STABILITY CONDITIONS FOR LINE BUNDLES ON NODAL CURVES

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ABSTRACT. We introduce the abstract notion of a *smoothable fine compactified Jacobian* of a nodal curve, and of a family of nodal curves whose general element is smooth. Then we introduce the notion of a combinatorial stability condition for line bundles and their degenerations.

We prove that smoothable fine compactified Jacobians are in bijection with these stability conditions.

We then turn our attention to *fine compactified universal Jacobians*, that is, fine compactified Jacobians for the moduli space $\overline{\mathcal{M}}_g$ of stable curves (without marked points). We prove that every fine compactified universal Jacobian is isomorphic to the one first constructed by Caporaso, Pandharipande and Simpson in the nineties. In particular, without marked points, there exists no fine compactified universal Jacobian unless $\gcd(d+1-g, 2g-2) = 1$.

CONTENTS

1. Introduction

A classical construction from the XIX century associates with every smooth projective curve X its Jacobian (the moduli space of degree 0 line bundles on X), a principally polarized abelian variety of dimension g. The construction carries on to smooth projective families of curves. One challenging problem arises when X ceases to be smooth. In this case the Jacobian can still be constructed, but in general it fails to be proper. A general problem from the mid XX century was to construct well-behaved compactifications of the Jacobian, whose boundary corresponds to degenerate line bundles of some kind.

Many different constructions have been pursued according to the particular generality required and the initial inputs (see for example [Igu56], [OS79], [AK80], [Cap94], [Sim94], [Pan96], [Est01]); some of these work in the relative case of families as well.

For simplicity here we restrict ourselves to the case where X is a nodal curve. Also, we will fix the horizon of all possible degenerations of line bundles to *torsion-free coherent sheaves of rank* 1. Since we will aim to construct *proper* moduli stacks of stable sheaves, without losing in generality we will additionally assume that all sheaves are *simple*. In this generality, the moduli space of sheaves was constructed as an algebraic space by Altman–Kleiman [AK80]. Esteves [Est01] later proved it satisfies the existence part of the valuative criterion of properness. (This moduli space is not of finite type, hence it is not proper, whenever X is reducible).

Most modular constructions of fine compactified Jacobians use some set of instructions (for example coming from GIT) to single out an open subset of the moduli space of simple sheaves choosing certain *stable* elements, to end up with a proper moduli stack. The construction is often followed by the observation that stability of a sheaf only depends on its multidegree and on its locally free locus in X, and then that these discrete data obey a collection of axioms (for example, the number of stable multidegrees of line bundles on a nodal curve X equals the complexity of the dual graph of X).

In this paper we introduce an abstract notion of a *fine compactified Jacobian* as an *open* subspace of the moduli space of rank 1 torsion-free simple sheaves of some fixed degree (not necessarily zero) on X, which is furthermore *proper* (see Definition 3.1).[1] It was observed in [PT22, Section 3] that fine compactified Jacobians can be badly behaved in the sense that they can fail to fit into a family for an infinitesimal smoothing of the curve. (This phenomenon already occurs when X has genus 1). Thus, we add a smoothability axiom to the objects that we aim to study. Note that our definition of smoothable fine compactified Jacobian includes

[1] The adjective "fine" classically refers to the existence of a Poincaré sheaf.

the modular fine compactified Jacobians constructed in the literature (e.g. those constructed by Esteves [Est01] and by Oda–Seshadri [OS79] and recently studied in [MV12a], [MRV17]) .

We then prove our first classification result, stating that smoothable fine compactified Jacobians correspond to a combinatorial datum that we call a *stability condition* (for smoothable fine compactified Jacobians), which keeps track of the multidegree and of the regular locus of the elements of the moduli space:

Theorem 1.1. *Let X be a nodal curve. Taking the associated assignment (see Definition 7.1) induces a bijection*

$$\left\{ \begin{array}{c} \textit{Smoothable fine compactified} \\ \textit{Jacobians of } X \end{array} \right\} \to \left\{ \begin{array}{c} \textit{Stability conditions for } X \textit{ as} \\ \textit{introduced in Definition 4.4} \end{array} \right\}$$

whose inverse is defined by taking the moduli space of sheaves that are stable with respect to a given stability condition (Definition 6.1).

This is Corollary 7.13 (in the case where S is a DVR and $\mathcal{X}/S$ is a regular smoothing of X). The most difficult part is the proof of properness of the moduli space of stable sheaves (Lemma 6.5).

In fact, Corollary 7.13 is an extension of Theorem 1.1 to the case of smoothable fine compactified Jacobians of *families* of nodal curves whose generic element is smooth (Definition 3.7). The combinatorial notion of a stability condition for a *family* is introduced in Definition 4.4, as the datum of a stability condition for each fiber, with an additional constraint of compatibility under the degenerations that occur in the family (which induce morphisms of the corresponding dual graphs).

A natural question is whether our abstract definition of fine compactified Jacobians produces new examples. The most general procedure to construct smoothable fine compactified Jacobians that we are aware of is by means of numerical polarizations. This was introduced by Oda–Seshadri [OS79] for the case of a single nodal curve, then further developed by Kass–Pagani [KP17], [KP19] for the case of the universal family over the moduli space of pointed stable curves (equivalent objects were constructed by Melo in [Mel19] following Esteves [Est01]). These definitions and constructions are reviewed in Section 8.

By Theorem 1.1, it is a completely combinatorial (but hard) question whether every smoothable fine compactified Jacobian is given by a numerical polarization. (Note that "smoothable" here is essential, due to the aforementioned genus 1 examples in [PT22, Section 3]. Those examples are not smoothable, whereas all compactified Jacobians obtained from numerical polarizations are smoothable). The case when the genus of X equals 1 was settled in the affirmative in [PT22, Proposition 3.15], and in Example 8.9 we discuss how to extend this to the case when the first Betti number of the dual graph of X equals 1. In Example 8.10 we discuss the numerical polarization that induces integral break divisors (slightly generalizing the analogous result by Christ–Payne–Shen [CPS] for the case when X is stable).

We resolve in the positive the similar question for the case of the universal curve over $\overline{\mathcal{M}}_g$. Without marked points, fine compactified universal Jacobians are all given by universal numerical polarizations:

Theorem 1.2. *Let* $\overline{\mathcal{J}}_g \to \overline{\mathcal{M}}_g$ *be a degree* d *fine compactified universal Jacobian. Then* $\gcd(d-g+1, 2g-2) = 1$ *and there exists a universal numerical polarization* Φ *such that* $\overline{\mathcal{J}}_g = \overline{\mathcal{J}}_g(\Phi)$ *(as defined in Section 8).*

This follows from Corollary 9.10. In particular, as we observe in Remark 9.11, without marked points, there are no more fine compactified universal Jacobians than the (essentialy equivalent) ones constructed in the nineties by Caporaso [Cap94], Pandharipande [Pan96] and Simpson [Sim94].

A similar result does not hold in the presence of marked points: in [PT22] the authors produce examples of fine compactified universal Jacobians for $\overline{\mathcal{M}}_{1,n}$ for all $n \geq 6$ that are not obtained from a universal numerical polarization, hence that do not arise from the methods by Kass–Pagani [KP19] or by Esteves and Melo [Est01, Mel19] (more details in Remark 8.18). An explicit combinatorial characterization of the collection $\Sigma^d_{g,n}$ of degree d fine compactified universal Jacobians for $\overline{\mathcal{M}}_{g,n}$ is available via Corollary 7.13 applied to the universal family over $\overline{\mathcal{M}}_{g,n}$ (see also Definition 4.10 and Remark 4.11). It would be interesting to interpret each element of $\Sigma^d_{g,n}$ as a (top-dimensional) chamber in some stability space, as was done in [KP19] for the case of compactified universal Jacobians arising from numerical polarizations. We explore these questions in Section 10.

Compactified universal Jacobians have recently played a role in enumerative geometry in the theory of the (k-twisted) double ramification cycle, see [BHP$^+$]. As realized in [HKP18], whenever a fine compactified universal Jacobian contains the locus Z of line bundles of multidegree zero, the double ramification cycle can be defined as the pullback of $[Z]$, via some Abel–Jacobi section. This perspective plays an important role in [HMP$^+$], where an extension to a *logarithmic double ramification cycle* is also defined. Because there are different fine compactified universal Jacobians containing Z, the double ramification cycle can be equivalently defined as the pullback of $[Z]$ from different spaces, potentially leading to different formulas, hence to relations in the cohomology of the moduli space of curves. Our classification leads to a complete description of *all* fine compactified universal Jacobians containing Z, whereas previously only those obtained via Kass–Pagani's method [KP19] were considered.

The problem of studying the stability space of complexes of sheaves on a projective variety X has attracted a lot of attention in the last two decades, after Bridgeland's breakthrough [Bri07] (extended to the case of families in [BLM$^+$21]). Most of the literature has been devoted to the case when X is nonsingular. It is natural to try to explicitly describe this stability space for X singular, and one place to start is assuming that X is a nodal curve. We expect that the combinatorics developed in Theorem 1.1 should be regarded as some kind of skeleton of that stability space.

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2. Notation

Throughout we work with Noetherian schemes over a fixed algebraically closed ground field k.

A **curve** over an extension K of k is a $\mathrm{Spec}(K)$-scheme X that is proper over $\mathrm{Spec}(K)$, geometrically connected, and of pure dimension 1. The curve X is a **nodal curve** if it is geometrically reduced and when passing to an algebraic closure $\overline{K}$, its local ring at every singular point is isomorphic to $\overline{K}[[x, y]]/(xy)$.

A coherent sheaf on a nodal curve X has **rank** 1 if its localisation at each generic point of X has length 1. It is **torsion-free** if it has no embedded components.

If F is a rank 1 torsion-free sheaf on a nodal curve X we denote by $\mathrm{N}(F)$ the subset of X where F fails to be locally free. Note that $\mathrm{N}(F)$ is contained in the singular locus of X. If F is a rank 1 torsion-free sheaf on X we say that F is **simple** if its automorphism group is $\mathbb{G}_m$, or equivalently if $X \setminus \mathrm{N}(F)$ is connected.

A **family of curves** over a k-scheme S is a proper, flat morphism $\mathcal{X} \to S$ whose fibers are curves. A family of curves $\mathcal{X} \to S$ is a **family of nodal curves** if the fibers over all geometric points are nodal curves.

If X is a nodal curve over K, we denote by $\Gamma(X)$ its **dual graph** i.e. the labelled graph where each vertex v corresponds to an irreducible component $X^v_{\overline{K}}$ of the base change of X to (the spectrum of) an algebraic (equivalently, a separable) closure $\overline{K}$, and edges corresponding to the nodes of $X_{\overline{K}}$. Note that if X^v_K is an irreducible component defined over K, then it is also defined over any extension L of K, and the corresponding vertices of the dual graphs $\Gamma(X_K)$ and $\Gamma(X_L)$ are canonically identified.

The dual graph is labelled by the geometric genus $p_g(X^v_{\overline{K}})$. The definition of dual graphs extends to the case where $(X, p_1, \ldots, p_n)$ is an n-pointed curve. In this case the dual graph $\Gamma(X)$ also has n half-edges labelled from 1 to n, corresponding to the marked points $p_1, \dots, p_n$. We refer to [ACG11] and [MMUV22] for a detailed definition and for the notion of graph morphisms.

Recall from [CCUW20, § 7.2] that if $\mathcal{X}/S$ is a family of nodal curves and s, t are geometric points of S, then every étale specialization of t to s (written as $t \rightsquigarrow s$) induces a morphism of dual graphs $\Gamma(X_s) \to \Gamma(X_t)$. (For the definition of étale specialization in this context we refer to [CCUW20, Appendix A]).

For a graph G and H a subgraph of G, we denote by $G \setminus H$ and by G/H the graph obtained from G by removing the edges of H and the graph obtained from G by contracting the edges of H, respectively.

We will denote by Δ the spectrum of a DVR with residue field K, and by 0 (resp. by η) its closed (resp. its generic) point. A **smoothing** of a nodal curve X/K over Δ is a flat family $\mathcal{X}/\Delta$ whose generic fiber $\mathcal{X}_\eta/\eta$ is smooth and with an isomorphism of K-schemes $\mathcal{X}_0 \cong X$. The smoothing is **regular** if so is its total space $\mathcal{X}$.

A **family of rank** 1 **torsion-free sheaves** over a family of curves $\mathcal{X} \to S$ is a coherent sheaf on $\mathcal{X}$, flat over S, whose fibers over the geometric points have rank 1 and are torsion-free.

If F is a rank 1 torsion-free sheaf on a nodal curve X with irreducible components X_i, we denote by $F_{\widetilde{X}_i}$ the maximal torsion-free quotient of the pullback of F to the normalization $\widetilde{X}_i$ of X_i, and then define the **multidegree** of F by

$$\underline{\deg}(F) := (\deg(F_{\widetilde{X}_i})) \in \mathbf{Z}^{\mathrm{Vert}(\Gamma(X))}.$$

We define the **(total) degree** of F to be $\deg_X(F) := \chi(F) - 1 + p_a(X)$, where $p_a(X) = h^1(X, \mathcal{O}_X)$ is the arithmetic genus of X. The total degree and the multidegree of F are related by the formula $\deg_X(F) = \sum \deg_{X_i} F + \delta(F)$, where $\delta(F)$ denotes the number of nodes of X where F fails to be locally free. [2]

If $X' \subseteq X$ is a subcurve (by which we will always mean a union of irreducible components), then $\deg_{X'}(F)$ is defined as $\deg(F_{X'})$, where $F_{X'}$ is the maximal torsion-free quotient of $F \otimes \mathcal{O}_{X'}$. The total degree on X is related to the degree on a subcurve by the formula

$$\deg_X(F) = \deg_{X'}(F) + \deg_{\overline{X \setminus X'}}(F) + \#(\mathrm{N}(F) \cap (X' \cap \overline{(X \setminus X')})). \tag{2.1}$$

where the overline denotes the (Zariski) closure.

From now on we fix an integer d once and for all.

2.1. **Spaces of multidegrees.** Here we define the space of multidegrees on a nodal curve as the collection of certain divisors on its dual graph and on its connected spanning subgraphs, suitably organised by degrees.

[2] note that in the Notation section of the papers [KP17, KP19, PT22] the last equation is incorrectly written with a minus sign: $-\delta(F)$ instead of the correct $+\delta(F)$.

Let Γ be a graph and let $\Gamma_0 \subseteq \Gamma$ be a connected spanning subgraph of Γ. We will denote by $n_\Gamma(\Gamma_0)$ or simply by $n(\Gamma_0)$ the number of elements in $\mathrm{Edges}(\Gamma) \setminus \mathrm{Edges}(\Gamma_0)$.

Define the space of multidegrees of total degree d of Γ at Γ_0 as the following collection of divisors on Γ:

$$S^d_\Gamma(\Gamma_0) := \left\{ \underline{\mathbf{d}} \in \mathbf{Z}^{\mathrm{Vert}(\Gamma)} : \sum_{v \in \mathrm{Vert}(\Gamma)} \underline{\mathbf{d}}(v) = d - n(\Gamma_0) \right\} \subset \mathbf{Z}^{\mathrm{Vert}(\Gamma)}. \tag{2.2}$$

According to our convention for the multidegree, if X is a nodal curve with dual graph Γ and $F \in \mathrm{Simp}^d(X)$, then $\Gamma \setminus \mathrm{N}(F)$ is a connected spanning subgraph $\Gamma_0(F)$ of Γ, and we have

$$\underline{\deg}(F) \in S^d_\Gamma(\Gamma_0(F)).$$

3. Fine compactified Jacobians

In this section we introduce the notion of a (smoothable) fine compactified Jacobian for a nodal curve (Definition 3.1), and for a flat family of nodal curves (Definition 3.7).

Let $\mathcal{X}/S$ be a flat family of nodal curves over a k-scheme S. Then there is an algebraic space $\mathrm{Pic}^d(\mathcal{X}/S)$ parameterizing line bundles on $\mathcal{X}/S$ of relative degree d (see [BLR90, Chapter 8.3]). By [AK80] and [Est01] the space $\mathrm{Pic}^d(\mathcal{X}/S)$ embeds in an algebraic space $\mathrm{Simp}^d(\mathcal{X}/S)$ parameterizing flat families of degree d rank 1 torsion-free simple sheaves on $\mathcal{X}/S$. The latter is locally of finite type over S and satisfies the existence part of the valuative criterion of properness. However, it can fail to be of finite type and separated. In the special case of $S = \mathrm{Spec}(K)$ and $X = \mathcal{X}/S$ we will simply write $\mathrm{Pic}^d(X)$ (resp. $\mathrm{Simp}^d(X)$) for $\mathrm{Pic}^d(\mathcal{X}/S)$ (resp. for $\mathrm{Simp}^d(\mathcal{X}/S)$).

Let X be a nodal curve over some field extension K of k. The main point of this paper is to describe well-behaved subspaces of $\mathrm{Simp}^d(X)$, generalizing existing notions of compactified Jacobians in the literature. Specifically, we study the following subschemes.

Definition 3.1. A **degree d fine compactified Jacobian** is a geometrically connected open subscheme $\overline{J} \subseteq \mathrm{Simp}^d(X)$ that is proper over $\mathrm{Spec}(K)$.

We say that the fine compactified Jacobian $\overline{J}$ is **smoothable** if there exists a regular smoothing $\mathcal{X} \to \Delta$ of X, where Δ is the spectrum of a DVR with residue field K, such that $\overline{J}$ is the fiber over $0 \in \Delta$ of an open and Δ-proper subscheme of $\mathrm{Simp}^d(\mathcal{X}/\Delta)$.

Note that the fiber over the generic point η of a nonempty, open and Δ-proper subscheme of $\mathrm{Simp}^d(\mathcal{X}/\Delta)$ is necessarily the moduli space of degree d line bundles $\mathrm{Pic}^d(\mathcal{X}_\eta/\eta)$. As openness and properness are stable under base change, the fiber over $0 \in \Delta$ is open in $\mathrm{Simp}^d(X)$ and K-proper. The axiom "geometrically connected" is redundant in the smoothable case, because the moduli space of degree d line bundles on the generic point is geometrically connected and dense in $\mathrm{Simp}^d(\mathcal{X}/\Delta)$.

Remark 3.2. It follows from Lemma 6.5 (combined with Proposition 7.2) that requiring the subscheme $\overline{J} \subseteq \mathrm{Simp}^d(X)$ to extend to *some regular* smoothing of the curve is equivalent to requiring that it extends to *all* smoothings.

In this paper, we will focus on smoothable fine compactified Jacobians, since they are better behaved and occur more often in applications.

Remark 3.3. When $K = k$ is algebraically closed and $\mathrm{char}(k) = 0$, Definition 3.1 coincides with [PT22, Definitions 2.1, 2.4] (by passing to the completion of the DVR).

In [PT22, Section 3] the authors give a complete classification of fine compactified Jacobians of curves of genus 1, showing in particular the existence of nonsmoothable examples.

Example 3.4. If X is a geometrically irreducible curve over K, then $\mathrm{Simp}^d(X)$ is proper over K, so the only degree d fine compactified Jacobian is $\mathrm{Simp}^d(X)$ itself. These Jacobians are always smoothable. (See Examples 4.6 and 8.7 for the corresponding unique stability condition).

Example 3.5. In the case when there are two geometrically irreducible components, fine compactified Jacobians are no longer irreducible. Assume for simplicity that X is a **vine curve of type** t, that is, the union of two nonsingular curves intersecting transversely at t nodes. We will later see in Example 7.5 that every fine compactified Jacobian of X is smoothable, and that it consists of t irreducible components whose generic points correspond to line bundles of consecutive bidegrees.

Remark 3.6. The moduli space $\mathrm{Simp}^d(X)$ of a nodal curve X admits a natural stratification (see for example [MV12a]) into locally closed subsets

$$\mathrm{Simp}^d(X) = \bigsqcup_{(\Gamma_0, \underline{\mathbf{d}})} \mathcal{J}_{(\Gamma_0, \underline{\mathbf{d}})}$$

where the union runs over all connected spanning subgraphs $\Gamma_0 \subseteq \Gamma(X)$ and all multidegrees $\underline{\mathbf{d}} \in S^d_{\Gamma(X)}(\Gamma_0)$. Each subspace $\mathcal{J}_{(\Gamma_0, \underline{\mathbf{d}})} \subset \mathrm{Simp}^d(X)$ is defined as the locus whose points are sheaves F that fail to be locally free on $\mathrm{N}(F) = \mathrm{Edges}(\Gamma) \setminus \mathrm{Edges}(\Gamma_0)$, and whose multidegree $\underline{\deg}(F)$ equals $\underline{\mathbf{d}}$.

We now extend the notion of a fine compactified Jacobian to the case of a family of nodal curves $\mathcal{X}/S$. Recall that the moduli space $\mathrm{Simp}^d(\mathcal{X}/S)$ is also defined in [Mel19] when $\mathcal{X}/S$ is the universal curve $\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}$ over the moduli stack of stable n-pointed curves of arithmetic genus g. In this case $\mathrm{Simp}^d(\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n})$ is a Deligne–Mumford stack representable (by algebraic spaces) and flat over $\overline{\mathcal{M}}_{g,n}$.

Definition 3.7. Assume that S is irreducible with generic point θ, and assume that the generic fiber $\mathcal{X}_\theta/\theta$ is smooth.

A **family of degree** d **fine compactified Jacobians** for the family $\mathcal{X}/S$ is an open algebraic subspace $\overline{\mathcal{J}} \subseteq \mathrm{Simp}^d(\mathcal{X}/S)$ that is proper over S.

We say that a degree d fine compactified Jacobian $\overline{\mathcal{J}}_{g,n} \subset \mathrm{Simp}^d(\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n})$ is a **degree** d **fine compactified universal Jacobian for the universal curve over** $\overline{\mathcal{M}}_{g,n}$. (We will often omit to specify "for the universal curve over $\overline{\mathcal{M}}_{g,n}$", when clear from the context).

Note that the assumption that the generic fiber is smooth implies that all fibers over S of a degree d fine compactified Jacobian are smoothable.

4. Stability conditions for smoothable fine compactified Jacobians

Here we define the combinatorial data identifying *smoothable* fine compactified Jacobians. We first do so for a single nodal curve, that is, for a fixed dual graph (Definition 4.4), and then we generalize the definition to families (Definition 4.10).

If X is a nodal curve over an algebraically closed field, a sheaf $F \in \mathrm{Simp}^d(X)$ has two natural combinatorial invariants, given by its multidegree and by the subset $\mathrm{N}(F) \subseteq \mathrm{Sing}(X) = \mathrm{Edges}(\Gamma(X))$ of points of the curve where F fails to be locally free. Hence it makes sense to study a fine compactified Jacobian $\overline{J}$ on X by looking at all pairs $\big(\Gamma_0(F) = \Gamma(X) \setminus \mathrm{N}(F), \underline{\deg}(F)\big)$ with $F \in \overline{J}$. Recall that, with the notation introduced in Equation 2.2, we can regard $\underline{\deg}(F)$ as an element of the space of multidegrees $S^d_{\Gamma(X)}(\Gamma_0(F))$.

For a single curve X, we identify, in Definition 4.4, the two properties characterizing the set of such pairs. One is related to properness, combined with the smoothability of the Jacobian, and it requires that the set of stable multidegrees should be a minimal complete set of representatives for the natural chip-firing action on the dual graph (see Definition 4.1). The other corresponds to openness. In combinatorial terms, this means that if we add an edge e to Γ_0, the set of stable multidegrees on $\Gamma \cup \{e\}$ should contain all multidegrees obtained by "adding a chip" to either endpoints of e.

For a family of curves, we further require compatibility with all contractions of the dual graphs involved.

4.1. Stability conditions for a single curve.

We start by introducing the twister group of a graph, which will play a role in characterizing smoothable compactified Jacobians.

Definition 4.1. Let G be a graph. For each $v \in \mathrm{Vert}(G)$, define the *twister of* G *at* v to be the element of $\mathbf{Z}^{\mathrm{Vert}(G)}$ defined by

$$\mathrm{Tw}_{G,v}(w) = \begin{cases} \#\text{ of edges of } G \text{ having } v \text{ and } w \text{ as endpoints} & \text{when } w \neq v, \\ -\#\text{ of nonloop edges of } G \text{ having } v \text{ as an endpoint} & \text{when } w = v. \end{cases}$$

The **twister group** (or chip-firing group) $\mathrm{Tw}(G)$ is the subgroup of $\mathbf{Z}^{\mathrm{Vert}(G)}$ generated by the set $\{ \mathrm{Tw}_{G,v} \}_{v \in \mathrm{Vert}(G)}$.

Recall from Equation (2.2) the definition of the space of multidegrees $S^d_G(G)$ of total degree equal to d. The twister group of G is contained in the sum zero submodule $S^0_G(G)$ of $\mathbf{Z}^{\mathrm{Vert}(G)}$. Hence the group structure on $\mathbf{Z}^{\mathrm{Vert}(G)}$ restricts to an action of $\mathrm{Tw}(G)$ on $S^d_G(G)$. The quotient group

$$J^d(G) := S^d_G(G)/\,\mathrm{Tw}(G) \tag{4.2}$$

is then a torsor over $J^0(G)$, which is a finite abelian group. The latter is also known as the **Jacobian** of the graph G. (It is also known by other names in the literature, such as the *degree class group*, or the *sandpile group*, or the *critical group* of the graph G).

Remark 4.3. Let $\mathcal{X}$ be a regular smoothing of X over some discrete valuation ring Δ with generic point η. Let T be the image under the restriction map $\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(X)$ of the kernel of the surjection $\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_\eta)$ (the restriction to the generic point). We claim that the restriction to T of the multidegree homomorphism $\mathrm{Pic}(X) \to \mathbb{Z}^{\mathrm{Vert}(\Gamma)}$ defines an isomorphism $T \to \mathrm{Tw}(\Gamma)$.

The linear combinations of irreducible components of X are then elements of $\mathrm{Pic}(\mathcal{X})$. Since $\mathcal{X}$ is regular, the irreducible components $\{X_v\}_{v \in \mathrm{Vert}(\Gamma)}$ of X are Cartier divisors on $\mathcal{X}$ and it is easy to check that the elements of T are of the form

$$\mathcal{O}_{\mathcal{X}}\left(\sum_{v \in \mathrm{Vert}(\Gamma)} d_v X_v \right) \otimes \mathcal{O}_X$$

with $\underline{\mathbf{d}} \in \mathbf{Z}^{\mathrm{Vert}(\Gamma)}$. One can explicitly compute that the restriction to T of the multidegree map $\mathrm{Pic}(X) \to \mathbf{Z}^{\mathrm{Vert}(\Gamma)}$ is given by

$$\mathcal{O}_{\mathcal{X}}\left(\textstyle\sum_{v \in \mathrm{Vert}(\Gamma)} d_v X_v \right) \otimes \mathcal{O}_X \quad \longmapsto \quad \textstyle\sum_{v \in \mathrm{Vert}(\Gamma)} d_v \, \mathrm{Tw}_{\Gamma,v}\,.$$

This homomorphism is injective by [Ray70, (5.2)], and it surjects onto $\mathrm{Tw}(\Gamma)$ by the latter's definition. This proves that $T \to \mathrm{Tw}(\Gamma)$ is an isomorphism. In particular, T is independent of the choice of the regular smoothing.

We are now ready to define our notion of a stability condition.

Definition 4.4. A **degree d smoothable fine compactified Jacobian stability condition**, or shortly a **degree d stability condition** for the graph Γ is a subset

$$\sigma = \{(\Gamma_0, \underline{\mathbf{d}}) : \ \underline{\mathbf{d}} \in S^d_\Gamma(\Gamma_0)\} \subset \{\text{connected spanning subgraphs of } \Gamma\} \times \mathbf{Z}^{\mathrm{Vert}(\Gamma)}$$

satisfying the following two conditions:

(1) For all edges e of $\mathrm{Edges}(\Gamma) \setminus \mathrm{Edges}(\Gamma_0)$ with endpoints v_1 and v_2 we have

$$(\Gamma_0, \underline{\mathbf{d}}) \in \sigma \Rightarrow (\Gamma_0 \cup \{e\}, \underline{\mathbf{d}} + \underline{\mathbf{e}}_{v_1}), (\Gamma_0 \cup \{e\}, \underline{\mathbf{d}} + \underline{\mathbf{e}}_{v_2}) \in \sigma,$$

where $\underline{\mathbf{e}}_{v_i}$ denotes the vector in the standard basis of $\mathbf{Z}^{\mathrm{Vert}(\Gamma)}$ corresponding to v_i.

(2) For every connected spanning subgraph Γ_0, the subset

$$\sigma(\Gamma_0) := \{\underline{\mathbf{d}} : \ (\Gamma_0, \underline{\mathbf{d}}) \in \sigma\} \subset S^d_\Gamma(\Gamma_0)$$

is a minimal complete set of representatives for the action of the twister group $\mathrm{Tw}(\Gamma_0)$ on $S^d_\Gamma(\Gamma_0)$.

If X is a nodal curve, a degree d stability condition on X is a degree d stability condition on its dual graph $\Gamma(X)$.

Note that the genus of each of the components of X does not play any role in the above definition.

Remark 4.5. The number of elements of $\sigma(\Gamma_0)$ equals the number of elements of the Jacobian $J^0(\Gamma_0)$. By the Kirchoff–Trent theorem, this number equals the complexity $c(\Gamma_0)$ of the graph Γ_0, defined as the number of spanning trees of Γ_0. (In particular, it is finite).

It is in general hard to classify all stability conditions on a given stable graph. However, the task is within reach when the number of vertices is small.

Example 4.6. If Γ only has 1 vertex (i.e. it is the dual graph of an irreducible curve), there is exactly 1 stability condition on Γ. If there are t edges, the unique degree d stability condition is

$$\sigma = \bigcup_{0 \le i \le t-1} \ \bigcup_{E \subseteq \mathrm{Edges}(\Gamma), |E| = i} \{\, (\Gamma \setminus E, d - i) \,\}.$$

Example 4.7. If instead Γ consists of 2 vertices v_1, v_2 connected by t edges, and no other edges (i.e. Γ is the dual graph of a vine curve of type t, see Example 3.5), let $\Gamma_1, \dots, \Gamma_t$ be the spanning trees. Then it follows from the definition that for every stability condition σ there exists a unique integer λ such that

(1) $\sigma(\Gamma_i) = \{\, (\lambda, d + 1 - \lambda - t) \,\}$ for all $i = 1, \dots, t$;
(2) $\sigma(\Gamma) = \{\, (\lambda, d - \lambda), (\lambda + 1, d - \lambda - 1), \dots, (\lambda + t - 1, d + 1 - \lambda - t) \,\}$.

4.2. **Stability conditions for families.** So far we have discussed the notion of a stability condition for a curve in isolation. The flatness condition for families of sheaves imposes an additional compatibility constraint.

Definition 4.8. Let $f \colon \Gamma \to \Gamma'$ be a morphism of graphs and let σ and σ' be degree d stabilities on Γ and Γ', respectively. We say that σ is f-compatible with σ' if for every connected spanning subgraph Γ_0 of Γ, we have

$$(\Gamma_0, \underline{\mathbf{d}}) \in \sigma \implies (\Gamma'_0, \underline{\mathbf{d}'}) \in \sigma',$$

where Γ'_0 is the image of Γ_0 under f (and therefore it is a connected spanning subgraph of Γ'), and $\underline{\mathbf{d}}'$ is defined by

$$\underline{\mathbf{d}}'(w) = \sum_{f(v)=w} \underline{\mathbf{d}}(v) + \#\{\text{edges of } \Gamma \setminus \Gamma_0 \text{ that are contracted to } w \text{ by } f\}. \tag{4.9}$$

Note that the notion of f-compatibility only depends upon the map that f induces on the set of vertices of the two graphs. We are now ready for the definition of the compatibility constraint.

Definition 4.10. Let $\mathcal{X}/S$ be a family of nodal curves. A **family of degree** d **stability conditions (for fine compactified Jacobians)** on $\mathcal{X}/S$ is the assignment of a degree d stability condition σ_s on $\Gamma(X_s)$ (as in Definition 4.4) for every geometric point s in S that is compatible for all morphisms $\Gamma(X_s) \to \Gamma(X_t)$ arising from any étale specialization $t \rightsquigarrow s$ occurring on S.

We will say that a family of degree d stability conditions for the universal family $\overline{\mathcal{C}}_{g,n}$ over $\overline{\mathcal{M}}_{g,n}$ is a **degree** d **universal stability condition of type** (g, n) (and often omit "of type (g, n)" when clear from the context). These stability conditions will be studied in Section 9.

A family of universal stability conditions for the universal curve is the assignment of a stability condition for each element of $G_{g,n}$, which is furthermore compatible with automorphisms and contractions:

Remark 4.11. In the case of universal stability conditions, the étale specializations of points of $\overline{\mathcal{M}}_{g,n}$ induce all morphisms in the category $G_{g,n}$ of stable graphs of genus g with n marked half-edges.

In particular, if σ is a degree d universal stability condition of type (g, n) and $\alpha\colon \Gamma(X_1) \to \Gamma(X_2)$ is an isomorphism of the dual graphs of two pointed curves $[X_1], [X_2] \in \overline{\mathcal{M}}_{g,n}$, then α identifies $\sigma_{[X_1]}$ with $\sigma_{[X_2]}$.

We conclude that a degree d universal stability condition of type (g, n) is a collection $\{\, \sigma_\Gamma \,\}_{\Gamma \in G_{g,n}}$ such that σ_Γ is a degree d stability condition on Γ and σ_Γ is f-compatible with $\sigma_{\Gamma'}$ for any morphism $f:\ \Gamma \to \Gamma'$ in $G_{g,n}$.

5. Combinatorial preparation: perturbations and lifts

This section contains combinatorial results that will be used in later proofs. First we introduce the notion of a perturbation for the multidegree of a sheaf that fails to be locally free at some nodes. This corresponds to deforming a simple sheaf into a line bundle (Lemma 5.2). Our main result here is Corollary 5.10, where we relate the chip-firing action on a graph with the chip-firing action on the graph obtained by blowing up each of its edges an equal amount of times.

Definition 5.1. Let $H \subseteq G$ be a connected spanning subgraph. An *H-perturbation* is an element in $S_G^{n(H)}(G)$ (for $n(H) = |\operatorname{Edges}(G) \setminus \operatorname{Edges}(H)|$) of the form

$$\sum_{e \in \operatorname{Edges}(G) \setminus \operatorname{Edges}(H)} \underline{\mathbf{e}}_{t(e)}$$

where t is some choice of an orientation on $\operatorname{Edges}(G) \setminus \operatorname{Edges}(H)$.

By the description in Remark 3.6 of the stratification of $\operatorname{Simp}^d(X)$, and by Lemma 5.2, if Γ_0 is a connected spanning subgraph of Γ, a Γ_0-perturbation is the same as the multidegree of a line bundle on X that specializes to a sheaf F with $\Gamma_0(F) = \Gamma_0$, $\underline{\deg}(F) = \underline{\mathbf{0}}$. We are now ready for the following technical lemma.

Lemma 5.2. *Let $\mathcal{X}$ be a family of nodal curves over the spectrum Δ of a DVR with algebraically closed residue field $\overline{K}$, and consider a section $\sigma : \Delta \to \operatorname{Simp}^d(\mathcal{X}/\Delta)$. Let us fix a geometric point $\overline{\eta}$ lying over the generic point of Δ and denote by Γ and Γ' the dual graphs of the geometric fibers X_0 and $X_{\overline{\eta}}$ of $\mathcal{X}$, respectively, and by $f : \Gamma \to \Gamma'$ the morphism of graphs induced by the specialization of $\overline{\eta}$ to 0. Write F_0 and $F_{\overline{\eta}}$ for $\sigma(0)$ and $\sigma(\overline{\eta})$, respectively, and set $\Gamma_0 = \Gamma_0(F_0) \subseteq \Gamma$ and $\Gamma'_0 = \Gamma_0(F_{\overline{\eta}}) \subseteq \Gamma'$ for the connected subgraphs where the corresponding sheaf is locally free. Then there exists a $\Gamma_0 \subseteq f^{-1}(\Gamma'_0)$-perturbation T such that the multidegrees $\underline{\mathbf{d}} = \underline{\deg}(F_0)$ and $\underline{\mathbf{d}}' = \underline{\deg}(F_{\overline{\eta}})$ satisfy the relation*

$$\underline{\mathbf{d}}' = f\left(\underline{\mathbf{d}} + T\right).$$

Note that if $\Gamma'_0 \subseteq \Gamma'$ is connected and spanning, then so is $f^{-1}(\Gamma'_0) \subseteq \Gamma$.

Proof. The section σ defines a rank 1 sheaf $\mathcal{F}$ over $\mathcal{X}/\Delta$. By [EP16, Proposition 5.5], the projectivization $\mathcal{Y} = \mathbf{P}_{\mathcal{X}}(\mathcal{F})$ is a family of curves over Δ and there exists a line bundle $\mathcal{L}$ over $\mathcal{Y}/\Delta$ such that $\mathcal{F} = \psi_* \mathcal{L}$ holds for the natural map $\psi : \mathcal{Y} \to \mathcal{X}$. Geometrically, the restriction of ψ to $Y_{\overline{\eta}} \to X_{\overline{\eta}}$ is the blow up that corresponds to adding a genus 0 vertex on all edges of Γ' that do not belong to Γ'_0. The description of $\psi_0 : Y_0 \to X_0$ in terms of Γ and Γ_0 is completely analogous. Moreover, the multidegrees of $L_{\overline{\eta}}$ and L_0 can be obtained from the multidegrees of $F_{\overline{\eta}}$ and F_0, respectively, by taking $\underline{\mathbf{d}}'$ (resp. $\underline{\mathbf{d}}$) and assigning degree 1 to the additional vertices. Then the claim follows from the fact that L_0 is a specialization of the line bundle $L_{\overline{\eta}}$. □

We will now prove some combinatorial ingredients that relate the chip-firing on a graph with the chip-firing on its blow up. We fix a graph Γ and let $\widetilde{\Gamma} = \widetilde{\Gamma}_m$ be the graph obtained by subdividing each edge of Γ into $m+1$ edges by adding m vertices.

We start by relating the complexities.

Lemma 5.3. *For every $m \geq 1$, the following formula relates the number of spanning trees (i.e. the complexity) of $\widetilde{\Gamma}_m$ with the complexity of the spanning subgraphs of Γ:*

$$c(\widetilde{\Gamma}_m) = \sum_{\substack{G\subseteq\Gamma \textit{ connected} \\ \textit{spanning subgraph}}} c(G)\cdot m^{|\mathrm{Edges}(\Gamma)\setminus\mathrm{Edges}(G)|}. \tag{5.4}$$

Proof. By applying [BMS06, Theorem 3.4] for each new vertex added, we deduce the equality

$$c(\widetilde{\Gamma}_m) = (m+1)^{1+e-v}c(\Gamma) = \sum_{n=1}^{1+e-v}\binom{e-v+1}{n}c(\Gamma)\, m^{e-v+1-n},$$

where e and v denote the number of edges and vertices of Γ, respectively. (It is easy to check that the formula in op. cit. also holds when the graph contains loops).

On the other hand, we have that each spanning tree T of Γ can be extended in $\binom{e-v+1}{n}$ ways to a subgraph $H\subseteq\Gamma$ with $v-1+n$ edges by choosing n edges in the set $\mathrm{Edges}(\Gamma)\setminus\mathrm{Edges}(T)$. This implies

$$\sum_{\substack{H\subseteq\Gamma \\ \text{spanning subgraph} \\ |\mathrm{Edges}(H)|=v-1+n}} c(H) = \binom{e-v+1}{n}c(\Gamma).$$

The claim then follows by combining the two equalities above. □

We now define the subset $\widetilde{\sigma}\subset S^d_{\widetilde{\Gamma}}(\widetilde{\Gamma})$ by lifting each element $(\Gamma_0,\underline{\mathbf{d}})\in\sigma$ to multiple elements in $S^d_{\widetilde{\Gamma}}(\widetilde{\Gamma})$ as follows. First of all, for every element $(\Gamma_0,\underline{\mathbf{d}})\in\sigma$ we can naturally identify $\underline{\mathbf{d}}$ with the element of $\mathbf{Z}^{\mathrm{Vert}(\widetilde{\Gamma})}$ obtained by extending $\mathbf{d}$ to zero on all exceptional vertices. Then a lift of $(\Gamma_0,\underline{\mathbf{d}})$ is any element $\widetilde{\underline{\mathbf{d}}}\in S^d_{\widetilde{\Gamma}}(\widetilde{\Gamma})$ of the form

$$\widetilde{\underline{\mathbf{d}}} = \underline{\mathbf{d}} + \sum_{e\in\mathrm{Edges}(\Gamma)\setminus\mathrm{Edges}(\Gamma_0)} \underline{\mathbf{e}}_{v(e)}, \tag{5.5}$$

for some choice of function v from the set of edges $\mathrm{Edges}(\Gamma)\setminus\mathrm{Edges}(\Gamma_0)$ to $\mathrm{Vert}(\widetilde{\Gamma})$ such that each vertex $v(e)$ is one of the m interior vertices of the rational chain in $\widetilde{\Gamma}$ that corresponds to the edge e.

Recall from Definition 5.1 the notion, for H a subgraph of G, of an H-perturbation. This terminology gives a way to relate the action of $\mathrm{Tw}(\widetilde{\Gamma}_m)$ with that of $\mathrm{Tw}(\Gamma)$.

Lemma 5.6. *Let G and G' be connected spanning subgraphs of Γ and let $\underline{\mathbf{d}}\in S^d_\Gamma(G)$ and $\underline{\mathbf{d}}'\in S^d_\Gamma(G')$. Assume that there exist two lifts $\tilde{\underline{\mathbf{d}}}$ and $\tilde{\underline{\mathbf{d}}}'$ in $\widetilde{\Gamma}$, defined as in Equation* (5.5)*, that are different and that belong to the same* $\mathrm{Tw}(\widetilde{\Gamma})$*-orbit.*

Then there exist a G-perturbation T and a G'-perturbation T' such that $\underline{\mathbf{d}}+T$ and $\underline{\mathbf{d}}'+T'$ are different and belong to the same $\mathrm{Tw}(\Gamma)$*-orbit.*

Proof. By the definition of the twister group we have

$$\tilde{\underline{\mathbf{d}}} - \tilde{\underline{\mathbf{d}}}' = \operatorname{div}(g) := \sum_{v \in \operatorname{Vert}(\widetilde{\Gamma}_m)} g(v)\, \operatorname{Tw}_{\widetilde{\Gamma}_m, v} \tag{5.7}$$

for some $g:\ \operatorname{Vert}(\widetilde{\Gamma}_m) \to \mathbf{Z}$ which we will now study in more detail.

By construction, the graph $\widetilde{\Gamma}_m$ comes with an inclusion $\operatorname{Vert}(\Gamma) \hookrightarrow \operatorname{Vert}(\widetilde{\Gamma}_m)$, which we use to define a function $f:\ \operatorname{Vert}(\Gamma) \to \mathbf{Z}$ by

$$f(v) = \left\lfloor \frac{g(v)}{m+1} \right\rfloor .$$

For later use, we wish to make sure that the function f is nonconstant. This can always be achieved by using that $\operatorname{div}(g)$ does not change if g is modified by adding the same constant to all vertices of $\widetilde{\Gamma}_m$. Since $\tilde{\underline{\mathbf{d}}}$ and $\tilde{\underline{\mathbf{d}}}'$ are distinct, the function $g:\ \operatorname{Vert}(\widetilde{\Gamma}_m) \to \mathbf{Z}$ is nonconstant, so that there exists an integer $0 \leq \ell \leq m$ such that $f_\ell(v) = \left\lfloor \frac{g(v)+\ell}{m+1} \right\rfloor$ is also nonconstant, and we may then use f_ℓ in place of f in what follows.

We would like to compare

$$\operatorname{div}(f) := \sum_{v \in \operatorname{Vert}(\Gamma)} f(v)\, \operatorname{Tw}_{\Gamma, v}$$

with $\underline{\mathbf{d}} - \underline{\mathbf{d}}'$. To this end, we study the behaviour of g at the endpoints of an arbitrary edge e of Γ.

By definition, the inverse image of e under the contraction morphism

$$\alpha:\ \operatorname{Edges}(\widetilde{\Gamma}_m) \to \operatorname{Edges}(\Gamma)$$

consists of a chain of $m+1$ edges. Fix the orientation for $\widetilde{\Gamma}_m$ so that g increases following the direction of the edges. For each $e \in \operatorname{Edges}(\Gamma)$, we then order the $m+2$ vertices of the chain $\alpha^{-1}(e)$ as $v_0, \ldots, v_{m+1}$ following the orientation. In particular, the vertices v_0 and v_{m+1} correspond to the two endpoints of e.

Because of the way we defined the lifts in (5.5), for $e \in G \cap G'$ we have $\operatorname{div}(g)(v_i) = 0$ for all $i = 1, \ldots, m$. For $e \in G' \setminus G$ (resp. $e \in G \setminus G'$) there exists a single vertex v_j with $1 \leq j \leq m$ such that $\operatorname{div}(g)(v_j) = 1$ (resp. $\operatorname{div}(g)(v_j) = -1$) and $\operatorname{div}(g)(v_i) = 0$ for all $i = 1, \ldots, m$, $i \neq j$. In the case $e \notin G \cup G'$, we either have $\operatorname{div}(g)(v_i) = 0$ for all $i = 1, \ldots, m$, or there exist two distinct indices $j_+, j_- \in \{1, \ldots, m\}$ such that $\operatorname{div}(g)(v_{j_+}) = 1$, $\operatorname{div}(g)(v_{j_-}) = -1$ and $\operatorname{div}(g)(v_i) = 0$ for $i \in \{1, \ldots, m\} \setminus \{j_+, j_-\}$.

As a consequence, the value of g at each vertex v_i in the chain $\alpha^{-1}(e)$ is uniquely determined by j (or by j_+, j_-) and by the value of g at the first two vertices of the chain, that is, by $a = g(v_0)$ and $b = g(v_1)$. For example, for the last two vertices we have

$$\begin{array}{lll} g(v_{m+1})-g(v_m) & g(v_{m+1}) & \\ b-a & (m+1)(b-a)+a & \text{if } e\in G\cap G' \\ b-a-1 & (m+1)(b-a)-(m+1-j)+a & \text{if } e\in G'\setminus G \\ b-a+1 & (m+1)(b-a)+(m+1-j)+a & \text{if } e\in G\setminus G' \\ b-a & (m+1)(b-a)+j_+-j_-+a & \text{if } e\notin G\cup G'. \end{array}$$

In particular, if we denote by v and w the endpoints of e corresponding to v_0 and v_{m+1} respectively, we have:

- $f(v)-f(w)=a-b$ for $e\in G\cap G'$ or $e\notin G\cup G'$ with $j_+=j_-$;
- $f(v)-f(w)\in\{a-b,a-b+1\}$ for $e\in G'\setminus G$ or $e\notin G\cup G'$ with $j_+<j_-$;
- $f(v)-f(w)\in\{a-b,a-b-1\}$ for $e\in G\setminus G'$ or $e\notin G\cup G'$ with $j_+>j_-$.

We use this to define three sets of directed edges of Γ. To this end, we use the same orientation of the edges of Γ that we chose earlier. For each $e\in \mathrm{Edges}(\Gamma)$, we denote the first endpoint of e by $v=v_0$ and the second by $w=v_{m+1}$.

Let now S be the following set of directed edges:

- all edges $e\in G'\setminus G$, oriented from v to w if $f(v)-f(w)=g(v_0)-g(v_1)+1$ and from w to v if $f(v)-f(w)=g(v_0)-g(v_1)$;
- all edges $e\notin G\cup G'$ with $f(v)-f(w)=g(v_0)-g(v_1)+1$, oriented from v to w.

Then let S' be the following set of directed edges:

- all edges $e\in G\setminus G'$, oriented from v to w if $f(v)-f(w)=g(v_0)-g(v_1)-1$ and from w to v if $f(v)-f(w)=g(v_0)-g(v_1)$;
- all edges $e\notin G\cup G'$ with $f(v)-f(w)=g(v_0)-g(v_1)-1$, oriented from v to w.

We define $N\subseteq \mathrm{Edges}(\Gamma)$ to be the set of edges that are neither in S or in S'. [3]

If one forgets the orientation, then S consists of the union of $\mathrm{Edges}(G'\setminus G)$ and of a subset $P\subseteq \mathrm{Edges}(\Gamma\setminus(G\cup G'))$. Analogously, the support of S' is the union of $\mathrm{Edges}(G\setminus G')$ and the same P as before, but with the edges in P taken with the opposite orientation than in the case of S. With this notation, we have $N=\mathrm{Edges}(G\cap G')\cup(\mathrm{Edges}(\Gamma\setminus(G\cup G')\setminus P)$.

Let us denote by $s,t:\ S\cup S'\to \mathrm{Vert}(\Gamma)$ the maps sending each oriented edge to its source and its target respectively. Then define

$$T'=\sum_{e\in S}\underline{\mathbf{e}}_{s(e)},\quad T=\sum_{e\in S'}\underline{\mathbf{e}}_{s(e)}.$$

We claim that $\mathrm{div}(f)=\underline{\mathbf{d}}-\underline{\mathbf{d}}'+T-T'$.

To prove the claim, we will check that both functions have the same value at each vertex v of Γ. If we view v as a vertex of $\widetilde{\Gamma}_m$, from $\underline{\mathbf{d}}-\underline{\mathbf{d}}'=\mathrm{div}(g)$ we deduce

[3]Note that the definitions of S, S' and N are independent of the chosen orientation.

the equality

$$\sum_{\substack{e \text{ edge of } \Gamma \\ \text{with endpoint } v}} (g(v) - g(v_1(e)) = \underline{\mathbf{d}}(v) - \underline{\mathbf{d}}'(v), \tag{5.8}$$

where we denoted by $v_1(e)$ the first internal vertex of $\alpha^{-1}(e)$ adjacent to $v = v_0$. Then by definition of S, S' and N we have the equalities

$$\begin{aligned}
\sum_{\substack{e\in N,\\ \{s(e),t(e)\}=\{v,w\}}} (g(v) - g(v_1(e)) &= \sum_{\substack{e\in N,\\ \{s(e),t(e)\}=\{v,w\}}} (f(v) - f(w)) \\
\sum_{e\in S\cup S',\ t(e)=v} (g(v) - g(v_1(e)) &= \sum_{e\in S\cup S',\ t(e)=v} (f(v) - f(s(v))) \\
\sum_{e\in S\cup S',\ t(e)=v} (g(v) - g(v_1(e)) &= \sum_{e\in S\cup S',\ t(e)=v} (f(v) - f(s(v))) \\
\sum_{e\in S,\ s(e)=v} (g(v) - g(v_1(e)) &= \sum_{e\in S,\ s(e)=v} (f(v) - f(s(v)) + 1) = T'(v) + \sum_{e\in S,\ s(e)=v} (f(v) - f(s(v)) \\
\sum_{e\in S',\ s(e)=v} (g(v) - g(v_1(e)) &= \sum_{e\in S',\ s(e)=v} (f(v) - f(s(v)) - 1) = -T(v) + \sum_{e\in S,\ s(e)=v} (f(v) - f(s(v))
\end{aligned}$$

Since all edges with endpoint v appear exactly once in each of the sums above, by adding them together and using (5.8) we obtain

$$\operatorname{div}(f)(v) + T'(v) - T(v) = \underline{\mathbf{d}}(v) - \underline{\mathbf{d}}'(v). \tag{5.9}$$

This proves the claim and ensures that $\underline{\mathbf{d}} + T$ and $\underline{\mathbf{d}}' + T'$ are in the same $\operatorname{Tw}(\Gamma)$-orbit. Finally, the fact that $\underline{\mathbf{d}} + T$ and $\underline{\mathbf{d}}' + T'$ are different follows from the fact that the function f is nonconstant.

□

We are now ready to state and prove our main result of this section.

Corollary 5.10. *Let σ be a subset of*

$$\{\textit{connected spanning subgraphs of } \Gamma\} \times \mathbf{Z}^{\operatorname{Vert}(\Gamma)}$$

such that the collection $\{\sigma(G)\}$, whose elements are defined by

$$\sigma(G) := \{\underline{\mathbf{d}} :\ (G, \underline{\mathbf{d}}) \in \sigma\} \subset S^d_\Gamma(G),$$

satisfies Part (1) of Definition 4.4.

Then let $\widetilde{\sigma}_m$ be the set whose elements are all lifts to the graph $\widetilde{\Gamma}_m$, via Equation (5.5)*, of all elements of $\sigma(G)$ for each $G \subseteq \Gamma$.*

Then the following are equivalent:

(1) the subset $\widetilde{\sigma}_m \subset S^d(\widetilde{\Gamma}_m)$ is a minimal complete set of representatives for the action of $\operatorname{Tw}(\widetilde{\Gamma}_m)$, and

(2) the collection σ satisfies Part (2) of Definition 4.4 (hence it is a degree d stability condition).

Proof. It is straightforward to check that the number of elements of $\widetilde{\sigma}_m$ equals the right hand side of (5.4), which equals the complexity of $\widetilde{\Gamma}_m$ by Lemma (5.3).

Therefore it suffices to prove that the following statements are equivalent:

(1) any two elements of $\widetilde{\sigma}_m$ are chip-firing equivalent on $\widetilde{\Gamma}_m$ if and only if they are equal, and
(2) for every connected spanning subgraph $G \subseteq \Gamma$, any two elements of $\sigma(G)$ are chip-firing equivalent on G if and only if they are equal.

The fact that (2) implies (1) follows immediately from Lemma 5.6 and the assumption that σ satisfies Part (1) of Definition 4.4.

For the converse implication, let $\underline{\mathbf{d}}$ and $\underline{\mathbf{d}}'$ be different elements in $\sigma(G)$, satisfying

$$\underline{\mathbf{d}} - \underline{\mathbf{d}}' = \sum_{v \in \mathrm{Vert}(G)} f(v)\, \mathrm{Tw}_{G,v}$$

for some $f\colon \mathrm{Vert}(G) \to \mathbb{Z}$.

First we show that we may assume $G = \Gamma$. Choose an orientation $s, t\colon \mathrm{Edges}(\Gamma) \setminus \mathrm{Edges}(G) \to \mathrm{Vert}(\Gamma)$ and define

$$\underline{\mathbf{e}} := \underline{\mathbf{d}} + \sum_{e \in \mathrm{Edges}(\Gamma) \setminus \mathrm{Edges}(G)} s(e), \quad \underline{\mathbf{e}}' := \underline{\mathbf{d}}' + \sum_{e \in \mathrm{Edges}(\Gamma) \setminus \mathrm{Edges}(G)} t(e).$$

Then by Part (1) of Definition 4.4 we have that $\underline{\mathbf{e}}, \underline{\mathbf{e}}'$ are different elements of $\sigma(\Gamma)$, and

$$\underline{\mathbf{e}} - \underline{\mathbf{e}}' = \sum_{v \in \mathrm{Vert}(\Gamma)} f(v)\, \mathrm{Tw}_{\Gamma,v}\,.$$

Then $\underline{\mathbf{e}}$ (resp. $\underline{\mathbf{e}}'$) admits a unique lift $\underline{\tilde{\mathbf{e}}}$ (resp. $\underline{\tilde{\mathbf{e}}}'$) to $\widetilde{\sigma}_m$ via Equation 5.5. Now we define $g\colon \mathrm{Vert}(\widetilde{\Gamma}_m) \to \mathbb{Z}$ to show that $\underline{\tilde{\mathbf{e}}}$ and $\underline{\tilde{\mathbf{e}}}'$ are in the same $\mathrm{Tw}(\widetilde{\Gamma}_m)$-orbit. As in the proof of Lemma 5.6, let $\alpha\colon \mathrm{Edges}(\widetilde{\Gamma}_m) \to \mathrm{Edges}(\Gamma)$ be the map induced by a contraction morphism. For a fixed $e \in \mathrm{Edges}(\Gamma)$, let v and w be the endpoints of e such that $f(v) < f(w)$. Then orient the $m+2$ vertices in the chain $\alpha^{-1}(e)$ in a consecutive way and so that α maps the first to v and the last to w, in other words $\alpha^{-1}(v) = v_0, v_1, \ldots, v_m, v_{m+1} = \alpha^{-1}(w)$. Then let $g(v_i) = f(v) + i \cdot (f(w) - f(v))$. This defines a function $g\colon \mathrm{Vert}(\widetilde{\Gamma}_m) \to \mathbb{Z}$, and with this definition we have

$$\underline{\tilde{\mathbf{e}}} - \underline{\tilde{\mathbf{e}}}' = \sum_{v \in \mathrm{Vert}(\widetilde{\Gamma}_m)} g(v)\, \mathrm{Tw}_{\widetilde{\Gamma}_m}, v.$$

This proves that $\underline{\tilde{\mathbf{e}}}$ and $\underline{\tilde{\mathbf{e}}}'$, which are different elements of $\widetilde{\sigma}_m$, are in the same $\mathrm{Tw}(\widetilde{\Gamma}_m)$-orbit, and thus it completes our proof that (1) implies (2). □

6. From stability conditions to smoothable fine compactified Jacobians

In this section we show how to construct a fine compactified Jacobian from a given stability condition. We define a sheaf to be *stable* if its multidegree is as prescribed by the stability condition, both in the case of a single nodal curve and for families. Then we show that the moduli space of stable sheaves thus defined is a fine compactified Jacobian as introduced in Definitions 3.1 and 3.7. The case of a curve in isolation is Corollary 6.4, and the more general case of families with smooth generic element is Theorem 6.3. The most difficult part is the proof of properness (Lemma 6.5).

Let X be a nodal curve over some field extension K of k, and let Γ be its dual graph.

Definition 6.1. Let σ be a degree d stability condition for Γ (see Definition 4.4). We say that $[F] \in \mathrm{Simp}^d(X)$ is **stable with respect to** σ if $(\Gamma_0(F), \underline{\deg}(F))$ is in σ, where $\Gamma_0(F) = \Gamma \setminus \mathrm{N}(F)$ is the (necessarily connected and spanning) subgraph of Γ that is dual to the normalization of X at the points (necessarily nodes) where F fails to be locally free.

We define $\overline{J}_\sigma \subseteq \mathrm{Simp}^d(X)$ as the subscheme of sheaves that are stable with respect to σ.

Let now $\mathcal{X}/S$ be a family of nodal curves over a k-scheme S. For every geometric point s of S, denote by Γ_s the dual graph of the fiber $\mathcal{X}_s$.

Definition 6.2. Let $\mathfrak{S} = \{\sigma_s\}_{s\in S}$ be a family of degree d stabilities for $\mathcal{X}/S$ as in Definition 4.10.

If F is a geometric point of $\mathrm{Simp}^d(\mathcal{X}/S)$ lying over $s \in S$, we say that F is **stable with respect to** $\mathfrak{S}$ if F is stable with respect to σ_s (see Definition 6.1).

We define $\overline{\mathcal{J}}^d_{\mathfrak{S}} \subseteq \mathrm{Simp}^d(\mathcal{X}/S)$ as the algebraic subspace whose geometric points are sheaves that are stable with respect to $\mathfrak{S}$.

The main result of this section is the following:

Theorem 6.3. *Assume that S is irreducible with generic point θ, and that the generic element of the family $\mathcal{X}_\theta/\theta$ is smooth. Let $\mathfrak{S}$ be a family of degree d stability conditions for $\mathcal{X}/S$. Then $\overline{\mathcal{J}}^d_{\mathfrak{S}} \subseteq \mathrm{Simp}^d(\mathcal{X}/S)$ is a degree d fine compactified Jacobian for the family $\mathcal{X}/S$.*

We immediately deduce:

Corollary 6.4. *The moduli space $\overline{J}_\sigma \subseteq \mathrm{Simp}^d(X)$ of σ-stable sheaves is a smoothable degree d fine compactified Jacobian for X.*

Proof. Apply Theorem 6.3 to a regular smoothing $\mathcal{X}/\Delta$ of X. □

The most delicate part of the proof of Theorem 6.3 is the following result, which builds on the combinatorial preparation of the previous section.

Lemma 6.5. *With the same assumptions as in Theorem 6.3, the moduli space* $\overline{\mathcal{J}}^d_{\mathfrak{S}}$ *is proper over* S.

Proof. By Definitions 4.4 and 6.1, the scheme $\overline{\mathcal{J}}^d_{\mathfrak{S}}$ is the union of finitely many locally closed strata and hence of finite type and quasi-separated over S. To prove properness, we apply the valuative criterion [Sta23, Lemma 70.22.3].

Given the spectrum Δ of a DVR and a commutative diagram (of solid arrows)

$$\begin{array}{ccccc} \eta = \Delta \setminus \{0\} & \xrightarrow{f} & \mathrm{Pic}^d(\mathcal{X}_\theta) & \hookrightarrow & \overline{\mathcal{J}}^d_{\mathfrak{S}} \\ \big\downarrow & & & \nearrow \cdots & \big\downarrow \\ \Delta & & \longrightarrow & & S, \end{array}$$

we need to prove that there exists exactly one dotted arrow that keeps the diagram commutative. (Note that $\mathrm{Pic}^d(\mathcal{X}_\theta)$ is open and dense in $\overline{\mathcal{J}}^d_{\mathfrak{S}}$).

Let us recall that morphisms to moduli spaces of sheaves (line bundles, simple sheaves) correspond to sheaves on the corresponding family of curves, but this correspondence is a bijection only after identifying two such sheaves whenever they differ by the pullback of a line bundle from the base. In the remainder of this proof we will slightly abuse the notation and assume this identification.

Let us denote by $\mathcal{X}_\Delta/\Delta$ the base change of $\mathcal{X}/S$ under $\Delta \to S$ and by $\overline{\mathcal{J}}^d_\Delta \subseteq \mathrm{Simp}^d(\mathcal{X}/\Delta)$ the base change of $\overline{\mathcal{J}}^d_{\mathfrak{S}}$. Let s be the image of $0 \in \Delta$ under $\Delta \to S$. Denoting by $X = \mathcal{X}_s = \mathcal{X}_{\Delta,0}$ the fiber over $0 \in \Delta$ of the family $\mathcal{X}_\Delta/\Delta$, by the hypothesis that $\mathcal{X}_\theta/\theta$ is smooth we deduce that $\mathcal{X}_\Delta/\Delta$ is a smoothing of X. Moreover, we have that $\overline{\mathcal{J}}^d_\Delta$ is the disjoint union of $\mathrm{Pic}^d(\mathcal{X}_{\Delta,\eta}) = \mathrm{Simp}^d(\mathcal{X}_{\Delta,\eta})$ and of $\overline{J}_\sigma \subseteq \mathrm{Simp}^d(X)$, where σ is the stability condition for $\mathcal{X}_s$.

The morphism $f\colon \Delta \setminus \{0\} \to \mathrm{Pic}^d(\mathcal{X}_\theta)$ corresponds to a line bundle $[\mathcal{L}_\eta] \in \mathrm{Pic}^d(\mathcal{X}_{\Delta,\eta}/\eta)$. We know from [Est01] that $\mathrm{Simp}^d(\mathcal{X}_\Delta/\Delta)$ satisfies the existence part of the valuative criterion of properness, hence $\mathcal{L}_\eta$ extends (possibly nonuniquely) to an element $[\mathcal{L}'] \in \mathrm{Simp}^d(\mathcal{X}_\Delta/\Delta)$. Our proof is concluded if we can show that $\mathcal{L}_\eta$ extends to a unique $[\mathcal{L}] \in \overline{\mathcal{J}}^d_\Delta$. When the total space $\mathcal{X}_\Delta$ of the smoothing $\mathcal{X}_\Delta/\Delta$ is regular, this follows immediately from Remark 4.3.

If the total space $\mathcal{X}_\Delta$ of the smoothing $\mathcal{X}_\Delta/\Delta$ is not regular, then each of the nodes of the central fiber X is a singularity of type A_n of $\mathcal{X}_\Delta$, for some $n \geq 0$. Thus each $e \in \mathrm{Edges}(\Gamma)$ corresponds to a singular point of type $A_{n(e)}$ for some $n(e) \geq 0$. Let m be least common multiple of the $\{n(e)\}_{e \in \mathrm{Edges}(\Gamma)}$, and consider the resolution of singularities $g\colon \widetilde{\mathcal{X}} \to \mathcal{X}_\Delta$ obtained by blowing up m times each node of $\mathcal{X}_\Delta$. The central fiber $\widetilde{X} \subset \widetilde{\mathcal{X}}$ is therefore obtained by replacing each node of the central fiber X of $\mathcal{X}_\Delta$ with a rational bridge of length m. Equivalently, the dual graph $\widetilde{\Gamma}$ of $\widetilde{X}$ is obtained by subdividing m times each edge of $\Gamma = \Gamma(X)$ by adding m additional genus 0 vertices. In other words, each edge of Γ is replaced in $\widetilde{\Gamma}$ by a rational chain of $m+1$ edges.

We have now achieved that $\widetilde{\mathcal{X}}$ is regular, and taking the direct image via g induces a surjection $g_*\colon \operatorname{Pic}^d(\widetilde{X}) \to \operatorname{Simp}^d(X)$ on the central fiber.

In order to prove the existence and uniqueness of an extension of $\mathcal{L}_\eta$ to $\mathcal{X}_\Delta/\Delta$ with a sheaf in $\overline{\mathcal{J}}^d_\Delta$, we prove that $g^*(\mathcal{L}_\eta)$ extends uniquely on $\widetilde{\mathcal{X}}/\Delta$ to a line bundle whose restriction to $\widetilde{X}$ has multidegree in $\widetilde{\sigma}$, where the elements of $\widetilde{\sigma}$ are defined by lifting elements of σ to $\widetilde{\Gamma}$ as in Equation 5.5. Note that taking the direct image via g maps line bundles on $\widetilde{X}$ with multidegree in $\widetilde{\sigma}(\widetilde{\Gamma})$ to rank 1, torsion free, simple sheaves on X with multidegree in $\sigma(\Gamma_0)$ for $\Gamma_0 \subseteq \Gamma$ the spanning subgraph whose edges e correspond to the rational bridges where the value of $\widetilde{\sigma}(\widetilde{\Gamma})$ is equal to zero on all vertices.

By Corollary 5.10 combined with the assumption that σ satisfies the second axiom of Definition 4.4, we deduce that $\widetilde{\sigma}$ is a minimal complete set of representatives for the action of the twister group $\operatorname{Tw}(\widetilde{\Gamma})$ on $S^d(\widetilde{\Gamma})$.

We conclude, as in the regular case, that $g^*(\mathcal{L}_\eta)$ extends to a unique line bundle $\widetilde{\mathcal{L}}$ on $\widetilde{\mathcal{X}}/\Delta$ whose multidegree on the central fiber $\widetilde{X}$ is an element of $\widetilde{\sigma}$. Therefore $g_*\widetilde{\mathcal{L}}$ is the unique extension of $\mathcal{L}_\eta$ whose multidegree on X is an element of σ. This completes the proof. □

We are now ready to complete the proof of Theorem 6.3.

Proof of Theorem 6.3. As $\overline{\mathcal{J}}^d_{\mathfrak{S}}$ is by definition constructible, openness is shown by proving that $\overline{\mathcal{J}}^d_{\mathfrak{S}} \subseteq \operatorname{Simp}^d(\mathcal{X}/S)$ is stable under generalization ([Sta23, Lemma 5.19.10]). That $\overline{\mathcal{J}}^d_{\mathfrak{S}}$ is stable under generalization follows from the first requirement of Definition 4.4, and from the requirement that the family of stability conditions is compatible for graph morphisms arising from étale specializations (as specified in Definition 4.10).

Properness is Lemma 6.5. □

7. From smoothable fine compactified Jacobians to stability conditions

In this section we show that a smoothable fine compactified Jacobian defines a stability condition in a natural way. We do so first for the case of a curve in isolation (Corollary 7.9), and then for the case of families (Proposition 7.10). We conclude by proving our main result, Corollary 7.13, which establishes that the operations described in this section and in the previous one are inverse to each other.

Throughout this section, we will consider a fixed nodal curve X over a field extension K of k, with dual graph Γ and denote by $\overline{J} \subset \operatorname{Simp}^d(X)$ a degree d fine compactified Jacobian of X.

Definition 7.1. We define the **associated assignment** $\sigma_{\overline{J}}$ of $\overline{J}$ as

$$\sigma_{\overline{J}} = \{(\Gamma_0(F), \underline{\deg}(F)) : \ [F] \in \overline{J}\},$$

where $\Gamma_0(F) = \Gamma \setminus \mathrm{N}(F)$ is the connected spanning subgraph of Γ obtained by removing the edges corresponding to the nodes where F fails to be locally free.

A key point is that, if a sheaf is an element of a given fine compactified Jacobian, then so are all other sheaves that fail to be locally free on the same set of nodes and that have the same multidegree:

Lemma 7.2. *Let $\overline{J} \subset \mathrm{Simp}^d(X)$ be a degree d fine compactified Jacobian, and assume that $(\Gamma_0, \underline{\mathbf{d}}) \in \sigma_{\overline{J}}$. Then $\overline{J}$ contains all sheaves $[F] \in \mathrm{Simp}^d(X)$ with $\mathrm{N}(F) = \Gamma \setminus \Gamma_0$ and $\underline{\deg}(F) = \underline{\mathbf{d}}$.*

Proof. By possibly passing to the partial normalization of X at the nodes in $\Gamma \setminus \Gamma_0$ we may assume that $\Gamma_0 = \Gamma$. Thus we aim to prove that if $\mathcal{J}^{\underline{\mathbf{d}}}(X) \cap \overline{J} \neq \emptyset$, then $\mathcal{J}^{\underline{\mathbf{d}}}(X) \subset \overline{J}$.

The moduli space of line bundles $\mathcal{J}^{\underline{\mathbf{d}}}(X)$ of multidegree $\underline{\mathbf{d}}$ is irreducible and $\overline{J}$ is open, so $\mathcal{J}^{\underline{\mathbf{d}}}(X) \cap \overline{J} \neq \emptyset$ implies that $\mathcal{J}^{\underline{\mathbf{d}}}(X) \cap \overline{J}$ is dense in $\mathcal{J}^{\underline{\mathbf{d}}}(X)$. For $[F]$ in $\mathcal{J}^{\underline{\mathbf{d}}}(X)$ we can then find a line bundle L on $X \times \Delta$, for some DVR Δ, whose generic element L_η is in $\mathcal{J}^{\underline{\mathbf{d}}}(X) \cap \overline{J}$ and whose special fiber L_0 equals F.

By properness of $\overline{J}$ there exists a sheaf L' on $X \times \Delta$ with the property that $L'_{X\times\eta} = L_{X\times\eta}$ (here η is the generic point of Δ) and whose special fiber L'_0 is in $\overline{J}$. Then $L' \otimes L^{-1}$ defines a morphism $\Delta \to \mathrm{Simp}^0(X)$ that maps the generic point η to a closed point, hence it must be the constant morphism. We conclude, in particular, that $[F] = [L'_0]$ and so $[F] \in \overline{J}$.

□

Our first goal is to show that if the fine compactified Jacobian is also smoothable, then its associated assignment does indeed satisfy the 2 conditions of a degree d stability condition given in Definition 4.4.

That the associated assignment to a fine compactified Jacobian satisfies the first condition of Definition 4.4 follows immediately from Lemma 5.2:

Corollary 7.3. *The associated assignment $\sigma_{\overline{J}}$ satisfies*

$$(\Gamma_0, \underline{\mathbf{d}}) \in \sigma_{\overline{J}} \Rightarrow (\Gamma_0 \cup \{e\}, \underline{\mathbf{d}} + \underline{\mathbf{e}}_{v_1}), (\Gamma_0 \cup \{e\}, \underline{\mathbf{d}} + \underline{\mathbf{e}}_{v_2}) \in \sigma_{\overline{J}}, \tag{7.4}$$

for all spanning subgraphs $\Gamma_0 \subseteq \Gamma$ and all edges e of $\mathrm{Edges}(\Gamma) \setminus \mathrm{Edges}(\Gamma_0)$ with endpoints v_1 and v_2.

Note that the above statement remains valid (and with the same proof) for an arbitrary *open* subscheme of $\mathrm{Simp}^d(X)$, properness and smoothability do not play any role here.

Our aim is now to prove, in Proposition 7.6, that smoothable fine compactified Jacobians also satisfy the second condition of Definition 4.4. Before we do that, we show that Corollary 7.3 allows us to conclude that fine compactified Jacobians of vine curves are automatically smoothable.

Example 7.5. Let X be a vine curve of type t (as defined in Example 3.5) over some algebraically closed field $K = k$. We claim that any fine compactified Jacobian $\overline{J}(X)$ of X is of the form $\overline{J}(X) = \overline{J}_\sigma(X)$ for some stability condition σ as in Example 4.7. In particular, by Corollary 6.4, $\overline{J}(X)$ is smoothable.

This is clear when $t = 1$, as in that case each geometrically connected component of $\mathrm{Simp}^d(X) = \mathrm{Pic}^d(X)$ is proper.

For the case of arbitrary $t > 1$, we work by induction on t, using only Part (1) of Definition 4.4 and the claim of Lemma 7.2 (neither of which relies on the hypothesis of smoothability).

Fix $e \in \mathrm{Edges}(\Gamma(X))$, let X_e be the partial normalization of X at the node e only. Because $t > 1$ we have that X_e is connected and therefore $\mathrm{Simp}^{d-1}(X_e) \neq \emptyset$. Let $j \colon \mathrm{Simp}^{d-1}(X_e) \to \mathrm{Simp}^d(X)$ be the morphism obtained by taking the pushforward along the partial normalization $X_e \to X$. Let $\overline{J}_e \subseteq j^{-1}\left(\overline{J}(X)\right)$ be a connected component. Then $\overline{J}_e$ is a degree $d-1$ fine compactified Jacobian of X_e, and X_e is a vine curve of type $t-1$. We apply the induction hypothesis to deduce that there exists some degree $d-1$ stability condition σ_e as described in Example 4.7 such that $\overline{J}_e = \overline{J}_{\sigma_e}(X_e)$.

Let σ be the stability condition on X of the type studied in Example 4.7 that is minimal among those that respect Condition (1) of Definition 4.4 and whose restriction to X_e equals σ_e. By Corollary 7.3 we have that $\overline{J}_e = \overline{J}_\sigma(X)$ is contained in $\overline{J}(X)$. By Corollary 6.4, $\overline{J}_\sigma$ is open in $\mathrm{Simp}^d(X)$, it is proper and geometrically connected. As $\overline{J}(X)$ also enjoys these three properties, we conclude that $\overline{J}(X) = \overline{J}_\sigma(X)$.

We now go back to our main line of reasoning, and show that the assignment associated to a *smoothable* fine compactified Jacobian also satisfies the second condition of Definition 4.4.

Proposition 7.6. *If* $\overline{J} \subseteq \mathrm{Simp}^d(X)$ *is also* smoothable*, then for all spanning subgraphs* $\Gamma_0 \subseteq \Gamma$*, the subset* $\sigma_{\overline{J}}(\Gamma_0) \subset S^d_\Gamma(\Gamma_0)$ *is a minimal complete set of representatives for the action of the twister group* $\mathrm{Tw}(\Gamma_0)$ *on* $S^d_\Gamma(\Gamma_0)$*.*

Proof. We first fix some notation for our proof. Let $\mathcal{X}/\Delta$ be a regular smoothing of X and let $\overline{\mathcal{J}}_{\mathcal{X}/\Delta}$ be an open and Δ-proper subscheme of $\mathrm{Simp}^d(\mathcal{X}/\Delta)$ whose special fiber equals $\overline{J}$. Then consider the degree 2 base change $\Delta \to \Delta$ of $\mathcal{X}/\Delta$, giving a nonregular smoothing $\mathcal{X}'/\Delta$ with A_2 singularities at all nodes of the special fiber X. By functoriality of the Picard functor, and by stability under base change of openness and properness, we have that the base change $\overline{\mathcal{J}}'_{\mathcal{X}'/\Delta}$ is also an open and Δ-proper subscheme of $\mathrm{Simp}^d(\mathcal{X}'/\Delta)$. Finally, let $f \colon \widetilde{\mathcal{X}} \to \mathcal{X}'$ be the blow up at all singularities. The family $\widetilde{\mathcal{X}}$ is then a regular smoothing of the special fiber $\widetilde{X}$, the curve obtained from X by replacing each node with an irreducible rational bridge.

We are now ready for the proof. First observe that, by the $m = 1$ case of Corollary 5.10, it is enough to prove that the collection $\widetilde{\sigma}_{\overline{J}}$ obtained by lifting every element of $\sigma_{\overline{J}}$ via Equation 5.5 is a minimal complete set of representatives for the action of $\operatorname{Tw}(\Gamma(\widetilde{X}))$.

Let $\tilde{\mathbf{d}}$ be the lift via (5.5) of some $\mathbf{d} \in S^d_\Gamma(\Gamma_0)$ for some $\Gamma_0 \subseteq \Gamma$. There exists $[L] \in \operatorname{Pic}^d(\widetilde{X})$ such that $\underline{\deg}(f_*(L)) \in S^d_\Gamma(\Gamma_0)$. By Hensel's lemma, L extends to a family $[\mathcal{L}] \in \operatorname{Pic}^d(\widetilde{\mathcal{X}}/\Delta)$. Since $\overline{J}_{\mathcal{X}'/\Delta}$ is universally closed over Δ, there exists $\mathcal{F}' \in \operatorname{Simp}^d(\mathcal{X}'/\Delta)$ such that $\mathcal{F}'|_{\mathcal{X}'_\eta} = f_*\mathcal{L}_{\mathcal{X}'_\eta}$ and $[\mathcal{F}'|_X] \in \overline{J}_{X'}$. We take $[\mathcal{L}'] \in \operatorname{Pic}^d(\widetilde{\mathcal{X}}/\Delta)$ such that $f_*(\mathcal{L}') = \mathcal{F}'$. Thus we have that $(\mathcal{L}' \otimes \mathcal{L}^{-1})_{\widetilde{\mathcal{X}}_\eta} = \mathcal{O}_{\mathcal{X}_\eta}$. We let then $\tilde{\mathbf{t}} = \underline{\deg}(\mathcal{L}' \otimes \mathcal{L}^{-1}|_{\widetilde{X}})$, and so $\tilde{\mathbf{d}} + \tilde{\mathbf{t}} = \underline{\deg}(\mathcal{L}'|_{\widetilde{X}})$ is in $\widetilde{\sigma}_{\overline{J}}$. This proves that $\widetilde{\sigma}_{\overline{J}}$ is a complete set of representatives.

To prove minimality, assume that there exist lifts $\tilde{\mathbf{d}}_\mathbf{1}, \tilde{\mathbf{d}}_\mathbf{2} \in \widetilde{\sigma}_{\overline{J}}$ such that $\tilde{\mathbf{d}}_\mathbf{2} = \tilde{\mathbf{d}}_\mathbf{1} + \tilde{\mathbf{t}}$ for some $\tilde{\mathbf{t}} \in \operatorname{Tw}(\Gamma(\widetilde{X}))$. By Hensel's lemma and by Lemma 7.2, there exist $[\mathcal{L}_1], [\mathcal{L}_2] \in \operatorname{Pic}^d(\widetilde{\mathcal{X}}/\Delta)$, coinciding on the generic fiber, and whose multidegrees on $\widetilde{X}$ equal $\tilde{\mathbf{d}}_\mathbf{1}, \tilde{\mathbf{d}}_\mathbf{2}$. The pushforwards $f_*(\mathcal{L}_1)$ and $f_*(\mathcal{L}_2)$ coincide on $\mathcal{X}'_\eta$, and because $\overline{J}_{\mathcal{X}'/\Delta}$ is separated over Δ, their central fibers must coincide, which implies that $\tilde{\mathbf{t}} = 0$. This proves minimality. □

Remark 7.7. In [Cap12] Caporaso introduced the notion of "being of Néron type" for a degree d compactified Jacobian of a nodal curve X. In our notation, this can be restated as the property that $\sigma_{\overline{J}}(\Gamma(X))$ is a minimal complete set of representatives for the action of $\operatorname{Tw}(\Gamma(X))$ on $S^d_{\Gamma(X)}(\Gamma(X))$.

Proposition 7.6 proves, in particular, that all smoothable fine compactified Jacobians are of Néron type. Similar results were obtained in [Kas13] and [MV12b] for fine compactified Jacobians obtained from some numerical polarizations (see Section 8 for the notion of a numerical polarization).

We refer the reader to [Kas13] for the definition of a Néron model and its relations with compactified Jacobians.

Remark 7.8. The smoothability assumption in the statement of Proposition 7.6 is crucial. In [PT22] the authors give an example, for X a nodal curve of genus 1, of degree 0 fine compactified Jacobians $\overline{J} \subset \operatorname{Simp}^0(X)$ whose collection of line bundle multidegrees contain an arbitrary number $r \geq 2$ of elements for each orbit of the action of $\operatorname{Tw}(\Gamma)$ on $S^0_\Gamma(\Gamma)$. Such compactified Jacobians can always be extended to a universally closed family over a regular smoothing, but such extensions are *separated* if and only if r equals 1.

We conclude with the main result:

Corollary 7.9. *The associated assignment (Definition 7.1) to a smoothable degree d fine compactified Jacobian of X is a degree d stability condition for the dual graph $\Gamma(X)$ (as in Definition 4.4).*

Proof. This is obtained as a combination of Corollary 7.3 and Proposition 7.6. □

We observe that the same result holds for families.

Proposition 7.10. *Let $\mathcal{X}/S$ be a family of nodal curves, and let $\overline{\mathcal{J}} = \overline{\mathcal{J}}_{\mathcal{X}/S}$ be a degree d fine compactified Jacobian for the family. Assume that for each geometric point s of S the fine compactified Jacobian $\overline{J}_s$ of the fiber X_s over $s \in S$ is* smoothable*.*

Then the collection $\left\{ \sigma_{\overline{J}_s} \right\}$ of the associated assignments of $\overline{J}_s$ for all geometric points s of S is a family of degree d stability conditions (as in Definition 4.10).

Proof. The case when S is a single geometric point is Corollary 7.9.

To complete our proof we will show that the assignment of $\sigma_{\overline{J}_s}$ is compatible with all morphisms $f\colon \Gamma(\mathcal{X}_s) \to \Gamma(\mathcal{X}_t)$ arising from étale specializations $t \rightsquigarrow s$.

Assume that F_t is a simple sheaf on the nodal curve $\mathcal{X}_t$ that specializes to F_s on $\mathcal{X}_s$. Suppose that the subcurve $\mathcal{X}_{t,0} \subseteq \mathcal{X}_t$ generalizes the subcurve $\mathcal{X}_{s,0} \subseteq \mathcal{X}_s$. By flatness and by continuity of the Euler characteristic, and because we are passing to the torsion-free quotients, the degrees are related by

$$\deg_{\mathcal{X}_{t,0}}(F_t) = \deg_{\mathcal{X}_{s,0}}(F_s) + n(F_s, f) \tag{7.11}$$

where $n(F_s, f)$ is the number of nodes of $\mathcal{X}_{s,0}$ that are smoothened in t and where F_s fails to be locally free. (See Equation (2.1)). Formula (7.11) is precisely the condition of Equation (4.9). □

Remark 7.12. Assume that the dual graph of the fibers of $\mathcal{X}/S$ is constant in S, and let S be irreducible with $\eta \in S$ its generic point. Then the family $\mathcal{X}/S$ induces a group homomorphism from the group of étale specializations of η to itself, which equals the Galois group $\mathrm{Gal}(k(\eta)^{\mathrm{sep}}, k(\eta)))$, to the automorphism group $\mathrm{Aut}(\Gamma(X_{k(\eta)^{\mathrm{sep}}}))$ of the dual graph of the generic fiber. Let G be the image of this group homomorphism. In this case, Proposition 7.10 implies that the associated assignment $\sigma_{\overline{J}_{k(\eta)}}$, which is defined as a collection of discrete data on the dual graph $\Gamma(X_{k(\eta)^{\mathrm{sep}}}) = \Gamma(X_{\overline{k(\eta)}})$, is invariant under the action of G.

Note that, in the particular case where S is a stratum of $\overline{\mathcal{M}}_{g,n}$ of curves whose dual graph is equal to a fixed graph Γ, we have $G = \mathrm{Aut}(\Gamma)$.

By combining Theorem 6.3/Corollary 6.4 with Proposition 7.10 and Lemma 7.2, we immediately obtain that the two operations of (1) taking the associated assignment to a fine compactified Jacobian, and (2) constructing the moduli space of stable sheaves associated with a given stability condition, are inverses of each other.

Corollary 7.13. *Let $\mathcal{X}/S$ be a family of nodal curves over an irreducible S, and assume that $\mathcal{X}_\theta/\theta$ is smooth for θ the generic point of S.*

If $\overline{\mathcal{J}} \subseteq \mathrm{Simp}^d(\mathcal{X}/S)$ is a degree d fine compactified Jacobian and $\sigma_{\overline{\mathcal{J}}}$ is its associated assignment (Definition 7.1), then the moduli space $\overline{\mathcal{J}}_{\sigma_{\overline{\mathcal{J}}}}$ of $\sigma_{\overline{\mathcal{J}}}$-stable sheaves (Definitions 6.1 and 6.2) equals $\overline{\mathcal{J}}$.

Conversely, if τ is a family of degree d stability conditions, and $\overline{\mathcal{J}}_\tau$ is the moduli space of τ-stable sheaves, then the associated assignment $\sigma_{\overline{\mathcal{J}}_\tau}$ equals τ.

8. Numerical polarizations

An example of a (smoothable fine compactified Jacobian) stability condition on a nodal curve X/K (as in Definition 4.4) comes from numerical polarizations, introduced by Oda–Seshadri in [OS79]. (In fact, the Oda–Seshadri formalism permits to also construct compactified Jacobians that are not necessarily *fine* in the sense of Definition 3.1).

In this section we review the notion of numerical stability for a single curve (Definition 8.1) and extend it to families (Definition 8.14) following [KP19].

We let Γ be the dual graph of X. Recall that a subgraph $\Gamma' \subseteq \Gamma$ is **induced** when for all pairs of vertices v, w it contains, it also contains all edges of Γ having v and w as endpoints.

Definition 8.1. Let $V^d(\Gamma) \subset \mathbb{R}^{\mathrm{Vert}(\Gamma)}$ be the sum-d affine subspace. Let $\phi \in V^d(\Gamma)$, and let $\Gamma_0 \subseteq \Gamma$ be a (not necessarily connected) spanning subgraph with edge set $E_0 = \mathrm{Edges}(\Gamma_0)$. We say that $\underline{\mathbf{d}} \in S^d_\Gamma(\Gamma_0)$ is **ϕ-semistable (resp. ϕ-stable)** on Γ_0 when the inequality

$$(8.2) \qquad \left| \sum_{v \in \mathrm{Vert}(\Gamma')} (\underline{\mathbf{d}}(v) - \phi(v)) + |\mathrm{Edges}(\Gamma') \setminus E_0| + \frac{e_{\mathrm{Edges}(\Gamma)\setminus E_0}(\Gamma')}{2} \right| \leq \frac{e_{E_0}(\Gamma')}{2}$$

(resp. $<$) is satisfied for all induced subgraphs $\emptyset \subsetneq \Gamma' \subsetneq \Gamma$. Here by $e_E(\Gamma')$ for some $E \subseteq \mathrm{Edges}(\Gamma)$ we denote the cardinality of the set of edges in E that do not belong either to Γ' or to the edge set of the induced subgraph on $\mathrm{Vert}(\Gamma) \setminus \mathrm{Vert}(\Gamma')$.

For $\phi \in V^d(\Gamma)$ we define the (numerical, Oda–Seshadri) ϕ-stability condition

$$(8.3) \qquad \sigma_{\Gamma,\phi}(\Gamma_0) := \left\{ \underline{\mathbf{d}} \in S^d_\Gamma(\Gamma_0) : \underline{\mathbf{d}} \text{ is } \phi\text{-semistable on } \Gamma_0 \right\} \subset S^d_\Gamma(\Gamma_0).$$

We define $\phi \in V^d(\Gamma)$ to be **nondegenerate** when for every spanning subgraph $\Gamma_0 \subseteq \Gamma$, all elements of $\sigma_{\Gamma,\phi}(\Gamma_0)$ are ϕ-stable.

Elements of $V^d(\Gamma)$ are called **numerical polarizations**.

Remark 8.4. If the spanning subgraph $\Gamma_0 \subseteq \Gamma$ is not connected, then $\mathrm{Edges}(\Gamma) \setminus \mathrm{Edges}(\Gamma_0)$ is a collection of edges that disconnects Γ. If we take for Γ' the induced subgraph of Γ on a connected component of Γ_0, the RHS of Inequality (8.2) equals zero. Therefore if ϕ is nondegenerate and the spanning subgraph Γ_0 is not connected, we always have $\sigma_{\Gamma,\phi}(\Gamma_0) = \emptyset$.

We conclude that, for nondegenerate ϕ's, the disconnected spanning subgraphs Γ_0 of Γ do not carry any additional information and they can be disregarded, as we have done in Definition 4.4.

Every nondegenerate numerical polarization gives a stability condition:

Proposition 8.5. *Let X be a nodal curve and let $\phi \in V^d(\Gamma(X))$ be nondegenerate. Then the ϕ-stability condition $\sigma_{\Gamma,\phi}$ defined in 8.1 is a degree d stability condition (as in 4.4) and the moduli space $\overline{J}_{\sigma_{\Gamma,\phi}}(X)$ of sheaves that are stable with respect to $\sigma_{\Gamma,\phi}$ (as defined in 6.1) is a degree d smoothable fine compactified Jacobian.*

Proof. The fact that the moduli space is a smoothable fine compactified Jacobian is [PT22, Proposition 2.9]. Thus $\sigma_{\Gamma,\phi}$ defines a stability condition by Corollary 7.9. □

As far as we know, the following question is open.

Question 8.6. Let σ_Γ be a degree d stability condition. Does there exist a $\phi \in V^d(\Gamma)$ such that $\sigma_\Gamma = \sigma_{\Gamma,\phi}$? (If one such ϕ exists, it is necessarily nondegenerate).

Below we give three classes of examples where we can answer Question 8.6 in the positive.

Example 8.7. (Irreducible curves). If X is irreducible, then there is a unique stability condition σ (see Example 4.6) and we have $\sigma = \sigma_\phi$ for ϕ the only element of $V^d(X) = \{d\}$.

Example 8.8. (Vine curves of type t). Let Γ consist of 2 vertices v_1, v_2 connected by t edges (and no loops). The complexity of Γ equals t. With the notation as in Example 4.7, it is straightforward to check that a stability condition σ_Γ determined by some $\lambda \in \mathbf{Z}$ as described in loc.cit. equals $\sigma_{\Gamma,\phi_\Gamma}$ for

$$\phi_\Gamma(v_1, v_2) := \left(\lambda - \frac{t-1}{2}, d - \lambda + \frac{t-1}{2}\right),$$

and that $\phi_\Gamma \in V^d(\Gamma)$ is nondegenerate.

Example 8.9. (Curves whose dual graph has genus 1). Let X be such that $b_1(\Gamma(X)) = 1$. A degree d stability condition σ on $\Gamma(X)$ is the same datum as a stability condition on Γ', where Γ' is obtained from $\Gamma(X)$ by setting the genus of each vertex to 0. Let X' be a curve whose dual graph is Γ'. Then by Corollary 6.4 the scheme $\overline{J}_\sigma(X')$ is a smoothable fine compactified Jacobian. Smoothable fine compactified Jacobians on X' have been classified in [PT22]: they all arise as $\overline{J}_{\sigma_\phi}(X')$ for some $\phi \in V^d(\Gamma') = V^d(\Gamma(X))$. Thus, by Corollary 7.9 we have that σ is also of that form.

Example 8.10. (Integral Break Divisors). Let Γ be a (not necessarily stable) graph and assume $d = b_1(\Gamma)$. Define the stability condition σ_{IBD} by

$$\sigma_{\mathrm{IBD}}(\Gamma_0) = \{\text{integral break divisors for } \Gamma_0\},$$

for all connected spanning subgraphs $\Gamma_0 \subseteq \Gamma$. Recall that $\underline{\mathbf{d}} \in S_\Gamma^{b_1(\Gamma)}(\Gamma_0)$ is an integral break divisor for Γ_0 if and only if it is of the form

$$\underline{\mathbf{d}}(v) = g(v) + \sum_{e \in \mathrm{Edges}(\Gamma_0) \setminus \mathrm{Edges}(T)} \underline{\mathbf{e}}_{t(e)}(v)$$

for some choice of a spanning tree $T \subseteq \Gamma_0$ and of an orientation $t \colon \mathrm{Edges}(\Gamma_0) \setminus \mathrm{Edges}(T) \to \mathrm{Vert}(\Gamma_0)$ (for $w \in \mathrm{Vert}(\Gamma_0)$ we denote by $\underline{\mathbf{e}}_w$ the function that is 1 on the vertex w and 0 elsewhere).

We claim that σ_{IBD} is indeed a degree $d = b_1(\Gamma)$ stability condition. Condition (1) of Definition 4.4 follows directly from the definition of an integral break divisor. The second axiom follows immediately from [ABKS14, Theorem 1.3] (earlier proved in [MZ08]).

When Γ is stable, it follows from [CPS, Lemma 5.1.5] that $\sigma_{\mathrm{IBD}} = \sigma_{\Gamma, \phi^\Gamma_{\mathrm{can}}}$, for

$$\phi^\Gamma_{\mathrm{can}}(v) := \frac{g(\Gamma)}{2g(\Gamma) - 2} \cdot (2g(v) - 2 + \mathrm{val}_\Gamma(v)) \tag{8.11}$$

where $\mathrm{val}_\Gamma(v)$ is the valence of the vertex v in Γ. (This follows from loc.cit. as $\phi^\Gamma_{\mathrm{can}}$ is induced from the canonical divisor, which is ample when Γ is stable).

When Γ is not necessarily stable, we define

$$\phi_{\mathrm{IBD}}(v) := \frac{g(\Gamma) + |\,\mathrm{Vert}(\Gamma)|}{2\,(g(\Gamma) + |\,\mathrm{Vert}(\Gamma)|) - 2} \cdot (2g(v) + \mathrm{val}_\Gamma(v)) - 1. \tag{8.12}$$

We claim that $\sigma_{\mathrm{IBD}} = \sigma_{\Gamma, \phi_{\mathrm{IBD}}}$.

Let Γ' be the graph obtained from Γ by increasing the genus of each vertex by 1. Then the integral break divisors of Γ' are the integral break divisors on Γ increased by 1 on each vertex and Γ' is stable. Therefore the integral break divisors on Γ are stable for the stability condition $\phi^{\Gamma'}_{\mathrm{can}} - 1$, which equals ϕ_{IBD}. This proves our claim.

We are now ready to define families of numerical polarizations, similarly to what was done in Definition 4.10 for families of stability conditions. We shall see that there are two ways of doing so, and that they are not equivalent.

Definition 8.13. Let $f \colon \Gamma \to \Gamma'$ be a morphism of stable graphs. We say that $\phi \in V^d(\Gamma)$ is f-compatible with $\phi' \in V^d(\Gamma')$ if

$$\phi'(w) = \sum_{f(v) = w} \phi(v).$$

Definition 8.14. Let $\mathcal{X}/S$ be a family of nodal curves, and let $\Phi = (\phi_s \in V^d(\Gamma(\mathcal{X}_s)))_{s\in S}$ be a collection of numerical polarizations, one for each geometric point of S. Assume also that Φ is **nondegenerate**, by which we mean that so is each of its coordinates ϕ_s for $s \in S$ (as per Definition 8.1).

The collection Φ is **weakly compatible** if the collection $\sigma_\Phi := \big(\sigma_{\Gamma(X_s),\Phi(\Gamma(X_s))}\big)_{s\in S}$ of degree d stability conditions defined via Proposition 8.5 is a family of degree d stability conditions as prescribed by Definition 4.10.

The collection Φ is **strongly compatible** if it is f-compatible for all morphisms $f\colon \Gamma(\mathcal{X}_s) \to \Gamma(\mathcal{X}_t)$ that arise from some étale specialization $t \rightsquigarrow s$ occurring on S.

For a strongly compatible collection, one can define the notion of a Φ-stable sheaf for all fibers as in Definition 8.1. We shall not do that, as this would be a repetition of our Definition 6.2. Instead, we observe the following:

Corollary 8.15. *Let $\mathcal{X}/S$ be a family of nodal curves over an irreducible scheme S with generic point θ, and assume that the generic element $\mathcal{X}_\theta/\theta$ is smooth. If Φ is a nondegenerate and weakly compatible family of numerical polarizations, the moduli space $\overline{\mathcal{J}}_{\sigma_\Phi}$ of sheaves that are stable with respect to σ_Φ is a family of degree d fine compactified Jacobians.*

Proof. By combining Proposition 8.5 with the fact that Φ is nondegenerate and weakly compatible, we deduce that σ_Φ is a family of stability conditions for $\mathcal{X}/S$. The result follows then from Theorem 6.3. □

In [KP19] the authors studied the theory of universal Oda–Seshadri stability conditions for compactified Jacobians. In loc.cit. they defined, for fixed (g, n) such that $2g - 2 + n > 0$, the space $V^d_{g,n}$ of **universal numerical polarizations**, i.e. elements $\Phi = (\phi_s)_{s\in\overline{\mathcal{M}}_{g,n}}$ that are strongly compatible for morphisms $f\colon \Gamma_1 \to \Gamma_2$ between any two elements of $G_{g,n}$ (a skeleton of the category of stable n-pointed graphs of genus g).

In loc.cit. the authors also constructed, for each such nondegenerate $\Phi \in V^d_{g,n}$, a compactified Jacobian $\overline{\mathcal{J}}_{g,n}(\Phi)$. In the language of this paper, this can be rephrased as follows.

Corollary 8.16. *If $\Phi \in V^d_{g,n}$ is a nondegenerate universal numerical stability condition, then $\overline{\mathcal{J}}_{g,n}(\Phi) \subset \mathrm{Simp}^d(\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n})$ is a fine compactified universal Jacobian.*

Remark 8.17. The space $V^d_{g,n}$ is never empty, as it always contains the *universal canonical polarization*, defined as $\Phi^d_{\mathrm{can}} := \frac{d}{2g-2}\, \underline{\deg}(\omega_\pi) \in V^d_{g,n}$. (When $d = g$ this coincides with the polarization defined in Equation (8.11)). It was observed in [KP19, Remark 5.13] that Φ^d_{can} is nondegenerate precisely when $d - g + 1$ and $2g - 2$ are coprime.

A natural question that we will address next is whether every fine compactified universal Jacobian arises as in Corollary 8.16, or if one can construct fine compactified universal Jacobian from nondegenerate weakly compatible numerical stability conditions that are not strongly compatible.

Remark 8.18. It is clear that a strongly compatible nondegenerate family of numerical polarizations is also weakly compatible. The converse is not true. Even more: on some families $\mathcal{X}/S$ there exist nondegenerate families of numerical polarizations $\Phi = (\phi_s \in V(\Gamma_s))$ that are weakly compatible, and such that there exists no nondegenerate $\Phi' = (\phi'_s)_{s\in S}$ that is strongly compatible and satisfies $\sigma_{\Gamma_s,\phi'_s} = \sigma_{\Gamma_s,\phi_s}$ for all $s \in S$.

Such examples are shown to exist in [PT22, Section 6], when $\mathcal{X}/S$ is the universal family $\overline{\mathcal{C}}_{1,n}/\overline{\mathcal{M}}_{1,n}$ and $n \geq 6$. In other words, for all $n \geq 6$ there are fine compactified universal Jacobians $\overline{\mathcal{J}}_{1,n}$ that are not of the form $\overline{\mathcal{J}}_{1,n}(\Phi)$ for any $\Phi \in V^d_{1,n}$.

Our main result in the next section, Theorem 9.7, settles in the affirmative the analogous question for the case of the universal family $\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g$ for all $g \geq 2$. More explicitly, when $n = 0$, every fine compactified universal Jacobian $\overline{\mathcal{J}}_g$ arises as in Corollary 8.16, i.e. it is of the form $\overline{\mathcal{J}}_g(\Phi)$ for some $\Phi \in V^d_g$.

9. Classification of universal stability conditions

The main result in this section is a classification of *universal* stability conditions, that is, the case when the family $\mathcal{X}/S$ is the universal family over $S = \overline{\mathcal{M}}_g$, the moduli stack of stable curves of genus g (and no marked points). As an intermediate step, we will also produce results that are valid for the case of the universal family over $S = \overline{\mathcal{M}}_{g,n}$ for arbitrary n.

One way to generate universal stability conditions is by means of strongly compatible (universal) numerical polarizations, see Definition 8.14 and Corollary 8.16. Our main result here is Theorem 9.7, where we classify all universal stability conditions with $n = 0$ and for every genus, by proving that they all arise from strongly compatible numerical polarizations, i.e. they all are of the form σ_Φ for some $\Phi \in V^d_g$ (see Corollary 8.15 and Corollary 8.16). Combining with the fact, shown in Proposition 7.10, that universal stability conditions classify fine compactified universal Jacobians, we deduce in Corollary 9.10 that, in the absence of marked points, there are no fine compactified universal Jacobians other than the classical ones constructed in the nineties by Caporaso, Pandharipande and Simpson.

As discussed in Remark 8.18, an analogous result does not hold, in general, when $n > 0$.

Recall from Example 3.5 that, for $t \in \mathbb{N}$, a **vine curve of type** t is a curve with 2 nonsingular irreducible components joined by t nodes. These curves and their dual graphs will play an important role in this section.

We will break down our argument in 3 parts.

9.1. **First part: universal stability conditions are uniquely determined by their restrictions to vine curves.** The main result of this section, summarized in the title, is a combination of Lemma 9.1 and Lemma 9.2 below. The first one establishes that a stability condition for a given curve is uniquely determined by its restriction to the spanning trees of the dual graph of that curve. In the second one we prove that an assignment of integers on all spanning trees of all elements of $G_{g,n}$ that is compatible with graph morphisms is uniquely determined by its values over the dual graphs of all vine curves.

Let us fix a degree d universal stability condition σ of type (g, n). Recall from Definition 4.10 (and Remark 4.11) that this is the datum of a collection

$$\sigma = \{ \sigma_\Gamma \}_{\Gamma \in G_{g,n}} .$$

that is compatible with all graph morphisms in $G_{g,n}$.

By Condition (1) of Definition 4.4, if $T \subseteq \Gamma$ is a spanning *tree*, then $\sigma_\Gamma(T) = \{ \underline{\mathbf{d}}_T \}$ contains a *unique* multidegree $\underline{\mathbf{d}}_T \in S^d_\Gamma(T)$. The following lemma shows that a stability condition is "overdetermined" by the collection of data obtained by extracting this unique value from all spanning trees.

Lemma 9.1. *Let Γ be a stable graph and let us fix, for all spanning trees $T \subseteq \Gamma$, a multidegree $\underline{\mathbf{d}}_T \in S^d_\Gamma(T)$. Then there exists* at most one *degree d stability condition σ_Γ whose value $\sigma_\Gamma(T)$ at each spanning tree $T \subseteq \Gamma$ equals $\{ \underline{\mathbf{d}}_T \}$.*

Proof. Let σ_Γ be a degree d stability condition that satisfies the hypothesis. Inductively define another collection M_Γ by defining $M_\Gamma(\Gamma_0) \subset S^d_\Gamma(\Gamma_0)$ for all spanning subgraphs Γ_0 of Γ. Starting from the assignments $M_\Gamma(T) := \{ d_T \}$ for all spanning trees $T \subseteq \Gamma_0$, let M_Γ be the minimal assignment that satisfies Condition (2) of Definition 4.4.

Because σ_Γ satisfies Condition (2) of Definition 4.4, for all spanning subgraphs $\Gamma_0 \subseteq \Gamma$ we have the inclusion $M_\Gamma(\Gamma_0) \subseteq \sigma_\Gamma(\Gamma_0)$. The inequality $|M_\Gamma(\Gamma_0)| \geq c(\Gamma_0)$ was proved by Barmak [Bar17] (we refer to Yuen's Phd thesis, [Yue18, Theorem 3.5.1], for a publicly available proof. Barmak's proof was originally shared with Yuen, who then filled in more details and published it with Barmak's permission.) By Condition (1) of Definition 4.4 we also have the equality $|\sigma_\Gamma(\Gamma_0)| = c(\Gamma_0)$. From this we conclude that $M_\Gamma(\Gamma_0)$ and $\sigma_\Gamma(\Gamma_0)$ must coincide. □

By applying the same idea as in [KP19, Lemma 3.9], we now see how compatibility with graph morphisms propagates an assignment of multidegrees on all nonsymmetric strata of vine curves to an assignment on all vertices of all spanning trees of all graphs $\Gamma \in G_{g,n}$. The reason for excluding the symmetric graphs is that compatibility under the automorphism exchanging the two vertices forces a unique possible value for each universal degree d stability condition on those graphs. Let $T_{g,n} \subseteq G_{g,n}$ be the collection of loopless graphs with 2 vertices, and let $T'_{g,n} \subseteq T_{g,n}$ be the subset of graphs such that each automorphism fixes the two vertices. For each element $G \in T_{g,n}$, fix an ordering of its two vertices, i.e. $\mathrm{Vert}(G) = \{v_1^G, v_2^G\}$.

Lemma 9.2. *For each function* $\alpha\colon T'_{g,n} \to \mathbb{Z}$*, there exists a unique collection of assignments* $\underline{\mathbf{d}}_{\Gamma,T} \in S^d_\Gamma(T)$ *for each spanning tree* $T \subseteq \Gamma$ *of each element* $\Gamma \in G_{g,n}$ *such that we have*

$$\sum_{v \in \operatorname{Vert}(\Gamma) : f(v) = v_1^G} \underline{\mathbf{d}}_{\Gamma,T}(v) = \alpha(G) \tag{9.3}$$

for all morphisms $f\colon \Gamma \to G$ *where* G *is in* $T'_{g,n}$*.*

Note that Equation (9.3) is the same as compatibility for graph morphisms defined in Definition 4.8, Equation (4.9).

Proof. This is an extension of the proof given in [KP19, Lemma 3.8]. The main point of the proof is that for a fixed spanning tree $T \subseteq \Gamma$, contracting all but one edge of T induces a bijection from $\operatorname{Edges}(T)$ to the set of morphisms from Γ to some graph G isomorphic to an element of $T_{g,n}$. The isomorphism is unique when $G \in T'_{g,n}$, and there are two isomorphisms when $G \in T_{g,n} \setminus T'_{g,n}$.

Each $e \in \operatorname{Edges}(T)$ induces a morphism $\Gamma \to \Gamma_e$ obtained by contracting all edges $\operatorname{Edges}(T) \setminus \{e\}$. There are two cases, depending on how each automorphism of Γ_e acts on $\operatorname{Vert}(\Gamma_e)$: (1) If each acts trivially then there exists a unique isomorphism from Γ_e to an element of $T'_{g,n}$. (2) if some act nontrivially, there are two isomorphisms from Γ_e to an element of $T_{g,n} \setminus T'_{g,n}$.

Now we view $\underline{\mathbf{d}}_{\Gamma,T}$ as a vector with $|\operatorname{Vert} T| = |\operatorname{Vert} \Gamma|$ unknown entries. In Case (1) we obtain an affine linear constraint among these unknowns by Equation (9.3). In Case (2) a linear constraint is given by compatibility for the extra automorphism, which imposes that the sum of the values of $\underline{\mathbf{d}}_{\Gamma,T}$ on the vertices on one side of e equals the sum of the values on the vertices on the other side of e. Altogether, this gives $|\operatorname{Edges}(T)| = |\operatorname{Vert}(\Gamma)| - 1$ affine linear constraints on the $|\operatorname{Vert}(\Gamma)|$ different entries of $\underline{\mathbf{d}}_{\Gamma,T}$. One more affine linear constraint is given by the fact that $\underline{\mathbf{d}}_{\Gamma,T} \in S^d_\Gamma(T) \subset \mathbb{Z}^{\operatorname{Vert}(\Gamma)}$. All in all, we obtain an affine linear system of $|\operatorname{Vert}(T)|$ equations in $|\operatorname{Vert}(T)|$ unknowns, which one can check has invertible determinant over $\mathbb{Z}$, hence a unique solution for $\underline{\mathbf{d}}_{\Gamma,T}$. □

Lemmas 9.1 and 9.2 allow us to regard every degree d universal stability condition of type (g,n) as an element of $\mathbb{Z}^{T'_{g,n}}$. However, not all elements of $\mathbb{Z}^{T'_{g,n}}$ give rise to a universal stability condition. For $\Gamma \in G_{g,n}$ it can happen that the collection on all spanning trees $\{\underline{\mathbf{d}}_{\Gamma,T}\}_T$ obtained from some $\alpha \in \mathbb{Z}^{T'_{g,n}}$ as in Lemma 9.2 is not the restriction of any stability condition on the graph Γ (see Lemma 9.1).

9.2. **Second part: when $n = 0$ over each vine curve with at least 2 nodes there is at most 1 stability condition that extends to a universal stability.** In this section we fix $g \geq 2$ and $n = 0$. The main result here is Corollary 9.5.

We first need some combinatorial preparation. Let GSym_g be the trivalent graph with $2g-2$ vertices $v_1, \ldots, v_{2g-2}$ of genus 0, where each vertex v_i is connected to v_{i-1}, v_{i+1} and v_{i+g-1} (indices should be considered modulo $2g-2$). For $i \in$

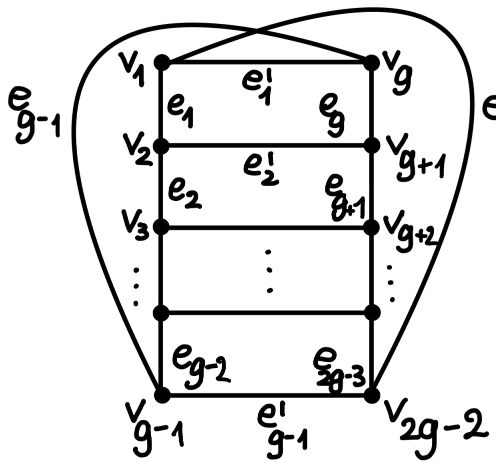

FIGURE 1. GSym_g

$\mathbf{Z}/(2g-2)\mathbf{Z}$ and $j = 1, \dots, g-1$, we shall denote by e_i the edge joining v_i and v_{i+1} and by e'_j the edge joining v_j and v_{j+g-1}. (See Picture 1). Let $\Gamma_g \subset \mathrm{GSym}_g$ be the maximal 1-cycle consisting of all edges of the form e_i.

Lemma 9.4. *For every $d \in \mathbf{Z}$ there is at most one assignment*

$$\sigma_{\mathrm{GSym}_g}(\Gamma_g) \subset S^d_{\mathrm{GSym}_g}(\Gamma_g)$$

that satisfies the two conditions of Definition 4.4, and that extends to a degree d universal stability condition of type $(g, 0)$. When such an assignment exists, the integers $d - g + 1$ and $2g - 2$ are necessarily coprime.

Proof. Assume that σ_{GSym_g} is the restriction to GSym_g of some universal stability σ. Since Γ_g is a graph of genus 1, we can apply the results from [PT22, Section 3] to describe the assignment $\sigma_{\mathrm{GSym}_g}(\Gamma_g)$. Namely, in genus 1 it is known that all such assignments are induced by a polarization ϕ whose value $\phi(v_i)$ at each vertex v_i is given by the average of the $2g-2$ admissible multidegrees in $\sigma_{\mathrm{GSym}_g}(\Gamma_g)$. In particular, the polarization ϕ should be invariant under the automorphisms of GSym_g preserving Γ_g, such as the cyclic automorphism mapping e_i to e_{i+1} for all i. From this we deduce that all $\phi(v_i)$ are equal, and since we have obtained Γ_g from GSym_g by removing the $g-1$ edges e'_i, we have $\sum_{i \in \mathbf{Z}/(2g-2)\mathbf{Z}} \phi_i = d - g + 1$. It follows that there exists at most 1 assignment, and that this assignment should be the one induced by $\phi = (\frac{d-g+1}{2g-2}, \dots, \frac{d-g+1}{2g-2})$. Then the claim follows from the fact that this polarization is nondegenerate if and only if $d - g + 1$ and $2g - 2$ are coprime. $\square$

We can now prove the following.

Corollary 9.5. *If $G \in G_g$ is a loopless graph with 2 vertices and at least 2 edges, there is at most one degree d stability condition for G that extends to a degree d universal stability condition.*

In the proof we will denote by $\Gamma(t, i, j)$ the object of G_g that consists of two vertices of genus i and j connected by t edges. (Note that $g = i + j + t - 1$).

Proof. The claim is obtained by proving that the value of an assignment on vine curves of type $t \geq 2$ can be obtained, using compatibility with graph morphisms (Definition 4.8), from the assignment calculated on $\Gamma_g \subset \mathrm{GSym}_g$ (described in Lemma 9.4). Recall that a degree d stability condition over a graph with 2 vertices has the same value on all spanning trees and is uniquely determined by this value (Example 4.7).

First we prove the claim when $G = \Gamma(t, i, 0)$ has one vertex of genus zero. Apply Lemma 9.4 to deduce the uniqueness of an assignment $\sigma_{\mathrm{GSym}_g}(\Gamma_g)$ that satisfies the two conditions of Definition 4.4 and automorphism-invariance. Then apply to Γ_g the first composition of contractions defined in [KP19, Lemma 3.9].

The statement for an arbitrary graph $G = \Gamma(t, i, j)$ is obtained by applying the second set of contractions used in the proof of [KP19, Lemma 3.9], to deduce the value of σ on $\Gamma(t, i, j)$ from the value of σ on $G = \Gamma(i - j + 2, t + 2j - 2, 0)$. More precisely, consider the graph G' with 4 vertices w_1, w_2, w_3, w_4 defined in the proof of [KP19, Lemma 3.9], and its spanning subgraph $G'_0 := w_1 - w_2 - w_4 - w_3 - w_1$. Let $\alpha \in \mathbf{Z}$ be defined by the condition

$$\sigma_{\Gamma(i-j+2,t+2j-2,0)}(G_0) = \{ (d - i + j - 1 - \alpha, \alpha) \}$$

for G_0 a spanning tree of $\Gamma(i - j + 2, t + 2j - 2, 0)$. By combining the axioms of a degree d stability with compatibility with graph morphisms (Definition 4.8), we find out

$$\sigma_{G'}(G'_0) = \{ (\beta + 1, \beta, \alpha, \alpha), (\beta, \beta + 1, \alpha, \alpha), (\beta, \beta, \alpha + 1, \alpha), (\beta, \beta, \alpha, \alpha + 1) \}$$

for the unique $\beta \in \mathbf{Z}$ such that $2\alpha+2\beta = d+1-t-i+j$ (observe that $d+1-t-i+j$ must be even since so is $d-g+1$, because by Lemma 9.4 it is coprime with $2g-2$, and by using $g = t+i+j-1$). From Condition (1) of Definition 4.4 we deduce that the unique assignment on the spanning tree $G''_0 \subset G'_0$ defined by $w_2 - w_1 - w_3 - w_4$ equals

$$\sigma_{G'}(G''_0) = \{ (\beta, \beta, \alpha, \alpha) \},$$

and applying the second set of contractions used in the proof of [KP19, Lemma 3.9] we deduce the value of σ on the graph $G = \Gamma(t, i, j)$. □

Remark 9.6. A degree d universal stability condition of type $(g, 0)$ always exists when d satisfies $\gcd(d-g+1, 2g-2) = 1$. Indeed, take $\sigma_{\Phi^d_{\mathrm{can}}}$ (as defined in Definition 8.14) for Φ^d_{can} the (universal) canonical polarization defined in Remark 8.17. As observed in loc.cit. Φ^d_{can} is nondegenerate if and only if $\gcd(d-g+1, 2g-2) = 1$.

9.3. **Third part: Conclusion.** We are now in a position to conclude. In this section, we fix $g \geq 2$.

Theorem 9.7. *Let τ be a degree d universal stability condition of type $(g, 0)$ (Definition 4.10 and Remark 4.11). Then $\gcd(d-g+1, 2g-2) = 1$ and there exists a nondegenerate $\Phi \in V_g^d$ such that τ equals σ_Φ (as in Definition 8.14). Moreover, if C is a stable curve without separating nodes, then $\Phi_{[C]} = \Phi^d_{\mathrm{can},[C]}$ is the canonical polarization defined in Remark 9.6.*

Proof. By combining Lemma 9.1 and Lemma 9.2 we deduce that proving the equality

$$\tau = \sigma_\Phi \tag{9.8}$$

for some nondegenerate $\Phi \in V_g^d$ is equivalent to proving the equality of the restrictions

$$\tau_G = \sigma_{G,\Phi(G)} \quad \text{for all loopless graphs } G \in G_g \text{ with 2 vertices.} \tag{9.9}$$

By Lemma 9.4 if $\gcd(d-g+1, 2g-2) \neq 1$ there exists no universal stability condition. From now on we will assume $\gcd(d-g+1, 2g-2) = 1$. Under this assumption, by [KP19, Corollary 3.6] there exists a nondegenerate $\Phi \in V_g^d$ (a modification over curves with separating edges of the canonical stability Φ^d_{can} introduced in Remark 8.17 and discussed in Remark 9.6) such that $\sigma_{G,\phi(G)}$ coincides with the restriction τ_G of τ to all loopless graphs G with 2 vertices and 1 edge. By Corollary 8.16, the assignment σ_Φ defines a degree d universal stability condition, which coincides with τ_G for all loopless graphs G with 2 vertices (by construction when G has 1 edge, and by Corollary 9.5 when G has at least 2 edges). We conclude that (9.9) holds, hence that $\tau = \sigma_\Phi$.

By construction and by compatibility for graph morphisms, we have that Φ and Φ^d_{can} coincide on graphs with no separating edges, equivalently on curves without separating nodes. □

By combining Theorem 9.7 and Proposition 7.10 (with $\mathcal{X}/S$ the universal family over $S = \overline{\mathcal{M}}_g$), we immediately deduce the following classification result.

Corollary 9.10. *If $\overline{\mathcal{J}}_g$ is a degree d fine compactified universal Jacobian for $\overline{\mathcal{M}}_g$, then d satisfies $\gcd(d-g+1, 2g-2) = 1$, and there exists $\Phi \in V_g^d$ such that $\overline{\mathcal{J}}_g = \overline{\mathcal{J}}_g(\Phi)$.*

We conclude by relating this result to compactified universal Jacobians constructed in the nineties by Caporaso [Cap94], Pandharipande [Pan96] and Simpson [Sim94]. By earlier work, this amounts to relating $\overline{\mathcal{J}}(\Phi)$, for any $\Phi \in V_g^d$, to the canonical fine compactified universal Jacobian $\overline{\mathcal{J}}(\Phi^d_{\mathrm{can}})$.

Remark 9.11. Let $\Phi \in V_g^d$ and assume $\gcd(d-g+1, 2g-2) = 1$. Then we claim that that there exists a line bundle $M \in \mathrm{Pic}^0(\overline{\mathcal{C}}_g)$ such that $\Phi = \Phi^d_{\mathrm{can}} + \underline{\deg}(M)$.

In particular, translation by M induces an isomorphism $\overline{\mathcal{J}}_g(\Phi^d_{\mathrm{can}}) \to \overline{\mathcal{J}}_g(\Phi)$ that commutes with the forgetful maps to $\overline{\mathcal{M}}_g$.

The claim follows from [KP19, Section 6]. In loc.cit. the authors study the natural translation action of $\mathrm{Pic}^0(\overline{\mathcal{C}}_{g,n})$ on $V^d_{g,n}$. This action induces an action on the collection of connected components of the nondegenerate locus, and by [KP19, Lemma 6.15 (2)] this action is transitive when n equals zero.

The relation between $\overline{\mathcal{J}}_g(\Phi^d_{\mathrm{can}})$ and the compactified Jacobians of Caporaso [Cap94], Pandharipande [Pan96] is discussed in [KP19, Remark 5.14]. The relation with Simpson's stability with respect to an ample line bundle is established in [KP19, Corollary 4.3] (see also [KP17, Section 3]).

10. Final remarks and open questions

In Definitions 4.4 and 4.10 we introduced the notion of families of degree d stability conditions for a single curve X and for the universal family $\overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$.

For X a nodal curve with dual graph Γ, let $\Sigma^d(\Gamma)$ be the set of all degree d stability conditions. In Section 8 we defined, following Oda–Seshadri [OS79], a stability space $V^d(\Gamma)$ with a subspace of degenerate elements (a union of hyperplanes). We define $\mathcal{P}^d(\Gamma)$ to be the set whose elements are the connected components (maximal dimensional polytopes) of the nondegenerate locus in $V^d(\Gamma)$. By Corollary 7.13 and Proposition 8.5, there is an injection $\mathcal{P}^d(\Gamma) \to \Sigma^d(\Gamma)$, and it is natural to ask when this map is surjective. This is the same as Question 8.6. The case when $g(X) = 1$ (or even $b_1(\Gamma(X)) = 1$) was settled in the affirmative in [PT22], see Example 8.9.

Then let $\Sigma^d_{g,n}$ be the set of all degree d stability conditions of type (g, n). The latter can be described as the inverse limit

$$\Sigma^d_{g,n} = \varprojlim_{G \in G_{g,n}} \Sigma^d(G)$$

where $G_{g,n}$ is the category of stable n-pointed dual graphs of genus g.

Similarly, for universal (and strongly compatible) numerical polarizations (see Section 8), in [KP19] the authors defined the affine space

$$V^d_{g,n} := \varprojlim_{G \in G_{g,n}} V^d(G)$$

(see Corollary 8.16). The latter is always nonempty, as it contains the canonical stability condition.

Let $\mathcal{P}^d_{g,n}$ be the set whose elements are the connected components of the complement of the degenerate stability conditions in $V^d_{g,n}$ (see Definition 8.14). By Corollary 7.13 and Corollary 8.5, we have a natural injection $\mathcal{P}^d_{g,n} \to \Sigma^d_{g,n}$, and one can ask under which assumptions the latter is surjective. The genus 1 case was settled in [PT22]: the map $\mathcal{P}^d_{1,n} \to \Sigma^d_{1,n}$ is a bijection if and only if $n \leq 5$ (see

Remark 8.18). The case without marked points is solved by Corollary 9.10: the map $\mathcal{P}_g^d \to \Sigma_g^d$ is a bijection for all g.

Here are some natural future problems to address.

(1) What is the analogue of Theorem 1.1 for the case of smoothable compactified Jacobians (not necessarily fine)? By this we mean a smoothable open substack of the moduli space of rank 1 torsion-free sheaves (not just the simple ones) that have a proper good moduli space.
(2) Is there a natural stability space with walls, containing $V_{g,n}^d$, and a natural bijection from the set of its maximal-dimensional chambers to $\Sigma_{g,n}^d$?

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