by a continuous surjective map $p : F \times [0,2\pi] \to S^3$ such that

(i) $p(x,t) = p(x,0) = x$ for each $x \in k$,

(ii) $p(x,2\pi) = p(h(x),0)$ for each $x \in F$, where $h : F \to F$ is a homeomorphism, called the <u>monodromy</u>,

(iii) no further identifications are made by p .

At each level t the map $p \mid F \times \{t\}$ gives a homeomorphism from F to a leaf F_t , with $F_0 = F_{2\pi} = F$, and otherwise two leaves meet only in their common boundary k .

We can view p as an isotopy in S^3 from i to $i \circ h$, which is 'strictly monotone', rel ∂F, in a sense to be defined. Notice that the path $p(\{f\} \times [0,2\pi])$, for $f \in F$ is either a single point, when $f \in \partial F$, or else an arc or a circle traversed strictly monotonically. As t increases, the surface F_t moves off itself in the direction of the positive normal.

I propose to show that, under suitable conditions, an isotopy p from i to $i \circ h$ in which certain parts of F_t remain fixed for an interval as t increases will still be enough to guarantee that F is a fibre for $k = \partial F$ with monodromy h. Such an isotopy, called monotone, rel ∂F, can be pictured as giving leaves which are stuck together in places, like badly cooked puff-pastry. These leaves must be separated to see the fibration.

<u>Definitions</u>. Let $p_0, p_1 : F \to Y$ be embeddings, and let $p : F \times I \to Y$ be an isotopy from p_0 to p_1 which is constant on a subset $A \subset F$. Then p is (a) <u>monotone rel A</u> or (b) <u>strictly monotone rel A</u> if it satisfies the conditions listed below :

(a) if $p(x,s) = p(y,t)$ and $s < t$ then either $s = 0$ and $t = 1$, or $x = y$ and $p(x,r) = p(x,s)$ for all $r \in [s,t]$.

(b) if $p(x,s) = p(y,t)$ and $s < t$ then either $s = 0$ and $t = 1$, or $x = y \in A$.

In such an isotopy the paths traced out by different points of F are disjoint, except possibly at their endpoints, and each path is traversed (a) monotonically, (b) strictly monotonically except for points in A.

<u>Lemma 1</u> Let F and Y be compact manifolds, and let $p : F \times I \to Y$ be a piecewise-differentiable isotopy from p_0 to p_1 which is monotone. Then there is an isotopy p' from p_0 to p_1 which has the same image as p, and is strictly monotone rel A, the subset on which p is constant.

<u>Corollary</u> Let $F \overset{i}{\subset} S^3$ be a compact surface with boundary k, and let $h : F \to F$ be a homeomorphism. Suppose that there is a p.d. monotone isotopy from i to $i \circ h$ which is constant precisely

on k , and has image S^3. Then $S^3 - k$ is fibred over S^1 with monodromy h .

Proof The strictly monotone isotopy which can be constructed using lemma 1 gives an appropriate map from $F \times I$ onto S^3.

Proof of Lemma 1 The map $\rho : F \times I \longrightarrow \mathbb{R}$ given by $\rho(f,t)$ = length of the curve $p(\{f\} \times [0,t]) = \int_0^t \left\| \frac{\partial p(f,t)}{\partial t} \right\| dt$ is well-defined and continuous. Then $\theta : F \times I \longrightarrow F \times \mathbb{R}$, given by $\theta(f,t) = (f, \rho(f,t))$ is continuous, with image X say. This map θ is an identification map, since F is compact, and p then factors through X as $p = \alpha \circ \theta$.

Since p is monotone, the continuous map $\alpha : X \longrightarrow Y$ is injective, except possibly on the image in X of $F \times \{0\} \cup F \times \{1\}$. Now let $\theta' : F \times I \longrightarrow X$ be the straight-line homotopy from θ_0 to θ_1. This map provides the same family of paths in X as θ, but the paths are reparametrised to eliminate unnecessary pauses, for θ' is injective, except on $A \times I$, where $\rho = 0$. The image of θ' is again X , and $p' = \alpha \circ \theta'$ then gives a strictly monotone isotopy, rel A, from p_0 to p_1, with the same image as p .

In many cases of the plumbing construction which follows a monotone isotopy can be seen fairly readily, hence a fibration and its monodromy. A similar idea underlies Thurston's construction of fibrations, [T1] , in which an embedded surface 'pops across' a sequence of tetrahedra, building up naturally a monotone isotopy.

Plumbing This construction starts from knots k_1 , k_2 spanned by oriented surfaces F_1 , F_2 and gives, with some further choices, a new knot k and spanning surface F .

Choose F_1 to lie entirely in the upper hemisphere $D_1 \subset S^3$, and meet $S^2 = \partial D_1$ in a closed 2r-gon G_0, in which r alternate edges form part of $k_1 = \partial F_1$. Choose F_2 similarly to lie entirely in the lower hemisphere D_2, also meeting S^2 in G_0, with the remaining r edges of G_0 lying in $k_2 = \partial F_2$. Suppose that the orientations of F_1, F_2 are consistent on G_0, with the positive normal pointing into D_1. Then $F = F_1 \cup F_2$ is an oriented surface, crossing S^2 with an r - fold monkey-saddle, (see figure 1), given by <u>plumbing F_1 and F_2 along G_0</u>.

A surface F may be seen to decompose in this way if, for example, there is one part of it, F_1, which in some projection apparently overlays the rest, apart from a common disc. In the case $r = 1$ the knot $k = \partial F$ is just the connected sum of k_1 and k_2, but generally the knot k will depend on the choice of disc G_0 in F_1 and F_2 as well as on k_1 and k_2.

Figure 2 shows a typical knot and spanning surface from Burde's examples, [B]. The surface is made of ribbons, each having a number of full twists as indicated, and can be built up by successive plumbing of a once-twisted annulus, g-1 copies of the surface L_s, (figure 3), for various s, and one further twisted annulus. One stage of the plumbing decomposition is illustrated in figure 4, where the shaded portion forms L_{c_2}, lying in the lower hemisphere, the cross-hatched part forms the polygon G_0 of overlap, lying in the equatorial level, while the rest of the surface lies in the upper hemisphere. The complete surface in figure 2 is built up inductively in this way.

Murasugi, [Mu1], [Mu2], used the plumbing construction, with a special choice of F_2, in his work on alternating knots. It also features in Stallings' work on fibred knots, [St2], [H]. The

The present description results from an independent investigation by myself and Paul Melvin in trying to picture geometrically why Burde's knots are fibred.

Murasugi needed firstly a result which is true for general plumbings.

__Theorem 1__ If F_1 and F_2 are plumbed to give F then a basis for $H_1(F)$ can be chosen so that F has Seifert matrix

$$A = \begin{pmatrix} A_1 & C \\ 0 & A_2 \end{pmatrix} \text{ where } A_1 , A_2 \text{ are Seifert matrices}$$

for F_1 , F_2 .

__Proof__ The basis required for $H_1(F)$ is simply the union of any chosen bases for $H_1(F_1)$ and $H_1(F_2)$.

Now if $\det A \neq 0$ then the constant term $\Delta_k(0)$ in the Alexander polynomial of $k = \partial F$ is equal to $\det A$, up to sign. In this case $|\Delta_k(0)| = |\Delta_{k_1}(0).\Delta_{k_2}(0)|$. Murasugi was able to show that the Seifert matrices for the natural plumbing components which come from an alternating knot diagram are non-singular, so that $\Delta_k(0)$ can be calculated in this way.

More striking is the second result.

__Theorem 2__ Let F_1 and F_2 be plumbed to give F . Suppose that F_1 , F_2 are both fibre surfaces, with monodromy h_1 , h_2 respectively. Then

(a) F is a fibre surface for $k = \partial F$,

(b) this fibration has monodromy $\overline{h}_2 \circ \overline{h}_1$, where $\overline{h}_i : F \longrightarrow F$ is the extension of h_i over F by the identity on $F - F_i$.

__Corollary__ To prove that Burde's knots are fibred it is enough to show that each L_s is a fibre surface for the link $K_s = \partial L_s$, since it is easily seen from the Hopf fibration that a once-twisted annulus is a fibre surface for its boundary, the Hopf link.

Theorem 2(a) was proved by Murasugi for his special cases, and by Stallings in general, using algebraic methods. Harer, [H] , proves theorem 2(b) for a special choice of F_2 .

Proof of Theorem 2 Construct a piecewise differentiable monotone isotopy, rel ∂F , from $i : F \subset S^3$ to $i \circ \overline{h}_2 \circ \overline{h}_1$, with image S^3 . The result then follows from the corollary to lemma 1.

Write H_0 for the complementary disc to G_0 in the equatorial sphere S^2 , and select a homeomorphism $j_0 : G_0 \longrightarrow H_0$, fixed on the boundary. Extend this by the identity over the rest of F to $j : F \longrightarrow F' = (F - G_0) \cup H_0$, and write j_1 , j_2 for the restriction of j to F_1, F_2 respectively.

When G_0 and H_0 are identified by j_0 the upper hemisphere D_1 becomes S^3 . In this S^3 the fibration with fibre F_1 and monodromy h_1 provides a strictly monotone isotopy, p_1 , in which $F_1 \times \{0\}$ and $F_1 \times \{1\}$ both have image F_1 . We can undo the identification by j_0 , to make p_1 yield a strictly monotone isotopy p_1' in D_1 . The image of $F_1 \times \{0\}$ under p_1' is still F_1 , but the image of $F_1 \times \{1\}$ is $F_1' = (F_1 - G_0) \cup H_0$; H_0 here plays the role of the negative side of G_0 in S^3 . The isotopy p_1' runs from $i_1 : F_1 \subset D_1$ to $j_1 \circ h_1$. We can assume that p_1' is piecewise differentiable. Extend it by the identity on the rest of F to a monotone isotopy, $\overline{p_1'}$, from $i : F \subset S^3$ to $j \circ \overline{h}_1 : F \longrightarrow F'$, with image $D_1 \cup F_2$.

This can be thought of as a half-way stage in the monodromy for F , with the image F' lying just to the negative side of F_1 and just to the positive side of F_2 . The isotopy then continues by marking time on $F_1 - G_0$ and sweeping round through D_2 from H_0 and the positive side of F_2 to G_0 and the negative side of F_2 .

As above, identify G_0 with H_0 and use the fibration coming from F_2 in D_2 to get a strictly monotone isotopy from $j_2 : F_2 \longrightarrow F_2'$ to $i_2 \circ h_2$. This is in keeping with the positive direction on F_2. Again, this extends by the identity over the rest of F to give a monotone isotopy from j to $i \circ \overline{h}_2$, and hence one from $j \circ \overline{h}_1$ to $i \circ \overline{h}_2 \circ \overline{h}_1$, with image $D_2 \cup F_1$. The composite of this isotopy and $\overline{p}_1'$ above gives an isotopy from i to $i \circ \overline{h}_2 \circ \overline{h}_1$, which is monotone and fixes only ∂F.

3. Twisting

To complete the proof that Burde's knots are fibred, and to construct the promised sequence of examples, I shall use the 'twist construction', one other major technique in building examples of fibred knots, [H], [St2].

Twist construction   Choose an unknotted curve c in a fibre surface F, whose neighbourhood in F is an untwisted ribbon. Twist s times about c, so that the knot $k = \partial F$ becomes the curve k_s.

Theorem 3   (a) k_s is fibred for each s,

(b) the monodromy h_s of k_s is given by

$$h_s = (\tau_c)^s \, h ,$$

where h is the monodromy of k, and τ_c is a Dehn twist about c.

Proof   To describe the twist, construct a small cylinder $c \times I$, with $c \times \{0\} = c$, meeting neighbouring fibres to F transversely in the curves $c \times \{t\}$. This forms part of the boundary of a solid cylinder $D \times I$, where D is a disc spanning c. The disc $D \times \{1\}$ will meet F again, and indeed k itself, unless c bounds a disc in F.

Cut S^3 open along $(c \times I) \cup (D \times \{1\})$ and twist $D \times I$ through

$2\pi st$ at level t before gluing back. Since $D \times \{1\}$ is glued back by the identity we get the curve k_s in place of k, spanned by a surface F_s homeomorphic to F. The cutting and gluing determines a 'shearing' of S^3 along $c \times I$, and a homeomorphism on its complement. Any of the original fibres which meet $c \times I$ still give a genuine surface after this shear, as the cut edges at $c \times I$ simply slide some way around one of the curves $c \times \{t\}$. The foliation of the complement of k_s by such spanning surfaces then ensures that k_s is fibred.

The homeomorphism on the complement of $c \times I$ which carries k to k_s can be used to describe the complement of k_s in terms of the complement of k. Think of a neighbourhood of $c \times I$ as a solid torus, with core c. Then $S^3 - k_s$ is homeomorphic to $S^3 - k$ with the solid torus V round c removed, and replaced by a solid torus V_s, whose meridian disc spans a curve running s times longitudinally and once meridianally around the boundary of V. The torus V_s, enclosing $c \times I$, can be viewed as the support for an isotopy of F_s which contributes $(\tau_c)^s$ to the monodromy of k_s. The monodromy of k_s is completed by composing this isotopy with the isotopy for k having monodromy h.

<u>Corollary</u> The surface L_s in figure 4 is a fibre surface for $K_s = \partial L_s$ for each s.

<u>Proof</u> L_s is given from L_0 by twisting s times about the boundary curve c shown in figure 4. This curve c, or one parallel to it in $\operatorname{int} L_0$, satisfies the construction requirements, consequently L_s is fibred for all s if and only if it is fibred for one s. Now L_1 is the connected sum of two once-twisted annuli, along the arc a in figure 5. Connected sum is a special case of plumbing, so L_1, and hence each L_s, is a fibre surface.

Note It is important to remember that in order to refer to a link, such as K_s , as fibred, the orientation on each component must be specified. It is possible to change orientation on some components of a fibred link to arrive at a non-fibred link.

4. Main Theorem

Theorem 4 For each of Burde's knots with genus $g > 1$ there are infinitely many fibred knots having the same Alexander polynomial and Seifert form.

Corollary Burde's knots are known to realise all possible Alexander polynomials of fibred knots, (see Theorem 5). Hence only the trefoil, figure-eight and trivial knot are determined among fibred knots by by their polynomial alone.

Proof Given one of Burde's knots k , I shall construct a sequence of knots, k_s , such that :

1. each k_s is fibred,
2. each k_s has the same Alexander polynomial and Seifert form as $k = k_0$,
3. infinitely many knots in the sequence are different.

0. Construction Let F be a fibre surface for k , with genus > 1 . select an unknotted curve c in F which separates F but does not bound a disc in F . A neighbourhood of c in F is then untwisted, since one edge of the ribbon bounds in the complement of the other.

It is possible to find such a curve c by first cutting F along the separating arc b shown in figure 2. One of the resulting components is a torus with one hole, whose boundary is unknotted. Choose c parallel to the boundary curve in this torus.

Twist s times about c to give the knot k_s .

1. Fibration By Theorem 3(a) each k_s is fibred, with fibre F_s, say.

2. Similarities The monodromy h_s induces an automorphism on $H_1(F_s)$, represented by $H_s \in GL(2g,Z)$. The Alexander polynomial of k_s is the characteristic polynomial of H_s.

We have $H_s = (T_c)^s H$, where T_c, H represent the automorphisms induced by τ_c, h respectively, relative to a suitable choice of basis, from Theorem 3(b).

For the homology class $\langle d \rangle \in H_1(F)$ represented by a closed curve d we have

$$\langle \tau_c(d) \rangle = \langle d \rangle + w \langle c \rangle ,$$

where w is the algebraic intersection number of c and d.

In this case $\langle c \rangle = 0$, so $\langle \tau_c(d) \rangle = \langle d \rangle$ for all $\langle d \rangle \in H_1(F)$. Thus $T_c = I$, and $H_s = H$, for all s. The Alexander polynomial and Seifert form of k_s are then independent of s.

3. Differences I shall show that some subsequence, at least, of the knots k_s are all different, by considering the volume, as a hyperbolic manifold, of the complement either of k_s or of some suitable companion.

The exterior of k_s, i.e. the complement of an open tubular neighbourhood of k_s, is given from the exterior of the link $k \cup c$ by attaching a solid torus, V_s, to the boundary torus C of the tubular neighbourhood of c so that the meridian of V_s is attached to (meridian + $s \times$ longitude) on C. Thus ext k_s arises from ext $(k \cup c)$ by Dehn surgery on C.

(a) If the complement of $k \cup c$ has a hyperbolic structure, with volume v, then, for all but finitely many s, so does the complement of k_s, with volume $v_s < v$. Under these conditions $\sup v_s = v$, so there is some strictly increasing subsequence of volumes. The corresponding sequence of knots k_s are then all different, since the volume v_s is a topological invariant. See Thurston, [T1, chapters 5 & 6] or [T2], for these results.

(b) The manifold ext $(k \cup c)$ is irreducible, since c does not bound a disc in ext k. Hence we can look at Johannson's decomposition of it into characteristic Seifert fibre space pieces and other atoroidal pieces. See [J], or the description of the same decomposition due to Jaco and Shalen, [JS].

Let N be the component in this decomposition which contains the boundary torus C of the tubular neighbourhood of c. Write the components of N as $C \cup T \cup S_1 \cup \dots \cup S_r$, where T separates c from k. The other components S_i, if any, are tori each leaving $c \cup k$ to one side and bounding a knot exterior on the other. N forms part of a 'splice decomposition' for ext $(k \cup c)$ in the sense of Neumann, [N], as indicated in figure 6.

If N is not a Seifert space, then its interior has a hyperbolic structure, [T3]. As in the special case (a), where $N = \text{ext}(k \cup c)$, the manifolds $N_s = N \cup V_s$ will be non-homeomorphic for infinitely many s. The Johannson decomposition of ext k_s will be similar to that of ext $(k \cup c)$, with N replaced by N_s, where N_s is hyperbolic.

Excluding at most those s for which N_s is homeomorphic to a Johannson component of ext $(k \cup c)$ leaves infinitely many ext k_s with different Johannson decompositions. Since a Johannson decomposition is preserved up to isotopy by a homeomorphism, it follows that there are infinitely many inequivalent knots k_s .

(c) The proof of Theorem 4 is completed by showing that N can not be a Seifert space.

Lemma 2 There is a curve t_k in the torus T with linking number $lk(t_k, k) \neq 0$.

Lemma 3 Each curve a in $\partial N - C$ has $lk(a,c) = 0$.

Corollary N is not a Seifert space.

Proof Suppose that N is a Seifert space, with boundary components $C_1, \ldots, C_k$, $k \geqslant 2$. Let $p : N \longrightarrow B$ be the fibre projection, where B is a surface with boundary curves $p(C_1), \ldots, p(C_k)$. The subgroup $V \subset H_1(B;Q)$ generated by $p(C_1), \ldots, p(C_k)$ has dimension $k-1$, as a vector space. The regular fibre generates $\ker p_*$, and each component C_i contains a regular fibre. Put $U = i_* H_1(C_2 \cup \ldots \cup C_k ; Q)$. Then $(p_*)^{-1} V = U$, by a dimension count, and so $i_* H_1(C_1) \subset U$.

Take $C = C_1$ in the case above, and consider the inclusion $j : N \subset S^3 - C$. Since all curves in $\partial N - C$ have zero linking number with c we have $j_*(U) = 0$, and thus $j_* i_* H_1(C) = 0$. This is impossible, since a meridian in C generates $H_1(S^3 - C) \neq 0$.

The corollary can also be proved by regarding N as homeomorphic to the exterior of a link with r+2 components, $\bar{c}, \bar{t}, \bar{s}_1, \ldots, \bar{s}_r$, corresponding to the boundary tori. The linking number of $\bar{c}$ with each of the others is 0 , by Lemma 3.

Now those links whose exteriors are Seifert fibre spaces can be listed quite briefly, [N] , [Sw] . In particular, if the linking number of one pair of components is zero then the linking number of any two regular fibres is zero, and the link is a 'necklace', figure 7. N can not then be a Seifert fibre space, since each component in a necklace has linking number ± 1 with at least one other.

<u>Proof of Lemma 2</u> Complete ext $(k \cup c)$ to S^3 by adding a solid torus with core k_s along the boundary of the tubular neighbourhood of k, and the solid torus V_s along C. Choose an embedded curve t_k in T which bounds on the side of T containing k_s. Then $lk(t_k, k_s) = lk(t_k, k)$ is independent of s, being calculated from the intersection of the surface which spans t_k avoiding V_s, and the core k_s. It is then enough to show that $lk(t_k, k_s) \neq 0$ for <u>some</u> s.

If T is incompressible in $S^3 - k_s$ then $\pi_1(T)$ is a subgroup of $\pi_1(S^3 - k_s)$, and its image in $H_1(S^3 - k_s) \cong Z$ is generated by $lk(t_k, k_s)$. If $lk(t_k, k_s) = 0$ then $\pi_1(T)$ lies in the commutator subgroup of $\pi_1(S^3 - k_s)$. This is impossible for a fibred knot k_s, since the commutator subgroup is free, while $\pi_1(T)$ is free abelian of rank 2. (More geometrically, T could be placed transverse to the fibration, as in [N], and the two sides analysed).

It remains to show that s can be found with T incompressible in $S^3 - k_s$.

(i) If N is not Seifert fibred we can choose s as above so that N_s has a hyperbolic structure, and forms part of a Johannson decomposition for ext k_s. Then T is one of the characteristic tori in this decomposition, and is incompressible.

For Theorem 4 we only need Lemma 2 in the next case.

(ii) If N is Seifert fibred then so is N_s, except when the meridian discs of V_s are glued along fibres in C. For some s in any case the space N_s will be Seifert fibred, and not a solid torus, unless $N \cong T^2 \times I$, which does not occur in a Johannson decomposition. For such s each boundary component of N_s is incompressible in N_s , and hence in $S^3 - k_s$.

<u>Proof of Lemma 3</u> The torus S_i must be compressible in $S^3 - c$, since c is unknotted. However it is incompressible on the side of S_i away from c . Hence there is a compression disc in N spanning an unknotted curve a_i in S_i . Now $H_1(S_i)$ can be generated by a_i and some curve b_i which bounds on the side away from c , and we have $lk(a_i,c) = lk(b_i,c) = 0$. It follows that $lk(c_i,c) = 0$ for every curve c_i in S_i .

For the torus T choose embedded curves t_k , as above, and t_c , bounding on the side in S^3 which contains k and c respectively, and so generating $H_1(T)$. The curve t_c generates the homology of the side of T which contains k , so k is homologous, on this side of T , to $lk(t_k,k) \times t_c$. Hence $lk(k,c) = lk(t_k,k) . lk(t_c,c)$.

Now c bounds in the complement of k by construction, so $lk(k,c) = 0$. From Lemma 2 $lk(t_c,c) = 0$. We already have $lk(t_k,c) = 0$ by construction. Thus $lk(t,c) = 0$ for every curve t in T.

5. <u>Remarks</u>

1. It is possible that an analysis of the 'stretch factor' of the monodromy h_s as $s \to \infty$ would give an alternative method for distinguishing enough of the knots k_s . Bonahon, [Bo] , has used this technique in looking at an explicit sequence of genus 2 knots. Some investigation would be needed to recognise and deal with cases where h_s is not pseudo-Anosov. In the other cases there is some hope that the stretch factor is a concave function of s . This would

give the stronger result that among the knots k_s for positive and negative s there are at worst pairwise repetitions.

2. The techniques actually used can be extended, with a closer analysis of the case where N is a Seifert fibre space, which must now be considered, to show that if c is any unknotted curve in the complement of a curve k, other than a meridian of k or spanning a disc disjoint from k, then infinitely many of the knots k_s given by twisting about c are different.

3. The calculation of the Alexander polynomial of Burde's knots given in B is most naturally related to the Conway-Kauffman version of this polynomial. I shall finish with a brief description of this version, and a direct calculation for Burde's knots.

Following Kauffman, [K], start with an orientable spanning surface M for a given oriented knot or link L, and construct a Seifert matrix A in the usual way, using linking numbers of curves pushed off M. Put

$$f(x,x^{-1}) = \det(xA - x^{-1}A^T),$$

and rewrite f as a polynomial $\nabla_L(z)$ in $z = x - x^{-1}$, which is possible by virtue of the symmetry of f in x and x^{-1}.

For a knot the Alexander polynomial is recovered, up to a power of t, by putting $x^{-1} = t^{\frac{1}{2}}$, and thus $z = t^{-\frac{1}{2}}(1 - t)$ in $\nabla_L(z)$. Indeed, for a knot $\nabla_L(z)$ is a polynomial in z^2, and has constant term 1, since $\det(A - A^T) = 1 = \nabla_L(0)$. In addition, for a <u>fibred</u> knot of genus g the leading term is $\pm z^{2g}$.

It is shown in [K] that $\nabla_L(z)$ depends only on L, given an orientation convention for linking numbers in S^3. The following lemmas often enable calculations to be made easily from knot diagrams.

Lemma 4 Let three links L_+, L_- and L_0 have knot diagrams identical except in the neighbourhood of one crossing, where, as shown in figure 8, the positive crossing in L_+ is cut out in L_0 and replaced by a negative crossing in L_-. Then

$$\nabla_{L_+}(z) = \nabla_{L_-}(z) + z\,\nabla_{L_0}(z)\,.$$

Lemma 5 $\nabla_L(z) = 0$ for a split link L.

It is then easy to establish the 'ring on a band' lemma.

Lemma 6 If L and K have diagrams identical except as shown in figure 9 then $\nabla_L(z) = -z^2\,\nabla_K(z)$.

Theorem 5 The knot $K(c_1,\dots,c_g)$ in figure 10 has Kauffman polynomial

$$\nabla(z) = 1 + \sum_{i=1}^{g} (-1)^{i-1} c_i\, z^{2i}\,.$$

Proof By induction on g and c_g, using the fact that $K(c_1,\dots,c_{g-1},0) = K(c_1,\dots,c_{g-1})$. The induction starts with $g = 0$, when the knot is trivial and $\nabla(z) = 1$.

Assume that $c_g > 0$. Alter one of the positive crossings in the g th ribbon as for Lemma 4, to get links $L_+ = K(c_1,\dots,c_g)$, $L_- = K(c_1,\dots,c_g - 1)$ and $L_0 = L(c_1,\dots,c_{g-1})$.

The link L_0, shown in figure 11, has a ring R on a band. Cut the band and lose the ring, as for Lemma 6, to get $L(c_1,\dots,c_{g-2})$. After cutting $g-1$ bands in this way we reach the positive Hopf link, with polynomial z, so, by Lemma 6 applied $g-1$ times,

$$\nabla_{L_0}(z) = (-z^2)^{g-1}\,z\,.$$

Then $$\nabla_{L_+}(z) = \nabla_{L_-}(z) + (-1)^{g-1} z^{2g} .$$

By the induction hypothesis we have $\nabla_{L_-}(z)$, so the theorem follows.

A similar calculation works for the case $c_g < 0$.

Corollary Burde's knots, $K(c_1,\ldots,c_{g-1},\pm 1)$, realise all Kauffman knot polynomials with leading coefficient ± 1 .

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References

[Bo] BONAHON, F. Ribbon fibered knots, cobordism of surface diffeomorphisms and pseudo-Anosov diffeomorphisms.
Preprint, 1982, Université de Paris-Sud, Orsay

[B] BURDE, G. Alexanderpolynome Neuwirtscher Knoten.
Topology, 5, (1966) 321-330.

[H] HARER, J. How to construct all fibred knots and links.
Topology, 21, (1982) 263-280.

[JS] JACO, W.H. and SHALEN, P.B. Seifert fibered spaces in 3-manifolds.
Memoirs Amer. Math. Soc. 220, (1979).

[J] JOHANNSON, K. Homotopy equivalence of 3-manifolds with boundaries.
Lecture Notes in Mathematics, 761, Springer-Verlag, Berlin, Heidelberg, New York, 1979.

[K] KAUFFMAN, L. The Conway polynomial.
Topology, 20, (1981) 101-108.

[M] MORTON, H.R. Infinitely many fibred knots having the same Alexander polynomial.
Topology, 17, (1978) 101-104.

[Mu1] MURASUGI, K. On the genus of the alternating knot, II.
J. Math. Soc. Japan, 10, (1958) 235-248.

[Mu2] MURASUGI, K. On a certain subgroup of the group of an alternating link.
Amer. J. Math. 85, (1963) 544-550.

[N] NEUMANN, W. These Proceedings.

[N1] NEUWIRTH, L.P. The algebraic determination of the genus of knots. Amer. J. Math. 82, (1960) 791-798.

[N2] NEUWIRTH, L.P. On Stallings fibrations. Proc. Amer. Math. Soc. 14, (1963) 380-381.

[N3] NEUWIRTH, L.P. The status of some problems related to knot groups. Topology conference VPISU, 1973, Lecture notes in Mathematics, 375, Springer-Verlag, Berlin, Heidelberg, New York, 1974, 209-230.

[Q] QUACH CAM VAN Invariants des noeuds classiques fibrés. Thesis. University of Geneva, 1981.

[QW] QUACH CAM VAN and WEBER, C. Une famille infinie de noeuds fibrés cobordants à zéro et ayant meme polynôme d'Alexander. Comm. Math. Helv. 54, (1979) 562-566.

[St1] STALLINGS, J. On fibering certain 3-manifolds. Topology of 3-manifolds and related topics, ed. M.K.Fort Jr. Prentice Hall, 1962, 95-100.

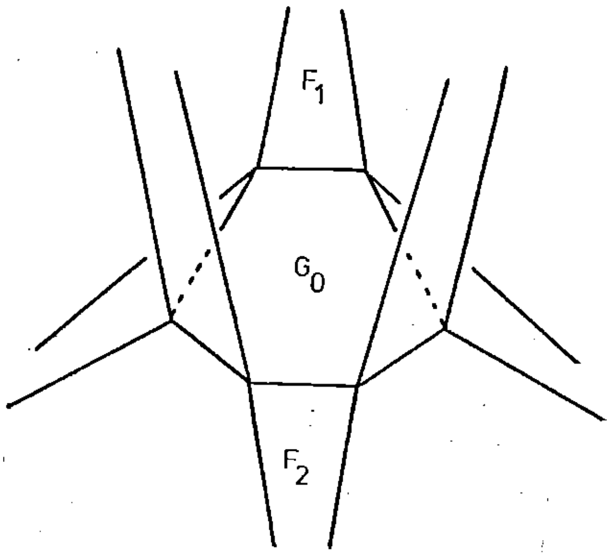
[St2] STALLINGS, J. Constructions of fibered knots and links. Proc. Symposium in Pure Math. of the Amer. Math. Soc. 32 part 2, (1978) 55-60.

[Sw] SWARUP, G.A. Cable knots in homotopy 3-spheres. Quart. J. Math. Oxford, (2), 31, (1980) 97-104.

[T1] THURSTON, W. The geometry and topology of 3-manifolds. To appear, Princeton University Press.

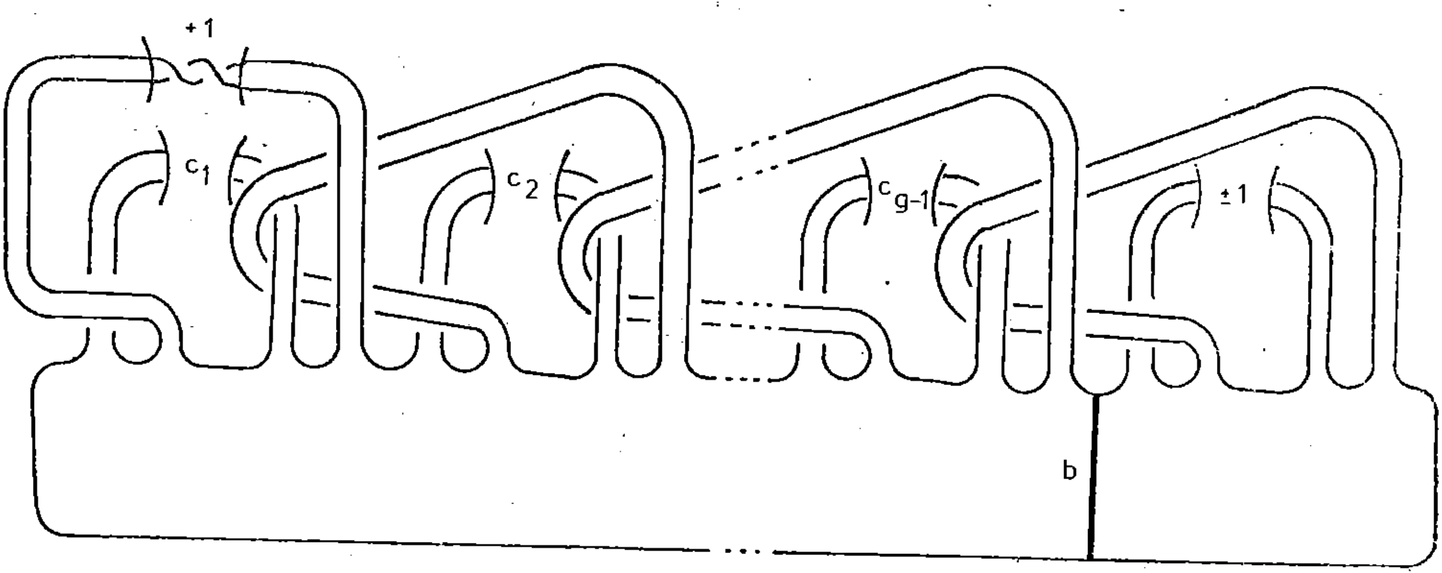
[T2] THURSTON, W. Three dimensional manifolds, Kleinian groups and geometry. Bull. Amer. Math. Soc. (New Series), 6, (1982) 357-381.

[T3] THURSTON, W. Hyperbolic structures on 3-manifolds, I: deformations of acylindrical manifolds. Preprint.



A 3-fold monkey saddle

Figure 1



Burde's fibred knots

Figure 2

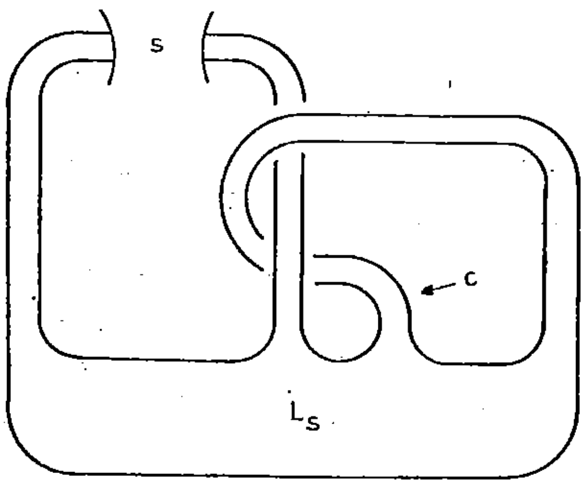


Figure 3

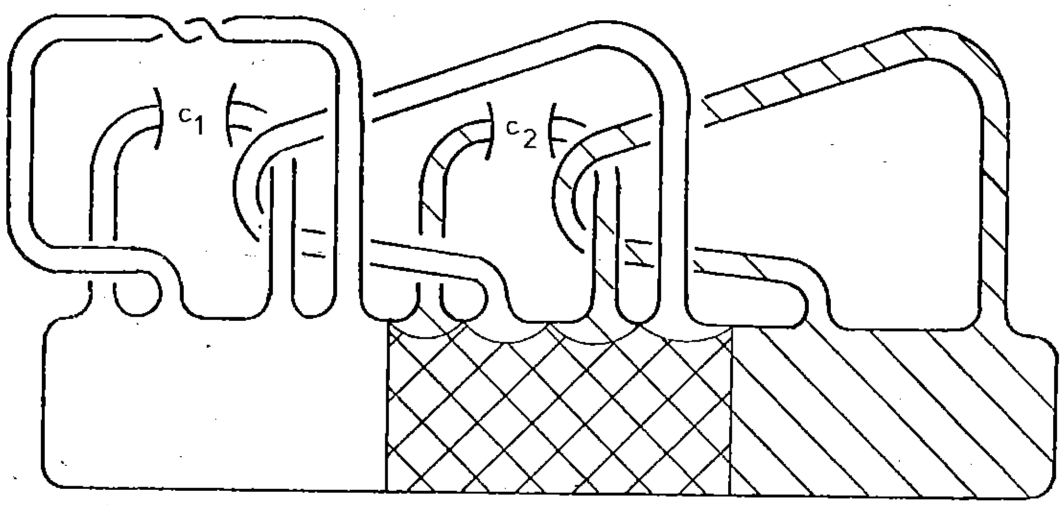


Figure 4

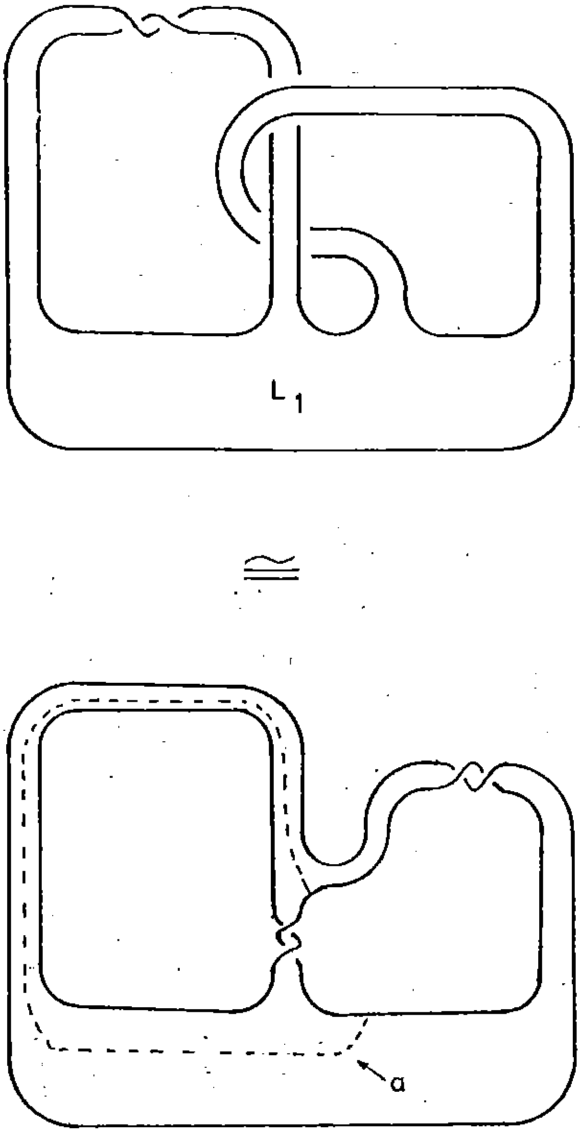
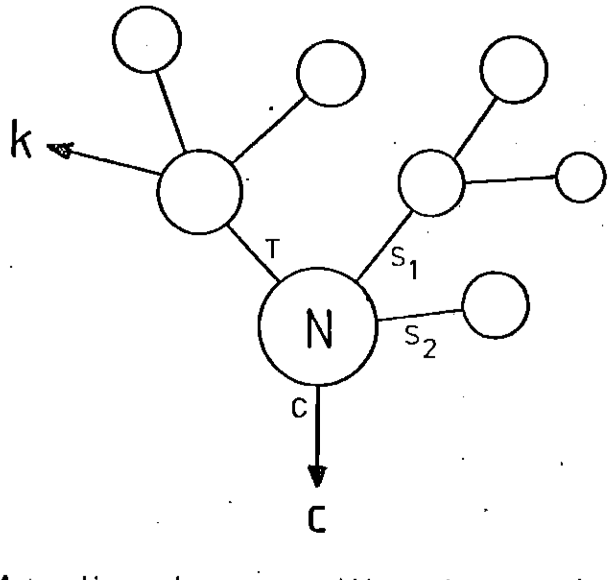
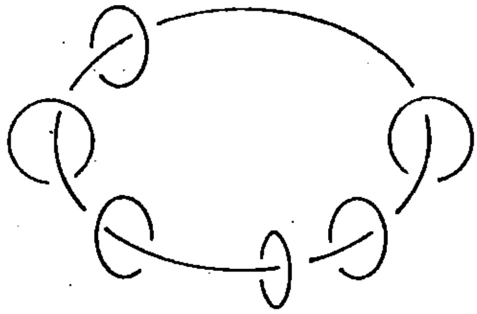


Figure 5



A splice decomposition for $ext(k \cup c)$

Figure 6



A necklace

Figure 7

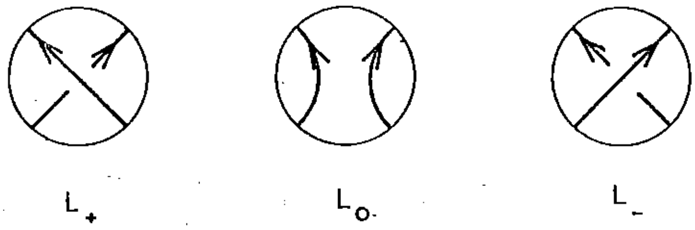


Figure 8

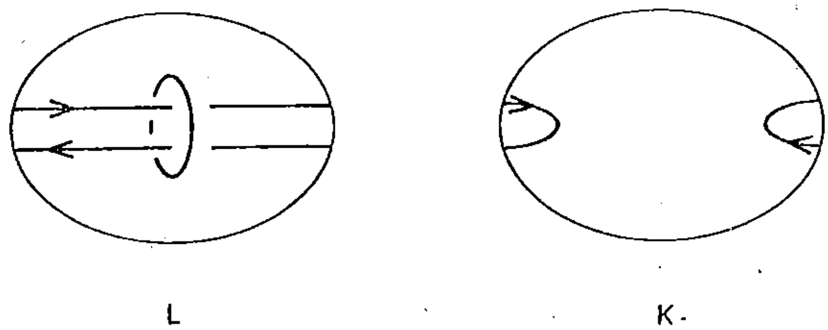
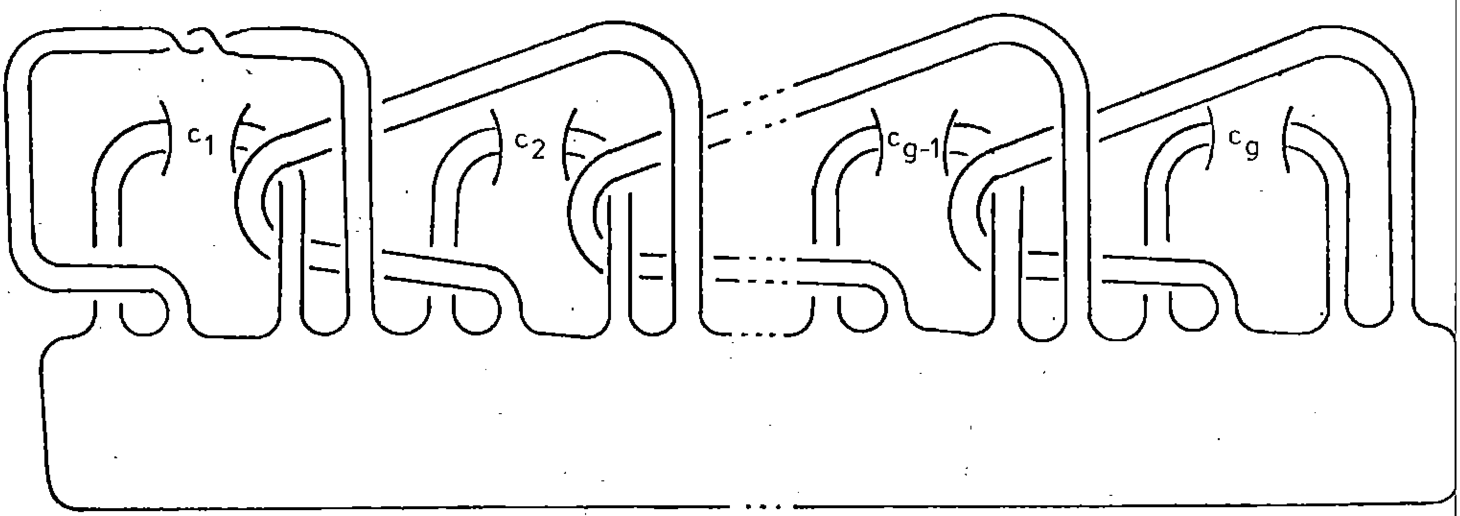
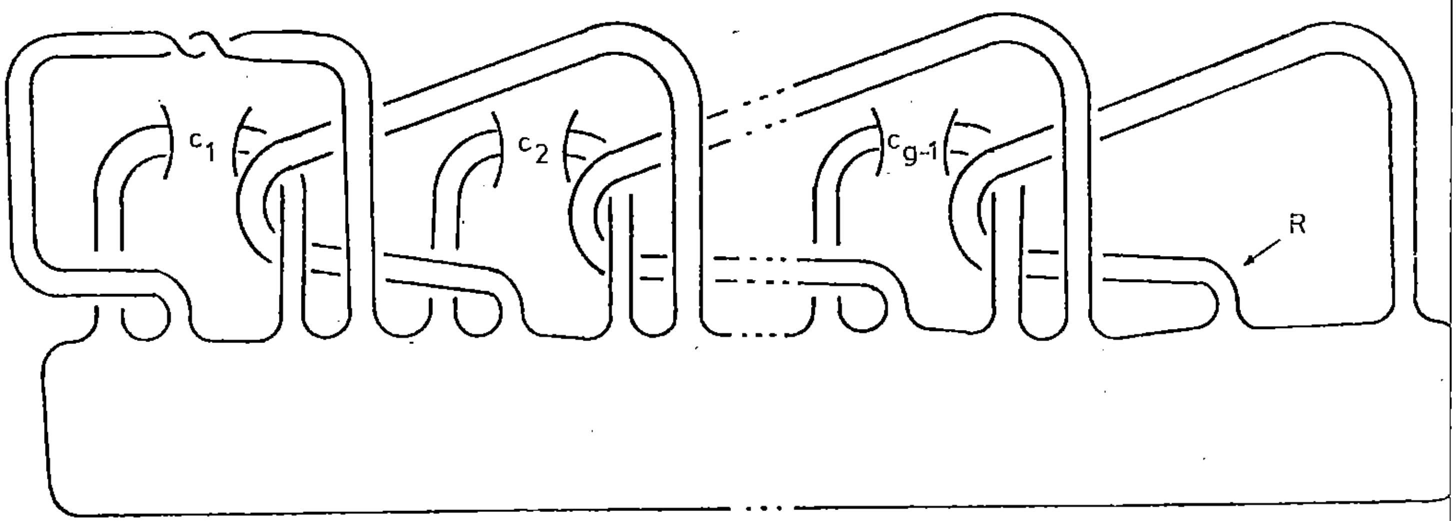


Figure 9



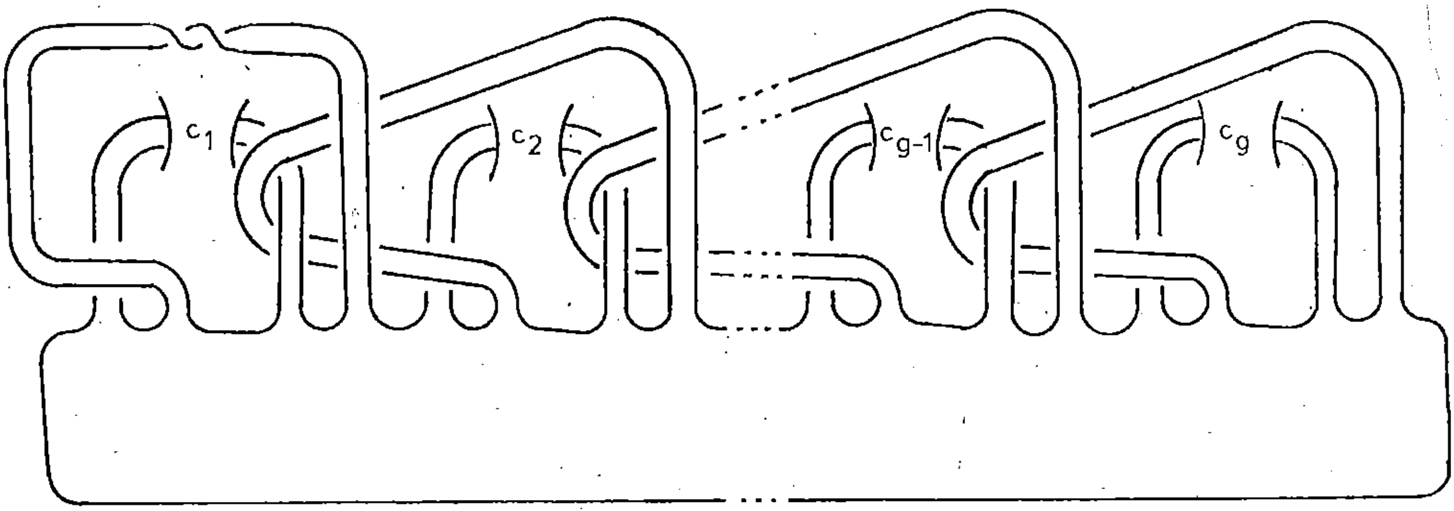
The knot $K(c_1, c_2, \ldots, c_g)$

Figure 10



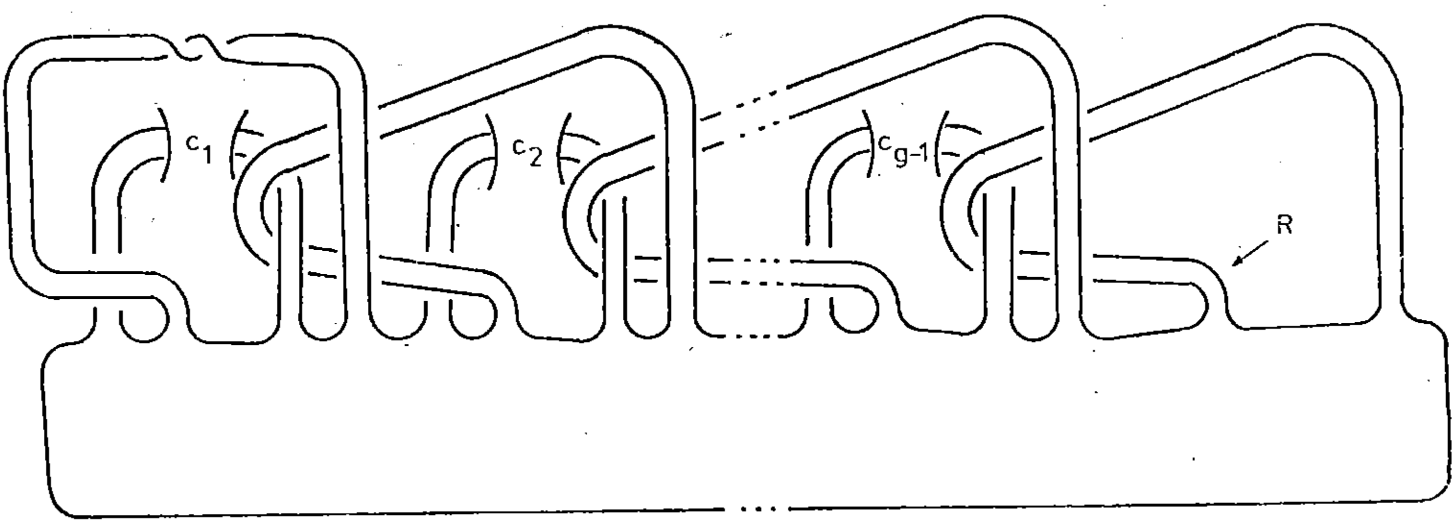
The link $L(c_1, c_2, \ldots, c_{g-1})$

Figure 11



The knot $K(c_1, c_2, \ldots, c_g)$

Figure 10



The link $L(c_1, c_2, \ldots, c_{g-1})$

Figure 11